March 20, 2018

SISSA Entrance Examination

Astroparticle Physics Curriculum

SOLVE two out of the four problems. If you hand in more than two, then state clearly which two problems you wish to be evaluated (otherwise, the two with the *lowest* score will be considered).
Answer two out of the six questions concisely (no more than one page per question).

Problem 1

A muon has a rest mass $m = 100 \text{ MeV}/c^2$ and a lifetime at rest $\tau_0 = 10^{-6}$s and the same electric charge as the electron: $q = -e = -1.6 \times 10^{-19}$ C. Throughout this exercise you can neglect the gravitational force on the muon.

1. Suppose a muon is produced in the atmosphere at height $h = 10^4$ m above Earth's surface and it is directed towards the Earth. Compute the minimum muon energy E_{min} such that the muon can reach the Earth's surface.

2. As a simplification, suppose the Earth has a constant and uniform magnetic field $\vec{B} = B_0 \hat{z}$ (with $B_0 = 10^{-4}$ T) pointing in the North direction of the Earth's rotation axis and extending only up to a distance $y_0 = 10^4$ m from the Earth's surface at equator. A muon of energy $E \geq E_{\text{min}}$ is produced at height y_0 above the equator and with velocity normally incident at the equator. The muon gets deflected by the magnetic field and reaches the Earth's surface.

   In what direction is the muon deflected? Write down the equations of motion for the muon.

   [Hint: it may be convenient to work in Cartesian coordinates with the origin on the Earth's surface at equator, $\hat{z}$ pointing north, $\hat{y}$ pointing vertically upward and $\hat{x}$ pointing west.]

3. Compute the deflection angle θ. Find the numerical value of θ for a muon of energy $E = E_{\text{min}}$ found in Part 1.

   [Hint: solve the equations of motion for the Cartesian coordinates $x(t), y(t)$, then find the trajectory $x(y)$ and the deflection distance on Earth $x(y = 0)$.]

Problem 2

Consider a $SU(2)$ gauge theory with gauge coupling g and gauge fields A^a_μ. The theory also contains a real scalar field ϕ (with components ϕ^a) transforming as the adjoint representation of $SU(2)$. The theory is described by the lagrangian

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi^a)^2 - \frac{1}{4}(F^a_{\mu\nu})^2 - V(\phi) \tag{1}$$

where the field-strength tensor is $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\nu + g\epsilon^{abc} A^b_\mu A^c_\nu$, ϵ^{abc} are the structure constants of $SU(2)$ (the totally anti-symmetric Levi-Civita tensor), and the covariant derivative acting on ϕ^a is $D_\mu \phi^a = \partial_\mu \phi^a + g\epsilon^{abc} A^b_\mu \phi^c$.

Suppose that the scalar potential $V(\phi)$ is minimized by the vacuum expectation value $\langle\phi\rangle$, which spontaneously breaks $SU(2)$.

1. Suppose

$$\langle\phi\rangle = v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad v \in \mathbb{R} \tag{2}$$

   How many gauge bosons are in the theory? How many gauge bosons acquire non-zero masses? What is the residual symmetry group after the breaking of $SU(2)$?

2. Compute the masses of all the gauge bosons of the theory.

3. Repeat Points 1. and 2. supposing that the symmetry is broken by the vacuum expectation value

$$\langle\phi\rangle = \frac{v}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v \in \mathbb{R} \tag{3}$$

Problem 3

Consider a free test particle moving in a circular orbit in Schwarzschild geometry:

$$ds^2 = -(1 - r_g/r)dt^2 + \frac{dr^2}{(1 - r_g/r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

By an appropriate rotation one can always choose the orbit to be at $\theta = \pi/2$.

1. First, derive the geodesic equation for a free particle trajectory $X^\mu(\sigma)$ moving in a curved background $g_{\mu\nu}$. Hint: Vary the point particle action:

   $$S = -\int d\sigma \sqrt{-g_{\mu\nu}(X)\frac{dX^\mu}{d\sigma}\frac{dX^\nu}{d\sigma}} \tag{2}$$

   with respect to X^μ, and parameterize it in terms of the proper time $d\tau^2 = -ds^2$.

   In the next three questions consider circular orbits in Schwarzschild.

2. Derive the two constants of motion associated to t and ϕ translations.

3. Derive relativistic Kepler's law ($\frac{d\phi}{dt}$ as a function of r and r_g). (Hint: You can start from the normalization of the 4-velocity vector and use the geodesic equation.)

4. Calculate the minimum radius of orbits.

Problem 4

The linear evolution of a density perturbations for a pressureless perfect fluid in an expanding universe without curvature and zero cosmological constant obeys

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\overline{\rho}\delta \tag{1}$$

with δ the density contrast $\delta(\mathbf{x}) = \rho(\mathbf{x})/\overline{\rho} - 1$, a the scale factor, $\overline{\rho}$ the mean density, G the Newton constant and the dot denotes time derivatives.

1. Derive the value of N for which the following relation holds for δ_1 and δ_2
$$(\delta_2\dot{\delta_1} - \delta_1\dot{\delta_2}) \propto a^{-N} \tag{2}$$
being δ_1 and δ_2 two solutions of Eq. 1.

2. Consider the Friedmann equation for the Hubble parameter $H = \dot{a}/a$:
$$H^2 = \frac{8\pi G}{3}\overline{\rho}\,. \tag{3}$$
Show that the Friedmann equation, complemented by the continuity equation for a pressureless perfect fluid in an expanding background, allows to derive a differential equation for which H satisfies Eq. 1 in place of δ.

3. Find the two independent solutions $\delta_1(t)$ and $\delta_2(t)$. Having set t_0 an arbitrary initial time at which $\delta_1 = \delta_2 = 1$, find the time t at which $\delta_2 = 100\delta_1$ (note: δ_2 grows with time).

Questions

1. Describe the relevance of Big Bang nucleosynthesis in the context of the standard cosmological model.

2. Global vs. local (gauge) symmetries in particle physics. Briefly discuss the differences, provide examples and mention results.

3. Explain briefly why in Einstein's theory of general relativity it is impossible to have monopole or dipole gravitational radiation.

4. Briefly discuss the likelihood function and its use as a statistical test.

5. What problems in cosmology does the inflationary scenario address?

6. Discuss the physical meaning of the event horizon of a black hole.