The Analogue Gravity Challenge

The bridge from theory to experiment

thesis submitted for the degree of

Doctor of Philosophy

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September 19, 2012
ABSTRACT

Analogue models of gravity have for some time been an excellent arena within which to improve our theoretical understanding of several crucial phenomena at the boundary of gravity and quantum field theory. Recently, however, the field has reached a level of maturity which opens up the possibility of directly probing the purely theoretical results experimentally.

In this thesis I describe the gravitational analogy in three different experimental scenarios and their aims to detect three different phenomena normally associated with wave propagation in strong gravitational fields.

To this end we shall review the basic features of the gravitational analogy, the basic tools and material useful for modeling and go into some detail to the theoretical basis for the analogy in the three systems discussed.

At the end of this thesis we shall discuss the current perspectives of the field and survey the most promising lines of research which can be pursued in the future.
Acknowledgements

I warmly thank my supervisor, Stefano Liberati, for his patience and guidance during the course of my PhD as well as all his good ideas for projects that resulted in such a varied thesis. I also would like to thank all the collaborators I have worked with while completing the work in this thesis. These people include Serena Fagnocchi, Matt Visser, Silke Weinfurtner, Mauricio Richartz, Jason Penner and Joe Niemela. I would especially like to thank Silke without whom the last two chapters of this thesis would have been missing!
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Chapter 1

Analogue Gravity Introduced

Gravity is hard. Specifically, experimenting with gravity is hard. In this chapter I will introduce the intriguing possibility that is very recently becoming a realization, that gravitational kinematics and the behaviour of matter and waves in strong gravitational fields be simulated in laboratory sized systems without experimenting directly with gravity itself.

This chapter serves to introduce the topic of analogue gravity from its historical first beginnings as a description of sound wave propagation in fluids in terms of a d’Alembert equation for a curved spacetime. This is done mainly in the first section of this chapter where we also explore the extent and completeness of the analogy, how to incorporate the vocabulary and how much of gravity can be simulated in this way. After introducing the nuts and bolts we motivate the utility of analogy, which goes beyond a simulation of Einstein gravity, to whet the appetite for the subsequent technical chapters in this thesis.

1.1 Analogue gravity, explained

In real spacetime we have the concept of a metric which describes the causal relationship between events where physical processes are carried out. I will now describe a situation in which a metrical relation which describes the causal and physical relationships present in an acoustic system which involves only sound waves.

The essential first example of an analogue geometry is the the geometry ‘experienced’ by propagating linearized sound waves in an inhomogeneous\(^1\) fluid.

The dynamical equation of motion for linearized perturbations (sound waves) \(\psi\) in a homogeneous fluid is given by the standard no-frills wave equation

\[
(\partial_t^2 - c^2 \nabla) \psi = 0,
\]  

(1.1)

\(^1\)Inviscid, irrotational barotropic (see below for details).
where \( c \) is the sound velocity. As will be demonstrated in detail below, the dynamical equation on motion for such sound waves in an inhomogeneous fluid is given by

\[
\Box_g \psi = \frac{1}{\sqrt{g}} \partial_a (g^{ab} \sqrt{g} \partial_b \psi) = 0, \tag{1.2}
\]

where \( \Box_g \) is the d’Alembertian differential operator associated with the metric

\[
g = \frac{\rho}{c_s} \left[ -(c_s^2 - v^2)dt^2 - 2v_i dt \, dx^i + \delta_{ij} dx^i \, dx^j \right],
\]

\[
= \frac{\rho}{c_s} \left[ -c_s^2 dt^2 + (dx - v dt)^2 \right], \quad g = |\text{Det} \, g|, \tag{1.3}
\]

where \( \mathbf{v}(\mathbf{x}) \) is the flow velocity field of the fluid and \( c_s \) is the sound speed. That is, the generalization for sound waves to an inhomogeneous background fluid is mathematically equivalent to a generalization of the background geometry on which the sound waves travel. The ‘game’ of acoustic geometry at this level is to construct fluid configurations which give rise to interesting acoustic metrics and to study small perturbations on these configurations which behave as if propagating in the acoustic metric geometry.

Below we derive equation (1.2) from the equations of motion for fluids.

### 1.1.1 Geometric acoustics in fluids

The geometric acoustics limit is entirely analogous to the geometric optics limit in optics whereby a sound wave is modeled as a sound ray. Of course in all wave phenomena, as in quantum mechanics\(^2\), there is an intrinsic ‘particle wave duality’ manifested by an uncertainty relation between position and velocity. The geometric acoustics limit is essentially the approximation provided by the particle picture alone. That is, wave-like perturbations are assumed to move along trajectories traced out by their definite position as a function of time.

\(^2\)Although in quantum mechanics this duality has far reaching consequences not implied by the same duality in ordinary wave dynamics.
1.1. Analogue gravity, explained

Figure 1.2: Non-trivial sound geodesics curve around a region of high flow rate. The flow of the river is from left to right in this image.

Concretely the geometric acoustics limit is imposed by considering only the high frequency sound waves in the fluid. Such high frequency modes follow the null geodesics of the acoustic geometry in a manner entirely analogous to the null trajectories of high frequency light rays in spacetime. As null trajectories, their form is independent of the conformal factor $\rho/c$ in the metric $1.3$.

Consider that a sound particle as it propagates at the sound speed $c_s$ in a flowing fluid with a position dependent flow speed $v(x)$. Then as far as the a stationary observer is concerned the sound particle’s motion in the unit $n$ direction is given by

$$\frac{dx}{dt} = v(x) + c_s n.$$ (1.4)

Hence the possible trajectories are defined by

$$||dx - v dt|| = c_s dt,$$ (1.5)

which are precisely the null geodesics of the metric $1.3$.

As an illustration of curved nature of effective geometry experienced by sound, consider a source of sound waves down-stream of a submerged boulder in a river schematically shown in Figure 1.2. One can imagine that the surface flow directly on top of the boulder is faster than the flow on either side. Intuitively, then, an observer upstream and co-linear with the source and boulder will receive first those sound waves that actually propagate around the boulder and not over the top of it. This observation is strongly reminiscent of the curvature of the ‘shortest path’ between points in spacetime by the presence of a non-trivial gravitational field. Therefore we can appreciate that, at
the very least, we should not be surprised to find utility in the methods of differential geometry for the description of sound waves in inhomogeneous flowing fluids.

Of course, at extremely high frequency / short wavelengths the continuum approximation for the fluid breaks down and the picture of dispersion-less sound rays propagating in the fluid should fail. In fact, it is in this limit that we may learn our most important lessons from the analogy; in the space of all possible modifications to wave propagation there should exist a subspace of ‘most plausible’ modifications which we might glimpse through the analogy.

1.1.2 Physical acoustics

Physical acoustics corresponds to the full dynamical description of the ‘field’ of sound fluctuations in the fluid including all the wave-like features in the propagation. The analogy for physical acoustics is therefore more comprehensive than for the null geodesics of geometric acoustics. In the physical acoustics picture we linearize perturbations around a solution to the full fluid mechanical equations of motion to obtain a scalar wave equation identical to that of a scalar field in a curved spacetime. This is done in the next section in full detail where we reproduce the full equation of motion (1.2).

1.1.3 The nitty gritty details of physical acoustics

Historically the first and also the simplest example an analogue geometry is the physical acoustics picture for sound propagation in fluids [1]. We will derive the result (1.2) in 3+1 dimensions, the result being dimension dependent in general.

It should be made clear as an aside at this point that the analogy is not restricted only to acoustics in flowing fluids. We will include at the end of this chapter a partial list of systems for which the analogy has been established and investigated.

Consider an irrotational barotropic perfect fluid. Barotropic means that the pressure is a function of only density and irrotational means \( dv = \nabla \times v = 0 \) where \( v \) is the velocity vector field. The Poincare lemma for the plane ensures then that \( v \) is the gradient of a scalar: \( v = df \) for some function \( f \). This is the simplifying assumption of irrotationality; the fluid dynamics reduces to that of a scalar field. The assumption of barotropicity is a statement that the pressure should depend only on the scalar density profile and prevents the generation of vorticity due to non-scalar pressure effects. A barotropic fluid has surfaces of constant pressure that coincide with surfaces of constant density. More generally in non-barotropic fluids the surfaces of constant pressure intersect the surfaces of constant density and the character of these intersections provides information about the fluid. The equations of motion for a invicid perfect fluid
are given by the coupled partial differential equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{(Continuity equation),}$$  \hspace{1cm} (1.6)

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p \quad \text{(Euler equation).}$$  \hspace{1cm} (1.7)

Which under the assumptions listed above reduce to

$$\frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot (\nabla f) + \rho \nabla^2 f = 0 \quad \text{(1.8)}$$

$$\rho \left( \frac{\partial}{\partial t} \nabla f + \frac{1}{2} \nabla (|\nabla f|^2) \right) = -\nabla p \quad \text{(1.9)}$$

These equations are supplemented by the barotropic equation of state $p = p(\rho)$. Of course the assumption of irrotationality can only be applied to the initial velocity configuration; it is up to the equations of motion to forbid the generation of a curl component under time evolution from an initially irrotational flow. We now demonstrate this fact. We have by (B.1)

$$\frac{\partial}{\partial t} \nabla \times \mathbf{v} = \nabla \times \frac{\partial \mathbf{v}}{\partial t} \quad \Rightarrow \quad \frac{\partial}{\partial t} \nabla \times \mathbf{v} = -\nabla \times \nabla \rho - \frac{1}{2} \nabla \times \nabla (|\nabla f|^2). \quad (1.10)$$

$$\frac{\partial}{\partial t} \nabla \times \mathbf{v} = -\nabla \times \nabla \rho - \frac{1}{2} \nabla \times \nabla (|\nabla f|^2). \quad (1.11)$$

Now, the second term vanishes by the well known fact that the curl of a gradient vanishes. For the first term we need to do some work. As mentioned above, for a barotropic fluid the surfaces of constant pressure coincide with the surfaces of constant density and hence that the vectors $\nabla p$ and $\nabla \rho$ are parallel. We have

$$\nabla \times \frac{\nabla p}{\rho} = \nabla \rho^{-1} \times \nabla p = \rho^{-2} \nabla \rho \times \nabla p = 0, \quad (1.12)$$

so that we have demonstrated the assertion.

So, we are ready to proceed. Consider an exact barotropic solution to (B.1) and \((f_0, \rho_0, p_0)\) and small perturbations thereof

$$f = f_0 + \epsilon f_1, \quad \rho = \rho_0 + \epsilon \rho_1. \quad (1.13)$$

We have

$$p(\rho_0 + \epsilon \rho_1) = p(\rho_0) + \epsilon \rho_1 p'(\rho_0) + O(\epsilon^2) =: p_0 + \epsilon p_1, \quad (1.14)$$

\[3\text{This is an expression of the general fact that } d^2 = 0\]
Chapter 1. Analogue Gravity Introduced

so that \( p_1 = p'(\rho_0) \rho_1 = c^2 \rho_1 \) where we define the sound speed \( c^2 = p'(\rho_0) \). Inserting this ansatz into (B.1) and (2.121) and collecting terms one obtains

\[
\begin{align*}
\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_1 \nabla f_0 + \rho_0 \nabla f_1) &= 0 \quad (1.15) \\
\frac{\partial f_1}{\partial t} + \nabla f_0 \cdot \nabla f_1 &= -\frac{p_1}{\rho_0} \quad (1.16)
\end{align*}
\]

The last equation is derived by using the expansion

\[
\nabla (p_0 + \epsilon p_1) = \nabla p_0 + \epsilon \nabla \left( \frac{p_1}{\rho_0} \right), \quad (1.17)
\]

and also we have neglected any zero mode constant fluctuations in the fluid by setting an integration constant to zero.

Eliminating both \( p_1 \) and \( \rho_1 \) we arrive at

\[
0 = \frac{\partial \rho_1}{c^2 \partial t} f_1 + \frac{\partial \rho_0}{c^2} \delta_{ij} v_i \partial_j f_1 + \frac{\partial \partial_{ij} p_0}{c^2} v_j \partial_i f_1 \partial_j \rho_0 \partial_j f_1 + \frac{\partial \partial_i p_0}{c^2} v_i v_j \partial_j f_1, \quad (1.18)
\]

which we rewrite, after dropping the subscript on \( f \), in the surprisingly familiar form

\[
\frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} \partial_b) f = 0, \quad (1.19)
\]

where we define the matrix of coefficients

\[
g^{-1} := \frac{1}{c \rho_0} \begin{pmatrix}
-1 & \cdots & -v \\
\cdots & \cdots & \cdots \\
-v & \cdots & c^2 1_3 - v \otimes v
\end{pmatrix}, \quad g = \frac{\rho_0}{c^2} \begin{pmatrix}
-(c^2 - v^2) & \cdots & -v \\
\cdots & \cdots & \cdots \\
-v & \cdots & 1_3
\end{pmatrix}, \quad (1.20)
\]

and the determinant \( g := \det g^3 \).

This result provides a demonstration of the claims laid in the introduction of this section, namely that traveling waves in a sufficiently idealized inhomogeneous fluid are exactly described by a Klein Gordon equation on a non-flat Lorentzian spacetime.

1.2 The motivation - what’s the problem with real gravity?

At first the fact that the causal structure of sound waves is encoded in a non-flat curved geometry might seem to be a curiosity with limited physics implications.

\footnote{In 1+D spacetime dimensions the metric differs in the power of the conformal factor appearing in the definition of \( g \). It can be shown that the conformal factor in 1+D dimensions is given by \((\rho_0/c)^{2/(1-D)}\) being undefined in 1+1 dimensions.}
1.2. The motivation - what’s the problem with real gravity?

By now the notions of analogue gravity or analogue geometry are mainstream in the gravity community. Many approaches to the problems faced by our current theory of gravity attempt to go beyond the continuum Lorentz invariant picture and replace spacetime with some discrete (and inherently quantum) structures. Indeed, it is widely expected that the continuum is manifested merely as a coarse grained representation of some more fundamental degrees of freedom in the deep ultraviolet. The subject of analogue gravity probes the realm between the discrete picture of quantum spacetime and the continuum ‘real world’ approximation. In particular analogue gravity aims to shed light on the semi-classical regime where matter fields are described by quantum field theories but gravity is assumed to be a classical background field. This regime is expected to cover a vast terrain and moreover is the fertile seed-bed for an array of exotic quant-gravitational phenomena, such as black hole radiance, cosmological particle creation and superradiant scattering, which serve as way-stones on the road to real quantum gravity.

Observability

One might ask why it is not enough to describe these phenomena directly, why we would want to simulate them in some complicated terrestrial experiments. The answer lies in the glaring fact that none of the results of semi-classical gravity, where gravity and quanta are simultaneously involved, have been probed experimentally. This lack of observability is due to many reasons:

- Many of the process are quantum spontaneous emissions from the vacuum occurring in far off places in the universe,
- They usually occur in strong gravitational fields (not Earth-strength ones) or highly accelerated frames,
- The fluxes are small,
- The spectra are usually quasi-thermal and the temperatures typically low

Analogues alleviate most if not all these difficulties:

- Powerful modern experimental techniques are capable of directly observing the quantum nature of matter

---

5 As we will see, in fact we only probe semi-classical gravity in the test-field approximation where the back-reaction of quantum fields on the classical metric, through the renormalised stress energy tensor is neglected.

6 Even non-gravitational spontaneous quantum emission from the vacuum has proved to be extremely difficult to probe experimentally – see [2] for a recent breakthrough in this area.

7 In particular we have in mind here Bose–Einstein condensates and quantum optical systems.
Analogue gravitational fields have nothing to do with large masses or high flow rates\(^8\).

Since analogue horizons are not causal horizons for the experimenter, he can use other means to directly observe ‘beyond the horizon’ allowing for correlation measurements between emitted partners\(^9\).

Cold atom experiments are capable of measuring very low temperature thermal ensembles.

**Separation of kinematic from dynamic gravity**

Instead of solving complicated non-linear dynamical equations for the gravitational field (for example the metric tensor and possibly also an independent connection or torsion), in analogue gravity one finds an effective geometry (read: gravitational field) directly in the equations of motion of some analogue system upon which some dynamical processes are taking place, usually wave propagation. Hence the analogy isolates only the kinematic aspects of gravity. We cannot expect, therefore, to find some equivalent of the Einstein equations describing the interaction between the matter fields and the analogue geometric tensors.

The central questions of the analogue gravity program are most succinctly summarized by Visser \([4]\)

- How much of general relativity can you carry over into these condensed matter analogues?

- How easy is it to actually build these things in the laboratory and do a few experiments?

The natural secondary but highly significant question to ask once we agree that at least some relativity is contained in the analogy is

- Can we model quantum fields in curved spacetime on these analogue geometries?

It turns out that, indeed, many of the important results of semi-classical gravity are not directly dependent on the dynamical form of the Einstein equations and are instead, really kinematic results putting us in a position to model them with analogues. Importantly, the Hawking emission falls into this category (see \[2.1.3\] for the details of this argument).

\(^8\)Very high surface gravities are evidently achievable – see \([3]\).

\(^9\)This is partly the theme of Chapter \([3]\).
The problem with black holes

There is also a significant quantity of theoretical work on these more general analogue spacetimes, and their implications regarding the Hawking effect [5, 6, 7, 8, 9, 10].

Probably the most significant semi-classical gravity result is the establishment of Hawking radiation first described by Hawking in 1974 [11]. Arguably, Hawking’s results on black hole radiation and their embedding into the more general framework of black hole thermodynamics serves as a benchmark for any quantum theory of gravity. However, the derivation of the existence of Hawking radiation is not strictly semi-classical: Hawking photons observed at late times at null infinity necessarily possessed an arbitrarily high frequency and hence energy in their past making the neglect of back-reaction questionable. These Hawking photons are virtual photons and hence one should carry out a more complete analysis of the renormalised stress energy tensor in order to understand back-reaction effects beyond the heuristic picture of the redshift argument.

In this sense Hawking radiation serves as a microscope which is sensitive, with arbitrary precision, to the microstructure of spacetime and quantum field theory. It is probably too much to believe in the validity of a classical smooth spacetime on arbitrarily short length scales: one expects both ordinary quantum field theory and our current geometric gravitational theories to break down or at least be significantly modified in the deep ultraviolet.

In fact there are good reasons to have doubts that physics is invariant under arbitrarily large boosts. Firstly there are the UV divergences of ordinary QFT – renormalisation is the idea that one can ignore very high energy processes during low energy experiments and it is well known that renormalisability of gravitational theories is problematic, possibly requiring a breakdown of boost invariance in the high UV [12, 13]. Secondly, and more in the specific context of black holes, there is an infinity in the entanglement entropy of a quantum field [14, 15] which again, would be tamed by some form of Lorentz violation in the UV [16, 17].

Therefore, the possibility of producing in the laboratory the geometric (kinematic) features pertaining to the onset of a Hawking flux is of great interest. Crucially, in the fluid and condensed matter analogues the continuum picture of sound quanta propagating on an effective geometry will break down for high frequency phonons in an entirely predictable way. That is, we know for sure that the fluid mechanical equations of motion should be replaced by molecular dynamical ones for disturbances approaching the inter-molecular spacing. Thus in analogue gravity we are able to isolate those features of black hole physics that depend on the unknown ultraviolet regime.
Quantum gravity phenomenology

The analogy between gravity and condensed matter systems is useful for two reasons: on the one hand it allows us to experimentally test some predictions of Lorentz invariant (LI) physics by analogy; on the other hand it provides for us the possibility of testing the physical and mathematical consequences of relaxing exact LI. As we will see, an analogue LI description is always approximate, applying only in a window of scales in the analogue system, with unavoidable violations near the edge of the window. It would be very convenient if we could model LI physics entirely inside the analogue LI window but, as we shall see, this is not always possible. This can either be seen as a stumbling block for the simulation of LI physics or as an opportunity to test, observe and experiment with Lorentz violating (LV) versions of some semi-classical gravity phenomena. The general wisdom would be that if we can see these phenomena in fluids, then their spacetime counterparts do not depend in a essential way on the details of physics in the UV.

The idea that LI might not be an exact symmetry of nature has a long and convoluted history, enjoying a strong revival recently [18]. However, while spacetime LV is not a topic which we shall be considering explicitly in this thesis, it suffices to say that analogue models are an arena where exact LI is broken and in an understandable, predictable and testable way. Indeed LV, black-hole physics, quantum gravity phenomenology all find a natural expression in analogue models and should be kept in the back of every analogue gravity-est’s mind.

1.3 The analogy developed

The next step in developing the analogy is to try to find background fluid configurations whose associated analogue geometry corresponds to a known exact solution to the Einstein gravitational field equations. This is the content of the rest of the following few short subsections.

1.3.1 Modeling spacetime with fluids

It is one thing to rewrite fluid mechanics in a fancy new way but is there any physics at stake here? In fact, as we will show below, it is possible to construct analogue (equatorial) Kerr and Schwarzschild geometries in fluids allowing us, in the very least, to import the techniques and known results from spacetime physics on these backgrounds to the study of fluid perturbations in the lab. Our real interest in this analogy, of course, is in the information flow in the opposite direction. The question is ‘what can we learn about spacetime physics by studying fluid perturbations?’ For example, it might be useful to find the analogue of the concepts of event horizon, apparent horizon
1.3. The analogy developed

or ergo-region in fluids, independently from their realisation in the exact Kerr and Schwarzschild cases, in order to elucidate the nature of the logical interconnectedness of the network of ideas that paint the picture of black hole physics - local Lorentz invariance, continuum approximation, causal boundaries, etc - in the ‘real world’.

