March 27, 2017

SISSA
Entrance
Examination

Astroparticle Physics Curriculum

SOLVE two out of the four problems. If you hand in more than two, then state clearly which two problems you wish to be evaluated (otherwise, the two with the *lowest* score will be considered).
Answer two out of the six questions concisely (no more than one page per question).

Problem 1

An electron of mass m and charge $-e$ (with $e > 0$) moves in a region of constant and uniform magnetic field $\vec{B} = \nabla \times \vec{A}$. Let $\vec{A} = -yB_0\hat{x}$, corresponding to a magnetic field along the z-direction: $\vec{B} = B_0\hat{z}$. Ignore the spin of the electron.

1. Write the Schrödinger equation in Cartesian coordinates (x, y, z).
2. Prove that the components p_x and p_z of the momentum are constants of motion.
3. Find the quantum energy levels of this system. Describe the motion of the electron.
4. Now consider the spin $\vec{s}$ of the electron. The magnetic dipole moment of the electron is $\vec{\mu} = e\hbar/(mc)\vec{s}$. What are the energy levels? Is there a degeneracy?

Problem 2

Cosmic ray protons whose energy is sufficient to induce the reaction

$$p + \gamma_{\mathrm{CMB}} \to p + \pi^0 \tag{1}$$

(inelastic scattering on CMB photons) loose energy very quickly and thus cannot reach the Earth. This is called the GZK cutoff.

1. What is the minimum energy E_p for which the reaction (1) is allowed? (Assume the CMB photons have energy $E_\gamma = 2 \cdot 10^{-4}$ eV. The masses of a proton and a pion are: $m_p = 938$ MeV, $m_\pi = 135$ MeV)

2. What is the fraction of its energy the proton looses in this collision (at energy E_p)?

3. The produced π^0 will (most likely) decay into two photons. Calculate the maximum and minimum energy of these photons (in the frame where the π^0 is moving) in terms of the energy and momentum of the pion, E_π and P_π.

   Show that the energy distribution of these photons dN_γ/dE_γ is flat, i.e. it does not depend on E_γ.

4. The π^0 does not decay into three photons. Why?

Problem 3

Consider an astronaut standing at fixed coordinates $(R, \bar{\theta}, \bar{\phi})$, with $R > 2GM/c^2$, outside a Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad (1)$$

1. Determine her world-line 4-velocity $u^\alpha \equiv dx^\alpha/d\tau$.

2. Determine the 4-acceleration $a^\alpha = u^\beta \nabla_\beta u^\alpha$. Calculate $|a|^2 = g_{\alpha\beta} a^\alpha a^\beta$ and describe its behaviour for $R \to \infty$ and $R \to 2GM/c^2$.

   Note: in the Schwarzschild metric $\Gamma^\mu_{tt} = -\frac{1}{2} g^{\mu\rho} \partial_\rho g_{tt}$.

3. Let us assume now that the astronaut stops accelerating and starts to free fall into the black hole with initial radial velocity equal to zero. Calculate the proper time she would take to reach the black hole singularity at $r = 0$, and show that it scales like $\sqrt{R^3/r_s}$, where $r_s = 2GM/c^2$.

   Hint: remember that in the Schwarzschild metric the conserved total energy E of a particle of mass m is given by

   $$\frac{E^2}{m^2 c^4} - 1 = \frac{\dot{r}^2}{c^2} - \frac{r_s}{r} + \frac{L^2}{m^2 c^2}\left(\frac{1}{r^2} - \frac{r_s}{r^3}\right), \quad (2)$$

   where L is the angular momentum.

   The following integral may be useful $\int_0^a \sqrt{\frac{x/a}{1-x/a}} dx = \pi a/2$, for $a > 0$.

4. Compute the tidal force to which our astronaut is subject, assuming that her head and feet are both distant $h/2$ from her center of mass, which is located at radial coordinate $R \gg h$. Hint: use the Newtonian formula to compute the difference in force excerpted at the head and feet of the astronaut.

Problem 4

Consider the Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)\,. \tag{2}$$

In the following, we are going to consider a spatially-flat Universe.

1. Derive new espressions in terms of derivatives with respect to conformal time τ, where $d\tau = dt/a$. What can be noticed about $d^2a/d\tau^2$ in a Universe filled with radiation only?

2. Consider now a Universe filled with matter and radiation. Write the total energy density $\rho \equiv \rho_{\rm m} + \rho_{\rm r}$ as a function of the scale factor a. Express the result in terms of the scale factor $a_{\rm eq}$ and energy density $\rho_{\rm eq}$ at the time of matter-radiation equality ($\rho_m = \rho_r$).

3. Calculate the redshift of matter-radiation equality as a function of the ratio $\rho_{\rm m}/\rho_{\rm r}$ at the present time.

4. Solve the system of differential equations to get an expression for $a(\tau)$ in terms of $a_{\rm eq}$ and $\rho_{\rm eq}$. Use the condition $a(\tau = 0) = 0$.

Questions

1. Discuss the CPT theorem, its relevance in quantum field theory and at least one application.

2. Provide a timeline of the thermal history of the Universe, highlighting the main events.

3. Explain briefly what is torsion and provide an interpretation of how it modifies parallel transport.

4. Provide a heuristic explanation of the presence of peaks in the angular power spectrum of the anisotropies of the Cosmic Microwave Background.

5. Discuss one possible dark matter candidate under the particle physics and astrophysical point of view.

6. Discuss the asymptotic freedom of the strong interactions of the Standard Model of particle physics.