SISSA
Entrance
Examination

Astroparticle Physics Curriculum

SOLVE at least two out of the four problems – the best two will count towards the evaluation of the written exam. Concisely answer two out of the six questions.

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Problem 1

Consider Einstein's field equations with a cosmological constant Λ,

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \ ,$$

where indices a, b, ... take values from 0 to 3, we use units in which the speed of light is equal to 1, the Ricci tensor is defined as $R_{ab} := R_{acb}{}^{c}$, and the Riemann tensor is given by the expression

$$R_{abc}{}^{d} = \partial_b \Gamma^{d}{}_{ac} - \partial_a \Gamma^{d}{}_{bc} + \Gamma^{d}{}_{be}\, \Gamma^{e}{}_{ac} - \Gamma^{d}{}_{ae}\, \Gamma^{e}{}_{bc}$$

in terms of the connection coefficients.

1. Deduce the equivalent form of the field equations,

   $$R_{ab} - \Lambda g_{ab} = 8\pi G \left(T_{ab} - \frac{1}{2} T g_{ab} \right) \ ,$$

   where T is the trace of the tensor T_{ab}.

2. Assuming a very small Λ and that spacetime is almost Minkowskian, with deviations from flatness only due to a weak and static Newtonian gravitational potential Φ, prove that $\Gamma^{i}{}_{00} \approx \partial_i \Phi$ and that $R_{00} \approx \nabla^2 \Phi$ (indices i, j, ... take values from 1 to 3).

   [**Hint**: The world-line of a freely-falling particle is a geodesic in spacetime.]

3. Under the same assumptions of the previous question, deduce the modified Poisson equation $\nabla^2 \Phi + \Lambda = 4\pi G \rho$, where ρ is the mass density of the sources, considered non-relativistic for consistency.

4. What is the change induced by the presence of Λ in the gravitational force that a spherically symmetric body with mass M exerts on a much lighter and smaller body with mass m?

Problem 2

Consider a quantum mechanical system made of two (interacting) identical, non relativistic, spin 1/2 fermions in their center of mass reference frame.

1. What are the symmetry properties of the space wave-function of the two fermions under the exchange of the two particles? Assuming that the total spin is zero, what are the values of the orbital angular momentum allowed by the Pauli exclusion principle?

   [**Hint**: the spherical harmonics with l even (odd) are even (odd) with respect to the inversion of the angular coordinates.]

2. Let $\mathbf{s}_1$, $\mathbf{s}_2$ be the two spin operators. Show that the operator $\mathbf{s}_1 \cdot \mathbf{s}_2$ (where $\cdot$ denotes the scalar product) is constant on the states with definite total spin. Using the above operator, construct the operators P_0, P_1 projecting on the spaces with total spin 0, 1 respectively.

3. Consider a scattering process between the two particles in the triplet total spin state. Show that the amplitude vanishes when the scattering angle is 90°.

   [**Hint**: the wave function for the relative coordinate $r = r_1 - r_2$, before taking into account the exclusion principle, is

   $$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta)\frac{e^{ikr}}{r} \, ,$$

   where $\mathbf{k}$ is wave vector of the incoming plane wave and θ is the scattering angle.]

Problem 3

The radiative emission of photons by high energy electrons is one of the key processes to understand several astrophysical phenomena.

1. Show that the process:

$$e^-(p) \to e^-(p') + \gamma(k)\,, \tag{1}$$

   where p, p' and k are the 4-momenta of, respectively, the incoming electron, outgoing electron and outgoing photon, is forbidden in the vacuum.

2. Consider now the case of photon emission in presence of the Coulomb field of a nucleus N with atomic number A and charge $Z \cdot e$. Assuming the limit of point nucleus and ignoring the effect of atomic electron screening, sketch the process (bremsstrahlung process):

$$e^-(p) + N(A,Z) \to e^-(p') + \gamma(k) + N(A,Z) \tag{2}$$

   with the Feynman diagram technique at the lowest order in perturbation theory. Using dimensional arguments and taking into account the couplings, try to guess an approximate expression for the cross section of this process in the limit of ultra-relativistic incident electron and infinite mass for the nucleus.