Consider the generic acoustic line element given above in equation (1.3) The first piece of vocabulary that can be imported to the fluid dynamics is that of the ergo region. We see that the metric coefficient \( g_{tt} \) becomes singular for \( x \) such that \( c^2(x) = v^2(x) \). It can be shown that the analogue spacetime curvature evaluated at these points is non-singular and hence, as in the case of the Schwarzschild horizon, these points are not true singularities of the geometry. We define the ergo region as the set of points

\[
E := \{ x \mid ||v(x)|| > c(x) \},
\]

the boundary of which is called an ergo surface. In the ergo region the co-moving frequency of a wave is negative when the laboratory frequency is positive and we will see in the next chapter that ergo-regions are associated with the phenomenon of superradiance.

Another important concept in mathematical relativity is that of a trapped surface. In the fluid analogues here we define a trapped surface as a two dimensional surface where the normal component of the fluid velocity is everywhere greater in magnitude than the local speed of sound. Compare this to the rather sophisticated definition of trapped surface in relativity [19]. The reason we are allowed such a simple definition here is that we have access to a preferred background ‘lab’ frame which provides an objective definition of velocity.

A horizon is defined in a similarly straightforward way as the boundary of the region from which null geodesics of the metric (1.3) can emerge. They are a kind of causal singularity beyond which sound waves cannot emerge.

Assuming that the flow \( v \) is time independent, the metric (1.2) can be brought into the manifestly static form

\[
ds^2 = \frac{\rho}{c} \left[ -(c^2 - v^2) d\tau^2 + \left( \delta_{ij} + \frac{v_iv_j}{c^2 - v^2} \right) dx^i dx^j \right],
\]

by way of the coordinate transformation

\[
d\tau = dt + \frac{v_i dx^i}{c^2 - v^2}.
\]

It is well known how to define the surface gravity of such a static geometry at a homogeneous space-like hyper-surface. The idea is to take the obvious time-like Killing vector possessed by this metric

\[
k = \partial_\tau,
\]
normalize it

\[ K = \frac{k}{\sqrt{ds(k,k)}} = \frac{k}{\sqrt{(\rho/c)(c^2 - v^2)}}, \]  

compute its acceleration,

\[ a = (K \cdot \nabla)K = \frac{1}{2} \left( \frac{\nabla(c^2 - v^2)}{c^2 - v^2} + \frac{\nabla(\rho/c)}{\rho/c} \right), \]  

and finally finish the arithmetic by taking the limit as one approaches the surface of interest

\[ \text{surface gravity} = \lim_{v^2 \to c^2} \sqrt{ds(a,a)} ds(k,k) \]  

\[ = \frac{1}{2} v \cdot \nabla(c^2 - v^2) \]

We can re-write this in terms of the normal derivative to the acoustic horizon

\[ \text{surface gravity} = c \frac{\partial(c - v)}{\partial n}. \]

Now that we have surface gravity (essentially, as mentioned above, a measure of the acceleration of the fluid at the acoustic horizon) and horizon we have all the ingredients to follow the Hawking derivation of Hawking radiation including an ignorance of the microphysics at extremely short distances.

### 1.3.2 Fluid black holes

Can we make a fluid flow such that the acoustics are those of a scalar field in a black hole geometry? Here we describe how the answer is ‘yes, morally’ for non-rotating black holes but ‘no’ for rotating ones changing to ‘yes’ when we relax the requirement to modeling only the equatorial plane of the Kerr solution.

**Schwarzschild geometry**

Assume a spherically symmetric velocity potential with a constant (that is position independent) speed of sound \( c \) and a background velocity field

\[ v = \sqrt{\frac{2GM}{r}} \partial_r. \]  

The continuity equation reduces to the divergence

\[ \nabla \cdot (\rho v) = 0, \]
which in polar coordinates and purely radial flow becomes

\[ \frac{2}{r} \rho \sqrt{\frac{2GM}{r}} + \frac{\partial}{\partial r} \sqrt{\frac{2GM}{r}} = 0, \]  

(1.32)

which is easily solved to yield

\[ \rho \propto r^{-3/2}. \]  

(1.33)

Integrating the constancy of the speed of sound equation \( c^2 = dp/d\rho \) we deduce the equation of state

\[ p = p_\infty - c^2 \rho \Rightarrow p - p_\infty \propto c^2 r^{-3/2}, \]  

(1.34)

providing us with a concrete expression for the metric coefficients as functions of \( r \)

\[ ds^2 \propto r^{-3/2} \left[ -dt^2 + \left( dr \pm \sqrt{\frac{2GM}{r}} dt \right)^2 + d\Omega^2 \right], \]  

(1.35)

where the ingoing and outgoing perturbations see the + and − forms of the metric respectively. Without specifying the exact solution to the continuity equation we are stuck with the proportionality equation. Now, as gravitational physicists we are supposed to recognize in this metric the Schwarzschild metric written in the slightly non-standard Paillevé-Gullstrand coordinate system multiplied by the conformal factor \( r^{-3/2} \).

We can see that we have an acoustic horizon at the usual place \( r = 2GM \) in the Schwarzschild geometry. There is only one caveat: As mentioned above, we have not really reproduced the Schwarzschild geometry but actually one that is conformally related to it.

It can be shown that the Hawking effect is conformally invariant, the temperature being related to the surface gravity of the horizon which is insensitive to the conformal factor. The only place where the conformal factor enters the physics is in the back scattering from the spacetime curvature, the so-called ‘grey-body’ factor. Hence, practically, the conformal factor should not be such a big problem if all we want to do is re-produce the Hawking effect in the lab.

**Kerr geometry**

The generalization of acoustic Schwarzschild to acoustic Kerr is highly non-trivial. In fact it has been shown [20] that the simulation of Kerr geometry in fluids, even up to a conformal factor, is impossible. The basic idea is that the acoustic metric must necessarily possess conformally flat spatial sections as can be seen in its form (1.20) whereas

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10 In a general coordinate system the divergence of a vector field is written as \( g^{-1/2} \left( \sqrt{g}^{1/2} \xi^a \right)_a \).
the Kerr metric does not possess this algebraic feature. However, it can be shown then the (2+1 dimensional) equatorial plane of the Kerr geometry is implementable as an analogue geometry as in (1.3) by some judicious coordinate changes.

In Boyer-Lindquist coordinates the equatorial sections are described by the metric

$$ds^2 = dt^2 + \frac{2m}{r} (dt - ad\phi)^2 - \frac{dr^2}{1 - \frac{2m}{r} + \frac{a^2}{r^2}} - (r^2 + a^2) d\phi^2,$$  \hfill (1.36)

which can be manipulated into the form (1.3) by the change of coordinates from \(r\) to \(\tilde{r}\) defined by [21]

$$\frac{1}{\tilde{r}} \frac{d\tilde{r}}{dr} = \frac{1}{\sqrt{(1 - \frac{2m}{r} + \frac{a^2}{r^2})(r^2 + a^2 + \frac{2ma^2}{r^2})}}.$$  \hfill (1.37)

One finds the form (1.3) in 2+1 dimensions with

$$v_r = 0$$

$$v_\phi = \frac{2am}{H^2(\tilde{r})r(\tilde{r})} \frac{1}{1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}},$$  \hfill (1.38)

where the function \(H(\tilde{r})\) describes the inverse transformation to (1.37) between \(r\) and \(\tilde{r}\).\footnote{To get a handle on \(H\) we note that when \(m \to 0\) we have \(r(\tilde{r}) = \sqrt{\tilde{r}^2 - a^2}\) in which case \(H(\tilde{r})\) approaches the function \(\sqrt{\tilde{r}^2 - a^2/\tilde{r}}.\)
}

$$r = \tilde{r}H(\tilde{r}).$$  \hfill (1.39)

We also have a position dependent sound speed \(c_s\)

$$c_s^2 = \frac{1 - \frac{2m}{r} + \frac{a^2}{r^2}}{H^2 \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}\right)},$$  \hfill (1.40)

where \(r\) is to be thought of as a function of \(\tilde{r}\) here.

The point of writing these complicated formulae down is to convince the reader that, indeed it is possible to find some fluid configuration which simulates the equatorial Kerr geometry but that its form is rather complicated and very likely not going to occur for naturally flowing fluids without external driving forces.

Crucially, however, the concepts of ergo-region and horizon do not depend on reproducing exactly the Kerr solution of GR: we expect to be able to reproduce ergo-regions and horizons in a ‘natural’ fluid configuration which does not match the Kerr solution and which can support an analogue Hawking or superradiance process. We will discuss such a possibility later in [6] in the case of a ‘naturally’ occurring bathtub vortex flow.
1.3.3 The expanding universe

We will make use of the analogy later on to model an expanding universe described by a homogeneous FRW line element

\[ ds^2 = -dt^2 + a^2(t)dx^2. \] (1.41)

One way to achieve this geometry experimentally is to find a homogeneous fluid in which the sound speed is controllable in time. We could then work with zero flow and make a coordinate transformation

\[ ds^2 = \Omega^2(t) \left[ -c_s^2(t)dt^2 + dx^2 \right] \]
\[ = -d\eta^2 + \Omega^2(\eta)dx^2, \quad \text{where,} \quad \eta(t) = \int^t \Omega^2(t')c_s^2(t')dt'. \] (1.42)

Indeed such fluids exist with a controllable sound speed, we will discuss them in Chapter 3. In that chapter we will also show how an inhomogeneous cosmological metric can arise from the naturally occurring flow of a particular kind of fluid associated with Bose Einstein condensation and its controlled release from a confining trapping potential.

1.3.4 Practical considerations

Naturalness – It should be stressed that, although we have presented the analogy and how it can be used to model real spacetime configurations of ‘practical’ interest to a gravity theorist, the problem of initializing and sustaining the fluid or condensed matter configuration which induces a particular acoustic geometry has been completely unaddressed to this point. We should not expect to be able find fluids in nature that behave in these non-standard ways under the influence of, say, gravity alone. In practice we should expect to have to force the fluid to behave as we require. In the presence of an external driving force the Euler equation (1.7) is modified to include the extra term on the right hand side

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \mathbf{F}_{\text{driving}}, \] (1.44)

which can be used to drive the fluid into a particular flow state such as those required above for the simulation of the Kerr equatorial plane.

Another potential difficulty in practice for these fluids is to initiate and sustain these high velocity flows without introducing turbulence. Throughout we have been using fluid equations suitable for laminar flow - flow without turbulence. It has been suggested that such flow conditions should be easiest to construct at extremely low temperatures.
Completeness – Immediately we can see from (1.3) that the analogue metric does not have enough degrees of freedom to simulate an arbitrary gravitational field. In general in $n$ spacetime dimensions the gravitational field is described by $n(n - 1)/2$ independent degrees of freedom which in 3+1 dimensions are 6. The analogue metric possesses only 4 degrees of freedom, the 3 components of the velocity $v$ and the density $\rho$.

Quanta – We have blithely been talking periodically up to this point about classical fluids and quantum spontaneous emissions in the same breath. We shall see below that there exists a wider class of models for analogue geometry than those for a simple flowing fluid presented up to this point, including models which utilize the quantum nature of the constituent matter in an essential way. Further on in this thesis we will be discussing the restricted problem of understanding some of the modified physics mechanisms that come part-and-parcel with broken (analogue) Lorentz invariance when it comes to the Hawking process or superradiance. This is part of a more general program of research which seeks to probe the principles, conditions and mechanisms necessary for a spontaneous emission, even if, in practice, there will be no spontaneous emission process in the system. In this direction one experiments directly with classical scattering in the inhomogeneous fluids (numerically or physically), inferring the physics of the quantum spontaneous emission by general arguments.

Another angle to the question of quanta is given in the recent article [22] where Unruh has demonstrated the equivalence between quantum and classical amplifiers in the context of analogue models. According to Unruh, a black hole is nothing but an amplifier of the quantum field vacuum fluctuations, the black hole (or white hole) radiance can be understood as a continual scattering of vacuum modes by the geometry. When dispersion is included, an additional mechanism for producing outgoing particles is available (‘mode conversion’) which intrinsically involves the dispersive nature of the propagation, but the analogy follows through identically: dispersive spontaneous emission and dispersive stimulated classical emission are two examples of the same amplification process. The question is, what can we hope to learn about the quantum spontaneous emission by looking at classical stimulated emission (scattering)? Unruh has answered this question by saying that the Bogoliubov coefficients in the classical scattering experiment match those for the quantum spontaneous emission since both are based on the same underlying amplifier model. Indeed, in the article [23], the authors study the scattering (and mode conversion) of an incident, classical water wave onto a white hole flow profile and claim to be confirming the ‘Hawking mechanism’. This is possible since the Bogoliubov coefficients of the scattering process $\beta_k$ measuring the amount of mode conversion and scattering by the flow exactly match the Bogoliubov coefficients measuring the spontaneously emitted ‘quanta’, even though we never hope to observe such quanta, nor do we really believe in the quantization of water waves; the quantum amplification is determined exactly by the classical behaviour of the amplifier.

Finally, there can be cases where a perfectly well defined analogue metric and geom-
etry emerges for the perturbations of a system but where the commutation relations for the fields associated with such perturbations are not the canonical ones (for example, see the article on the ‘slow-light’ proposal \cite{24} where the highly non-linear dispersion relation can interfere with the emergence of the correct commutation relations for the quantum theory to pass through the analogue). In general, one requires both the metric and commutation relations to emerge correctly in an analogue in order to carry out the analogue derivation of a QFT process on the curved geometry.

1.4 Summary

In this chapter we have introduced the analogy in simple and intuitive terms deriving the basic first example of the analogy for sound in a moving fluid.

The motivation and larger picture into which the analogue program should be embedded has been pointed out and we have entered into the discussion of the practical aspect of analogue gravity which, after all, is the crucial element and perhaps only reason to be thinking about analogues at all.

Cast in a different light, this possibility of describing terrestrial experiments using a curved spacetime analogue provides one with a new set of tools to tackle difficult problems in condensed matter physics and is able to suggest new and interesting non-equilibrium quantum effects. Moreover, a successful application of these alternative techniques and consequent observation of the effects predicted would serve as further theoretical support for the validity of the semiclassical theory.
Chapter 2

A Toolbox for Analogue Gravity

In this chapter we collect and dissect a toolbox for the aspiring analogue gravity-est. Often we point the reader to the relevant literature where a complete discussion is unnecessary. Due to the interdisciplinary nature of analogue gravity we necessarily require a very general toolbox touching on a wide range of topics from fluid mechanics, Bose-Einstein Condensation (BEC), nonlinear optics, semi-classical gravity phenomena and non-linear wave phenomena. In no way is this toolbox intended to be complete, however, and we include a partial list of excluded topics at the end of the chapter. The choice of topics reflects the necessary background for what is presented in the following chapters.

In the first section we introduce some phenomena from quantum field theory in curved spacetime. These are the main actors in analogue gravity since these are the phenomena which we are trying to simulate\(^1\) in condensed matter systems through the analogy. We discuss the Dynamical Casimir effect (DCE) / cosmological particle creation (isolating a conflict of terminology that exists in the literature), Hawking radiation and superradiant scattering. In the second section we dissect Bogoliubov coefficients a little, expanding on some alternative ways to compute them. In the following three sections we present basic material on three different analogue systems and how there can exists a notion of an effective metric for excitations. In the final section we discuss modified dispersion in general, the useful tools and some of the dynamical consequences.

2.1 Quantum fields and gravity together

In this section we discuss three separate phenomena of quantum fields in curved spacetime. General references for quantum fields in curved spacetime are [25, 26, 27, 28].

\(^1\)Equivalently, generalise.
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For black hole radiation the contemporary treatments of Jacobson is most illuminating [27]; for superradiance the reviews [29] and the works of [30] are recommended; for dynamical Casimir effect / cosmological particle production the text of Birrell and Davies is still the best reference [25].

2.1.1 Dynamical Casimir Effect - Cosmological particle creation

Within the framework of quantum field theory in curved spacetime where fields propagate on a fixed background, some interesting – and sometimes surprising – quantum instabilities are predicted. Among them a crucial role has been played by the so called cosmological particle creation: a quantum emission of particles as a consequence of an expansion of spacetime.

Also known as Dynamical Casimir effect (DCE), the perhaps confusingly named quantum emission is a very general phenomenon, being associated with wave propagation over a dynamical space and not (when back-reaction is neglected) to a property of the Einstein equations, it is not confined to gravitational physics. Originally discussed in the gravitational context by Parker [31, 25] who studied particle production in FLRW spacetimes, the Dynamical Casimir Effect (DCE) has found applications in an extremely diverse range of subjects (see for example the excellent text [32] and the review article [33] which is mainly focussed on the moving boundary type DCE which we do not discuss in this thesis).

Very simply put, as far as we are concerned here DCE is the observation that a wave equation with some time dependent parameters do not admit global plane waves as solutions. This is a very trivial and unsurprising fact but the physical implications are rather far-reaching when it is embedded into the language and formalism of QFT.

To illustrate the physics at stake here let us take the very simple example of the ordinary wave equation in flat spacetime

$$\partial_t^2 f = c^2 \nabla^2 f.$$  (2.1)

The dynamics described by (2.1) is equivalent to that of a collection of harmonic oscillators. Expanding $f$ in eigenfunctions of the spatial Laplacian $\nabla^2$

$$f(t, x) = \sum_k A_k \lambda_k(t, x), \quad \lambda_k(x, t) = f_k(t) e^{i k \cdot x},$$  (2.2)

each coefficient $f_k$ satisfies the equation of motion for a simple harmonic oscillator

$$\left(\partial_t^2 + \omega_k^2\right) f_k = 0,$$  (2.3)
where $\omega_k^2 = c^2 k \cdot k$ depends only on the magnitude $k = \sqrt{k \cdot k}$ of the wavenumber $k$. Crucially, when $c$ is space and time independent, Eq. (2.3) admits global plane wave solutions

$$ f_k = e^{\pm i\omega_k t}, \quad (2.4) $$

providing the global solution to Eq. (2.1) as a linear combination of traveling plane waves

$$ f(t, x) = \sum_{k, \pm} A_{k, \pm} e^{\pm i\omega_k t + i x \cdot k}, \quad (2.5) $$

the constant coefficients being determined by the initial condition. Note that for each $k$ there are two frequencies $\pm \omega_k$ (where we always assume $\omega > 0$ and indicate the negative partner with an explicit minus sign) — this doubling of the degrees of freedom will be crucial for the physics of particle production. That an arbitrary initial condition can be reconstructed is a non-trivial functional analytic fact made possible by the completeness of the plane wave ‘basis’ $\{e^{i x \cdot k}\}$ which we can express as a ‘decomposition of unity’ relation

$$ \sum_k e^{i k(x-x')} = \delta(x - x'). \quad (2.6) $$

**Frequency or norm?** — Associated with the equation of motion (2.1) is an inner product

$$ (f, g) = i \int dx \left[ f^* \partial_t g - g \partial_t f^* \right], \quad (2.7) $$

which is independent of time when both $f$ and $g$ satisfy the equation of motion (2.1). This inner product is not really an inner product; however, since it is not positive definite: for fixed constant frequency $k$ the solutions with time dependence $\exp(+i\omega t)$ have *negative norm* whereas for $\exp(-i\omega t)$ the norm is positive. Hence we usually abuse language a bit and say that such negative norm modes are *negative frequency*. In some interesting situations, for example in the ergo-region of a rotating black hole, the frequency can be positive while the norm be negative. The splitting of the Hilbert space is always done with respect to the positive and negative norm states. Hence we will use negative norm to describe the components such as $\exp(i\omega_k t)$ here and in what follows in this thesis but occasionally write ‘negative frequency’ when no confusion is possible.

Now, when $c$ is time dependent (which could happen, for example, when the background spacetime is expanding), the steps leading up to Eq. (2.3) remain valid but the equation of motion now includes the time dependent frequency

$$ \left( \partial_t^2 + \omega_k^2(t) \right) f_k = 0, \quad (2.8) $$

where $\omega_k^2(t) = c^2(t) k \cdot k$. Usually we are interested in situations where the time dependence is confined to a finite time interval so that Eq. (2.8) admits solutions which in
the asymptotic past and future look locally (in time) like plane waves with frequencies $\omega^\text{in}_k$ and $\omega^\text{out}_k$. Global solutions to (2.8) in that case are not plane waves but are typically expressible in terms of Hypergeometric functions. We can, however, choose boundary conditions either in the past or future of the time dependent region such that the global solution (for each $k$) and its first derivative converge point-wise, as $t \to \pm \infty$ appropriately, to a single complex exponential plane wave $\exp(i\omega^\text{in/out}_k t)$. The fact that this plane wave is a not global solution to the equation of motion implies that in the future region it cannot be simply $\exp(i\omega^\text{out}_k t)$ but must contain also a component with $\exp(-i\omega^\text{out}_k t)$ – the only other function which is a solution in the future region, the plane wave of opposite frequency.

