July 17, 2006

SISSA
PhD
Entrance
Examination

Astroparticle Physics
Curriculum

Solve 2 out of the 5 problems. Concisely answer 2 out of the 5 short questions

Problem 1. Particle decay

A charged pion ($m_\pi = 140\,\text{MeV}$) decays into a muon ($m_\mu = 105\,\text{MeV}$) and an antineutrino ($m_{\bar{\nu}} = 0$ for our purposes): $\pi^- \to \mu^- \bar{\nu}$.

1. Determine the maximum energy of the neutrino as a function of the pion momentum.

2. Determine the minimum energy of the muon as a function of the pion momentum

3. Suppose the pion is at rest. What is the angle between the muon spin and momentum? What is the behavior of the decay rate in the limit in which the muon is massless?

   [Reminder: the pion has spin = 0, the muon and the neutrino have spin = 1/2, the neutrino is considered massless, the pion decay interaction only involves left-handed fields. Hint: use angular momentum conservation]

Problem 2. Dark energy

In cosmology, distances are proportional to a scale factor a, increasing function of the time t. The Friedmann equation relates the cosmological expansion to the energy density ρ and pressure p of the spatially homogeneous cosmic fluid. It is written as

$$H^2 = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3c^2}\rho + \frac{Kc^2}{a^2} \ , \tag{1}$$

where G is the Newton constant, c the speed of light, and K a constant related to the global spacetime curvature; ρ and p obey the conservation equation

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0 \, . \tag{2}$$

1. The data available today in cosmology say that the expansion is accelerated, $d^2a/dt^2 > 0$. Find the condition that ρ and p must satisfy in order to achieve acceleration today.

2. Suppose that the cosmic energy density is made of two components, a non-relativistic one ρ_m with $p_m = 0$ and a dark energy with constant equation of state w, $p = w\rho$. At the present, $\Omega_{de} = \rho_{de}/(\rho_m + \rho_{de}) = 0.75$. Find the condition that w must satisfy in order to achieve acceleration today.

3. The conservation equation (**??**) is supposed to hold for both matter and dark energy separately. Find the ratio a/a_0 between the scale factor when dark energy and matter abundances were equal and the present one, as a function of w, assuming $\Omega_{de} = 0.75$, $\Omega_m = \rho_m/(\rho_m + \rho_{de}) = 0.25$ today.

Problem 3. Neutrinos in the early universe

One typical process maintaining neutrinos in thermal equilibrium in the early Universe is:

$$\nu_e + e^+ \to \nu_e + e^+ \; . \tag{1}$$

1. Sketch the process with the Feynmann diagram technique. Neglecting numerical factors, but taking couplings into account, use dimensional arguments to give an approximate expression for the cross section of this process σ, in the limit of relativistic initial and final state particles (here and in the estimates below, it is useful to use natural units $c = \hbar = 1$).

2. As a rule of thumb, the temperature at which a particle decouples from thermal equilibrium in the Early Universe can be estimated as the temperature at which the interaction rate per particle Γ ($\Gamma \equiv n\sigma|v|$, where n is the number density of target particles and $|v|$ the relative velocity) becomes smaller than the Universe expansion rate H. Derive an order of magnitude estimate of the neutrino decoupling temperature implementing the expression for sigma derived above and assuming that the Universe is in a radiation-dominated epoch; dimensional analysis can be used to infer the expression for Γ and H, neglecting numerical factors, but keeping track of couplings and finding the correct power law scalings in temperature (example: it can be assumed that the average energy of a relativistic particle in a thermal environment is of order T).

3. Suppose the setup we have considered can be extrapolated back in time to very large temperatures; give an estimate of the largest temperature at which neutrinos are expected to be in thermal equilibrium.

For reference, in natural units: the Fermi constant is $G_F = g^2_{\text{weak}}/\left(4\sqrt{2}\, M_W^2\right) \sim 10^{-5}\,\text{GeV}^{-2}$, where $M_W \sim 100\,\text{GeV}$ is the W boson mass; Newton's gravitational constant is $G_N = 1/M^2_{Pl}$, with the Planck mass $M_{Pl} \sim 10^{19}\,\text{GeV}$.

Problem 4. Cosmological scalar fields

The Klein–Gordon action of a real scalar field in a generic gravitational background is

$$S = \int d^4x \sqrt{-\det(\mathrm{g}_{\mu\nu})} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} m^2 \psi^2\right] . \quad (1)$$

Suppose that the field is embedded in the flat Friedmann Robertson Walker metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2) \ , \quad (2)$$

where $a(t)$ is the scale factor (natural units $c = \hbar = 1$ are assumed).

1. Derive the Klein–Gordon equation of motion in terms of a and the Hubble expansion rate $H = (da/dt)/a = \dot{a}/a$.
2. In the limit of an homogeneous field, $\psi(t, \vec{x}) \to \psi(t)$, and for a constant and positive H, show that the solution is always damped in time.
3. Get back to the inhomogeneous case, and write the Klein Gordon equation in the Fourier space by posing $\psi(t, \vec{x}) = \psi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}}$. By using the growing solution of the Friedmann equation

   $$\left(\frac{\dot{a}}{a}\right)^2 = H^2 \quad (3)$$

   with H constant and positive, show that asymptotically in time the equation for $\psi_{\vec{k}}(t)$ becomes formally identical to the one obtained above in the case of homogeneity.

Problem 5. Harmonic oscillators

Consider the following system of two harmonic oscillators, whose Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{8m} + \frac{m\omega^2}{2}x^2 + 2m\omega^2 y^2 \,. \tag{1}$$

1. Compute the energy levels of the system.
2. Add to H a perturbation of the form $V = 4\epsilon m\omega^2 xy$, where $\epsilon \ll 1$. Compute the new energy of the first three levels at the leading order in perturbation theory.
3. Compute the energy levels of all states of $H + V$ exactly and compare the result with the perturbed one.

Question 1. Symmetries and particles

In the context of quantum field theory, point out the difference between local (gauge) and global symmetries in terms of properties of the particle spectrum

Question 2. Time reversal

Explain in a few lines why in quantum mechanics the Time reversal symmetry is described by an anti-unitary operator.

Question 3. Homogeneity and isotropy

The cosmological principle is based on the assumptions of homogeneity and isotropy. Discuss why homogeneity does not imply isotropy, and give a counterexample.

Question 4. Anisotropies in the Cosmic Microwave Background (CMB)

Explain why the anisotropies in the CMB possess a linearly polarized component.

Question 5. Dark matter

Explain the meaning of “dark” in the name, in terms of interactions that the dark matter may or may not have.