July 4, 2016

SISSA
Entrance
Examination

Astroparticle Physics Curriculum

SOLVE two out of the four problems. If you hand in more than two, then state clearly which two problems you wish to be evaluated (otherwise, the two with the *lowest* score will be considered).
Answer two out of the six questions concisely (no more than one page per question).

Problem 1

The Schrödinger equation for the wavefunction ψ of an electron of mass m_e and energy E in a spherically symmetric potential $V(r)$ is written, in spherical coordinates, as:

$$\left[-\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{1}{2m_e r^2}\vec{L}^2 + V(r)\right]\psi(\vec{r}) = E\psi(\vec{r}) \ , \tag{1}$$

where $\vec{L}$ is the angular momentum operator. In the following, we will ignore the spin of the electron.

1. Consider the hydrogen atom with the potential $V(r) = -e^2/r$, where e is the electron charge. Assume the electron is in S-wave, so that its wavefunction is of the form:

$$\psi(\vec{r}) = Ne^{-\beta r} \ , \tag{2}$$

   with N being a normalization constant and $\beta \in \mathbb{R}$. Find β and the energy eigenvalue E in terms of $\hbar, m_e, e$.

2. An electron is in the ground state of tritium $^3_1$H ($A = 3, Z = 1$). By a nuclear reaction, the nucleus instantaneously transforms to a nucleus of Helium-3 $^3_2$He ($A = 3, Z = 2$). Compute the probability that, after the reaction, the electron is in the ground state of Helium-3.

3. An electron is in a hydrogen atom with orbital angular momentum $\ell = 1$, and assume its wavefunction is of the form:

$$\psi(\vec{r}) = \tilde{N} r e^{-\tilde{\beta} r} Y_{1m}(\theta, \phi), \tag{3}$$

   where $\tilde{N}$ is a normalization constant and $Y_{lm}(\theta, \phi)$ is a spherical harmonic function. Find the normalization constant $\tilde{N}$, the energy eigenvalue and $\tilde{\beta}$ in terms of $\hbar, m_e, e$.

4. In the full discrete spectrum of the hydrogen atom, how many states have the same energy as computed in part 3?

[*The following integral may be useful:*

$$\int_0^\infty r^2 \exp(-\gamma r) dr = 2/\gamma^3 \ , \tag{4}$$

for $\gamma \in \mathbb{R}$].

Problem 2

There are experimental indications that neutrinos have a non-zero mass; the size of such mass scale is known to be much smaller than that for charged leptons, but has not been measured yet.

1. The dominant decay mode of charged pions is a muon/muon-neutrino pair:
$$\pi^+ \to \mu^+ \nu_\mu \, . \tag{1}$$
Suppose to have an ideal experiment in which π^+ are decaying at rest; write the relation between the neutrino mass and the momentum of the outgoing muon. [*Here and everywhere in the following, you can neglect the phenomenon of neutrino oscillations, and assume that* ν_μ *is just a particle of given mass* m_{ν_μ}].

2. Consider now a burst of ν_μ emitted in a supernova explosion in the Large Magellanic Cloud (LMC; distance $\sim 1.6 \cdot 10^5$ light years (lyr)). Assume that a flux of ν_μ in the energy range between 10 MeV and 30 MeV is detected on Earth within a time interval $\Delta t_{obs} \sim 10$ sec, and that the intrinsic duration of the emission was shorter than Δt_{obs}; estimate the upper bound on m_{ν_μ} one can derive.

3. Estimate the accuracy one would need in measuring the momentum of the outgoing muon in the ideal experiment introduced above at point [1] to be sensitive to a value of the mass m_{ν_μ} at the level of the upper bound just found at the point [2]. [*You can estimate such accuracy as the relative difference between the momenta in the case* $m_{\nu_\mu} \neq 0$ *with respect to the case* $m_{\nu_\mu} = 0$*; the mass of* μ^+ *is about 100 MeV and* μ^+ *is about 25% lighter than* π^+].

4. A neutrino burst - with energy range and within a time interval comparable with those indicated above - was detected in coincidence with a supernova explosion in the LMC in 1987. Rather than ν_μ, the detected neutrinos were electron antineutrino, $\bar{\nu}_e$. Model detectors as: *a)* objects whose detection targets are free protons and electrons; *b)* and in which scattering events are identified only in case the final state contains a relativistic charged lepton. Apply the appropriate conservation rules to find the scattering processes for ν_e, $\bar{\nu}_e$, ν_μ and $\bar{\nu}_\mu$ that are allowed and

relevant for detection. Draw the corresponding Feynmann diagrams; knowing that, at these energies, the flux at Earth for each of the four neutrino kinds is comparable, identify the process making $\bar{\nu}_e$ the easiest to detect. Can you explain the reason why this is indeed the case?