Now comes the interesting part: at times when $c(t)$ is constant particles are defined in QFT with respect to a plane wave basis of the wave equation at that time, the field operator $\hat{\phi}$ at some time $t$ being expressed as

$$\hat{\phi} = \sum e^{i k \cdot x} f_k a_k + e^{-i k \cdot x} f^*_k a^*_k,$$

(2.9)

where $f_k$ are the positive norm $\omega^\text{in/out}_k > 0$ solutions to (2.3) and the $n$-particle states of momentum $k$ being defined as the states $a_k^n |0\rangle$. What is important here is that the particle content is tied to a decomposition of the solution space into positive and negative norm solutions to (2.3), here indicated by the complex conjugate for the negative norm. When this decomposition is modified, the definition of the $n$ particle states is also modified.

This can happen, for example, when an initially positive frequency $f_k$ propagates through a time dependent region emerging on the other side with a negative frequency component

$$e^{-i \omega^\text{in}_k t} \leftarrow f_k(t) \longrightarrow \alpha e^{-i \omega^\text{out}_k t} + \beta e^{i \omega^\text{out}_k t},$$

(2.10)

where $\alpha$ and $\beta$ are some constants. Note that the solution must be of this form on the right hand side since the only solutions to (2.8) when $\omega_k$ is constant are these two exponentials. This structure (2.10) is the basic crucial ingredient of particle production effects, its significance for DCE cannot be overstated.

Since we work in the Heisenberg picture of quantum evolution we need to choose the vacuum state once, which we do at $-\infty$ in terms of the solutions $f_k$ which are purely positive norm at $-\infty$, that is, with the time dependence $\exp(-i \omega^\text{out}_k t)$. An observer who decomposes his field in to positive frequencies $\omega^\text{out}_k$ at $+\infty$ should use a different decomposition in terms of solutions $g_k$ to (2.8) which converge to positive frequency plane waves at $+\infty$

$$g_k(t) \longrightarrow e^{-i \omega^\text{out}_k t}, \quad \text{as} \quad t \to +\infty.$$

(2.11)

Of course the is only one unique quantum field and its unique quantum evolution, it is only its interpretation in terms of particle content which changes from $-\infty$ to $+\infty$. 

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Hence we can write $\hat{\phi}$ in one of two ways

$$
\hat{\phi} = \sum_k e^{ik\cdot x} f_k a_k + e^{-ik\cdot x} f_k^* a_k^\dagger
$$

(2.12)

$$
= \sum_k e^{ik\cdot x} g_k b_k + e^{-ik\cdot x} g_k^* b_k^\dagger
$$

(2.13)

the first representing a decomposition in terms of $-\infty$ particles the second in terms of $+\infty$ particles.

In terms of the $g_k$ (2.10) becomes

$$
f_k = \alpha_k g_k + \beta_k g_k^*, \text{ for } t \to \infty,
$$

(2.14)

one arrives at

$$
\hat{\phi} = \sum_k g_k \left( \alpha_k a_k e^{ik\cdot x} + \beta_k^* a_k^\dagger e^{-ik\cdot x} \right) + g_k^* \left( \beta_k a_k e^{ik\cdot x} + \alpha_k^* a_k^\dagger e^{-ik\cdot x} \right)
$$

(2.15)

so that we may identify

$$
b_k = \alpha_k a_k - \beta_k^* a_k^\dagger.
$$

(2.16)

and its adjoint relation. Here one uses the fact that the mode functions are parametrized only by the magnitude of the wavenumber $k$ so that the coefficients $\alpha_k$ and $\beta_k$ also only depend on the magnitude and that, in particular, $\beta_{-k} = \beta_k$ but the exponential factor changes sign along with an over all sign change from the integration measure. Importantly the creation operators depend on the angle of $k$ as well as its magnitude. This kind of relationship between different sets of annihilation and creation operators is known as a Bogoliubov transformation, here presented in this simplified context.

The number of $b_k^\dagger$ quanta contained in the vacuum state at late times is easily computed to be

$$
\text{Number}_k = \langle 0 | b_k^\dagger b_k | 0 \rangle = |\beta_k|^2 = |\beta_k|^2,
$$

(2.17)

since in this case $\beta_k$ only depends on the magnitude $k$ of the vector wavenumber $k$. The spectrum is usually quoted in terms of the number density $\text{Number}_k$ at fixed norm $k$, given by including the phase space integral over the angles in $k$ space, for example in $3+1$ dimensions we have

$$
\text{Number}_k = k^2 |\beta_k|^2.
$$

(2.18)

The spectrum $\text{Number}_k$ typically has the bell shape shown in Figure 2.1 which can be understood intuitively by considering two separate limits: the adiabatic and sudden limits. The adiabatic limit applies to frequencies much higher than the typical timescale of the variation of $c(t)$ in (2.8) — these frequencies should not notice the slow time dependence of the parameter and hence should not be excited in the spectrum. This is the exponentially decaying tail to the spectrum in 2.1. The other extreme
Figure 2.1: The characteristic ‘bell shape’ for DCE emission spectrum $k^2|\beta_k|^2$ in 3+1 dimensions with three regions clearly visible: low $k$ where the sudden approximation is accurate and the spectrum is approximately quadratic in $k$; the turnover region where the particle production is most abundant; the exponentially decays adiabatic region at high $k$.

applies to frequencies for which the variation of $c(t)$ is very fast with respect to the period of oscillation for the mode. These modes, ‘see’ the time variation as essentially instantaneous. In that case the equation of motion can be solved exactly by assuming continuity at the jump giving

$$e^{-i\omega_{k\text{out}} t} \leftrightarrow f_k \rightarrow \frac{1}{2} \omega_{k\text{out}} \frac{1}{\omega_{k\text{out}}} \frac{e^{-i\omega_{k\text{out}} t}}{e^{-i\omega_{k\text{out}} t}} - \frac{1}{2} \omega_{k\text{in}} - \omega_{k\text{out}} e^{+i\omega_{k\text{out}} t}, \quad (2.19)$$

which, using the dispersion relation $\omega_k^2 = c^2 k^2$ associated with (2.8), leads to the momentum independent Bogoliubov coefficient

$$|\beta_k|^2 = \frac{1}{4} \left( \frac{c^\text{in} + c^\text{out}}{c^\text{in}} \right)^2. \quad (2.20)$$

and therefore the quadratic $\text{Number}_k \propto k^2$. This is the quadratically increasing part of the low $k$ (that part for which the frequencies are low and hence the sudden limit applies) end of the spectrum shown in 2.1.

The peak wavenumber of emission is the wavenumber most efficiently produces by the time dependence. Intuitively this peak wavenumber should be such that the
corresponding frequency is approximately equal to the relative time change in the time dependent parameter

\[ \omega_{\text{peak}} \simeq \frac{\dot{c}}{c} \]  

(2.21)

This ‘rule of thumb’ generally plays out but there can exist deviations from this rule in certain situations [33].

It can be shown a complete description of \(|0\rangle\) in terms of the \(g_k\) basis and the state annihilated by all the \(b_k\), that is, \(b_k|\Omega\rangle = 0\) is given by

\[
|0\rangle = \prod_k \mathcal{N}_k \exp \left[ -\left( \frac{\beta_k^*}{2\alpha_k} \right) b_k^* b_k \right] |\Omega\rangle \\
= \prod_k \mathcal{N}_k \sum_n \sqrt{\frac{2^n}{n!}} \left( \frac{\beta_k^*}{2\alpha_k} \right)^n |2_k\rangle,
\]

(2.22)

where the product is also supposed to indicate a tensor product of decoupled \(k\) states \(|k\rangle = b_k^*|\Omega\rangle\). Here the \(\mathcal{N}_k\) are normalisation factors. We note that the initial vacuum state is expressed as a product state of excited states in the future region and that each \(k\) eigenmode is doubly excited. This structure of the quantum state defines is as a so-called ‘squeezed state’, typically found in many quantum emission processes. We will see later that this double excitation corresponds to a back-to-back emission of oppositely traveling quanta. This kind of phenomenon is known as an \textit{in-equivalence of vacua} and is the source many mathematical difficulties in the problem of trying to define what QFT is in a general curved spacetime.

We point out here that the only initial assumption required to derive all of what we have above was the relation (2.10). Many and diverse physical systems are capable of giving rise to such a frequency mixing, including the cosmological particle creation already mentioned, a variation of the refractive index for electromagnetic waves (equivalent to a time varying wave speed) or time varying boundary conditions for a wave problem. It is this latter example which is often assumed to be the extent of the terminology ‘Dynamical Casimir effect’ due to its close relationship with the original Casimir effect and Casimir force between two conducting plates for the electromagnetic field. In the case of cosmological particle creation commonly referred to, we note here for completeness that the scalar wave equation on an expanding background spacetime can be cast in the form

\[
(\partial_t^2 + a^2(t)\nabla^2) f = 0,
\]

(2.23)

where \(a\) is the scale factor of cosmological expansion entering the metric tensor

\[
ds^2 = -d\tau^2 + a^2(\tau)d\mathbf{x}^2,
\]

(2.24)

(\(\tau\) being a modified time coordinate) so that the time dependent wave speed in (2.8) is directly related to the expansion (or contraction) of the background space itself.
We will see (see 3) that particle production in this way can arise also in Bose–Einstein condensates and we will have time to discuss DCE in more detail there as well as in a latter chapter (4) on optics.

2.1.2 Bogoliubov coefficients - more details

In this short section we provide a more technical motivation for the relation (2.34) based on the inner product, show that it is equivalent to the intuitive definition and discuss some alternative method for their characterisation based on some simple local measurements which do not require a global solution to the wave equation.

Define the asymptotic wave speeds as \( c_i = c(t \to -\infty) \) and \( c_f = c(t \to +\infty) \) and let the equation of motion be

\[
\frac{d^2 \phi}{dt^2} - c^2(t) \nabla^2 \phi = 0. \tag{2.25}
\]

The equation (2.25) is solved by linear combinations of eigenfunctions to the spatial Laplacian

\[
\phi_k = f_k(t)e^{ikx}. \tag{2.26}
\]

The Bogoliubov coefficients are defined as inner products between different collections of exact solutions to the wave equation (2.25). For example between the two collections (one for each k) of Hypergeometric global solutions to (2.25), one set which converges to a single plane positive frequency plane waves in the past \( \{\phi_k^{\text{in}}\} \) and one set which converges similarly in the future \( \{\phi_k^{\text{out}}\} \).

The Bogoliubov coefficient is a function of two momenta and is defined in the time dependent case as

\[
\beta_{ij} = (\phi_i^{\text{out}*, \phi_j^{\text{in}}}), \tag{2.27}
\]

where the inner product \((\ ,\ ,\ )\) between two functions \(\phi_1\) and \(\phi_2\) is defined slightly differently than the time independent case as

\[
(\phi_1, \phi_2) = -i \int_{\Sigma} c^2(t) \left( \phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^* \right), \tag{2.28}
\]

and can be non-zero even if the momenta \(i\) and \(j\) differ in magnitude. The correctly normalised plane waves, for example in the ‘in’ region \(t \to -\infty\) should satisfy

\[
(\psi_k^{\text{in}}, \psi_{k'}^{\text{in}}) = \delta(k - k') \tag{2.29}
\]

so that they are given by

\[
\psi_k^{\text{in}}(t, x) = \frac{e^{ikx-i\omega_i t}}{c_i \sqrt{2\omega_i}}, \tag{2.30}
\]
where $\omega_i^2 = c_i^2 k_i^2$. We have, for example

$$\phi_k^\text{in}(t) \to \psi_k^\text{in}(t), \quad \text{as} \quad t \to -\infty.$$  

(2.31)

Then at a time $t_f$ when $c(t)$ has relaxed to the constant value $c_f$ the inner product between an out solution with momenta $k_{\text{out}}$ and an in solution with momentum $k_{\text{in}}$ is given by

$$\beta_{k_{\text{in}}, k_{\text{out}}} = -i e^{-i\omega_{\text{out}} t} - c_f (\partial_t f_{k_{\text{in}}} + i\omega_{\text{out}} f_{k_{\text{in}}}) \int_\Sigma dx \: e^{i(k_{\text{in}} + k_{\text{out}}) x}$$

$$= -i e^{-i\omega_{\text{out}} t} - c_f (\partial_t f_{k_{\text{in}}} + i\omega_{\text{out}} f_{k_{\text{in}}}) \delta(k_{\text{in}} + k_{\text{out}})$$

$$= : \beta_k \: \delta(k_{\text{in}} + k_{\text{out}}),$$

(2.32)

where we defined the isotropic Bogoliubov coefficient $\beta_k$ with $k = |k_{\text{in}}| = |k_{\text{out}}|$. That is

$$\beta_k := -i e^{-i\omega_{\text{out}} t} - c_f (\partial_t f_{k_{\text{in}}} + i\omega_{\text{out}} f_{k_{\text{in}}}),$$

(2.33)

where $\omega_{\text{out}}^2 = n_f^2 k_{\text{out}}^2$. We see that for this simple wave equation, $k_{\text{in}}$ must be equal to $-k_{\text{out}}$ as we anticipated earlier when we discussed the squeezed nature of the vacuum state and the back-to-back particle production.

This form of the Bogoliubov coefficient is particularly useful for numerical computations for example: Choose an appropriately normalised initial condition $v_{k_{\text{in}}}$ propagate it through the non-trivial time dependent region and compute the derivative $\partial_t f_{k_{\text{in}}}$.

Consider the global solution which behaves as in (2.10) converging to the a single particle (positive frequency) state in the asymptotic past but which is a mix of positive and negative frequency properly normalised plane waves in the future (necessarily of the same magnitude as the initial frequency as constrained by the $\delta$ functional in the general form for $\beta_{k_{\text{in}}, k_{\text{out}}}$)

$$\frac{e^{-i\omega_{\text{in}} t}}{c_i \sqrt{2\omega_{\text{in}}}} \quad \leftrightarrow \quad f(t) \quad \to \quad A \frac{e^{-i\omega_{\text{out}} t}}{c_f \sqrt{2\omega_{\text{out}}}} + B \frac{e^{i\omega_{\text{out}} t}}{c_f \sqrt{2\omega_{\text{out}}}}.$$  

(2.34)

Then

$$\beta_k = -i e^{-i\omega_{\text{out}} t} n_f (\partial_t f_{k_{\text{in}}} + i k c_f f_{k_{\text{in}}})$$

$$= -i e^{-i\omega_{\text{out}} t} \frac{c_f}{c_f \sqrt{2\omega_{\text{out}}}} (i \omega_{\text{out}} (A e^{-i\omega_{\text{out}} t} + B e^{i\omega_{\text{out}} t}) + i \omega_k (A e^{-i\omega_{\text{out}} t} + B e^{i\omega_{\text{out}} t}))$$

$$= -i e^{-i\omega_{\text{out}} t} \frac{2 i \omega_{\text{out}} B e^{i\omega_{\text{out}} t}}{2\omega_{\text{out}}}$$

$$= B,$$  

(2.35)
which is the intuitive definition of the Bogoliubov coefficient (as the component of the negative frequency part of the asymptotic waveform) that we gave at the beginning of this section.

We see that when the initial condition is a global solution to the wave equation (there is no non-trivial time dependence in the wave speed, \( c_f = c_i \) and the initial condition remains of plane wave form) the term in round brackets on the second line of (2.32) vanishes and \( \beta = 0 \) identically. When there is a time dependence the initial condition will evolve out of the monochromatic plane wave form, the term in round brackets will be non-vanishing and \( \beta_k \) will be non-vanishing, indicating particle production.

It is important to note that any initial condition will propagate through the time dependent region emerging in the future static region as a function which necessarily oscillates with frequency \( \omega^{\text{out}} \) since the frequencies are constrained by the future dispersion relation \( \omega^2 = c_f^2 k^2 \). That is, the initial positive norm solution (2.34) is required to oscillate at the ‘out’ frequency in the future region but in a way in which the real and imaginary parts possibly do not recombine to constitute a pure exponential plane wave. An ‘in’ solution will be deformed only in its relationship between the real and imaginary parts of the waveform, a seemingly trivial deformation which is responsible for all the particle production effects.

If \( \alpha = a + i\tilde{a} \) and \( \beta = b + i\tilde{b}, \ a, \tilde{a}, \ b, \tilde{b} \) all real, then

\[
\alpha e^{-i\omega t} + \beta e^{i\omega t} = \sqrt{(a + b)^2 + (\tilde{a} - \tilde{b})^2 \cos (\omega t + \phi_1)}
+ i \left[ \sqrt{(\tilde{a} + \tilde{b})^2 + (a + b)^2 \cos (\omega t + \phi_2)} \right], \tag{2.36}
\]

where

\[
\phi_1 := \arctan \left( \frac{\tilde{a} - \tilde{b}}{a + b} \right), \tag{2.37}
\]

\[
\phi_2 := \arctan \left( \frac{-a + b}{\tilde{a} + \tilde{b}} \right). \tag{2.38}
\]

Therefore by reading the phase shift and amplitude difference with respect to a pure cosine of the real and imaginary parts of an ‘in’ state in the ‘out’ region we can read directly off the Bogoliubov coefficients without doing any complicated integrals like (2.28)!

We can invert these equations to express the Bogoliubov coefficient components as functions of the amplitudes and phase shifts as follows: For a real amplitude \( A_1 \),
imaginary amplitude $A_2$, and phases $\phi_1$ and $\phi_2$ we have that

\begin{align}
4ab &= \frac{A_1^2}{1 + \tan^2 \phi_1} - \frac{A_2^2}{1 + \tan^{-2} \phi_2}, \quad (2.39) \\
4\tilde{a}\tilde{b} &= \frac{A_2^2}{1 + \tan^2 \phi_2} - \frac{A_1^2}{1 + \tan^{-2} \phi_1}. \quad (2.40)
\end{align}

Writing this as $a = \Gamma_1 / b$ and $\tilde{a} = \Gamma_2 / \tilde{b}$ we have

\begin{align}
\frac{\Gamma_2}{b} - \tilde{b} &= \tan \phi_1 \left( \frac{\Gamma_1}{b} + b \right), \quad (2.41) \\
- \frac{\Gamma_1}{b} + b &= \tan \phi_2 \left( \frac{\Gamma_2}{b} + \tilde{b} \right), \quad (2.42)
\end{align}

and hence

\begin{align}
\left( \tan \phi_2 - \frac{1}{\tan \phi_1} \right) \tilde{b}^2 - 2\tilde{b}b + \Gamma_2 \left( \tan \phi_2 + \frac{1}{\tan \phi_1} \right) = 0, \quad (2.43)
\end{align}

so that,

\begin{align}
\tilde{b} &= \frac{b \tan \phi_1}{\tan \phi_1 \tan \phi_2 - 1} \pm \frac{1}{2} \frac{\tan \phi_1}{\tan \phi_1 \tan \phi_2 - 1} \sqrt{4b^2 - 4\Gamma_2 (\tan^2 \phi_2 - \tan^{-2} \phi_1)}, \quad (2.44)
\end{align}

leaving us with the last equation to solve for $b$

\begin{align}
- \Gamma_1 + b^2 - b \tan \phi_2 \left( \frac{\Gamma_2}{b(b)} + \tilde{b}(b) \right) = 0, \quad (2.45)
\end{align}

which can be done numerically. After obtaining $b$ we can back substitute to get $\tilde{b}$ and hence $|\beta|^2$. In this way one could write a very simple numerical recipe to get the Bogoliubov coefficients from just the data of the amplitudes and phases of the ‘in’ mode functions when propagating in the ‘out’ region. The hard part of this algorithm would be obtaining the phase and amplitude shifts for each $k$ separately in order to plot $|\beta_k|^2$ as a function of $k$.

From the general Bogoliubov $\beta_{k_{\text{in}},k_{\text{out}}}$ we can compute the number density, total number and total energy

\begin{align}
\frac{dN}{d^3k_{\text{out}}} &= \int |\beta_{k_{\text{in}},k_{\text{out}}}|^2 d^3k_{\text{in}}, \quad (2.46) \\
N &= \int \frac{dN}{d\omega_{\text{out}}} d\omega_{\text{out}}, \quad (2.47) \\
E &= \hbar \int \omega_{\text{out}} \frac{dN}{d\omega_{\text{out}}} d\omega_{\text{out}}. \quad (2.48)
\end{align}
Spatial variation of propagation speed

In the case of a spatially varying wave speed the analysis is somewhat more involved. Firstly we wish to find the equivalent of the plane wave basis in the asymptotic regions. In the case when the parameters are both time and space independent in the asymptotic regions then the analysis is very similar to above. The only real difference is that the assumption of separability is modified to the assumption that

$$\phi_{kn}(t,x) = v_{kn}(t)\psi_{kn}(x),$$  \hspace{1cm} (2.49)

where the functions $\psi_k$ are eigenfunctions of the differential operator

$$D = \frac{1}{c^2(x)} \frac{d^2}{dx^2}.$$  \hspace{1cm} (2.50)

Under mild assumptions on the function $c^2(x)$ these spatial mode functions satisfy the usual completeness relations

$$\int dx \psi_n(x)\psi_n'(x) = \delta_{n,n'}, \hspace{0.5cm} \phi(x) = \sum_n (\phi, \psi_n)\psi_n(x),$$ \hspace{1cm} (2.51)

where the simplified summation notation is understood to represent an integral in the case of non-compact regions and it can be shown that the eigenvalues are strictly negative whence we label them as $-n^2$. For example, exact solutions are known for many different choices of $c(x)$ including tanh, sech$^2$, square wells and delta functions. It is the functions $\psi_n$ which play the role of the plane waves $\exp(ikx)$ in the Fourier expansion, allowing for an exact solution to be found.

