July 8, 2014

SISSA
Entrance
Examination

Astroparticle Physics Curriculum

SOLVE two out of the four problems. If you hand in more than two, then state clearly which two problems you wish to be evaluated (otherwise, the two with the *lowest* score will be considered).
Answer two out of the six questions concisely (no more than one page per question).

Problem 1

1. Consider a quantum system described by the Hamiltonian H, with eigenstates $|i\rangle$ with eigenvalues E_i: $H|i\rangle = E_i|i\rangle$, with $E_0 \leq E_1 \leq E_2 \leq \cdots$.

   Prove that
$$E_0 \leq \langle\psi|H|\psi\rangle \tag{1}$$
   for any normalized state $|\psi\rangle$.

2. Now suppose that a particle of mass m is subject to the one-dimensional Hamiltonian
$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \tag{2}$$
   where the potential $V(x)$ is defined by
$$V(x) = \begin{cases} +\infty & (x < 0) \\ 0 & (0 \leq x \leq a) \\ V_0 & (x > a) \end{cases} \tag{3}$$
   where V_0, a are positive constants.

   Under what condition on the parameters V_0, m, a bound states exist?

3. Consider as an Ansatz for the ground state wavefunction: $\psi(x) \sim \sin(\pi x/L)$ for $0 \leq x \leq L$ and $\psi(x) = 0$ otherwise. For $V_0 = \infty$, L coincides with a; for large $V_0 \gg \hbar^2/(ma^2)$, the wavefunction penetrates the region $x > a$, and the parameter L is slightly larger than a. Let
$$\Delta \equiv L - a \ll a\,. \tag{4}$$
   Compute the ground state energy, at leading order in Δ/a.

[*The following integral may be useful:* $\int \sin^2 y\, dy = \frac{1}{2}(y - \sin y \cos y)$].

Problem 2

In the early Universe photons interact with free electrons via Thomson scattering

$$\gamma + e^- \to \gamma + e^- \ . \tag{1}$$

1. Draw the Feynman diagram of this process and estimate its cross section using dimensional analysis (assume the energy of the photon is much smaller than the electron mass) and including the correct powers of the coupling constant.

2. Write the rate of interaction Γ of a photon due to Thomson scattering if the density of electrons is n_e. Estimate its value taking $n_e = 0.22\ \mathrm{m}^{-3}$, the present electron density in the Universe.

3. Our expanding Universe is described by the metric

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \tag{2}$$

   where the scale factor $a(t)$ increases as a function of time and it is normalized to unity today. How does the interaction rate Γ depends on a?

4. Photons remain in thermal equilibrium with electrons if the rate Γ is larger than the rate of expansion of the Universe $H \equiv \dot{a}/a$. The present value of this quantity is $H_0 = 2.3 \times 10^{-18}\mathrm{s}^{-1}$. Assuming $H = H_0 a^{-3/2}$ (valid for a matter-dominated Universe), determine at what value of a the photons stop being in thermal equilibrium with free electrons.

5. Actually in the real Universe photons stop being in equilibrium long before what one finds above. Can you guess why?

[*Useful constants:* $m_e = 0.51$ MeV*;* $\hbar = 6.6 \times 10^{-16}$ eV $\cdot$ s*;* $c = 3.0 \times 10^8$ m/s]

Problem 3

A particle of rest mass m_0 moves in a plane perpendicular to a magnetic field B with velocity v.

1. Determine the angular frequency of the orbit. Compare the classical answer (known as the cyclotron frequency) and the relativistic one.
2. Compute the radius of the orbit as a function of the velocity. Sketch the form of this function.
3. Calculate the classical angular frequency for a proton in a field $B = 1$ T.
   What are the maximum momentum and the maximum total energy of protons that can be achieved in a cyclotron with radius $r = 2$ m and a magnetic field of 1 T?
   (Use $q = 1.6 \times 10^{-19}$ C, 1 MeV=1.6×10^{-13} J, $c = 3 \times 10^{8}$ m/s.)

Problem 4

Consider a real scalar field $\phi(x^\mu)$ with Lagrangian (the metric signature is $+,-,-,-$):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{4}\left(\phi^2 - \sigma^2(T)\right)^2 \tag{1}$$

where $\sigma^2(T)$ is a parameter whose dependence on temperature T can be sketched as:

$$\begin{aligned} \sigma^2(T) &\simeq 0 \qquad \text{if} \qquad T \gg T_c \\ \sigma^2(T) &\simeq \sigma_0^2 \qquad \text{if} \qquad T \ll T_c \end{aligned} \tag{2}$$

with T_c some reference temperature and σ_0^2 a positive constant.

1. What is the internal symmetry group of this Lagrangian? What are the vacua of this theory in the cases $T \gg T_c$ and $T \ll T_c$?

2. Consider this theory in the early Universe context, and assume that in the cooling along with the Universe expansion from $T \gg T_c$ to $T \ll T_c$, two disconnected region of non-equivalent vacua are generated; to smoothly connect these two vacua, a region where ϕ is away from the vacuum must exist: assume for simplicity that such a region takes the form of a "wall", namely a static, infinite plain of finite thickness. Find an approximate estimate of the thickness l of the wall by minimizing the approximate estimate for the total energy per unit surface area associated to the scalar field.

3. In the Cartesian coordinate system $\{x, y, z\}$, assume that the wall extends to infinity in the $x-y$ directions and is centered at $z = 0$ in the direction transverse to the wall. Write the scalar field equation for this theory and implement the appropriate boundary conditions ensuring that the energy stays finite at spatial infinity. Show that the wall can be described by a solution of the form: $\phi(z) = \phi_0 \tanh(z/l)$. Find ϕ_0 and l.

4. Compute the energy momentum tensor for the solution just found, and discuss the implication following from the linearised equation for the Newtonian potential Φ: $\Delta\Phi = 4\pi G_N(T_{00} + T_{11} + T_{22} + T_{33})$.

Questions

1. Black holes are characterized by event horizons and singularities. Discuss their definition and their physical meaning.

2. The baryon number in the Standard Model

3. The Cosmological Constant Problem.

4. Sources and propagation of cosmic rays in the Galaxy.

5. Discuss the motivation and the properties of the inflationary paradigm.

6. Describe the possible observational methods for detecting gravitational waves.