SISSA
Entrance
Examination

Astroparticle Physics Curriculum

SOLVE two out of the four problems. Concisely answer two out of the five questions.

September 3, 2007

PROBLEM 1.

CONSIDER the Hamiltonian of a one-dimensional quantum harmonic oscillator with mass m and frequency Ω:

$$H = \frac{p^2}{2\,m} + \frac{m\Omega^2 q^2}{2}\,, \tag{1}$$

with p the momentum operator and q the position coordinate.

1. Find the energy of the ground state as the minimum energy allowed by the position-momentum uncertainty relation $\Delta q \cdot \Delta p \geq \hbar/2$, with Δq and Δp the root-mean-square deviations of the corresponding operators.

2. Assume an initial state with the system in its ground-state configuration, and consider a sudden change in the oscillation frequency from the initial value Ω to a new value ω, being $\omega < \Omega$ and the timescale for such transition much shorter than the oscillation periods in the initial and final states. Determine the probability for the system to be in its ground-state configuration after the transition, as a function of the ratio ω/Ω. (Recall that: $\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2) = \sqrt{\pi/\alpha}$.)

3. One can show that the expectation value for the energy of the oscillator in the final state is:

   $$E_f = \hbar\omega \left(\frac{1}{2} + \frac{(\Omega - \omega)^2}{4\omega\Omega} \right)\,. \tag{2}$$

   Suppose that the initial state represents the minimum energy configuration in a early stage of the Universe, and let the sudden transition be an inflationary phase making the Universe expand by a factor of about 10^{27}. The number of quanta in the initial state is $n_i = 0$; what is the number of quanta n_f in the final state?

PROBLEM 2.

CONSIDER the electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in vacuum, i.e. without sources.

1. Write the symmetric, gauge invariant energy-momentum tensor for this field.
2. Show that if $\vec{E} \cdot \vec{B} \neq 0$, the conservation of the energy-momentum tensor implies the equations of motion.
3. Note that the energy-momentum tensor is traceless. Use this to derive the equation of state (the relation between pressure and density) of a homogeneous and isotropic photon gas.
4. The cosmic microwave background has a blackbody spectrum with temperature 2.7 K. Estimate the number density of photons in the background. Use for Stefan's constant $\sigma = 5.67 \cdot 10^{-5}$ erg $\mathrm{s}^{-1}\,\mathrm{cm}^{-2}\ \mathrm{K}^{-4}$, for Boltzmann's constant $k = 1.38 \cdot 10^{-16}\,\mathrm{erg}\,\mathrm{K}^{-1}$, for the speed of light $c = 3 \cdot 10^{10}\ \mathrm{cm}\,\mathrm{s}^{-1}$.

Problem 3.

The 24 satellites of the Global Positioning System (GPS) orbit the Earth in circular orbits, with orbital periods of 12 hours. Each satellite carries an atomic clock on board.

1. Calculate the distance of each satellite from the surface of the Earth (i.e. the orbital altitude) assuming for simplicity a purely circular orbit.

2. Calculate the time shift (in microseconds per day) due to the special relativistic time dilation of a clock on a GPS satellite with respect to a clock at the north pole on the Earth's surface.

3. A prediction of General Relativity is that clocks tick at different rates depending on their distance from a massive object. Calculate the time shift (in microseconds per day) due to this other relativistic effect (assume Schwarzschild geometry for simplicity). How does this compare, in magnitude and sign, to the previously calculated special relativistic effect?

4. If the navigation accuracy required for the GPS system is about 15 meter then the clock ticks from the GPS satellites must be known to an accuracy of 50 nanoseconds. If the two relativistic effects discussed above were not taken into account, how long would it take for the system to fail its stated goal (i.e. how long would it take for the corrections calculated above to equal the required accuracy of the system)?

Problem 4.

The thermodynamical temperature T of the Cosmic Microwave Background (CMB) possesses small fluctuations $\delta T \ll T$ obeying the approximate evolution equation

$$\frac{d^2}{dr^2}\left(\frac{\delta T}{T}\right) + \frac{k^2}{3}\frac{\delta T}{T} = F \tag{1}$$

in the Fourier space, where k is the module of the wavevector $\vec{k}$, F is a constant. The independent variable r, playing the role of time, is actually the distance traveled by photons since the Big Bang at $r = 0$.

1. Find the general solution $(\delta T/T)(k, r)$ of (1) as a function of r, for each given k. By imposing either one of the two sets of initial conditions
$$\frac{d}{dr}\left(\frac{\delta T}{T}\right) = 0 \quad \text{for } r = 0 \text{ (case A)}\ , \tag{2}$$
$$\frac{\delta T}{T} = \frac{3F}{k^2} \quad \text{for } r = 0 \text{ (case B)}\ , \tag{3}$$
determine the values of ks at which $\delta T/T$ has local maxima and minima, at a fixed value of the distance r_d.

2. The origin of CMB occurs at an epoch characterized by a cosmic time $t_d \simeq 300,000$ years, while the present age of the universe is $t_0 \simeq 15$ billions years. Calculate the distance $r_0 - r_d$ traveled by photons between these two epochs, knowing that the null geodesics are characterized by $ds^2 = 0 = c^2dt^2 - a(t)^2dr^2$, where $c = 3 \cdot 10^5$ km/sec and the scale factor describing the cosmic expansion is $a(t) = (t/t_0)^{2/3}$.

3. Assume that r_d, i.e. the distance traveled by photons between the Big Bang and the epoch of origin of the CMB, is approximately 225 Mpc, where 1 Mpc$\simeq 3 \cdot 10^6$ light years. Consider the solutions of (1) with both sets of initial conditions (cases A and B). Determine the angles subtended in the sky by the wavelengths $\lambda = 2\pi/k$ corresponding to the maxima and minima in the two cases, as seen from a distance $r_0 - r_d$. The observations indicate that the pattern of $\delta T/T$ across the sky shows a peak at a scale subtending 1 degree. Which set of initial conditions is supported by this evidence?

Questions:

1. Explain the concept of comoving coordinates and peculiar velocities in cosmology.
2. Discuss the possible formulations and implications of the Equivalence Principle.
3. Discuss gauge invariance in Quantum Electrodynamics.
4. Discuss the relation among the Schrödinger, Heisenberg and Interaction representations of quantum mechanics.
5. Discuss a mechanism for very high energy particle emission in astrophysics.