Under decomposition of the general type in (2.51) the equation of motion, in the asymptotic past when the time dependence not yet acting, becomes

$$\partial^2_t(v\psi) - \partial^2_x(v\psi) = \left(\partial^2_t v + n^2 v\right) \psi = 0 \Rightarrow \partial^2_t v + n^2 v = 0,$$  \hspace{1cm} (2.52)

and we are back to the same mathematical problem as before. Everything follows through identically except for the fact that ‘single particle states’ are now represented by wave functions which are not necessarily plane monochromatic waves but instead are eigenfunctions to the operator $D$.

### 2.1.3 Hawking effect

The term ‘black hole’ was coined by John Wheeler in 1967 to describe the exact spherically symmetric vacuum solutions to Einstein’s equations which possess trapped regions such as the Schwarzschild, Kerr and the electromagnetically charged generalisations as well as more general black holes. Since we do not explicitly compute any Hawking
radiation in this thesis, this description is intentionally brief but included for completeness due to the significant role played by the Hawking process in the development of analogue gravity.

Classically nothing can come out of a black hole: it is black because the event horizon is a one-way membrane allowing only ingoing geodesics. When one includes the quantum effects, however, the black hole is no longer black but, for example in the case of Schwarzschild hole, glows with a color corresponding to its temperature $T_H$ which is related to the surface gravity $\kappa$ by (in unite where $c = \hbar = G = 1$)

$$T_H = \frac{\kappa}{2\pi}. \quad (2.53)$$

However, the Hawking effect can also be understood on the level of classical functions and ordinary differential equations and it is the purpose of this small subsection to provide such a presentation in order to isolate the non-trivial details from standard methods which differentiate the hawking process from other effects which persist also in Minkowski space. We consider here the more physical description in terms of gravitational collapse.

Consider a wave packet which travels through the surface of a collapsing star, just managing to escape the other side as the surface falls to within the Schwarzschild radius. Such a packet will be ‘almost’ trapped by the hole as it forms but manages to escape after spending some time close to the horizon $^2$. It can be shown that the relationship between the outgoing null coordinates $u$ before the star collapse (at $I^-$) and the outgoing null coordinate $U$ after the collapse, far from the star (at $I^+$) is given by the particular exponential form

$$U \simeq U_H + A e^{-\kappa u}, \quad (2.54)$$

where $U_H$ labels the outgoing null curve which sits exactly on the horizon (the generator of the horizon), $A$ is some constant and $\kappa$ is the surface gravity. We see the large red-shifting effect of the horizon for those modes which eventually escape to $I^+$ where a large change in $u$ gives rise to a small change in $U$.

In Figure 2.2 schematically how in-falling null geodesics which escape to $I^+$ tearing away from the horizon formed from gravitational collapse. The relationship between the affine coordinates at $I^-$ and those on $I^+$ is approximately exponential at late times (large $u$) giving rise to the Hawking spectrum as encoded in the Bogoliubov coefficients.

In the article in reference $^{[35]}$ it is shown that this exponential relation is all that is required in order to see a thermal flux at $I^+$, generalising the derivation of Hawking

$^2$This is the apparent horizon: the real event horizon might in fact be further if in the future, a large chuck of matter falls into the hole, increasing its mass and hence its Schwarzschild radius.
radiation to the case of a slowly varying $\kappa$ or a finite lifetime horizon for which the exponential relation applies only for a finite time. In this way the idea of Hawking radiation can be seen to not strictly depend on the existence of an absolute horizon but can also apply, for example, to apparent horizons.

Another way to understand Hawking radiation is through the difference between the frequencies as measured by freely falling observers and the Killing frequency as measured far from the hole. This viewpoint brings the discussion closer to that of DCE, discussed above. Very simply, the idea is that, to a freely falling observer an outgoing mode close to the horizon has a time dependence which drops to zero as the observer passes the event horizon. Such a function, to the observer, cannot contain only positive frequencies (positive norm modes, we are not in a case where these two concepts diverge) since a function which vanishes in space on an interval must contain all frequencies (see [27] for a detailed account). This highlights that the quantum vacuum as defined by freely falling observers and their notion of positive norm modes

Figure 2.2: Schematic diagram of the geodesics which propagate from $\mathcal{I}^-$ to $\mathcal{I}^+$ by passing close to the formed horizon around a collapsed star. These geodesics are classically the ones followed by any wavepacket which are observed at late times from the black hole.
differs from the vacuum state as defined with respect to the Killing frequencies. Hence, as in 2.1.1 there is a Bogoliubov transformation between the creation and annihilation operators which define the two vacua and Fock spaces. It can be shown that the relation (2.54) implies that

$$|\beta_k|^2 \propto \frac{1}{1 - e^{2\pi \omega/\kappa}},$$

(2.55)

which is a Planckian spectrum of temperature $T = \kappa/2\pi$.

### 2.1.4 The many faces of superradiance

As a third and final QFT process amenable to experimental probes through the Analogy and which we will discuss at some length in a subsequent chapter in the context on the analogy, we introduce here the phenomenon of superradiant scattering, again initially found to pertain to wave scattering by a rotating Kerr black hole [36, 37] but later understood to be a very general physical process [29]. It is related to various instabilities discovered earlier known as Klein instabilities or Schiff-Snyder-Weinberg effects [26]. We will have time to discuss how the phenomenon is robust against modifications to the linear dispersion relation for wave propagation, paving the way for a future experimental observation in real world dispersive flowing fluids.

Put simply, the phenomenon of superradiance is the occurrence of a wave scattering process in which the total reflected amplitude of wave is greater than the total incident amplitude. At the quantum level it represents an instability in a quantum field if the scattering equation which gives rise to superradiance is related to the field’s equation of motion.

The traditional scattering problem is defined as follows: one has an equation of motion with some potential

$$\frac{d^2 f}{dx^2} + (\omega^2 + V(x)) f = 0, \quad V(x) = \begin{cases} V_i, & x \to -\infty \\ V_f, & x \to +\infty \end{cases}$$

(2.56)

to which is associated the functional

$$W[f] = f'f^* - ff'^*,$$

(2.57)

which is conserved in space when acting on solutions to (2.56). The equation of motion often arises after the separation of the time dependence $\exp(-i\omega t)$ of a stationary 1+1 dimensional wave equation where $\omega$ is interpreted as frequencies. Then the ‘scattering solution’ $f_s$ representing an incoming wave from the right being both transmitted and reflected by the potential decomposes asymptotically as

$$Te^{ik_t x} \leftrightarrow f_s(x) \longrightarrow e^{ik_u x} + \Re e^{ik_v x},$$

(2.58)
where the momenta $k_i, k_r$ and $k_f$ satisfy the dispersion relation in each region obtained by inserting the plane wave ansatz into the equation of motion (2.56)

$$- k^2 + \omega^2 + V_{i,f} = 0, \quad \implies k = \pm \sqrt{\omega^2 + V_{i,f}}, \quad (2.59)$$

in the two asymptotic regions. The signs of the momenta are determined by the requirement that the reflected and transmitted modes have positive and negative group velocities respectively. For example

$$0 < \frac{d\omega}{dk_r} = \frac{k_r}{\omega}, \quad \implies k_r = \sqrt{\omega^2 + V_i} > 0 \quad (2.60)$$

Similarly we have $k_t, k_{in} < 0$ determined by the negativity of their group velocities.

The conservation of $W[f_s]$ implies the relation between the reflection and transmission coefficients $R$ and $T$ respectively

$$1 - |R|^2 = -\frac{k_t}{k_r} |T|^2. \quad (2.61)$$

From relation (2.61) and the fact that $k_t < 0$ and $k_r > 0$ we can conclude that in the ordinary scattering process we must have $|R| < 1$.

The important ingredients are the scattering solution and the dispersion relation: we choose which modes to include consistently in the scattering solution by their group velocity $d\omega/dk$ in the asymptotic regions, the mode with the coefficient $R$ must be right moving at $+\infty$ while the the mode with the coefficient $T$ must be left moving at $-\infty$.

Superradiant scattering must therefore have an unusual kind of potential $V$ in order to avoid the arguments here leading to $|R| < 1$.

**Spacetime superradiance – Equatorial Kerr metric**

As we saw in the introduction, the geometry of the equatorial sections of a rotating Kerr black hole are described in Boyer-Lindquist coordinates (in units where $G = c = 1$) by the metric

$$ds^2 = dt^2 + \frac{2M}{r} (dt - ad\phi)^2 - \frac{dr^2}{1 - \frac{2M}{r} + \frac{a^2}{r^2}} - (r^2 + a^2) d\phi^2, \quad (2.62)$$

where $a$ is the rotation parameter and $m$ is the mass which completely characterise the solution.

The wave equation for a massless scalar field $f$ in the full 3+1 dimensional Boyer-Lindquist coordinates including the polar angular coordinate $\theta$ is complicated and we do not write it down here. However a coordinate change [38] of the Kerr metric results in a separated equation of motion for radial modes from the polar angular part justifying
the dimensional reduction to the equatorial $\theta = \pi/2$ sections. That is, expanding the field $f$ in eigenmodes $f_{\omega,m} = R(r)\Theta(\theta)e^{-i\omega t}e^{im\phi}$ of constant angular momentum $m$ and frequency $\omega$, the resulting equation of motion for the function $R$ separates from the equation of motion of $\Theta$. Furthermore, under the redefinition of $R$ and change of variables to the radial coordinate $r^*$ as

$$u(r) = \sqrt{r^2 + a^2}R(r),$$

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{r^2 - 2Mr + a^2},$$

the equation of motion for $u(r^*)$ is written in the simple form

$$\frac{d^2u}{dr^{*2}} + V_{\omega,m}(r^*)u = 0,$$

where $V_{\omega,m}$ is a function whose precise form is rather complex

$$V_{\omega,m} = -\frac{4Mar\omega - a^2m^2 + \Delta (\lambda_{m,t} + \omega^2) a^2}{(r^2 + a^2)^2} - \frac{\Delta (3r^2 - rMr + a^2)}{(r^2 + a^2)^3} + \frac{3\Delta^2r^2}{(r^2 + a^2)^4} + \omega^2.$$

Here $\Delta = r^2 + 2Mr + a^2$ and the $\lambda_{m,t}$ are eigenvalues for the $\Theta$ angular part of the wave equation. The role of the variable $r^*$ is similar to a tortoise coordinate, covering only the region $r > r_h^+$, where $r_h^+ = M + \sqrt{M^2 - a^2}$ is the outer horizon radius (see [39] for a concise summary of the Kerr solution) where the component $g_{rr}$ of the metric becomes singular. This region $r \in (r_h^+, \infty)$ is mapped in terms of $r^*$ onto the real line $r^* \in (-\infty, +\infty)$.

We note, however, that as $r \to \infty$ the function $V_{\omega,m}$ tends to a constant $V_{\omega,m} \to \omega^2$. The coordinate $r^*$ has a nice interpretation as a generalised ‘tortoise’ coordinate such that $r^* \to -\infty$ as $r$ approaches the event horizon. The other asymptotic is given by

$$V_{\omega,m} \to \left(\omega - \frac{am}{2M(M + \sqrt{M^2 - a^2})}\right)^2$$

as $r \to r_h^+$ or equivalently $r^* \to -\infty$. The quantity subtracting $\omega$ inside the bracket can be expressed in terms of the angular velocity of the Kerr black hole $\Omega_H = \Omega_H = -(g_{t\phi}/g_{\phi\phi})|_H$ as simply $m\Omega_H$ giving the simple form of the potential

$$V_{\omega,m} = \begin{cases} \omega^2, & r^* \to +\infty \\ (\omega - m\Omega_H)^2, & r^* \to -\infty \end{cases}$$

The allowed wave-numbers and those chosen for the scattering solution can be concisely understood using a dispersion diagram as shown in Fig.2.3. We note that
the scattering solution chooses only a subset of the set of all possible wave-numbers allowed by the dispersion relation

\[ k^2 = \begin{cases} \omega^2 & r^* \to +\infty \\ (\omega - m\Omega_H)^2 & r^* \to -\infty \end{cases} \]  

in such a way that the solution can be interpreted as a scattering process: A localised wave packet with momentum centered around \( k_{in} \) traveling from \(+\infty\) converting into a reflected wavepacket with momentum centered around \( k_r \) which travels back to \(+\infty\) and a transmitted wavepacket with momentum centered around \( k_t \) which travels on to \(-\infty\). The direction of motion is given by the group velocity \( d\omega/dk \) of the packet and the momenta are determined by the dispersion relation. In the \( r^* \to -\infty \) region we have

\[ \frac{d\omega}{dk} = \frac{k}{\omega - m\Omega_H}, \]  

and the sign does not coincide with the sign of \( k \) when \( \omega < m\Omega_H \): crucially the group and phase velocities are of opposite sign in the \( r^* \to -\infty \) region when \( \omega < m\Omega_H \).

Explicitly solving the dispersion relation we have

\[ k_{in} = -\omega \]
\[ k_r = \omega \]
\[ k_t = -(\omega - m\Omega_H). \]  

In this case, the Wronskian \((2.57)\) is still conserved when acting on solutions to the equation of motion and acting on the scattering solution \((2.58)\) with the above defined momenta gives

\[ 1 - |R|^2 = \frac{1}{\omega} (\frac{\omega - m\Omega_H}{\omega}) |T|^2, \]  

and hence

\[ |R|^2 = 1 - \frac{1}{\omega} (\frac{\omega - m\Omega_H}{\omega}) |T|^2. \]  

Since we choose positive frequencies \( \omega > 0 \) there is a window of frequencies \( \omega < \Omega_H \) for which \( |R| > 1 \) is possible.

This is one of the historical routes to the discovery of superradiance \([36, 37]\) but the basic key ingredients can be distilled by the equation of motion \((2.65)\) with the potential like \((2.69)\) and the requirement of a scattering solution of the type defined above.

**Electromagnetic potential formulation**

An equation of the form \((2.65)\) can also arise as the wave equation in flat spacetime for a charged scalar field in the presence of a electromagnetic vector potential \( V_\mu \) which
is non-zero only in the time direction $V_{\mu} = \delta_{\mu}^0 eV$ [30]. The potential modifies the derivative operators $\partial_{\mu}$ to $D_{\mu} = \partial_{\mu} - iV_{\mu}$ and consequently the wave equation to

$$0 = D_{\mu}D^{\mu}f = \partial_x^2 f - (\partial_t - ieV)^2 f,$$

leading to

$$\frac{d^2 f_{\omega}}{dx^2} + (\omega - eV(x))^2 f_{\omega} = 0,$$

when acting on fixed frequency modes.

If one choses the potential to have the correct form

$$eV(x) = \begin{cases} 
0 & x \to +\infty \\
eV_0 & x \to -\infty 
\end{cases},$$

Figure 2.3: The dispersion relation (2.70) shown in the two asymptotic regions, the green dash-dotted line at $+\infty$ and the blue dashed line at $-\infty$. We have shown the case where $\omega < m\Omega_H$, specifically $\omega = 0.4$ and $m\Omega_H = 0.7$. Note that the intersections of the blue dashed line (the effective frequency at $-\infty$) lie below the $k$ axis, allowing for a negative norm transmitted mode and consequently superradiant scattering back to $+\infty$. 

2.1. QUANTUM FIELDS AND GRAVITY TOGETHER
we are led directly, from the analysis above to the conclusion that

\[ |R|^2 = 1 - \frac{\omega - eV_0}{\omega} |T|^2 > 1, \] (2.78)

and hence superradiance for \( \omega < eV_0 \).

**Acoustic superradiance**

Alternatively we can understand the the equation of motion

\[ \frac{d^2 f_\omega}{dx^2} + (\omega - eV(x))^2 f_\omega = 0, \] (2.79)

as the equation of motion for a dimensionally reduced analogue model. For the 2+1 dimensional fluid metric given in the previous chapter (1.20) with a flow vector field \( V = (0, V) \) which is zero in the \( x \) direction but which is non-zero and a function of only \( x \) in the, say, \( z \) direction, the analogue metric is given by

\[ g \propto \begin{pmatrix} 1 - V^2(x) & 0 & V(x) \\ 0 & -1 & 0 \\ V(x) & 0 & -1 \end{pmatrix}. \] (2.80)

As long as \( V < 1 \) we can always make a change of frame to the frame in which \( V = 0 \) and the metric is revealed as that of flat space. The determinant is 1 and the wave equation is

\[ \left[ (\partial_t + V \partial_z)^2 + \partial_z^2 + \partial_x^2 \right] f = 0, \] (2.81)

reducing for fixed frequency \( \omega \) and fixed momenta \( p \) in the \( z \) direction to

\[ \frac{d^2 f_{\omega,p}}{dx^2} + [(\omega - pV)^2 - p^2] f_{\omega,p} = 0 \] (2.82)

If the flow is such that

\[ V(x) = \begin{cases} 0 & x \to +\infty \\ V_0 & x \to -\infty \end{cases}, \] (2.83)

then we are in a situation very similar to that of the Kerr black hole once again but this time with the additional ‘mass’ term \( p^2 \).

According to the analysis of [30] superradiance still exists for such a wave equation but only in the regime known as the Klein region, when,

\[ |p| \leq \omega \leq pV_0 - |p|, \] (2.84)

if such a region exists. We see that the frequencies that are superradiated must not be too small for the mass term to effect their propagation but must not be too large for the \( y \) flow to be negligible.

Note that we cannot choose \( p = 0 \) since \( p \) also comes as a pre-factor on the velocity \( V \) inside the squared bracket which is ultimately responsible for the superradiance.
Generalised non-dispersive superradiant scattering

In the article in Reference [40] superradiant scattering is generalised to include also the case when the potential contains an imaginary part

$$\frac{d^2 f_\omega}{dx^2} + [V(x) + i\Gamma(x)] f_\omega = 0$$

(2.85)

where $V(x) \to \omega^2$ as $x \to +\infty$.

In this case, the ordinary Wronskian (2.57) is not conserved and its derivative with respect to $x$ can be related to the imaginary part of the potential

$$i \frac{d}{dx} W[f] = 2\Gamma |f|^2$$

(2.86)

Assuming only the form of the scattering solution at large $x$

$$f_s \to e^{ik_0 x} + Re^{ik_x},$$

(2.87)

we can integrate the condition (2.86) up to an arbitrary point $x_0$ (which is supposed to be analogous to an horizon) finding

$$|R|^2 = 1 - \frac{i}{2\omega} W[f] \bigg|_{x=x_0} - \frac{1}{\omega} \int_{x_0}^{\infty} \Gamma(x)|f(x)|^2 dx$$

(2.88)

The authors conclude that superradiance is possible in such cases when the right hand side is greater than 1, implying certain conditions on the form of the imaginary part of the potential and the evaluation of the Wronskian at the ‘horizon’ $x_0$.

The quantum spontaneous emission

Superradiance can be understood alone as a classical scattering phenomena. However, if one considers a quantum field whose fixed frequency modes satisfy an equation of motion such as

$$\frac{d^2 f_\omega}{dx^2} + (\omega - eV(x))^2 f_\omega = 0,$$

(2.89)

with the appropriate asymptotic behaviour of the potential, then one might ask what superradiance implies for the stability of such systems. For example the zero-point vacuum fluctuations could possibly be continually amplified in this manner leading to an instability of the vacuum and particle production.

This line of reasoning indeed plays out as expected [41, 42, 36], leading to the idea that quantum fields are capable of extracting angular momentum from the spinning black hole through amplification of the vacuum fluctuations.
2.2 Bose Einstein Condensation

The physics of Bose Einstein Condensation (BEC) is complex and was first described by Bose and Einstein in 1924. Nevertheless it may be discussed using very basic physics as we will show below. As with all good theories, by making use of carefully chosen approximations, one can get to the underlying physics of BEC relatively easily. The limits of these approximations mark the boundary of the validity of the analogue description, of which we will make use later, and it is for this reason that we shall spend some time here discussing them.

The main aim of this short discussion is to remove the veil of mystery from the physics of BEC sufficiently to render transparent the description of elementary BEC excitations in terms of an analogue metric. All too often the important message from an analogue gravity computation is shrouded in the details of the background physics. We make no attempt here to use hyper-precise terminology and stick to intuitive, basic descriptions as far as possible.

