SISSA
Entrance
Examination

Astroparticle Physics Curriculum

2008 spring session

SOLVE two out of the four problems. Concisely answer two out of the six questions.

April 16, 2008

PROBLEM 1.

CONSIDER a 1-dimensional non-relativistic quantum mechanical system formed by two coupled harmonic oscillators with mass M and potential:

$$V(x_1, x_2) = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2 x_2^2 + h(x_1 - x_2)^2,$$

where x_1 and x_2 are the positions of the two oscillators. Find the energy levels of the system in the following cases:

1. $k_1 = k_2$.
2. In the presence of a constant electrical field E_0, assuming both particles have the same charge e and $k_1 = k_2$ as before.
3. $k_1 \neq k_2$, $E_0 = 0$.

Problem 2.

Consider a real scalar field $\phi(x)$ with Lagrangian:

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)\,,$$

being the potential V some function of the field ϕ.

1. Apply Noether's theorem to an infinitesimal translation and find the energy momentum tensor of the field ϕ.

2. Consider the field ϕ close to a minimum of the potential $V(\phi)$. Write down in this case the explicit expression for the energy momentum tensor $T_{\mu\nu}$.

3. Take the expansion of the field in a single plane wave

   $$\phi(x) = \phi_0 \sin(\omega t - \vec{k}\cdot\vec{x})$$

   and average T_{00} and T_{ij} under the assumptions of isotropy and homogeneity. Having identified the energy density and pressure as $\rho \equiv \langle T_{00}\rangle$ and $p \equiv \langle T_{11}\rangle = \langle T_{22}\rangle = \langle T_{33}\rangle$, find the limits in which the field has an equation of state $w = \rho/p$ as appropriate for radiation, dust (matter) and a cosmological constant.

Problem 3.

You are the navigator of a spaceship and you are requested by your captain to place the vessel in an equatorial circular orbit around a black hole of mass M. For simplicity we shall approximate the black hole as a Schwarzschild one and your orbit will be characterized by the following values (in Schwarzschild coordinates (t, r, θ, ϕ)):

$$r = R > 2M, \quad \theta = \pi/2, \quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega \neq 0\,.$$

1. Calculate for the given orbit how the spaceship angular velocity ω must be bound, taking into account that the spaceship must travel slower than light.

2. For a given orbital radius R, what is the value of ω needed for the spaceship to be able to orbit the black hole in free fall? Use the pieces of information obtained so far to derive the range of radii for which there can be a free falling motion.

3. Once in orbit around the black hole at a safety distance $R = 12M$, you receive a S.O.S signal from a spaceship that is as well equatorially orbiting the black hole in free fall. Once corrected for the effects due to the spaceships relative motion, the standard S.O.S. frequency arrives to your ship with a purely gravitational redshift of $\sqrt{5}/2$ (i.e. $\nu_{\text{observed}} = (2/\sqrt{5})\,\nu_{\text{standard}}$) at which radius is located the other spaceship?

Useful info

For the Schwarzschild metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2,$$

the non-zero connections involved in the problem are:

$$\Gamma^r_{tt} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) \qquad \Gamma^r_{rr} = -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1}$$

$$\Gamma^r_{\theta\theta} = -r\left(1 - \frac{2M}{r}\right) \qquad \Gamma^r_{\phi\phi} = -r\left(1 - \frac{2M}{r}\right)\sin^2\theta$$

Problem 4.

Consider a flat Friedmann-Robertson-Walker (FRW) metric described by

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \ ,$$

where $a(t)$ is the cosmological scale factor. It obeys the Friedmann equation

$$H = \sqrt{\frac{8\pi G}{3}\rho} \ ,$$

where G is the Newton constant, ρ the energy density, $H = \frac{1}{a}\frac{da}{dt}$ the Hubble expansion rate.

1. Determine the causal horizon, i.e. the distance r traveled by light from $t = t_1$ to $t = t_2 > t_1$ on a trajectory characterized by constant θ and ϕ, when the energy density is given by non-relativistic particles characterized by $\rho = \rho_1(a/a_1)^{-3}$, where a_1 is the value of the scale factor at t_1.

2. Consider the quantity known as Hubble radius, and defined as H^{-1}. Show that, at the time t_2 and in the limits $t_2 \gg t_1$ and $t_2 \gg H^{-1}(t_1)$, the proper distance associated to the causal horizon, $a \cdot r$, is proportional to the Hubble radius with a proportionality factor of order one.

3. Consider an Universe dominated by a cosmological constant Λ ($\rho = \Lambda/(8\pi G)$). Determine the causal horizon r as in question 1. Argue that at late times, i.e. in the same limits for t_2 as in question 2, $a \cdot r \gg H^{-1}$.

Questions:

1. Explain why, in an expanding Universe, a given cosmological component, which is initially in thermal equilibrium with the rest of the system, becomes eventually decoupled.
2. Use the equivalence principle to argue that light rays are bent in a gravitational field.
3. Briefly discuss an evidence for physics beyond the standard model of particle physics.
4. Compare briefly the main feature of the cold dark matter paradigm versus the hot dark matter model.
5. Briefly discuss one process relevant for the interaction of relativistic electrons with an electromagnetic background in astrophysics.
6. Discuss the high-energy behavior of a two-body scattering cross section.