July 4, 2013

SISSA
Entrance
Examination

PhD in Astroparticle Physics

SOLVE at least two Exercises and answer at least two Questions.

Problem 1

Suppose the symmetry SU(2) $\times$ U(1) is broken by three real scalar fields H_a, $a = 1, 2, 3$. The three fields can be combined in a 2×2 hermitian, traceless matrix $H \equiv H_a \sigma_a$, where σ_a, $a = 1, 2, 3$, are the Pauli matrices. H transforms as $H \to U H U^\dagger$ under $U \in$ SU(2) and has a definite value Y_H of the U(1) generator Y (hypercharge). Let us denote by T_3 the third generator of SU(2), $T_3 \equiv \sigma_3/2$.

1. What are the values that T_3 can assume on H? Write the form of H corresponding to each of those eigenvalues.

2. For which values of Y_H there exists a non-vanishing value of the vev $\langle H \rangle$ such that $Q \equiv T_3 + Y$ is unbroken by that vev?

3. For which values of Y_H there exists a non-vanishing value of the vev $\langle H \rangle$ such that Q, up to a constant, is the only generator unbroken by that vev?

Problem 2

1. Consider the infinitesimal transformation $x^\mu \to x^\mu + \xi^\mu$. Show that if the generator of the transformation ξ^μ satisfies the equation (where the semicolon denotes the covariant derivative)

$$\xi^{\mu;\nu} + \xi^{\nu;\mu} = 0, \tag{1}$$

   then the metric is invariant under the transformation. This equation is called Killing equation and its solutions are called Killing vectors. Argue why the existence of Killing vectors in a spacetime implies the presence of symmetry (hint: consider how you could define suitable coordinates systems where this becomes manifest).

2. Show that if λ is an affine parameter and x^μ a geodesic, then $d(u^\mu u_\mu)/d\lambda = 0$, where $u^\mu = dx^\mu/d\lambda$.

3. Show that $\xi^\nu u_\nu$ is conserved along the geodesic.

4. Consider the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{2}$$

   where M is the mass of the central object (in units where $G = c = 1$). Can you immediately identify at least two Killing vectors, compute their norms and briefly explain what symmetries they refer to?

Problem 3

The emission of relativistic protons in an astrophysical source can be traced via inelastic scatterings of the outflowing protons on the interstellar gas surrounding the source; such process can produce neutral pions, which in turn decay into photons that can be detected by a gamma-ray telescope.

1. The dominant decay channel of neutral pion is the electromagnetic decay $\pi^0 \to 2\gamma$. Argue, based on conservation laws, that the electromagnetic decay $\pi^0 \to 3\gamma$ is forbidden.

2. Consider the decay $\pi^0 \to 2\gamma$ for monochromatic pions of energy E_π, assuming that the photon angular distribution is isotropic in the π^0 rest frame; what is the resulting energy spectrum dN/dE_γ of the emitted photons in the observer (laboratory) frame?

3. For a given energy spectrum of outflowing protons, the total energy spectrum of the emitted photons follows from the superposition from the decay spectra from π^0s of different energies. Can you identify a distinctive spectral feature for such a convolution? (*Hint: consider the geometric mean between maximal and minimal energy in the decay at given E_π.*) Argue that such feature allow to discriminate the hypothesis of a proton outflow from the source from a electron outflow, which in turn can be traced with a gamma-ray telescope via Bremsstrahlung emission on the surrounding interstellar gas.

Problem 4

Consider a spatially flat Friedmann-Robertson-Walker (FRW) universe

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \tag{1}$$

in the presence of a time dependent scalar field $\phi(t)$.

1. In the slow-roll approximation the Friedmann equation and the equation of motion for the field are

$$\begin{aligned} H^2 &\simeq \frac{V(\phi)}{3M_P^2} \\ 3H\dot{\phi} &\simeq -\frac{dV(\phi)}{d\phi} \ , \end{aligned} \tag{2}$$

   where $H \equiv \dot{a}/a$. Taking a quadratic potential $V = \frac{1}{2}m^2\phi^2$, find the scale factor $a(t)$.

2. It is customary to define a slow-roll parameter

$$\epsilon(\phi) = \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 \ . \tag{3}$$

   What is the constraint on ϵ to allow a period of accelerated expansion: $\ddot{a} > 0$?

3. Explain (for this system or in general) why acceleration is possible in General Relativity, despite of Gravity being always attractive in Newtonian physics. In particular, which condition has to be satisfied by the pressure and density of a perfect fluid in order to drive an accelerated expansion of the universe?

4. The action for a scalar field in a gravitational background is

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) \ . \tag{4}$$

   Derive from this action the equation of motion for ϕ in the FRW metric above (without making the slow-roll approximation).