2.2.1 Non-interacting Bosons

At its root a BEC is nothing but a collection of Bosons in their lowest energy state for a given temperature. Bosons, of course, distribute themselves throughout the energy levels $\epsilon_i$ according to the Bose distribution

$$f(\epsilon_n) = \frac{1}{\exp \beta(\epsilon_n - \mu) - 1},$$

(2.90)

where $\mu$, the chemical potential, is the increase in energy of a system of particles when an additional particle is added and $\beta = 1/T$, is the inverse temperature. The energies $\epsilon_i$ are kinematically determined by the properties of the Hamiltonian operator. For example, in the time independent hard potential well the $\epsilon_n$ are the eigenvalues with the well known discrete spectrum

$$\left( -\frac{\hbar^2}{2m} \nabla^2 + V \right) \psi_n = \epsilon_n \psi_n$$

(2.91)

$$\Rightarrow \epsilon_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}.$$  

(2.92)

From the distribution (2.90) one can calculate the mean number of particles and mean energy

$$N = \sum_n f(\epsilon_n), \quad E = \sum_n \epsilon_n f(\epsilon_n).$$

(2.93)
The generalisation to include an arbitrary spectrum we introduce the density of states
\[ g(\epsilon) = \sum_j \delta(\epsilon - \epsilon_j). \] (2.94)
For example, for bosons in a harmonic potential well in 3D the energy levels are
\[ \epsilon_{ijk} = \hbar \sum_i n_i \omega_i, \] (2.95)
so that for small \( \epsilon \) the density of states \( g(\epsilon) \) is quite discrete and becomes more continuous at higher \( \epsilon \) and is approximated by the integral
\[ g(\epsilon) \approx \frac{1}{\hbar^3 \omega_0^3} \int_0^\infty du dv dw \delta(\epsilon - u - v - w) = \frac{\epsilon^2}{2\hbar^3 \omega_0^3}. \] (2.96)
For a general potential \( V \) the energy eigenvalues are difficult if not impossible to obtain explicitly so one usually resorts to asymptotic, scaling, or inequality-based data usually resulting in a thermodynamic (high occupation number) description. However, we will discuss a particular class of time dependent potentials later on (see Section 3.2.1 in the following chapter) for which we can compute the eigenvalues exactly and which constitutes an analogue model for an expanding spacetime.

The key physics result that underpins the entire phenomenon of BEC is that there is a phase transition in the occupation of the ground state associated with the lowest energy eigenvalue \( \epsilon_0 \) at some point as one lowers the temperature of the bosons. At this phase transition the ground state attains a macroscopic occupation number and the BEC is well approximated by a perfect fluid.

More precisely, it is known that the chemical potential \( \mu \) is a complicated function of temperature \( T = \beta^{-1} \) and particle number \( N \) although it is known that at high temperature \( \mu \) is large and negative, reducing the Bose distribution (2.90) to the classical Boltzman factor, while below a certain critical temperature \( T_c \) the chemical potential is nothing but the ground state energy of a Boson \( \epsilon_0 \). From (2.90) we see that this implies a singular occupation of the ground state \( \psi_0 \) when \( T < T_c \). This singularity is resolved when interactions between the bosons are included which are neglected in (2.90).

### 2.2.2 Interactions change things slightly

Fundamentally we have a Hamiltonian operator \( H \) describing the N-body dynamics of the collections of bosons written as a function of creation and annihilation operators. This Hamiltonian does not diagonalise into N free particles but instead there is an interaction term with two annihilation and two creation operators and hence there is
a scattering of the atoms from one another. These interactions will introduce several non-trivial new features which will turn out to have important interpretations in terms of curved spacetime kinematics.

**First approximation** – Dilute Gas approximation. This approximation allows us to neglect the full dynamics of scattering, keeping only the two body s-wave scattering interaction. This interaction is modeled by the ‘pseudo-potential’

\[ V(r) = g\delta(r) \quad (2.97) \]

It is called a ‘pseudo’-potential since there really is no potential function for the particles, they are free particles that interact ‘at a point’.

Then the Hamiltonian can finally be written down simply as

\[ H = \sum_n \epsilon_n a_n^\dagger a_n + \frac{g}{2V} \sum_n a_n^\dagger a_n^\dagger a_n a_n \delta_{n1+n2+n3+n4} \quad (2.98) \]

**Second approximation** – Macroscopic occupation of ground state. This assumption is essentially the assumption that the occupation number of the ground state is so high that the state is an approximate eigenstate of the creation \( a_0^\dagger \) and annihilation operator \( a_0 \). That is, at high occupation adding one particle or removing one makes ‘practically’ no difference to the state. Hence weakly (that is, under expectation values), we replace these operators by the eigenvalues \( \sqrt{N_0} \). After this substitution the Hamiltonian is a sum over excited state quantum numbers and takes the general form

\[ H = A + \sum_n B_n a_0^\dagger a + N_0 A \sum_n (a_n^\dagger a_{-n}^\dagger + a_n a_{-n}) \quad (2.99) \]

which contains particle non-conserving terms in the last sum, where \( A \) is a number valued constant. However, perhaps miraculously, it is possible to diagonalize this Hamiltonian with the non-perturbative ‘Bogoliubov’ transformation, mixing creation and annihilation operators in new linear combinations of them

\[ a_n = u_n \alpha_n + v_{-n}^* b_{-n}^\dagger, \quad a_n^\dagger = u_n^* b_n^\dagger + v_n \alpha_{-n} \quad (2.100) \]

Then,

\[ H = \text{const} + \sum_{n \neq 0} E_n b_n^\dagger b_n, \quad (2.101) \]

with the famous Bogoliubov dispersion relation energy eigenvalues

\[ E_n = \sqrt{\epsilon_n^2 + 2\epsilon_n gN_0} = \sqrt{\frac{gN_0}{m} p^2 + \frac{p^4}{(2m)^2}}. \quad (2.102) \]
Here we have included the atomic Bosonic dispersion $\epsilon(p) = p^2/(2m)$ in the last line. The coefficients of the Bogoliubov linear combinations (2.100) are not arbitrary, of course, but are constrained to satisfy

$$v_n^2 = u_n^2 - 1 = \frac{1}{2} \left( \frac{\epsilon_n + gN_0}{E_n} - 1 \right),$$

which uniquely determine them as

$$u_n, v_{-n} = \pm \sqrt{\frac{N_0^2/2m + gN_0}{2\epsilon_n}} \pm \frac{1}{2}.$$  

This is similar to what we described in 2.1.1 where the choice of vacuum state uniquely determines the mode functions. This constraint is exactly the condition on the mode functions which cancel the off diagonal terms in the Hamiltonian. It turns out that these mode functions are in fact real and even functions of $n$ so that, for example, $u_n = u_{-n} = u_n^*$ as can be seen from (2.104).

Now it is interesting to note that the dispersion relation (2.102) exhibits relativistic behaviour for low momenta

$$E(p) \approx cp \quad \text{for small } p,$$

where the wave propagation speed $c$ is given by $c^2 = gN_0/m$, but exhibits non-relativistic free particle like behaviour for high $p$

$$E(p) \approx \frac{p^2}{2m} \quad \text{for high } p.$$  

In this way, the quantum theory described by (2.101) describes elementary quantum excitations on top of the ‘classical’ background of the macroscopically occupied ground state. These elementary excitations, somewhat surprisingly, are not single particle excited states, but instead are collective degrees of freedom which behave relativistically at low energies. It is these excitations whose dynamics are subject to a description in terms of an analogue spacetime and metric as we will in the next sections.

### 2.2.3 A BEC as an analogue spacetime

In a later chapter we will consider a weakly interacting BEC trapped in an attractive harmonic potential and the computation of a correlation signal for the produced quanta.

Here in this sub-section we derive the analogue metric for low momentum BEC perturbations which we will make use of later on and describe the approximations necessary for such a geometric description. We introduce here the fluid dynamical description of a BEC and show how the basic perturbations of the canonical variables,
fluid flow and density, are described by a wave equation equivalent to that which
describes the propagation of a scalar field on a curved spacetime.

We will not discuss any of the manifold phenomena associated with dispersion; these are ‘advanced topics’ which we will have time to look at in separate sections and chapters. General references for this by now standard material can be found in References [43, 44, 45].

In the dilute gas approximation, one can describe a Bose gas through a second quantised field \( \hat{\Psi} \) satisfying
\[
\frac{i\hbar}{\partial t} \hat{\Psi} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{x}) + g(\bar{a}) \hat{\Psi}^\dagger \hat{\Psi} \right) \hat{\Psi}.
\] (2.107)

Here \( m \) is the mass of the constituents, \( g \) introduced in (2.97), parametrizes the strength of the interactions between pairs of bosons in the gas and \( \bar{a} \) is the scattering length of the atoms in an s-wave approximation. The former can be re-expressed in terms of the scattering length as
\[
g(\bar{a}) = \frac{4\pi\bar{a}\hbar^2}{m}.
\] (2.108)

Being a non-relativistic field, \( \hat{\Psi} \) satisfies the commutation
\[
\left[ \hat{\Psi}(\mathbf{x}), \hat{\Psi}^\dagger(\mathbf{x}') \right] = \delta(\mathbf{x}, \mathbf{x}'), \quad \left[ \hat{\Psi}(\mathbf{x}), \hat{\Psi}(\mathbf{x}') \right] = 0.
\] (2.109)

In a mean field approximation, the quantum field can be separated into a macroscopic (classical) condensate (called the wave function of the condensate) and a fluctuation: \( \hat{\Psi} = \psi + \hat{\varphi} \), with \( \langle \hat{\Psi} \rangle = \psi \). Collecting the leading order terms in \( \psi \), equation (2.107) leads to the Gross–Pitaevskii (GP) equation of motion for the wave function of the condensate
\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V_{\text{ext}} \psi + g(\bar{a})|\psi|^2 \psi.
\] (2.110)

Neglecting the terms proportional to the fluctuation \( \hat{\varphi} \) amounts to neglecting back-reaction effects of the quantum fluctuations on the condensate.

Representing the wave function of the condensate as [46]
\[
\psi(t, \mathbf{x}) = \sqrt{\rho(t, \mathbf{x})} \exp\left[ -i\theta(t, \mathbf{x})/\hbar \right],
\] (2.111)
where \( \rho = \psi^* \psi \) is the condensate density, and defining an irrotational ‘velocity field’ by \( \mathbf{v} := \nabla \theta/m \), the Gross–Pitaevskii equation can be rewritten as a continuity equation plus an Euler equation reminiscent of the fluid dynamical equations of motion:
\[
\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{v}) = 0,
\] (2.112)
\[
m \frac{\partial}{\partial t} \mathbf{v} + \nabla \left( \frac{mv^2}{2} + V_{\text{ext}}(t, \mathbf{x}) + \frac{\hbar^2}{2m} \nabla^2 \sqrt{\rho} \right) \left( \rho - \frac{\hbar^2}{2m} \nabla^2 \sqrt{\rho} \right) = 0.
\] (2.113)
These equations are completely equivalent to those of an irrotational and inviscid fluid apart from the presence of the so-called quantum potential

\[ V_{\text{quantum}} = -\frac{\hbar^2}{2m} \nabla^2 \sqrt{\rho / (2m \sqrt{\rho})}, \tag{2.114} \]

which has the dimensions of an energy. When relatively small gradients of the background density \( \rho \) are involved, this term can be safely neglected and a hydrodynamical approximation (also known as the Thomas Fermi (TF) approximation) holds.

This constitutes the derivation of the perfect fluid description of the background BEC dynamics on top of which are played out the dynamics of the quantum perturbations. Near the edges of the BEC, where the density is low, the quantum pressure becomes non-negligible and the fluid description breaks down.

Inserting the ansatz \( \Psi = \psi + \hat{\varphi} \) into Eq. (2.107) and again neglecting back-reaction, one alternatively obtains the equation for the operator valued perturbation

\[ i\hbar \frac{\partial}{\partial t} \hat{\varphi}(t, \mathbf{x}) = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{x}) + 2g\rho \right) \hat{\varphi}(t, \mathbf{x}) + g\rho \hat{\varphi}^\dagger(t, \mathbf{x}). \tag{2.115} \]

The elementary excitations of this dynamical system are described by variables obtained by a Bogoliubov transformation from the fundamental field. This can be also shortcut by adopting the so called quantum acoustic representation [45]

\[ \hat{\varphi}(t, \mathbf{x}) = e^{-i\theta/\hbar} \left( \frac{1}{2\sqrt{n_c}} \hat{\rho}_1 - i \frac{\sqrt{\rho}}{\hbar} \hat{\theta}_1 \right), \tag{2.116} \]

where \( \hat{\rho}_1, \hat{\theta}_1 \) are real quantum fields which describe the excitations in density and phase respectively. Inserting the acoustic representation ansatz above into Eq. (2.115) one again obtains a pair of hydrodynamic-like equations for the quantum excitations

\[ \partial_t \hat{\rho}_1 + \frac{1}{m} \nabla \cdot \left( \rho_1 \nabla \theta + \rho \nabla \hat{\theta}_1 \right) = 0, \tag{2.117} \]

\[ \partial_t \hat{\theta}_1 + \frac{\nabla \theta \cdot \nabla \hat{\theta}_1}{m} + g(\bar{a}) \rho_1 - \frac{\hbar^2}{2m} D_2 \hat{\rho}_1 = 0, \tag{2.118} \]

where \( D_2 \) represents a second-order differential operator derived from the linearization of the quantum potential [45]. As mentioned earlier, we will be concerned with the acoustic excitations which are wave-like. These excitations are of sufficiently low frequency as to render the quantum potential negligible, allowing for a bona-fide perfect fluid description and equations (2.117) and (2.118) are identical to the equations (1.16) with \( \theta_1 \) and \( \rho_1 \) playing the roles of \( \rho_1 \) and \( f_1 \) in (1.16). Hence the derivation of the equivalence between this set of equations and a Klein Gordon equation for \( \theta_1 \) in a curved background follows immediately as in section 1.1.2. The metric in this case is given by

\[ ds^2 = \sqrt{\frac{\rho}{gm}} \left[ -(c_s^2 - v^2) dt^2 - 2v_i dx^i dt + dx^2 \right]. \tag{2.119} \]
Here $c_s^2(t, x) = g \rho(t, x)/m$ is the squared local sound speed of the perturbations and $v(t, x)$ is the background flow velocity of the BEC introduced above [6, 43]. It is then clear that in this long wavelength regime the quasi-particles satisfy a relativistic, phononic dispersion relation $\omega^2 = c_s^2 k^2$ (with $\omega$ and $k$ equal respectively to the energy and wavenumber of the quasi-particle).

Alternatively one could adopt an eikonal (short wavelength) approximation in the equations (2.117) and (2.118), isolating also the high frequency part and show that the complete dispersion relation is indeed the well known Bogoliubov dispersion given by $\omega_{Bog}^2 = c_s^2 k^2 + \hbar k^4/(2m)$ as is done explicitly in ???. It is evident at this point that in this context the gravitational analogy is valid only in the long wavelength approximation. More specifically, for wavelengths of the order of the healing length $\xi = \hbar/\sqrt{mg\rho}$, when the quartic term becomes comparable to the quadratic one in $\omega_{Bog}^2(k)$, a geometrical description is no longer available.

### 2.3 Theory of surface waves on fluids

In fluid dynamics, gravity waves are waves created at the interface between two fluids as a result of a gravitational restoring force. The system considered in this section and that which will be the subject os Chapter 6 is the interface between an incompressible fluid with density $\rho$ and another incompressible fluid with negligible density $\tilde{\rho} (\ll \rho)$, specifically the air-water interface in an open channel flow.

The subject of open channel flows is classical, dating back to Airy in the 19th Century [47]. The plethora of gravity wave phenomena is not entirely captured by any one single wave equation where various different approximations can be adopted resulting in different partial differential equations for some restricted dynamics. To name a few, Boussinesq approximation results in the Boussinesq-type equations from which the Kortewegde Vries equation results in the approximation to unidirectional propagation. The Airy wave theory we will use here is a linear theory, essentially making use of the first order perturbation theory of surface disturbances. Even in the simplified context of Airy wave theory, the diversity of phenomena is extremely rich and includes dispersive effects.

Linearised gravity waves have received considerable attention recently from the analogue community being the subject of at least two analogue gravity experiments [48, 49, 23] and a series of theoretical works [50, 51, 52].

In general, the open channel flow problem consists of two steps (i) Solving for the background flow velocity profile and free surface and (ii) solving the equation of motion for perturbations on top of this background.

Below we will demonstrate the meaning of ‘long wavelengths’ and show that they satisfy a wave equation in an analogue curved spacetime while also discussing the non-linear dispersion for shorter wavelengths.
2.3. Theory of surface waves on fluids

2.3.1 Background flow

We start our analysis again here with the basic equations of fluid dynamics as we wrote them in (1.1.3), but now with the additional assumption of incompressibility

$$\nabla \cdot \mathbf{v} = 0,$$

(2.120)

and the Euler equation,

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} - g\hat{z},$$

(2.121)

where $g$, $\mathbf{v}$ and $P$ are, respectively, the gravitational acceleration, the velocity and the pressure of the fluid. Note that we are assuming that the gravitational force is the only external body force acting on the fluid.

In Refs. [50, 52] the authors relax the assumption of flat bottom and height as well as that of incompressibility. In Ref. [53] the authors study gravity waves when the assumption of irrotationality of the background flow is relaxed. For simplicity here we assume incompressibility, irrotationality and flat bottom.

Let $\mathbf{x}_\parallel = (x, y)$ and $z$ (or, equivalently, $\mathbf{x}_\parallel = (r, \phi)$ and $z$) be the spatial coordinates and assume the fluid tank to be flat such that the $z$-direction is perpendicular to the bottom of the tank ($z = 0$). Also, let $h (t, \mathbf{x}_\parallel)$ be the height of the fluid at $x_\parallel$. Besides the basic equations above, to determine the flow of water in an open channel, we need three boundary conditions, namely:

1. The normal speed must vanish at the bottom of the tank, i.e. $\mathbf{v}_z|_{z=0} = 0$;

2. The rate of change in the height of the fluid must be equal to the vertical velocity of the fluid at the surface,

$$v_z|_{z=h} = \frac{dh}{dt}\bigg|_{z=h} = \frac{\partial h}{\partial t} + (\mathbf{v}_\parallel|_{z=h} \cdot \nabla_\parallel) h;$$

(2.122)

3. The pressure must be continuous at the interface. Since $\tilde{p}$, the pressure of the air component, is negligible, the pressure at the surface of the fluid must vanish (observe that we are neglecting surface tension effects here),

$$P(z = h) = 0 \Rightarrow P = P_1(z - h) + \mathcal{O}(z - h)^2.$$  

(2.123)

As usual (recall the first example in [1.1.3]) the flow is assumed to be irrotational, so that the velocity of the fluid $\mathbf{v}$ can be written as the gradient of a scalar field $\psi$,

$$\mathbf{v} = \nabla \psi.$$  

(2.124)
Chapter 2. A Toolbox for Analogue Gravity

The continuity equation, therefore, reduces to Laplace’s equation for the field,
\[ \nabla \cdot \mathbf{v} = \nabla^2 \psi = \nabla_{\parallel}^2 \psi + \partial_z^2 \psi = 0. \] (2.125)

Euler’s equation under the irrotational assumption can be written as Bernoulli’s equation,
\[ \frac{\partial \psi}{\partial t} + \frac{1}{2} \nabla \psi \cdot \nabla \psi = -\frac{P}{\rho} - gz. \] (2.126)

Expanding the equation above in powers of \( z - h \), we obtain from the 0th-order term,
\[ 0 = \frac{\partial \psi}{\partial t} \bigg|_{z=h} + v_{||}^2 \bigg|_{z=h} + v_z^2 \bigg|_{z=h} + gh, \] (2.127)
where we have used the boundary condition (3).

### 2.3.2 Wave propagation

When the free surface of an open channel flow is perturbed, gravity acts as a restoring force resulting in oscillations around the background flow. The mathematical description of these oscillations, called gravity waves, is given in terms of perturbations of the background quantities, \( \psi \) and \( h \), in the following way.

