July 6, 2017

SISSA
Entrance
Examination

Astroparticle Physics Curriculum

SOLVE two out of the four problems. If you hand in more than two, then state clearly which two problems you wish to be evaluated (otherwise, the two with the *lowest* score will be considered).
Answer two out of the six questions concisely (no more than one page per question).

Problem 1

A body of rest mass M_0 moves with velocity $\vec{v}$ in the lab frame. At a given time it emits in the forward direction a light ray of energy E_0 in the rest frame of the body. This light ray gets reflected by a perfect mirror placed orthogonally to the direction of motion of the light ray and at rest in the lab frame, and then it gets absorbed by the body which emitted it. [You can work in natural units where $c = 1, \hbar = 1$.]

1. Compute the energy E_1 of the light ray in the lab frame.
2. Compute the rest mass M_1 and the 3-momentum $\vec{P}_1$ of the body after the light ray emission in the lab frame.
3. Compute the final 3-momentum $\vec{P}_2$ and the final velocity $\vec{v}_2$ in the lab frame, and the final rest mass M_2 of the body.
4. Discuss whether there is an upper bound on the energy E_0 of the emitted light ray.

Problem 2

Consider the Friedmann equation governing the dynamics of a FLRW Universe having as dominant components a matter term (equation of state $w_M = p_M/\rho_M \simeq 0$) and a dark energy term ($w_{DE} < -1/3$ and w_{DE} not dependent on time):

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G_N}{3}\left(\rho_M + \rho_{DE}\right) - \frac{k}{a^2}\ ; \tag{1}$$

in this expression $a(t)$ is the Universe scale factor, $\dot{a}$ its time derivative, while k is the curvature parameter, taking on values +1, 0, or -1 for, respectively, a positively curved, a flat, and a negatively curved Universe.

1. Show that Eq. (1) can be rewritten in a form resembling the condition of energy conservation for a particle in a 1-dimensional potential well:

   $$\frac{1}{2}\left(\frac{dy}{d\tau}\right)^2 + V(y) = E_{tot} \tag{2}$$

   with the "spatial coordinate" $y \equiv a(t)/a(t_0)$, being $a(t_0)$ the scale factor today. Choosing the normalization of the "total energy" as $E_{tot} = 1$, find the definition of τ in the "kinetic term" and the expression for "potential energy" $V(y)$.

2. Consider the case for $w_{DE} < -1$; argue that for $\rho_{DE} < 0$ the Universe would evolve from a current expansion phase to a contracting solution, while in the case $\rho_{DE} > 0$ and $k = 0$ the expansion would not stop.

3. Focussing on the expanding solution found at the previous point, show that the scale factor becomes infinite at a finite time in the future.

Problem 3

Consider a spacetime characterised by the following metric:

$$ds^2 = \left(1 - \frac{M}{r}\right)^2 dt^2 - \left(1 - \frac{M}{r}\right)^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \qquad (1)$$

1. Find a transformation to new coordinates (v, r, θ, ϕ), with $v = t + r + f(r)$, so that $g_{rr} = 0$ and show that the geometry is not singular at $r = M$.

2. Determine the outgoing and ingoing light rays trajectories and sketch them and the associated light cones on a $(\tilde{t}, r)$ diagram, where $\tilde{t} \equiv v - r$.

3. Is this the geometry of a black hole?

Problem 4

Consider a real self-interacting scalar field in four dimensions, with action:

$$S_M = \int d^4x \left[-\frac{1}{2}(\partial\phi)^2 + V(\phi^2) \right] . \quad (1)$$

[Here the signature is $- + ++$.]

1. Write the corresponding Euclidean action.

2. The effective potential V_{eff} is defined by:

   $$\Gamma(\bar{\phi}) = \int d^4x V_{eff}(\bar{\phi}) \quad (2)$$

   where Γ is the effective action and $\bar{\phi}$ is constant. Show that at 1-loop Γ is given by the formal expression:

   $$\Gamma(\bar{\phi}) = S(\bar{\phi}) + \frac{1}{2}\mathrm{Tr}\log(\Delta/\mu^2) . \quad (3)$$

   where μ is an arbitrary mass scale. [Hint: this can be obtained from the functional integral:

   $$e^{-\Gamma(\bar{\phi})} = \int (d\eta) e^{-S(\bar{\phi}+\eta)} \quad (4)$$

   by expanding the action to second order in η around a classical solution $\bar{\phi}$, and evaluating the Gaussian integral.]

3. Now assume the classical potential:

   $$V = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 . \quad (5)$$

   Using a momentum cutoff, calculate the logarithmically divergent part of the effective potential.

4. Define the renormalized quartic coupling as the coefficient of $\phi^4/4!$ in the effective potential. Calculate the beta function, which is the logarithmic derivative of the renormalized coupling with respect to μ. [Assume that the effects of the mass can be neglected.] What does this say about the asymptotic behavior of the theory at very large energies?

You may need the following result:

$$\int_0^x du\, u \log(u + A) = \frac{1}{4}\left[2\left(x^2 - A^2\right)\log\left(A + x\right) + 2A^2\log(A) + 2Ax - x^2\right] \ .$$

Questions

1. Discuss Bose-Einstein condensates.

2. Briefly describe the form and the main properties of the Cabibbo-Kobayashi-Maskawa matrix, and its importance in particle physics.

3. Using the equivalence principle, argue that light must be deflected in a gravitational field.

4. Describe briefly the Casimir effect.

5. Discuss the spectrum of cosmic rays and possible explanations for its main features.

6. Discuss the current observational evidence for Dark Energy.