Question 1

The role of chirality in particle physics.

Question 2

Resorting to dimensional analysis define a mass and a length associated with the gravitational constant (Planck mass and Planck length). Briefly discuss their meaning.

Question 3

Concisely discuss a cosmological or astrophysical test of the properties of neutrinos.

Question 4

Describe one experimental evidence that the Universe was in a hot thermal phase at early times.

Question 5

The Friedman–Lemaître–Robertson–Walker line element contains the parameter k, which denotes the curvature of the spatial slices. Explain why the cosmological principle implies that k has to be constant and why one can restrict k to only take the values -1, 0, 1 without loss of generality.

Question 6

Argue for the necessity of antiparticles in relativistic Quantum Field theory.

Solution to Problem 1

1. Expanding $U = \mathbf{1} + i\alpha_a T_a$, one finds the action of T_3 on H: $H \to [T_3, H]$. The values of T_3 on H can be either found by using the eigenvalue equation or just by noticing that the value of T_3 on a SU(2) triplet are $T_3 = +1, 0, -1$. The corresponding eigenvectors are $H \propto \sigma_1 + i\sigma_2, \sigma_3, \sigma_1 - i\sigma_2$.

2. Need to find the values Y_H such that $[T_3, \langle H \rangle] + Y_H \langle H \rangle = 0$ has a solution for a $\langle H \rangle \neq 0$. Those are nothing but the opposite of the eigenvalues of T_3. Using the previous result: $Y_H = +1, 0, -1$.

3. From the previous results, we know that the candidate values of Y_H are $Y_H = +1, 0, -1$ and that the vevs leaving Q unbroken are, respectively, $\langle H \rangle \propto \sigma_1 - i\sigma_2, \sigma_3, \sigma_1 + i\sigma_2$. Let us consider the three cases in turn and recover the generators unbroken by the vev in each case. Let us write the generic generator as $T = \alpha_a T_a + \beta Y$. If $Y_H = 1$ and $\langle H \rangle = (\sigma_1 - i\sigma_2)/2$ (up to a constant), the unbroken generators are the ones such that $0 = T(\langle H \rangle) = [\alpha_a T_a, \langle H \rangle] + \beta \langle H \rangle$. That happens if and only if $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \beta$ i.e. if $T \propto Q$. Therefore in this case Q, up to a constant, is the only generator unbroken by the vev. Analogously for $Y_H = -1$. On the other hand, if $Y_H = 0$ and $\langle H \rangle = \sigma_3$ both T_3 and Y (and all their linear combinations) are obviously unbroken. Therefore, Q is the only generator unbroken only if $Y_H = \pm 1$.

Solution to Problem 2

1. We have

$$g'^{\mu\nu}(x'^{\lambda}) = \frac{\partial x'^{\mu}}{\partial x^{\sigma}}\frac{\partial x'^{\nu}}{\partial x^{\rho}} g^{\sigma\rho}(x^{\lambda}) \quad (1)$$

$$g'^{\mu\nu}(x'^{\lambda}) = g'^{\mu\nu}(x^{\lambda}+\xi^{\lambda}) = g'^{\mu\nu}(x^{\lambda}) + g^{\mu\nu}_{,\lambda}\xi^{\lambda} + O(\xi^2) \quad (2)$$

   Combining the two in order to calculate the change at fixed coordinate, one gets

$$g'^{\mu\nu} = g^{\mu\nu} + g^{\mu\sigma}\xi^{\nu}_{,\sigma} + g^{\nu\sigma}\xi^{\mu}_{,\sigma} - \xi^{\lambda}g^{\mu\nu}_{,\lambda} + O(\xi^2) \quad (3)$$

   Using the definition of the covariant derivative and after some manipulations this can take the form

$$g'^{\mu\nu} = g^{\mu\nu} + \xi^{\nu;\mu} + \xi^{\mu;\nu} + O(\xi^2) \quad (4)$$

   So, if the Killing equation is satisfied the metric remains invariant.

   Now, if one chooses a coordinate system such that ξ^{μ} has unit length and is aligned with the coordinate x^{λ}, then the Killing equation takes the simple form $g_{\mu\nu,\lambda} = 0$. That is, there is an adapted coordinate system in which the metric is independed of this coordinate.

2. We have $d(u^{\mu}u_{\mu})/d\lambda = u^{\nu}\nabla_{\nu}(u^{\mu}u_{\mu}) = 2u^{\mu}u^{\nu}\nabla_{\nu}u_{\mu}$, which vanishes by merit of the geodesic equation.

3. $u^{\nu}\nabla_{\nu}(\xi^{\mu}u_{\mu}) = u_{\mu}u_{\nu}\nabla^{\nu}\xi^{\mu} + \xi^{\mu}u^{\nu}\nabla_{\nu}u_{\mu}$. The first term vanishes because ξ^{μ} is killing, the second because of the geodesic equation.