Taylor expanding \( \psi \) around \( z = 0 \),
\[ \psi(t, x_{||}, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \psi_n(t, x_{||}), \] (2.128)
and substituting into Laplace’s equation (2.125) above yields the recursion relation for the \( \psi_n \)
\[ \psi_{n+2} = -\nabla_{\parallel}^2 \psi_n. \] (2.129)

The boundary condition (1) implies that \( \psi_1 = 0 \) and, therefore, from the recursion relation, we write the field \( \psi \) only in terms of \( \psi_0 \) [54],
\[ \psi(t, x_{||}, z) = \sum_{n=0}^{\infty} \left( -1 \right)^n \frac{z^{2n}}{(2n)!} \nabla_{\parallel}^{2n} \psi_0(t, x_{||}). \] (2.130)

Substituting equation (2.130) in the boundary condition for stationary surface height equation (2.122), we obtain, after some algebraic manipulation, an equation involving only \( h \) and \( \psi_0 \),
\[ 0 = \frac{\partial h}{\partial t} + \nabla_{\parallel} \cdot \int_0^h \mathbf{v}_{||} dz = \frac{\partial h}{\partial t} + \nabla_{\parallel} \cdot \left( \sum_{n=0}^{\infty} \left( -1 \right)^n \frac{h^{2n+1}}{(2n+1)!} \nabla_{\parallel}^{2n+1} \psi_0(t, x_{||}) \right) \] (2.131)
In terms of the variables $h$ and $\psi_0$ by using eq. (2.130) the Bernoulli equation (2.126) becomes,

$$\sum_{n=0}^{\infty} \frac{(-1)^n h^{2n}}{(2n)!} \frac{\partial}{\partial t} \nabla_\parallel^{2n} \psi_0 + \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{(-1)^n h^{2n}}{(2n)!} \nabla_\parallel^{2n+1} \psi_0 \right)^2$$

$$+ \frac{1}{2} \left( \frac{\partial h}{\partial t} + \nabla_\parallel h \cdot \sum_{n=0}^{\infty} \frac{(-1)^n h^{2n}}{(2n)!} \nabla_\parallel^{2n+1} \psi_0 \right)^2 = -gh.$$ (2.132)

Hence we may describe surface perturbations by the functions $h$ and $\psi_0$ alone. Let

$$\psi_0 \to \psi_0 + \delta \psi_0, \quad h \to h + \delta h,$$ (2.133)

where $\delta \psi_0 \ll \psi_0$ and $\delta h \ll h$. By substituting this ansatz into the equations which determine the background flow, i.e. equations (2.131) and (2.132) assuming that $\psi_0$ and $h$ are exact solutions to the background equations, we obtain equations that describe the perturbations themselves,

$$\frac{\partial \delta h}{\partial t} + \nabla_\parallel \cdot \left( \nabla_\parallel \cdot \delta \psi_0 \right) = 0,$$ (2.134)

and

$$\left( \frac{\partial}{\partial t} + \nabla_\parallel \cdot \nabla_\parallel \right) \left( \sum_{n=0}^{\infty} \frac{(-1)^n h^{2n}}{(2n)!} \nabla_\parallel^{2n+1} \psi_0 \right) = -\tilde{g} \delta h,$$ (2.135)

where $\tilde{g}$ is given by,

$$\tilde{g} = g + \left( \frac{\partial}{\partial t} + \nabla_\parallel \cdot \nabla_\parallel \right)^2 h.$$ (2.136)

Assuming that the height of the fluid is much smaller than the horizontal scales on which the flow profile changes significantly, we can keep only the lowest order term in the sums appearing in the perturbation equations above. In other words, if the height $h$ is much smaller than the wavelength $\lambda$ of the waves, we can neglect higher order terms since $\nabla_\parallel^2 = O(1/\lambda^2)$ and $h/\lambda \ll 1$, see ref. [50]. Therefore, eqs. (2.134) and (2.135) reduce, respectively, to

$$\frac{\partial \delta h}{\partial t} + \nabla_\parallel \cdot \left( \nabla_\parallel \cdot \delta \psi_0 \right) = 0,$$ (2.137)

and

$$\left( \frac{\partial}{\partial t} + \nabla_\parallel \cdot \nabla_\parallel \right) \delta \psi_0 = -\tilde{g} \delta h.$$ (2.138)

Eliminating $\delta h$ we arrive at an equation of motion for $\delta \psi_0$ which compactly written as a Klein Gordon equation

$$\frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g}\gamma^{\mu\nu} \partial_\nu \delta \psi) = 0,$$ (2.139)
with an effective metric, dropping the $|z=h$ subscripts, given by,

$$\frac{g_{\mu\nu}}{\tilde{g}} = \frac{h}{g} \left( \begin{array}{ccc} -\left(c^2 - v^2\right) & -v \\ \cdots & \cdots \\ -v & 1_3 \end{array} \right) \quad (2.140)$$

Shallow water gravity waves, therefore, propagate in the analogue spacetime above with a velocity given by $c^2 = \tilde{g}h$. Note that when the surface is exactly flat and time independent then $\tilde{g} = g$ and we recover the standard result that the surface wave propagation speed is $\sqrt{gh}$.

### 2.4 Non-linear optics, superluminal pulses

This section deals with some elementary optics theory which will be necessary later on in Chapter 4. Most of the material from this section comes from the thesis [53].

When an electromagnetic wave travels in a dielectric medium, it excites the atoms in the medium, altering the way it propagates. Importantly, the reaction of the atoms to the wave is not instantaneous. This implies that the action of the atoms back on the electromagnetic wave depends on how the wave was interacting with the atoms some time prior. This is a non-local interaction and is responsible for some very interesting non-linear effects in optics.

The polarization of a dielectric material by an time varying electric field can be modeled as the linear but non-instantaneous in time form

$$P = \epsilon_0 \int_{-\infty}^{t} \chi(t-t')E(t')dt' \quad (2.141)$$

where $\epsilon_0$ is the dielectric constant and $\chi$ is a response function modeling the relaxation time of the bound charges.

Using the Maxwell equations this implies equation of motion for the electric field

$$\nabla^2 E_\omega + \frac{\omega^2}{c^2} (1 + \chi_\omega) E_\omega = \nabla^2 E_\omega + \frac{\omega^2}{c^2} n_\omega E_\omega = 0 \quad (2.142)$$

the $\omega$ subscript refers to the time Fourier transform which acts simply on the convolution (2.141) yielding a product. Here we have defined the refractive index $n_\omega = 1 + \chi_\omega$.

The assumption (2.141), however, is only accurate for low power electric fields and should be replaced by a higher order effect depending on multiple powers of the electric field

$$P = P^{(1)} + P^{(2)} + \ldots \quad (2.143)$$
where
\[
P^{(n)} = \epsilon_0 \int t^1 \ldots \int t^n \chi_{a_1\ldots a_n} \, E_{a_1}(t_1) \ldots E_{a_n}(t_n)
\]  
(2.144)

It can be shown \[55\] that the next non-zero contribution comes at order $P^{(3)}$ which induces a modification to the equation of motion (2.142).

These higher order effects play a role when the electric power of a signal is strong. We will be interested later on in this thesis in the case where an independent field mode propagates in a medium in which a second, high power signal is also propagating. The non-linear effect of the high power signal, through relation (2.143) is to locally modify the refractive index experienced by the first mode. This effect, known as the **non-linear Kerr effect** is the focus of a concentrated experimental effort, which will will discuss later in Chapter 4 seeking to directly observe an analogue of the Hawking effect.

### 2.5 Wave dispersion, tools and basics

In this section we discuss the role of modified dispersion relations in analogue gravity. The motivation is twofold. On the one hand real analogue systems are linearly dispersive only in a limited frequency band, being subject to an analogue description only in this restricted sense \[56\]. On the other hand, there is a growing voice in the gravity community that local Lorentz invariance (LLI) might be broken in the UV for various reasons \[12, 13\]. In fact one of the main motivations for considering modified dispersion comes from black hole physics where many of the ‘UV problems’ arise (recall our discussion in Sec. 1.2) such as the Transplanckian problem and infinite entanglement entropy.

#### 2.5.1 Motivation for considering dispersive theories

When waves travel on a medium which possesses some microstructure at small scales their propagation senses this microstructure and this is reflected in a modified dispersion relation. The prototypical example of this phenomenon is wave propagation on a regular 1D lattice of spacing $a$ \[57\]; solutions to the finite difference wave equation are plane waves $\exp(-i\omega t + ikx)$ with

\[
\omega^2 = \frac{2}{a^2} (1 - \cos(ka)) \simeq k^2 - \frac{a^2}{12} k^4 + \mathcal{O}(k^6).
\]  
(2.145)

We see that when the wavelength $2\pi/k$ becomes comparable with the spacing $a$ the waves sense the lattice and slow down i.e: the group velocity $d\omega/dk$ becomes less than 1 but when $|ka| \ll \sqrt{6}$ (the turning point of the quartic dispersion relation) the waves propagate as in the continuum with linear dispersion. We also saw above that
BEC perturbations satisfy according to the non-linear Bogoliubov dispersion, indicating that the very short wavelength perturbations sense the atomic nature of the superfluid background. The situation in optics is slightly different in that the electromagnetic field does not depend on the presence of the background to be well defined as is the case in BEC and the lattice example given above: at long wavelengths and 'short' frequencies there is a chance for resonances in the material and the modes see a non-trivial background structure while at short wavelengths modes do not see the dielectric and propagate as in flat space. This is reflected in the Sellmeier dispersion relation given above in [2.4] which converges to linear dispersion for very high frequencies.

Of course it would be desirable to have at hand a micro-theory of spacetime and quantum fields but we are not so lucky yet to have a consistent theory including both gravity and quantum theory at short length scales. One approach taken in the black hole literature, motivated by the analogue gravity picture, is to remain ignorant to the true microphysics while modeling it with a modified dispersion relation in the UV. One then can go about testing the predictions for black hole radiance of the modified theory, comparing them to the linear dispersive Hawking result.

One major result from the analogue gravity community has been the demonstration that the inclusion of a modified dispersion relation in the UV changes dramatically the derivation of Hawking’s result but leaves the conclusion unscathed [58, 59, 60, 61, 62, 63, 64]. This has been essential groundwork necessary before real experimental attempts are made to directly observe an analogue of Hawking radiation in dispersive media which naturally possess such a cut-off scale where the linear dispersion breaks down (the length scale where the picture of a fluid dissolves into that of propagating atoms). On the other hand this conclusion about the robustness of the Hawking spectrum settles an old question about whether black hole evaporation depends on the UV completion of GR, necessarily involving some quantum or discrete properties of spacetime, showing that one can have black hole radiance in these modified spacetime scenarios. We will have more to say on this later on in the chapter on DCE in optical systems as well as chapter 5 where we explicit work out a dispersive model.

2.5.2 Dispersion diagrams

Consider the polynomial dispersion relations

\[ \omega^2 = k^2 \pm \frac{k^4}{\Lambda^2}, \]  

(2.146)

where the ± refers to super- and subluminal dispersion and \( \Lambda \) is some momentum scale controlling the onset of the non-linearity. Physical propagating degrees of freedom possess a frequency and wavenumber pair \( (\omega, k) \) which satisfies the dispersion relation.

One can visualise the allowed wave-numbers and frequencies by drawing the dispersion diagram, shown in Figure 2.4. We note that the subluminal dispersion relation
Figure 2.4: A superluminal 2.4(a) and subluminal 2.4(b) dispersion relation with the physical degrees of freedom marked with black dots. In both plots the blue dashed line represents a line of constant frequency $\omega$.

admits 4 physical degrees of freedom as long as the frequency $\omega$ is small enough compared with $\Lambda$.

When one is dealing with a non-linear dispersion relation there arises a distinction between group and phase velocity defined respectively as

$$\text{group velocity} = \frac{d\omega}{dk}, \quad (2.147)$$

$$\text{phase velocity} = \frac{\omega}{k}. \quad (2.148)$$

The group velocity can be read from the dispersion plot as the gradient of the red curves at the intersections with the blue dashed line. We see that the group velocity can point in the opposite direction to the phase velocity. It is the group velocity which is the carrier of information in a wave form and the quantity which satisfies normal causality requirements.

2.5.3 Moving media

The existence of a discrete substrate implies that the frame of reference which is at rest with respect to the substrate play a special role; it is the one in which the modified
Figure 2.5: A superluminal 2.5(a) and subluminal 2.5(b) dispersion relation when the medium is moving with respect to an observer. In both plots the blue dashed line represents a line of constant observer frequency $\omega$.

dispersion is defined. Transforming to a moving frame, moving at speed $V$ with respect to the substrate introduces a doppler factor in the dispersion relation as

$$ (\omega - kV)^2 = \Omega^2 = k^2 \pm \frac{k^4}{\Lambda^2}, $$

which modifies the dispersion diagram as shown in the two frames: 2.5 shows the dispersion relation as seen in the frame of the moving observer while in 2.6 we show the same plots from the frame of the substrate.

In the figures 2.5 it is possible to obtain many data associated with the propagating degrees of freedom. The intersections, as discussed, correspond to physical degrees of freedom. The gradient of the dispersion curve at this point provides us with the group velocity in the frame of the medium. The difference in the gradients of the dispersion curve and the straight line given by $d\Omega/dk - (\pm V)$ give us the group velocity in the moving frame since

$$ d\Omega = d\omega - Vdk \Rightarrow \frac{d\Omega}{dk} = \frac{d\omega}{dk} - V. $$

The group velocities in the frame of the medium are given by the sums
2.5. Wave dispersion, tools and basics

Figure 2.6: A superluminal 2.6(a) and subluminal 2.6(b) dispersion relation when the medium is moving with non-zero speed $V$ with respect to an observer. In both plots the blue dashed line represents a line of constant observer frequency $\omega$.

2.5.4 Hawking ‘scattering’ with modified dispersion

Classically, nothing can come out of a black hole — they are truly black. As we saw, with the introduction of quantum effects black holes radiate at a temperature proportional to their surface gravity. Quantum modes are observed at $\mathcal{I}^+$ whose history places them arbitrarily close to the horizon at early times but not inside.\[3\]

However, when one tries to generalise the derivation of Hawking radiation to the case of non-linear dispersion, one discovers the fact that, due to a momentum dependent group and phase velocity for dispersive waves the ‘absolute blocking’ property of the non-dispersive horizon no longer exists.

Let us assume that the dispersion is modified in the frames freely falling towards the black hole. These frames move with velocity $V(x)$ at different distances $x$ from the black hole, defining a position dependent doppler shift as in (2.149). Then, assuming a WKB approximation where the velocity changes are mild over lengths scales large with respect to the wavelength of the mode, we can define the position dependent dispersion.

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\[3\]It should be mentioned that there exists a consistent derivation of the Hawking effect from the quantum tunneling of modes from behind the horizon. See [65] and references therein for a description of that interpretation of Hawking radiation.
Chapter 2. A Toolbox for Analogue Gravity

Figure 2.7: The propagation, backwards in time of a mode escaping a region of non-zero and negative flow. The black dot indicating the wave number and effective frequency in the frame of the medium slides up the curve approaching the point where the blue line is tangent to the dispersion curve and the group velocity with respect to the medium is zero. This is where the mode conversion takes place.

\[ \omega - kV(x) = k^2 - \frac{k^4}{\Lambda^2}. \]  

(2.151)

A mode which is propagating away from the black hole at large distances, with positive group velocity can be shown to have originated, if one traces the history back in time, as an \textit{incoming} mode at \( \mathcal{I}^+ \) before approaching the hole and being ‘converted’ into an outgoing mode by the spacetime curvature! This is illustrated in Figure 2.7 where as we track back in time, the group velocity in the frame of the hole goes to zero since the straight line becomes tangent to the red curve at some point. This point is unstable \cite{60, 61, 62} and in fact, modes turn around at this point, propagating back away from the hole (backwards in time). Hence we see that the outgoing mode originated far from the black hole, propagated as an in-coming mode towards the hole before being converted to an outgoing mode at late times.

Therefore for dispersive fields, the possibility of an entirely classical analogue of the
Hawking process is provided, corresponding to the *scattering* of a classical incoming wave through the mode conversion process into an outgoing mode.

For superluminal dispersion, the mode can come from *behind* the horizon and arbitrarily close to the singularity prompting the study of analogue models with a flow rate which tapers off to a constant value inside the horizon, extending the interior high flow region to arbitrarily large negative positions and avoiding the singularity present in spacetime black holes.
Chapter 3

Particle Creation and Correlations in Expanding Bose Einstein Condensates

In this chapter we will be looking at one of the most studied analogue systems, Bose-Einstein Condensates (BEC) and an analogue of cosmological particle creation in them. Although BEC have been extensively studied by the analogues community, the novelty of the material in this chapter is in the use of quantum correlations as an observable, and in particular the quantum correlations as are created by analogue cosmological particle production.

We consider both a 3+1 isotropically expanding BEC as well as the experimentally relevant case of an elongated, effectively 1+1 dimensional, expanding condensate. In this case we include the effects of inhomogeneities in the condensate, a feature rarely included in the analogue gravity literature.

In both cases we link the BEC expansion to a simple model for an expanding spacetime and then study the correlation structure numerically and analytically (in suitable approximations). We also discuss the expected strength of such correlation patterns and experimentally feasible BEC systems in which these effects might be detected in the near future.

3.1 Introduction

As described in sec[11] the propagation of linearized acoustic perturbations on an inviscid and irrotational fluid can be shown to be described by the same equation of motion as that which describes the propagation of a scalar field on a curved spacetime. Of course such ideal systems are not readily available in nature but it is possible to produce
systems which, to a very good degree of approximation, do actually behave as perfect fluids. In recent years a special role in this sense has been played by Bose-Einstein condensates, the basic observation being in this case that phase perturbations of the wave function describing a weakly interacting BEC are, in a certain limit, described by a massless quantum scalar field propagating on a non-trivially curved background Lorentzian space-time [43, 44, 45]. This observation stems from the demonstration, given below, that the macroscopically occupied ground state of a BEC can be thought of as a perfect fluid under some appropriate approximations.

A BEC can therefore be used to simulate special gravitational backgrounds and the associated particle production such as Hawking radiation of black holes [66, 67, 68, 69], or cosmological particle production. For example, releasing the trapping potential of a BEC appropriately [70, 71, 72, 73, 74, 75, 76] or modulating external parameters of the experimental setting [77, 78, 79, 80, 81], it is possible to induce the necessary dynamics which give rise to interesting background geometries. Due to the non-trivial time evolution, the system undergoes a non-equilibrium phase to which are associated quantum emissions whose spectrum strictly depends on the parameters modulating the evolution. This opens up the possibility of experimentally accessing new kinds of phenomena directly involving the quantum nature of cold atom systems as well as providing additional insights on the general mechanisms of such emissions in the gravitational case.

We highlight at this point the dual nature of the analogy: on one hand, the analogy provides us with useful mathematical tools with which to describe interesting condensed matter phenomena while on the other, the analogy provides us with models upon which the dynamics of some semiclassical gravity phenomena are played out.

As the title suggests, the central focus of this chapter will be on dynamical particle production in an expanding BEC. This emission is of the dynamical Casimir type, arising from a time dependence in some parameters in the equation of motion. It could be said that such an emission is a ‘regular’ QFT emission: the predicted spectra and physics mechanisms involved, generally speaking, occur at the same energy scale. In this way a modification of the high energy part of the physics of a low energy process will not effect the obtained results\(^1\). In contrast, the Hawking emission in its regular guise could be said to be an ‘exceptional’ emission, seemingly depending on arbitrarily high frequencies in the prediction of a low frequency emission: although the spectrum of Hawking quanta is peaked, under ordinary circumstances, at sensible frequencies while decaying to zero emission in the UV, the black hole horizon\(^2\) and the exponential

\(^1\)An notable counterexample to this would be if the time dependence arose from an expansion of the physical background, either spacetime or condensate itself. In those cases a produced particle might have a wavelength which, going back in time through the contraction, have been ‘Trans-Planckian’ at early times. However, the idea that the particle was ‘produced’ at a certain moment in the past when its wavelength should have been short is problematic, with particles being well defined only in the static regions after the expansion has ceased.

\(^2\)Note that strictly speaking, a trapping horizon is not a necessary ingredient for a Hawking emis-
blue-shifting to the past it entails ensures that each Hawking quanta observed at $\mathcal{I}^+$ spent a part of its lifetime as an exponentially large frequency. The lifetime of the black hole (or apparent horizon) determines the magnitude of exponential blue-shifting of the Hawking quanta. In this way a black hole in the ordinary sense (with linear dispersion) acts as an arbitrarily strong microscope, sensitive to all scales in the background. Now, the cosmological emission being, morally speaking, insensitive to the microphysics of the background geometry (be it spacetime or an analogue thereof), we will not be concerned in this chapter with the modifications induced by a non-linear dispersion relation, preferring to restrict our attention to the linear dispersive low frequency part of the Bogoliubov spectrum, keeping track of the UV scale of the modifications and checking that the results lie within the linear regime (see (2.105)). As mentioned above, at the end of this chapter we relax this simplification and briefly show how the spectrum really is unaffected by a high frequency modification to the dispersion for ‘slow’ processes.

The issue of how to practically observe these quantum emissions and verify their origin is a crucial point in the condensed matter literature. From the perspective of analogue models a direct observation of the created particles seems quite difficult in the light of the fact that often they possess a quasi-thermal and spectrum, somehow observable on top of a background thermal noise. There has been recently a large interest in the calculation and measurement, instead, of correlation patterns in expanding condensates to access these quantum effects (for example see [82, 83, 84]). In the article [85] the author proposes studying the density correlations in a toroidally shaped BEC with time varying interaction strength as a method to uncover some micro-physics, in particular the character of the quantum squeezing of phononic modes as the interaction strength is varied. In [67] it has been proposed that evidence for analogue Hawking emission in a sonic black hole manifested in a BEC might be solicited by measuring the two point correlation structure associated with the quantized density or phase perturbation fields. The idea is that the correlation pattern should display a peculiar signature of the type of emission expected from black holes. In [68] a numerical study of the full dispersive dynamics confirmed the robustness of the results of [67] outside of the linear dispersive regime while in [66, 69] a more general method was presented for studying the correlations due to analogue Hawking radiation in BEC. Following the same line of reasoning, we apply this idea in this chapter to analogue cosmological emission and attempt to isolate a cosmological signal in the correlation pattern in an expanding BEC.