   [**Hint**: use, as intermediate step, a guess for the cross section for the elastic scattering of an electron and a photon in quantum electrodynamics (QED).]

3. Having written the amplitude for the elastic scattering process $e^-(p) + N(A,Z) \to e^-(p') + N(A,Z)$ in the form:

$$\mathcal{M}_s := \bar{u}(p')\mathcal{A}_0(p',p)u(p)\,, \tag{3}$$

   where $u(p)$ and $u(p')$ are the 4-component spinors for the electrons, respectively, in the initial and final state, apply the QED Feynman rules and the definition of $\mathcal{A}_0$ to write the amplitude for the bremsstrahlung process. Assume that the radiated photon in the final state is soft, i.e. $|\vec{k}| \ll |\vec{p}' - \vec{p}|$, and show that this amplitude can be factorized as $\mathcal{M}_s$ times a factor accounting for the emission of the photon.

Problem 4

Consider the generalized homogeneous Poisson equation in a flat Friedmann-Robertson-Walker (FRW) Universe:

$$\ddot{\Delta} - \left[3\left(2w - c_s^2\right) - 1\right]\mathcal{H}\dot{\Delta} + \left[\left(\frac{3}{2}w^2 - 4w + 3c_s^2 - \frac{1}{2}\right)\mathcal{H}^2 + k^2 c_s^2\right] 3\Delta = 0\,, \tag{1}$$

where $\Delta(\vec{k}, t)$ is a generic Fourier mode of a density contrast perturbation field in linear regime $\Delta \equiv \rho(\vec{x}, \tau)/\rho_0(\tau) - 1$, as a function of the conformal cosmic time τ, and $\mathcal{H} = (da/d\tau)/a$ is the Hubble rate obeying the Friedmann equation:

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \rho_0\,; \tag{2}$$

w and c_s^2 represent the equation of state and sound velocity of the medium with mean cosmological density ρ_0, which evolves through the continuity equation

$$\dot{\rho}_0 + 3\mathcal{H}(1 + w)\rho_0 = 0\,. \tag{3}$$

1. Consider the limit of large scales in Eq. (1), $k \to 0$. Demonstrate that for both Radiation Dominated Era (RDE, w=c_s^2 = 1/3) and Matter Dominated Era (MDE, w=c_s^2 = 0), there exist growing perturbation modes in time.

2. Assuming RDE and considering in Eq. (1) the relation determining when the term proportional to k^2 dominates over the one proportional to $\mathcal{H}^2$, find the time behavior of the solution at late times after such transition. Suppose that the latter occurs at a given time τ_{HC} and is instantaneous. Find the appropriate initial conditions for the solution at $\tau > \tau_{HC}$, as a function of the growing mode at $\tau \leq \tau_{HC}$.

3. Suppose now that, at late times during MDE, corresponding to a given scale factor and time a_Λ, τ_Λ, the cosmic expansion becomes driven by a Cosmological Constant Λ, characterized by $w = c_s^2 = -1$. By considering large scales, $k \to 0$, determine the time evolution of Δ during the Cosmological Constant regime, neglecting matter. On the basis of the results, discuss what is the effect of Λ on the growth of perturbations.

Questions:

1. Discuss the pattern of inhomogeneities in the cosmic microwave background radiation, and how they are related to the physical processes in the early Universe.

2. Discuss one of the viable frameworks for dark matter in the Universe, and strategies on how to probe it.

3. CP violation in the theory of elementary particles and its relevance for the generation of an asymmetry between baryons and antibaryons in the very early Universe (baryogenesis).

4. Sketch one of the mechanisms for particle acceleration in astrophysical objects.

5. Give an example of the relevance of the superposition principle in elementary particle physics.

6. Can you describe any observational/experimental evidence, direct or indirect, for the existence of gravitational radiation?