4. The obvious Killing vectors (metric components independent from related coordinate) are $\partial/\partial t$, $\partial/\partial\phi$ with norms respectively equal to g_{00} and $g_{\phi\phi}$ corresponding to the fact that the spacetime is static and spherically symmetric.

SOLUTION TO PROBLEM 3

1. $\pi^0 \to 3\gamma$ violates charge conjugation: the charge parity of the photon is $\eta_C(\gamma) = -1$, say you can argue saying the C conjugation reverses the electric field, or writing the QED vertex $A_\mu \bar{\psi}\gamma^\mu\psi$ apply the charge operator on $\bar{\psi}\gamma^\mu\psi$ and find that it goes into $-\bar{\psi}\gamma^\mu\psi$. From the process I am saying that is dominant you find, $\eta_C(\pi^0) = \eta_C(\gamma)\eta_C(\gamma) = +1$; since the 3γ final state has $\eta_C = -1$ is this electromagnet decay is forbidden.

2. Assume for instance that the π^0 propagates in the z direction; let θ^* be the angle between the direction of one of the photons and the z axis in the pion rest frame; you are told to assume isotropic decay in that frame:
$$\frac{dN}{dcos\theta^*} = \frac{1}{2}, \tag{1}$$
while you want to compute:
$$\frac{dN}{dE_\gamma} = \frac{dN}{dcos\theta^*}\frac{dcos\theta^*}{dE_\gamma} = \frac{1}{2}\frac{dcos\theta^*}{dE_\gamma}. \tag{2}$$
Obviously you also have $E^*_\gamma = m_\pi/2$, which boosted to the lab frame gives:
$$E_\gamma = \gamma\frac{m_\pi}{2}(1 \pm \beta\cos\theta^*) \tag{3}$$
with the usual Lorentz factors $\gamma = E_\pi/m_\pi$ and $\beta = p_\pi/E_\pi$. So the distribution is flat:
$$\frac{dN}{dE_\gamma} = \frac{1}{p_\pi} \tag{4}$$
with limiting values $(E_\pi \pm p_\pi)/2$.

3. For given E_π, $E_{\gamma,\mathrm{min}}{\cdot}E_{\gamma,\mathrm{max}} = (m_\pi/2)^2$, i.e. $1/2\,[\log(E_{\gamma,\mathrm{min}})+\log(E_{\gamma,\mathrm{max}})] = \log(m_\pi/2)$: when plotted versus $\log(E_\gamma)$ the spectra from individual decays pile symmetrically around $\log(m_\pi/2)$. The Bremsstrahlung process, for relativistic electrons, has no mass scale involved, hence for an electron power low spectrum translates into a power-law photon spectrum (with the same spectral index, I am not asking to show this, extra points if they do that).

Solution to Problem 4

1. Combining the two equations eliminating a, the equation for ϕ becomes

$$3\dot{\phi} = -m\sqrt{6}M_P \ , \tag{1}$$

which has the solution

$$\phi = -m\sqrt{2/3}M_P t + \phi_0 \ . \tag{2}$$

By plugging this into the Friedmann equation, one gets

$$\frac{\dot{a}}{a} = \frac{m}{\sqrt{6}M_P}\left(-m\sqrt{2/3}M_P t + \phi_0\right) \ , \tag{3}$$

which has the solution

$$a = a_0 \exp\left[-\frac{m^2}{6}\left(t - t_0\right)^2\right] \ . \tag{4}$$

2. By directly calculating $\ddot{a}/a$ from the explicit solution the condition for acceleration is

$$m^2(t - t_0)^2 > 3 \ , \tag{5}$$

which is equivalent, calculating ϵ from the explicit solution, to

$$\epsilon < 1 \ . \tag{6}$$

3. In GR pressure gravitates and it can be negative. From the second Fridman equation one can easily see that it is enough that $\rho - 3p$ is negative.

4. The extremization of the action with respect to ϕ implies

$$\int d^4\sqrt{-g}\left[-g^{\mu\nu}\partial_\mu\phi\delta(\partial_\nu\phi) - V'\delta\phi\right] = 0 \ . \tag{7}$$

The above expression is equivalent to

$$\int d^4x\left[-\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\delta\phi\right) + \partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right)\delta\phi - \sqrt{-g}V'\delta\phi\right] = 0 \ . \tag{8}$$

By assuming that the total derivative integral in the expression above vanishes one obtains the motion equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - V^{'} = 0 \ , \tag{9}$$

which by using the FRW metric and the Lagrangian definition becomes

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2}{a^2}\phi + V'(\phi) = 0 \ . \tag{10}$$