Problem 3

Consider a particle in orbit in a Schwarzschild geometry ($c = 1$):

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \ .$$

For simplicity, assume that the orbit lies in the equatorial plane (i.e., $\theta = \pi/2$).

1. Show that along geodesics there are the two quantities $\bar{E}$ and $\bar{J}$ which are constant of motion associated, respectively, to the Euler-Lagrange equation in t and ϕ (i.e. $\dot{\bar{E}} = \dot{\bar{J}} = 0$, where we denoted by a dot the derivative with respect to the geodesic affine parameter λ). HINT: the Lagrangian for a free particle is:

$$L = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \ .$$

2. For photons orbits, show that these quantities are related as:

$$\bar{E}^2 = \left(\frac{dr}{d\lambda}\right)^2 + \frac{\bar{J}^2}{r^2}\left(1 - \frac{2GM}{r}\right) \equiv \dot{r}^2 + V_{\rm eff} \ .$$

   The effective potential $V_{\rm eff}$ defined above yields information about the orbits of particles. Show that for photons there can exist an *unstable circular* orbit of radius $3r_s/2$, where $r_s = 2GM$ is the Schwarzschild radius, and find its $\bar{E}^2$ value on such circular ($\dot{r} = 0$) orbit.

3. Compute the proper time (not just the coordinate time) for the photon to complete one revolution of the circular orbit as measured by an observer stationed at $3r_s/2$. What orbital period does a very distant observer assign to the photon?

Problem 4

The relation between cosmological redshift z and scale factor a is:

$$1 + z(t, t_e) \equiv \frac{a(t)}{a(t_e)}$$

with t the time of observation and t_e the time at which a photon is emitted.

1. Derive an expression for the quantity

$$\dot{z}(t_0, t_e) \equiv \left.\frac{dz}{dt}\right|_{t=t_0}$$

as a function of z and of the Hubble rates at the present time t_0 and t_e. [*The Hubble rate is* $H(t) \equiv (da/dt)/a(t)$].

2. Using the Friedmann equation in a flat Friedmann-Lemaître-Robertson-Walker Universe with only matter and a cosmological constant (assume zero radiation contribution):

$$H^2 = \frac{8\pi G}{3}\rho \, ,$$

write $\dot{z}(t_0, t_e)$ as a function of H_0, z and the cosmological parameter $\Omega_{m,0}$ [*Recall:* $H_0 \equiv H(t_0)$; $\Omega_{i,0} \equiv \frac{\rho_{i,0}}{\rho_{crit,0}}$, *with* $\rho_{crit,0} \equiv 3H_0^2/8\pi G$; *also:* $\rho_m(t) \propto a(t)^{-3}, \rho_\Lambda(t) = const$].

3. Study and sketch the function $\dot{z}(t_0, t_e)$ as a function of z for the following three cases: a) $\Omega_{m,0} = 1$ and $\Omega_{\Lambda,0} = 0$; b) $\Omega_{m,0} = 0$ and $\Omega_{\Lambda,0} = 1$; c) $\Omega_{m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$.

4. What is the physical interpretation of $\dot{z}(t_0, t_e)$? If you interpret $\dot{z}(t_0, t_e)$ in terms of a spectral shift by using $\delta\lambda/\lambda = \delta v/c$, can you roughly estimate the expected velocity shift at $z = 3$ for the case *a*) above at two different epochs separated by 10 years? [*Note:* $H_0 \simeq 70$ *km s*$^{-1}$ *Mpc*$^{-1}$ $\sim 2 \times 10^{-18}$ *s*$^{-1}$].

Questions

1. In the periodic table of chemical elements, up which element is expected to be synthesised in the Early Universe, in nuclear fusions in stars or in supernova events?

2. Describe the content of the Goldstone theorem and briefly discuss one application of it.

3. Explain briefly why in Einstein's theory of general relativity it is impossible to have monopole or dipole gravitational radiation.

4. Discuss the difference between luminosity and angular distances in cosmology, as well as the consistency relation between them.

5. Explain the reason why Standard Model neutrinos cannot be the dominant component of the dark matter in the Universe.

6. Discuss the definition of entropy in different physical systems.