In particular this chapter is mainly concerned with the experimentally accessible correlations between the phase and density perturbations in a BEC which are born during the expansion of the underlying gas both in the widely studied effectively 1+1 dimensional “cigar” shaped geometry and in a homogeneous expanding BEC. Such

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3 See chapter 4 for further discussion on this point.
Chapter 3. Particle Creation and Correlations in Expanding Bose Einstein Condensates

BEC systems are of theoretical interest in their own right and correlation measurements are experimentally accessible with the current technology [82, 83, 84]. Therefore it is possible that these systems will provide in the near future an arena within which to experimentally address interesting and new non-equilibrium quantum effects while at the same time shed light on the physics of quantum fields in expanding spacetime.

In the next section we collect and contextualize some results relevant for the study of correlations in an expanding BEC, reviewing how the analogy is applied to the dynamics of the system. In the third section we compute the correlation structure for a massless scalar field in a particular expanding background spacetime in 3+1D and show how to interpret such a result in terms of both an isotropically expanding 3+1D condensate and a BEC with a time dependent scattering length. We shall develop general formulas for probing the effect of particle production on correlations as embodied in the equal-time Wightman functions. We shall derive results which are completely generic to particle production in any spatially translation invariant system, which can be applied to standard cosmologies as much as to analogue spacetimes.

In the fourth section which represents the main result of this chapter using both analytic and numerical methods we discuss the structure of correlations in an anisotropically expanding, effectively 1+1-dimensional, BEC. In particular we shall derive the general, finite size, 1+1-dimensional, equal time Wightman function relevant for the “cigar” shaped expanding BEC presently realized in experiments. We conclude with some proposals for improving the observability in practice of the correlation signal and some comments on how such emissions are related to Hawking radiation.

3.2 The ingredients for cosmology in the lab

In this section we give a more detailed look at the analogue description of a BEC, and in particular that which pertains to an expanding condensate relevant for modeling, through the analogy, phenomena in an expanding universe.

We refer the reader to the introductory section 2.2 for the basic theory as well as the appendix D.3 for additional details and a glossary of terms and concepts relevant to BEC physics.

3.2.1 Scaling solution

Condensates undergoing expansion or contraction in $D$ spatial dimensions as a result of a time varying confining harmonic potential $V_{\text{ext}}$ admit a dynamic generalization to the static Thomas Fermi approximation for the Gross Pitaevskii equation (see equation (2.110) in chapter 2) known as the scaling solution [86, 87]. In this case the external
3.2. The ingredients for cosmology in the lab

potential is of harmonic type with time varying trapping frequencies

$$V_{\text{ext}}(t, x) = \frac{1}{2} \sum_{i} m \omega_i^2(t) x_i^2.$$  \hfill (3.1)

It can be shown that with the density undergoes a simple scaling transformation in each spatial dimension $x_i$ parametrized by the so-called scaling functions $\lambda_i(t)$ as

$$\rho(t, x) = \frac{\rho_s(x_i/\lambda_i(t))}{\prod_i \lambda_i(t)},$$  \hfill (3.2)

where $\rho_s$ is the static TF ground state solution

$$\rho_s = \begin{cases} \rho_0 \left(1 - \sum_i \bar{\omega}_i x_i^2\right), & \text{for } \sum_i \bar{\omega}_i x_i^2 < 1, \\ 0, & \text{otherwise} \end{cases},$$  \hfill (3.3)

where $\bar{\omega}_i = m \omega_i^2 / 2\mu$ ($\mu$ being the chemical potential) and $\rho_0$ is the maximum density at the centre of the BEC. The scaling functions are determined by the dynamics of the varying harmonic frequencies $\omega_i(t)$, controlled by the experimenter, as we will see below.

In the static TF approximation the density and chemical potential can be expressed algebraically in terms of the physical parameters as

$$\rho_0 = \frac{\mu}{g(\bar{a})} = \frac{1}{8\pi} \left(\frac{15m^3N \prod_i \omega_i}{\bar{a}^{3/2} \hbar^3}\right)^{2/5},$$  \hfill (3.4)

and

$$\mu = \frac{1}{2} \hbar \bar{\omega} \left(15N \bar{a} \sqrt{\frac{m \bar{\omega}}{\hbar}}\right)^{2/5}.$$

respectively, where $N$ is the number of condensed bosons. Here we have introduced the notation $\bar{\omega} = (\prod_i \omega_i)^{1/d}$ in $d$ spatial dimensions, the geometric mean of the trapping frequencies. For the phase, the scaling transformation takes the form

$$\theta(t, x) = \frac{1}{2} m \sum_i x_i^2 \dot{\lambda}_i / \lambda_i,$$

where the dot \ represents the lab time derivative $\partial_t$.

This scaling induces a background flow velocity profile in the condensate given by

$$v(t, x) = \frac{\dot{\lambda}(t)}{\lambda(t)} x.$$  \hfill (3.7)

\footnote{Note that the TF approximation is not a good approximation near the boundary of the condensate where the density drops to zero. The condensate boundary, here given by the parabola sharply dropping to zero, in the full solution will be smooth.}
Chapter 3. Particle Creation and Correlations in Expanding Bose Einstein Condensates

With this velocity and the evolution of the density given in (3.2) one can start studying the analogue geometry associated with such a scaling background solution using the metric (2.119), reproduced here for reference

\[ ds^2 = \sqrt{\frac{\rho}{g m}} \left[ -(c_s^2 - v^2) dt^2 - 2v_i dx^i dt + d\mathbf{x}^2 \right]. \]  

(3.8)

Written in terms of co-moving coordinates \( \tilde{x}_i = x_i/\lambda_i(t) \) the metric (3.8) for the scaling solution BEC in D spatial dimensions reads

\[ ds^2 = \left( \frac{\rho(t, \mathbf{x})}{g m} \right)^{\frac{1}{D-1}} \left( -c_s^2(t, \mathbf{x}) \, dt^2 + \lambda_i(t) d\tilde{x}_i^2 \right), \]  

(3.9)

from which a number of standard representations for a FRW background are easily obtained. The cosmological assumption of homogeneity is satisfied for a BEC in the central region where the density is almost constant. Working in such a central region, \( \rho \) and \( c_s \) become functions only of \( t \) and, assuming an isotropic expansion \( \lambda_i(t) = \lambda(t) \) for all \( i \), the metric can be written in standard FRW form as

\[ ds^2 = -d\tau^2 + a^2(\tau) d\mathbf{x}^2, \quad \tau(t) = \int_t^t dt' \rho^{1/(D-1)}(t') \left( c(t') \lambda(t') \right)^{1/2} \frac{\rho(t')}{g m} \]  

(3.10)

with the scale factor defined by

\[ a^2(\tau) = \left( \frac{\rho(\tau)}{g m} \right)^{\frac{1}{D-1}} \lambda^2(\tau) \propto \lambda^{2-D/(D-1)} = \begin{cases} \lambda(\tau)^0 & \text{in 2+1 dimensions} \\ \lambda(\tau)^{1/2} & \text{in 3+1 dimensions} \end{cases} \]  

(3.11)

where we have used the scaling solution \( \rho(\eta) = \rho_0/\lambda^D(\eta) \) for the density.

What one can see by (3.11) is that in 2+1 dimensions, the effective geometry is a scaled Minkowski space and all particle creation effects are hidden in the definition of the time coordinate \( \tau \). Of course in 1+1 dimensions, the effective geometry does not exist (as shown in Sec. 1.1.3), so that 3+1 dimensions is the first dimension when we obtain an interesting geometry.

Alternatively, still within the assumptions of homogeneity and isotropy, we define a conformal time coordinate

\[ \eta(t) = \int_t^t dt' \frac{c(t')}{\lambda(t')}, \]  

(3.12)

with respect to which the metric is written

\[ ds^2 = \left( \frac{\rho(\eta)}{g m} \right)^{\frac{1}{D-1}} \lambda^2 (d\eta^2 - d\mathbf{x}^2). \]  

(3.13)

Note that we require a homogeneous sound speed \( c \) for this definition.
3.2. The ingredients for cosmology in the lab

On the other hand, if the expansion is explicitly anisotropic we may define the function

$$\tau(t) = \int^t dt' \left( \frac{\rho(t')}{gm} \right)^{\frac{1}{2(D-1)}} c(t'),$$

(3.14)

also requiring a homogeneous sound speed but with respect to which the metric takes on the anisotropic Bianchi I or Kasner form

$$ds^2 = d\tau^2 - \left( \frac{\rho}{gm} \right)^{\frac{1}{D-1}} \lambda^2 \cdot d\tilde{x}^2 =: d\tau^2 - \sum_i a_i^2(\tau)(d\tilde{x})^2,$$

(3.15)

where we have defined the independent FRW scale factors $a_i$.

Note that here, with the analogue picture available, we have two notions of proper coordinates in the game. On the one hand, there are the lab coordinates, the true proper coordinates given by $x_l = \lambda(t)x_c$, and the analogue geometric ‘proper’ coordinates $x_p^i := a_i(\tau)\tilde{x}^i$.

3.2.2 Dictionary of spacetimes and trapping frequencies

As stated above, for time dependent harmonic traps the scaling functions $\lambda_i(t)$ are not free parameters but are constrained to be solutions to the auxiliary equations of motion

$$\ddot{\lambda}_i(t) + \omega_i^2(t)\lambda_i(t) = \frac{\omega_i^2(0)}{\lambda_i(t) \prod_j \lambda_j(t)},$$

(3.16)

with boundary conditions $\lambda_i(0) = 1$ and $\dot{\lambda}_i(0) = 0$, determined by a self consistency condition on the dynamics of the expanding condensate [86].

Equation (3.16) represents a kind of dictionary mapping scaling functions to trapping potentials; in practice, as an analogue-gravity-ist, one has in mind some particularly interesting spacetime one wishes to simulate by an expanding BEC. The scaling parameters $\lambda_i(t)$ determine the time dependence of the analogue spacetime metric completely, albeit in a variety of ways depending on things such as the dimension of the BEC and analogue spacetime, so that in order to get a BEC to behave like your favourite spacetime one needs to control $\lambda_i$ which are in turn are completely determined by $\omega_i(t)$ through (3.16). In this way, one can custom-make some time dependent analogue geometry though experimental control of the trapping frequencies $\omega_i(t)$.

This dictionary is one-to-one and not always intuitive, the inverse transformation being

$$\omega_i(t) = \sqrt{\frac{\omega_i^2(0)}{\lambda_i^2(t) \prod_j \lambda_j(t)}} - \frac{\ddot{\lambda}_i(t)}{\lambda_i(t)},$$

(3.17)
An example of a counterintuitive translation is as follows. Intuitively, the most natural development for the scaling functions is that of complete and sudden release of the trapping potential. The characteristic expansion rate in that case $\dot{\lambda}$ is given by the initial frequency $\omega(0)$. If we ask ‘What is the trapping potential which gives a smooth (e.g., $\tanh(\cdot)$) expansion for the $\lambda_i$?’ the answer depends on the steepness of the tanh: to force a condensate to expand faster than the free expansion rate requires a repulsive potential while slower expansions are possible with a nice slowly decaying potential. Indeed if the term inside the square root in (3.17) becomes negative, we understand the associated $\lambda(t)$ to be unphysical as it would require a complex trapping frequency.

The example of a forced expansion will be pertinent later on for analytical considerations but at this stage we focus on the sudden release scenario since this is closest to what is being performed in experiments.

### 3.2.3 Cigar shaped expanding and suddenly released BEC

Of particular experimental interest is the ‘cigar’ shaped condensate possessing one elongated dimension (‘||’) and two tightly trapped orthogonal dimensions (‘\(\perp\)’). In what follows, unless otherwise stated, we work with the cigar background geometry. Since our main focus later on will be on such elongated condensates and in particular on experimentally realistic condensates, we list here the following realistic values for the physical parameters describing the cigar BEC

$$\omega_{||} = 10^2 \text{Hz}, \quad \omega_{\perp} = 10^3 \text{Hz}$$

$$m = 1.57 \times 10^{-25} \text{kg} \text{ (Rubidium 87)}$$

$$\bar{a} = 42 \text{ Bohr} = 2.221 \times 10^{-9} \text{ m}$$

$$N = 10^5.$$  

These are respectively the trapping frequencies, atomic mass, scattering length and number of condensed atoms. Assuming these numbers, the following derived quantities are obtained

$$L \approx 5.45 \times 10^{-5} m$$

$$\mu \approx 5.8 \times 10^{-31} \text{kgm}^2\text{s}^{-2} \approx 5500 \hbar \text{s}^{-1}$$

$$\rho_0 \approx 2.9 \times 10^{20} \text{ m}^{-3}$$

$$c_{s,0} \approx 2 \text{ mms}^{-1}$$

corresponding to the transverse condensate size (where, in the TF approximation the density drops to zero), the (static) chemical potential, the maximum condensate density and the maximum initial sound speed respectively.

Starting from such a cigar shaped initial geometry, let us consider the case when the ‘||’ potential is suddenly switched off while keeping the ‘\(\perp\)’ potentials constant. A
Figure 3.1: The solution of the scaling problem for an anisotropic expansion of an initially cigar shaped configuration for both the radial/orthogonal (blue-dashed line) and axial/parallel (red-solid line) directions to the axis of release. Note that the scaling function of the direction in which the trapping frequency remains fixed (blue, dashed) in fact decreases during the expansion. The parameters used in this solution are $\omega_\parallel = 100\text{Hz}$, $\omega_\perp = 10^3\text{Hz}$ and a sudden release of the $\parallel$ direction only.

similar problem has been studied in [83] and observed in [82] where the authors instead consider a BEC freely expanding in all the three spatial dimensions after completely releasing the trapping potential of an initially cigar shaped condensate. In these analyses, due to the expansion in all three dimensions, the density drops very quickly and the gas becomes collision-less signaling the failure of the hydrodynamical description shortly after the release of the trap. In the case we consider here instead, the perpendicular trapping is kept tight and the density drops more slowly and the condensate remains longer in a collisional regime where the spacetime analogy pertains.

In Fig. 3.1 we show the numerical solution to (3.16) for cigar-like initial geometry
Chapter 3. Particle Creation and Correlations in Expanding Bose Einstein Condensates

after the potential in the ‘||’ direction is suddenly switched off while keeping the ‘⊥’ potentials constant. Notice that the tightly constrained ‘⊥’ dimension in fact pulls even tighter during the expansion in the ‘||’ dimension as indicated by the decay of the scaling functions \( \lambda_\perp \). In this way one can understand how the volume of the cigar geometry, scaling as \( (\lambda_{||}\lambda_\perp^2)^{-1} \), decays much more slowly than in an isotropically expanding BEC which scales as \( \lambda_{||}^{-3} \).

3.2.4 Validity of the hydrodynamic approximation during expansion

As already mentioned above, it is also important to keep track of the hydrodynamical approximation throughout the expansion. Since the quantum pressure scales differently with time than the density, it is of interest to understand whether the hydrodynamic approximation improves or degrades as the expansion proceeds and, if it loses accuracy, when the approximation breaks down. In Fig. 3.2 we display the quantum pressure \( V_{\text{quantum}} \) and interaction \( V_{\text{interaction}} (= g\rho) \) contributions to GP before the release of the axial trapping potential. In Fig. 3.3 we display the ratio \( V_{\text{quantum}}/V_{\text{interaction}} \) as a function of co-moving axial position \( \tilde{z} = z\lambda_{||} \) for four different times after the expansion.

Note that although there is always a region near the edge of the condensate where the ratio is greater then 1, as time progresses the proportion of the condensate which remains within the hydrodynamical regime gets smaller until approximately \( t = 0.25 \)s when the quantum pressure becomes important globally in the condensate. The failure of the hydrodynamic approximation near the boundary of the condensate is expected in general since the TF approximation is known to break down on scales on the order of a healing length from the boundary where the density becomes exponentially small. Indeed the true exact solution, being a smooth exponential decay to zero, differs from the TF approximate solution, given in (3.3), near the edge which sharply goes to zero there. Our analysis in the following will be concentrated on the more central part of the condensate where the TF approximation holds.

3.3 Quantum excitations and correlations during expansion

We now intend to consider particle creation associated with the expansion of the condensate from a sudden release in the cigar configuration and its signature in the correlation structure. Before doing so however, one might wonder if such expansion is sufficiently rapid to lead to any relevant particle production at all. Also, given that we shall work in the analogue gravity framework, one would also like to be sure that the ostensible excitations can be meaningfully described as phonons rather than having
3.3. Quantum excitations and correlations during expansion

Figure 3.2: The quantum pressure and interaction energy expressed in Joules as a function of axial distance in meters. Note that the quantum pressure is certainly non-negligible near the boundary of the TF approximate ground state density wave-function (at \( z \approx 2.7 \times 10^{-5} \) m).

to deal with the complicated issues associated with non-linear dispersion. We shall discuss these issues first and then introduce the correlation function that will be the subject of our studies in subsequent sections.

### 3.3.1 Timescales

Let us now analyze and compare the relevant inverse time scales involved in the problem of an expanding condensate to make an order of magnitude estimate of whether excitation effects can be expected to be observable.

Firstly there is an intrinsic infrared cutoff given by the size of the condensate. This length scale translates into a frequency \( \omega_s \) using the (central maximum) sound speed \( \omega_s = c_s / (L \lambda(t)) \).

Secondly there exists an intrinsic ultraviolet cutoff associated with the healing length, the healing frequency \( \omega_h = 2\pi c_s / \xi \) where \( \xi = \hbar / (mc_s) \) is the healing length. The
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Figure 3.3: The ratio $R = \frac{V_{\text{quantum}}}{V_{\text{interaction}}}$ as a function of co-moving axial position $\tilde{z} = z\lambda ||$ at various times $t$ after release of the trapping potential. Note that the axial half-length of the condensate at time $t = 0$ (which of course is the co-moving length for all times) is $L_z \approx 2.7 \times 10^{-5}$ m.

healing length, as already mentioned, is the length scale at which the quantum potential, neglected in the hydrodynamical approximation, becomes intrinsically comparable with the interaction energy (which is proportional to the density of the condensate and hence becomes less relevant at lower densities). This is also the scale at which the Bogoliubov dispersion relation enters in the quadratic regime and hence the description in terms of non-interacting phonons breaks down. The healing frequency constraint is a constraint on the speed of expansion before the quantum potential begins to be significant for the dynamical process.

Thirdly there is the characteristic frequency associated with the expansion itself, $\omega_e \approx \dot{\lambda}/\lambda$, which fixes the frequency of the most abundantly created excitations in an expanding condensate. This is generic to any particle creation effect by a time-varying external field (see e.g. [77] and Section 2.1.1).

It is of interest to compare these three frequencies in order to test the validity of our approximations and to estimate if the expected excitations are created within the natural ultraviolet and infrared cutoffs. In Fig. 3.4 we compare these three inverse
3.3. Quantum excitations and correlations during expansion

Figure 3.4: A comparison of the relative magnitudes of $\omega_e$ and $\omega_s$ (red-solid and blue-dashed respectively) and with the healing frequency (inset plot) in $s^{-1}$ as a functions of time (in s). Note that, for the majority of the duration of the expansion, $\omega_s$ is much smaller than the expansion frequency $\omega_e$ which is much smaller than the healing frequency. Hence the bulk of the particle production spectrum lies within the low frequency cutoff provided by the total size of the condensate and the high frequency cutoff provided by the healing length.

We see that indeed there exists a regime in which the typical excitation frequency lies entirely within the natural ultraviolet and infrared cutoff frequencies. Quantitatively this regime begins after about 0.005s. Let us also note the vastly higher healing frequency representing the UV cutoff for a phononic treatment. This is shown in the smaller subfigure in Fig. 3.4 for comparison.

Therefore we conclude that there is a viable regime within which the excitations have a phononic (linear) dispersion relation and wavelengths much shorter than the size of the BEC.
3.3.2 Measuring quantum correlations - Wightman functions

Now that we have attained that the expanding BEC system is of the right size and character to dynamically produce particles which propagate on the BEC and with linear dispersion, we would like to know exactly how to detect such quantum emissions. As mentioned in the introduction, a direct measurement of produced particles seems difficult to achieve given the quasi-thermal nature of the spectrum and the low fluxes involved. A key observation is that quantum particle creation tends to produce particles in pairs which propagate in opposite directions. This was shown in Section 2.1.1 by equation (2.22). These pairs of particles are naturally entangled and will carry a particular correlation signature.

