SISSA
Entrance
Examination

Astroparticle Physics Curriculum

SOLVE two out of the four problems. Concisely answer two out of the four questions.

July 17, 2008

2008 summer session

Problem 1.

Consider a particle of mass m in a three-dimensional harmonic oscillator potential. Let the Hamiltonian for this configuration be in form:

$$H = \frac{|\vec{p}|^2}{2\,m} + \frac{1}{2} m\omega^2 |\vec{r}|^2 \ , \tag{1}$$

with $\vec{p}$ the particle momentum and $\vec{r}$ its position vector.

1. After introducing a Cartesian coordinate system, identify a complete set of eigenvectors of the Hamiltonian, compute energy levels and their degeneracy.

2. Consider the weak perturbation to the system obtained by adding to the harmonic oscillator potential the extra term:

   $$V_\epsilon = \epsilon\, |\vec{r}|^4 \ , \tag{2}$$

   where ϵ is some small constant. Use perturbation theory to compute the shift in the ground state to first order in ϵ.

3. The small perturbation of Eq. (2) mixes, to first order in ϵ, the ground state of the unperturbed Hamiltonian with other states. What are the (unperturbed) energy levels of these states?

see continuation in the next page:

Useful formulae:

You can use, without proof, that, for a *one-dimensional* harmonic oscillator of mass m and frequency ω, annihilation and creation operators are defined, respectively, as:

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right) \quad \text{and} \quad a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right) \ ,$$

where x is the position variable and p the momentum, and that the ground state wave function is:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \ .$$

This wave function is properly normalized, as it can be shown remembering that:

$$\int_0^\infty dx \exp\left(-ax^2\right) = \frac{1}{2}\sqrt{\frac{\pi}{a}} \ .$$

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Problem 2.

Consider a massless scalar particle ϕ coupled to electrons and muons through the interaction:

$$\mathcal{L} = g\,\phi\,(\bar{\psi}_e\psi_e + \bar{\psi}_\mu\psi_\mu)\ . \tag{1}$$

1. The interaction in Eq. (1) contributes to the cross section of the process:

   $$e^+ + e^- \to \mu^+ + \mu^-\ . \tag{2}$$

   Draw the Feynman diagrams corresponding to this term as well as the standard interaction term in Quantum Electro-Dynamics (QED).

2. Experimentally, the cross section for $e^+ + e^- \to \mu^+ + \mu^-$ matches the QED prediction at better than 1%. This sets a bound on the coupling g in Eq. (1). Give an order of magnitude estimate for this bound. (The full evaluation of the cross section is not requested; the electromagnetic coupling constant is about $e \simeq 0.3$.)

3. Explain why the exchange of the scalar ϕ induces, in the non-relativistic limit, a potential inversely proportional to the distance: $V \propto 1/r$.

4. The exact form of the potential generated by the exchange of ϕ is (in natural units, i.e. for $\hbar = c = 1$)

   $$V(r) = -\frac{g^2}{4\pi r}\ . \tag{3}$$

   This potential generates a force between a hydrogen atom and a source like the Earth. Find the value of g for which the ratio between this force and the gravitational attraction for a hydrogen atom on the surface of the Earth is equal to 1. (Assume that the Earth is composed by the same number of protons and neutrons and, obviously, that the number of electrons is very close to the one of protons. In natural units, $m_p \simeq 1$ GeV and the Newton's constant $G_N \simeq 7 \cdot 10^{-39}$ GeV^{-2}.)

Problem 3.

Assume an expanding, flat, Friedmann-Robertson-Walker solution as a model for the Universe.

1. The number density of electrons at the present epoch is about 0.2 m^{-3}. Estimate the number density of electrons when the Universe was one millionth of its present size. Determine whether, at such epoch, electrons were relativistic, given that the present temperature of the Universe is about 10^{-4} eV and the electron mass is $m_\mathrm{e} = 0.511$ MeV.

2. The cross-section for the scattering of photons off electrons is given by the Thomson cross-section $\sigma_\mathrm{T} = 6.7 \times 10^{-29}\ \mathrm{m}^2$. Using this and the electron density derived above, estimate the mean free path of photons λ through the electron gas when the Universe was one millionth of its present size. Compute the typical time between interactions, and compare it to the age of the Universe in this phase (around 10,000 years): what can you deduce from this comparison?

3. As rule of thumb to estimate the relevance of the interaction, one can also compare λ^{-1} to the Universe expansion rate H. Taking into account the answer to the previous question, and using the Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\rho \tag{1}$$

   in the limit when the energy density of the Universe ρ is dominated by the radiation component, deduce the evolution of the photon-electron system at later times. (Suppose electrons always remain free and do not form atoms). How the situation would change if we assume that the Universe is dominated by non-relativistic matter, instead of radiation?

Problem 4.

Consider the scattering process involving a photon and an electron:

$$\gamma + e^- \to \gamma + e^- \, . \tag{1}$$

1. In Compton's experiment a photon scatters off an electron initially at rest in the laboratory frame. Derive the relation between the change in the wavelength of the photon and the angle θ between the direction of the incident and the outgoing photon in the laboratory frame.

2. Suppose now that the process in Eq. (1) describes the scattering of a ultra-relativistic electron (ratio between electron energy and rest mass energy equal to 10^4) on an isotropic background of starlight photons (assume for simplicity a mono-energetic population of temperature $T = 0.3$ eV). Estimate the maximum energy of the final state photons.

Questions:

1. The Schwarzschild solution is usually written as:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

   which is not well behaved at $r = 0$ and $r = 2GM$. Discuss the nature and the physical meaning of the two points.

2. Discuss distance measures in cosmology.

3. Is the baryon number conserved in the Standard Model of Particle Physics? Why?

4. Discuss an experimental result relevant for special relativity.