July 8, 2015

SISSA
Entrance
Examination

Astroparticle Physics Curriculum

SOLVE two out of the four problems. If you hand in more than two, then state clearly which two problems you wish to be evaluated (otherwise, the two with the *lowest* score will be considered).
Answer two out of the six questions concisely (no more than one page per question).

Problem 1

Consider a particle of mass m in one dimension in a quadratic potential, with Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}kx^2 \,. \tag{1}$$

Initially, the particle is in the ground state $|0\rangle$ of H_0. For $t > 0$, a small time-dependent perturbation H_1 is introduced so that the full Hamiltonian is $H = H_0 + H_1(t)$, where

$$H_1(t) = \epsilon\, e^{-\lambda t} x^2 \qquad (\lambda > 0) \tag{2}$$

1. Construct creation and annihilation operators $a^\dagger$ and a, as linear combinations of x and p, such that $[a, a^\dagger] = 1$ and $H_0 = \hbar\omega(a^\dagger a + 1/2)$, where $\omega \equiv \sqrt{k/m}$. Express the perturbation $H_1(t)$ in terms of creation and annihilation operators.

2. Based on symmetries of the Hamiltonian, find a selection rule for the transition to the excited states of H_0.

3. To the lowest order in ϵ, find the probabilities that at late times ($t \gg 1/\lambda$) the particle will be found:

   a) in the 1st excited state $|n = 1\rangle$ of H_0;

   b) in the 2nd excited state $|n = 2\rangle$ of H_0;

4. Extend the result in [3.] to compute, to the lowest non-vanishing order in ϵ, the probability that at late times the particle will be found in the nth excited state $|n\rangle$ of H_0, for any value of n.

[*The following formulae may be useful:*

$$(a^\dagger)^n|0\rangle = \sqrt{n!}|n\rangle$$

and

$$e^{i\alpha G} A e^{-i\alpha G} = A + i\alpha[G, A] + \frac{(i\alpha)^2}{2!}[G, [G, A]] + \cdots + \frac{(i\alpha)^n}{n!}[G, [G, \cdots, [G, A]\cdots]]$$

for $\alpha \in \mathbb{R}$.].

Problem 2

An electromagnetic air shower is a cascade of photons, electrons and positrons generated by the interaction of a primary high energy cosmic-ray photon, electron or positron entering the atmosphere of the Earth.

1. Prove that particle production processes involving photons γ, electrons e^- and positrons e^+, namely:

$$\gamma \to e^- + e^+ \qquad e^- \to e^- + \gamma \qquad \text{and} \qquad e^+ \to e^+ + \gamma\,, \tag{1}$$

are forbidden in the vacuum.

In the atmosphere of the Earth such processes occur in the field of atomic nuclei. Consider the pair-production process of a high energy photon (photon energy E_γ much larger than the electron mass $m_e \simeq 500$ keV) stimulated by a nucleus N of atomic number A and charge Z:

$$\gamma + N(A,Z) \to e^- + e^+ + N(A,Z)\,. \tag{2}$$

In the following, ignore any role of atomic electrons and take the limit of infinite nucleus mass.

2. Sketch the process in Eq. (2) at leading order with the Feynman diagram technique. Find the dependence of the scattering cross section for this process on the fine structure constant α, Z and A.

3. Argue that the leading term in the scattering cross section of the process in Eq. (2) does not depend on E_γ.

4. Consider an electromagnetic air shower initiated by a 1 TeV photon. Let λ_γ be the average distance in the atmosphere over which a photon splitting occurs. To model the development of the electromagnetic air shower, one would need to include also elastic collisional losses, which dominate for low energies; consider the simplified picture in which there is a sharp energy cutoff $E_c \simeq 80$ MeV, such that above it only splitting processes take place, while below it collisional processes are efficient and splitting processes are turned off. Derive the total number of particles in the shower (photons, electrons and positrons) as a function of the depth of the shower in units of λ_γ.

Problem 3

A flat FLRW universe is described by the metric

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 \tag{1}$$

with the scale factor a normalized to unity today: $a(t_0) = 1$. The observed redshift z of an object is related to the scale factor at the time when light was emitted by $1 + z = 1/a$, so that the evolution of the universe is encoded in the function $H(z)$, where H is the Hubble parameter $H \equiv \frac{1}{a}\frac{da}{dt}$.

Consider a kind of astrophysical objects with fixed and known luminosity L (total energy emitted by the object per unit time). These objects are dubbed *standard candles*. The luminosity distance of an object is defined as:

$$d_L \equiv \sqrt{\frac{L}{4\pi F}}\ , \tag{2}$$

where F is the flux of energy (energy per unit time and unit area) one observes on the Earth.

1. Explain why it makes sense to call this quantity 'distance'.
2. Derive the expression for $d_L(z)$ where z is the observed redshift of the standard candle. This relation depends on the cosmological evolution $H(z)$.
3. Explain why standard candles at a given redshift appear fainter in a Universe with a positive cosmological constant compared to one with matter only (keeping fixed the present value of the Hubble constant).

Another definition of distance in cosmology is the angular-diameter distance d_A. Suppose we have an astrophysical object of known physical size ℓ (*standard yardstick)*. This is observed to subtend a certain (small) angle on the sky $\delta\theta$ and we can define

$$d_A \equiv \frac{\ell}{\delta\theta}\ . \tag{3}$$

4. Show that $d_A(z) \leq d_L(z)$ at any z.
5. Write the approximate expression for $d_L(z)$ and $d_A(z)$ valid for $z \ll 1$. Comment on the obtained result.

Problem 4

The Lagrangian for the electromagnetic field in the vacuum is:

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}\,, \qquad (1)$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field strength, given in terms of the scalar potential V and the vector potential $\vec{A}$ via the four-vector potential $A^\mu = (V, \vec{A})$.

1. Write the Lorentz-covariant, gauge invariant, symmetric, conserved energy-momentum tensor of the electromagnetic field.

2. Consider a source of electromagnetic radiation and choose a coordinate system such that the source is at rest in the origin. Then take a point far from the source on the z-axis. Using the properties of electromagnetic waves, determine which components of the energy-momentum tensor are nonzero at that point and write them in terms of electric and magnetic fields, $\vec{E}$ and $\vec{B}$.

3. Knowing that the power emitted by the Sun is approximately 1.4×10^{26}W, use these formulae to determine the root-mean-squared electric field of the solar radiation on the Earth.

4. Calculate the gravitational redshift, $z = \frac{\lambda - \lambda_0}{\lambda_0}$, where λ_0 is the wavelength at emission and λ the wavelength of the radiation arriving on the Earth.

[*The following quantities may be useful:*

the mass of the Sun is about 2×10^{30} kg;
Newton's constant is 6.7×10^{-11} m^3 kg^{-1} s^{-2};
the radius of the Sun is about 7×10^5 km;
the Earth orbit has a radius of about 1.5×10^8 km.]

Questions

1. Illustrate two observational evidences for the existence of the dark matter component of the Universe.

2. Explain the importance of the synthesis of light elements within the Standard Model of Cosmology.

3. Discuss an example of violation of a discrete symmetry in the Standard Model of Particle Physics.

4. Discuss Killing vectors in General Relativity.

5. Discuss Fermi-Dirac and Bose-Einstein statistics.

6. Dirac and Majorana masses for fermions: properties and issues.