The Wightman function $G(x, y) := \langle \phi(x)\phi(y) \rangle$ for a quantum field $\phi$ is a measure of the correlation between two separate points $x$ and $y$ in spacetime. If the field $\phi$ were a truly random variable the correlation would be zero everywhere. In general the equal-time Wightman function for a massless scalar field on a flat 3+1 dimensional FRW spacetime written in the form

$$ds^2 = -a^6(\eta)d\eta^2 + a^2(\eta)d^2x,$$

(3.26)

with scale factor $a$ (which includes Minkowski space as the $a = 1$ special case) only depends on the magnitude of the co-moving spatial difference $x = ||x - y||$ and (possibly) on time. Since any function on such a spacetime decouples into independent co-moving momenta $k$ modes we have

$$G(\eta, x, y) = \int \frac{d^3kd^3k'}{(2\pi)^3}\langle 0|\phi_k\phi_{k'}|0\rangle e^{i(k\cdot x + k'\cdot y)}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i(k\cdot (x - y))} u_k u_k^*$$

$$= \int_0^\infty \frac{dk}{2\pi^2} \frac{1}{x} \sin (kx) u_k(\eta) u_k^*(\eta)$$

$$= G(\eta, x),$$

(3.27)

where $k = \sqrt{k\cdot k}$, $u_k$ are the mode-function solutions to $\Box u = 0$ and

$$\phi_k = (b_ku_k + b_k^\dagger u_k^*)/\sqrt{2},$$

(3.28)

is the quantum degree of freedom associated with $k$. In Minkowski spacetime the integral [3.27] is computed exactly as $G(\eta, x) = \hbar/(4\pi^2x^2)$ for non-zero $x$ while on the light-cone the Wightman function also possesses a singular imaginary part [26]. In 1+1 dimensions the equal time Wightman function becomes logarithmic and time independent $G(x, y) = -\hbar/(4\pi \ln (x - y)^2)$ and, owing to the conformal invariance in 1+1 dimensions for the massless scalar field and the conformal flatness of FRW spacetimes, also takes this form in an arbitrary 1+1 FRW background in terms of co-moving coordinates and the conformal time variable [25].
3.3. Quantum excitations and correlations during expansion

3.3.3 Exactly Soluble Expanding Spacetime

The main purpose of this section is to gain an intuitive understanding of correlations due to particle production and provide a useful toolbox for the more challenging investigation of the anisotropic scaling solution. In order to do so we shall study the ideal case of a homogeneous 3+1D isotropically expanding BEC. In this case, the analogue metric is that of a simple FRW spacetime geometry. We shall choose a scale factor that increases from an initial constant value $a_i$ to a final constant value $a_f$ smoothly as in this case the structure of correlations can be solved for analytically. Our interest in these exactly soluble models derives from the fact that in these systems one can see the appearance in the Wightman function of the characteristic features expected from the creation of particles in a transparent way. We shall also discuss how to simply interpret such models in terms of a BEC with a varying scattering length.

We will work with the form for the metric given by (3.26) which can be obtained from the more traditional form (3.10) by the coordinate transformation

$$d\eta = \left( \frac{gm}{\rho_0} \right)^{3/2} \frac{1}{\lambda^{3/2}(\tau)} d\tau.$$

(3.29)

The functional relationship between the scale factor $a$ the scaling function $\lambda$ is unchanged (see (3.11)) in this representation as $a^2(\eta) \propto \sqrt{\lambda(\eta)}$.

Furthermore we will consider the specific form for the scale factor

$$a^4(\eta) = \frac{a_i^4}{2} + \frac{a_f^4 - a_i^4}{2} \tanh \left( \frac{\eta}{\eta_0} \right),$$

(3.30)

where $\eta_0$, $a_i$ and $a_f$ are constant numerical parameters. Such a metric describes an isotropic homogeneous spacetime with flat spatial sections which expands by a finite amount over a finite time period.

This spacetime has been studied previously in the literature for example in [77] where it was shown, to arise as the analogue geometry experienced by phase perturbations in a BEC with time varying interaction strength $g$. In that article the particle content of the initial Minkowski vacuum state after expansion is studied in detail and the observability of the created particles discussed. This spacetime is also studied in [75] where it is shown to arise as the analogue geometry associated with a ring of trapped ions. A very similar geometry is studied in [81] where instead of tanh, the authors employ an error function temporal profile for the variable scattering length and the correlation structure is studied semi-analytically in the so-called “sudden limit” (which we will discuss later), and numerically in the general case, with the full Bogoliubov spectrum. In [88] the author studies the controlled release of a condensate.

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5Recall that the error function is defined as $\text{Erf}(t) \propto \int_t^\infty \exp(-x^2) dx$ and smoothly interpolates between constant values.
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such that the trapping frequency varies exponentially in time and the associated particle production in terms of the number of particles produced. The separate case of an exponential variation of a time dependent coupling constant $g(t)$ is also studied as well as the mixed analysis where both the trapping frequency and coupling constant are varied again in terms of the number of particles produced. In all those analyses, however, the correlation structure is not computed instead they focus solely on the spectrum of produces particles alone (given by the magnitude of $\beta_k$ only).

The field equation for a massless minimally coupled scalar field $\phi$ propagating on the background geometry (3.26) decouples into independent modes and is solved by the ansatz

$$\phi(\eta, x) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i k \cdot x} \left( b_k u_k(\eta) + b_k^\dagger u_k^*(\eta) \right),$$  

(3.31)

where $u_k(\eta)$ satisfy the $k$-isotropic mode equation

$$\partial^2_\eta u_k + a^4 k^2 u_k = 0,$$  

(3.32)

and $k^2 = k \cdot k$ is the squared magnitude of $k$.

For $a(\eta)$ as (3.30), the field equation (3.32) is solved by a rather complicated product of elementary and hypergeometric functions [77]. Conveniently for us, it is not necessary to work explicitly with these cumbersome functions: For each co-moving momentum $k$ the physics of cosmological particle production is contained only in inner products (which are labelled traditionally as $\alpha_k$ and $\beta_k$) between two particular solutions $\phi_k^\text{in}$ and $\phi_k^\text{out}$ to (3.32), solutions which converge respectively in the past ($\eta \to -\infty$) and future ($\eta \to +\infty$) to plane waves as was shown in Section 2.1.1.

Physically, these two particular solutions represent a description of two different vacuum states, $|\text{in}\rangle$ and $|\text{out}\rangle$, in the two asymptotic regimes where the function $a(\eta)$ becomes constant and the definition of particle state is unambiguous. For example, the quantum state $|\text{in}\rangle$ described by the choice $\phi_k^\text{in}$ (the vacuum state for $\eta \to -\infty$) is no longer a vacuum (zero particle) state as $\eta \to \infty$ since that state, $|\text{out}\rangle$, is described by the mode functions $\phi_k^\text{out}$ and in general one has

$$\phi_k^\text{in} = \alpha_k \phi_k^\text{out} + \beta_k \phi_k^{\text{out}*},$$  

(3.33)

with $\beta_k \neq 0$. The spectrum of particle content of the state $|\text{in}\rangle$ at late times in terms of the late time particle states is given by the modulus $|\beta_k|^2$ which has the characteristic bell shape shown in Fig. 3.5 here shown as a function of the isotropic modulus $k = \sqrt{k \cdot k}$ appropriate for the isotropic mode equations (3.32) and hence isotropic coefficients $\alpha_k$ and $\beta_k$.

---

6Specifically $\alpha_k$ is the innerproduct between positive norm (in this case equivalent to positive frequency, see the DCE section in Chapter 2) plane waves whereas $\beta_k$ is the inner product between one positive and one negative norm (frequency) plane wave.
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Figure 3.5: The characteristic shape of the spectrum of produced particles (scaled to the maximum flux at the peak) in an expanding spacetime described by (3.26), as a function of the modulus $k = \sqrt{k \cdot k}$. Recall that the number spectrum is proportional to the volume of space $V/2\pi$ as $N_k = V k^2 / \beta_k^2 / 2\pi$ so that what we plot here is in fact particle number density. The peak wave number is fixed by the typical timescale of the expansion $k_{\text{peak}} \propto 1/\eta_0$.

Of course, at $\eta \to +\infty$ the solutions $\phi_k^{\text{out}}$ are nothing but plane waves at the appropriate frequency (modulated by the scale factor $a$, and normalised by that same frequency) so we have

$$\phi_k^{\text{in}}(\eta, x) = \frac{1}{\sqrt{2ka_f^2}} \left( \alpha_k e^{-ika^2_0 \eta} + \beta_k e^{ika^2_0 \eta} \right) \quad \text{for} \quad \eta \to \infty. \quad (3.34)$$

The inner products $\alpha_k$ and $\beta_k$ are given exactly in this model by the simple expressions

$$\alpha_k = \frac{2\sqrt{AB}}{A + B} \frac{\Gamma(-iA)\Gamma(-iB)}{\Gamma^2 (-i(A + B)/2)},$$

$$\beta_k = \frac{-2\sqrt{AB}}{B - A} \frac{\Gamma(-iA)\Gamma(iB)}{\Gamma^2 (i(B - A)/2)}, \quad (3.35)$$

where $\Gamma$ is the Euler function and the dimensionless variables $A$ and $B$ are

$$A = k\eta_0 a_i^2, \quad B = k\eta_0 a_f^2. \quad (3.36)$$
They are constrained to satisfy $|\alpha_k|^2 - |\beta_k|^2 = 1$ by the constancy of the Klein Gordon norm of the (unique) solution $\phi$ written in the two different mode function bases.

Since the mode-functions are simple linear combinations of plane waves long after the expansion has taken place, the equal time Wightman function \([\text{3.27}]\) there can be written down in a relatively simple way in terms of $\alpha_k$ and $\beta_k$ alone as

$$G(\eta, x) = \frac{4\pi}{2a_f^2} \int_0^\infty dk \left[ \frac{\sin kx}{x} (1 + 2|\beta_k|^2) + I + I^* \right], \quad (3.37)$$

where

$$I = \frac{\sin kx}{x} \alpha_k \beta_k^* e^{-2ika_f^2\eta}. \quad (3.38)$$

Of course, such an expression cannot be considered exact for a realistic finite-sized BEC but will break down nearby the boundaries due to finite size effects. Furthermore, in the case of a finite volume “cigar”-shaped BEC and anisotropic expansion the expression \([3.37]\) will have to be replaced by a more complicated formula as we shall see in sect. 3.4.

We note a few things about relation \([3.37]\). Firstly, when $\beta_k = 0$ the expression reduces to the familiar flat-space Wightman function. Secondly, due to the presence of the sum $\eta + \eta'$ of the times in the non-equal time Wightman function the equal time two point function is not time independent whereas, as is clear from the expression, the equal position two point function is spatially homogeneous. In general one expects the phases as well as the magnitudes of $\alpha_k$ and $\beta_k$ to be relevant for the correlations. The Bogoliubov coefficients have time reversal symmetry in the magnitudes but not in the phases.

The expression \([3.37]\) is the sum of three physically distinct terms. The 1 term represents the background zero point fluctuations, the $|\beta_k|^2$ term an enhancement of the background correlations due to particle production \([7]\) and the terms labelled $I$ and $I^*$ represent additional non-trivial structure propagating on top of the (enhanced) Minkowski-like correlations due to the entanglement between the created pairs of particles. The $I + I^*$ term can be integrated exactly as an unwieldy infinite sum of functions which we calculate in appendix ??.

In Fig. 3.6 we plot the result of numerically integrating the contribution from $I + I^*$

$$\text{Ent} = \frac{4\pi}{2a_f^2} \int_0^\infty dk (I + I^*), \quad (3.39)$$

in $G$ for four successive times as a function of separation $x = ||x - x'||$. We see that going backwards in time a bump of correlations merges into the correlation singularity \([8]\).

---

7This enhancement of background fluctuations from a non-vacuum excited state is discussed in \([89]\). There the author, instead of using the dynamically produced non-vacuum state, shows such the enhancement from the choice of an initial thermal state.

8Since we always work under the assumption that the field theory is an effective field theory valid only above a small length scale, for example the healing length in the case of a BEC, the coincidence singularity does not exist in practice. In the BEC example, higher order contributions to the dispersion relation become non-negligible and in fact dominate at very short length scales resolving the singularity.
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Figure 3.6: The exact numerical integration of (minus) the entanglement contribution to $G$ as a function of point separation $x$ for four successive times $\eta = 0$ (red-solid), $\eta = 1$ (green-dashed), $\eta = 2$ (blue-dotted) and $\eta = 4$ (black-dot-dashed) for the choice of parameters $a_t^2 = 1$, $a_f^2 = \pi$ and $\eta_0 = 1$. at the origin. The (coordinate) speed of the bump is determined by the final value of the scale factor $a_f^2$ which in the displayed plot was taken to be $\pi$. We note that the bump moves with a velocity of close to 7.5 units per second which is approximately the speed at which two particles moving at speed $\pi$ units per second would be receding from one another. Note that the coordinate speed is not constrained to be the speed of sound $c_s$, the coordinate speed in lab variables.

The contribution to the Wightman function $G$ from the 1 and $|\beta_k|^2$ is non-propagating and contribute a divergence at the coincidence limit $x = 0$ which decays in $x$.

In the next section we show that the essential structure of these correlations are present already in the so-called adiabatic approximation to the integrand $I + I^*$.
Adiabatic and sudden approximations

The adiabatic approximation for a mode \( k \) is accurate when \( k\eta_0 \gg 1 \). It is a good approximation for high wave number, or short wavelength modes. These are the modes which feel the expansion as a ‘slow’ process. In the adiabatic limit one can show, from (3.35) and (3.38), that

\[
I + I^* \simeq \frac{2}{x} \sin (kx) \sin \left[ k \left( \Phi - 2a_i^2 \eta \right) \right] e^{-\pi a_i^2 k\eta_0},
\]

where the phase is given, when \( a_f > a_i \), by (see Appendix D.2)

\[
\Phi = \ln \left( \frac{a_i^2 + a_f^2}{a_f^2 - a_i^2} \right) a_i^2 \eta_0 + \ln \left( \frac{a_f^4 - a_i^4}{4a_i^4} \right) a_i^2 \eta_0.
\]

On the other hand, the sudden approximation is accurate for a given \( k \) whenever \( k\eta_0 \ll 1 \). Hence it should be a good approximation to the exact result for low wave-numbers. In this sudden limit one can similarly show (again from (3.35) and (3.38)) that

\[
\begin{align*}
\alpha_k &= \frac{a_f^2 + a_i^2}{2a_i a_f}, \\
\beta_k &= \frac{a_f^2 - a_i^2}{2a_f a_i}.
\end{align*}
\]

In this limit the contribution \( I + I^* \) becomes

\[
I + I^* \simeq -\frac{1}{2} \frac{a_f^4 - a_i^4}{a_f^2 a_i^2} \sin \frac{kx}{x} \cos 2k a_f^2 \eta.
\]

which represents an unbounded contribution to \( G \) if assumed accurate over the entire positive \( k \) axis (as would be the case in the formal limit \( \eta_0 \to 0 \)). In Fig. 3.7 we compare the exact, adiabatic approximate and sudden approximate forms of \( I + I^* \) for a representative choice of \( x \) and \( \eta \).

The Wightman function in the sudden limit may be calculated assuming the sudden approximation for all \( k \). By comparison with the flat space computation, we have for \( x - y \) non-null and \( \eta \neq -\eta' \),

\[
\langle \phi(x)\phi(y) \rangle = -\frac{1 + 2\beta^2}{4\pi[-(\eta - \eta')^2 + (x - x')^2]} - \frac{\alpha \beta}{4\pi^2[-(\eta + \eta') + (x - x')]}.
\]

We see the effect of particle creation \( \beta^2 \) and a time dependent correlation \( \alpha \beta \) even in the equal time Wightman function, predicting a propagating singularity. Alternatively
3.3. **Quantum excitations and correlations during expansion**

Figure 3.7: A comparison of the exact expression (in red-solid) with both the adiabatic approximation (in blue-dots) and the sudden approximation (in green-dashes) near the turn over region \((k\eta_0 \approx 0.3)\) where neither approximation is very accurate. We use the representative values of \(t = 2\) and \(x = 3\). Note that the sudden approximation is most accurate for small \(k\) and the adiabatic for large \(k\). Also note that the adiabatic approximation remains relatively accurate (at least in mimicking the qualitative behavior of the exact function) all the way down to \(k = 0\) where the approximation in principle should be poor.

we note that the adiabatic approximation is accurate almost uniformly over all \(k\), even close to the origin, where it is superseded by the sudden approximation while still capturing the qualitative behavior of the \(I + I^*\) term.

Integrating the adiabatic approximation \(I + I^*\) over \(k\) we observe a ‘bump’ of correlations traveling out from the origin and decaying as displayed in Fig. 3.8. This is in close agreement with the exact result as shown in Fig. 3.6. We note that the only feature of the exact result not captured by the adiabatic approximation is the negative “tail” to the bump as it propagates outwards from the origin.

The role of the parameter \(\eta_0\) is visible in Fig. 3.9 where we again show the result of integrating the adiabatic approximation\(^9\) to \(I + I^*\) for the same values \(a_i^2 = 1\) and \(a_f^2 = \pi\) and times \(\eta = 0, 2, 4\). One sees that as the rate of expansion increases (as \(\eta_0\) becomes small) the shape of the bump becomes sharply peaked and that the peak occurs slightly closer to the origin. This is consistent with our intuitive understanding of the bump as arising from propagating particles created during the expansion and is

\(^9\)For computational reasons we restrict our attention to the adiabatic approximation in this section.
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Figure 3.8: The integrated $I + I^*$ contribution to the Wightman function in the adiabatic approximation at three successive times $\eta = 0$ (red-solid), $\eta = 2$ (blue-dots) and $\eta = 4$ (green-dashes). Note that since the expression (3.37) is only valid after the expansion has taken place (the midpoint of which occurs around $\eta = 0$) the contribution for $t < 0$ are not shown. To make these plots and for comparison with the exact result again we have used the values $a_i^2 = 1$, $a_f^2 = \pi$, and $\eta_0 = 1$. Also plotted (in black-dash-dot) is the standard flat spacetime correlations $(4\pi^2 x^2)^{-1}$.

also consistent with the singular sudden result (3.44).

For small $\eta_0$ the expansion and hence the particle production takes place over a short time interval. Assume for the purposes of illustration that all the particle production occurs at a one point in space, say at $X$. (Particle production occurs at every point but this simplification serves to capture the relevant physical mechanism) Then, two observers at a fixed spatial separation symmetric to $X$ will simultaneously observe a burst of particles passing by. For large $\eta_0$ the particle production is spread out over a longer time interval since the scale factor $a$ is changing significantly over a longer period and hence the observers will see a more diffuse burst of particles. In this way it is not surprising that attempting to integrate the sudden approximate solution over all $k$ (that is, in the formal $\eta_0 \rightarrow 0$ limit) one finds a propagating singularity at approximately $x = 2a_f^2 \eta$ as can be inferred from the structure of (3.44).
3.3. Quantum excitations and correlations during expansion

Figure 3.9: Comparison of the bump structure of correlations for three different values of the parameter \( \eta_0 = 2, 1, 1/2 \) and \( 1/4 \) (from top left to bottom right) at three different times \( \eta = 0 \) (red-solid), \( \eta = 2 \) (blue-dots) and \( \eta = 4 \) (green-dashes). Also plotted (in black-dash-dot) is the standard flat spacetime correlations for comparison.

Varying scattering length interpretation

As mentioned at the beginning of this section, the metric (3.26) can be used to describe phase perturbations in a BEC with a time dependent scattering length \( \bar{a}(t) \) (and hence coupling parameter \( g(t) \) and sound speed \( c_s \)) as discussed in the article [77]. In such varying scattering length models one works in a homogeneous approximation where the background condensate density \( \rho \) is assumed constant and the flow velocity \( \mathbf{v} \)
everywhere zero. The FRW scale factor (which we temporarily write as \( a_s \) for clarity) is related to the scattering length \( \bar{a} \) by

\[
a_s(\eta) = \left( \frac{\rho}{4\pi \hbar^2} \right)^{1/4} \frac{1}{a^{1/4}(\eta)}.
\]

(3.45)

Hence a decrease in the scattering length corresponds to an expansion of the FRW spacetime and vice versa. In [77] the authors use this form for the metric and also the tanh scale factor profile (3.30) described above.

We interpret our result in this context as a propagating bump of correlations when the scattering length is varied according to

\[
\bar{a}(\eta) = \frac{\rho}{4\pi \hbar^2} \left( \frac{a^4_i + a^4_f + a^4_i - a^4_f}{2} \right) \text{tanh} \left( \frac{\eta}{\tau_0} \right)^{-1}.
\]

(3.46)

Note that this time dependence is not determined by any internal dynamics of the BEC but instead is a free experimental choice. This is in contrast with the expanding BEC case where the time dependent parameter, the scaling function \( \lambda \), is determined by the internal dynamics of the BEC.

### 3.4 Axial correlations in an expanding elongated condensate

We shall now consider the experimentally relevant case of an anisotropic harmonically trapped BEC, where the trap is released only along the \( z \) dimension. This section constitutes the main new results in this chapter.

Included in the analysis are the inhomogeneities of the condensate, so that although we only use the results of the previous section on the homogeneous condensate as a guide for the intuition we expect the results to qualitatively match.

We write \( \lambda(t) \) := \( \lambda_z(t) \) and \( \lambda_{\perp}(t) \) := \( \lambda_x(t) = \lambda_y(t) \). The simple form and cylindrical symmetry of the scaling solution in this case allows us to separate the variables in the background velocity as \( c_s(t, z, r) = \tilde{c}(z, \tilde{r}) / \sqrt{\lambda(t) \lambda_{\perp}^2(t)} \) where \( \tilde{c} = c_s \sqrt{1 - \tilde{\omega}_z z^2 - \tilde{\omega}_r \tilde{r}^2} \) and \( \tilde{\omega}_i \) was defined in section 3.2.1.

Let us now introduce new coordinates through the exact differential expressions

\[
dT = \frac{dt}{\lambda^{3/2} \lambda_{\perp}}, \quad dZ = \frac{d\tilde{z}}{\tilde{c}(\tilde{z}, \tilde{r})} + f \, d\tilde{r},
\]

(3.47)

with \( f(\tilde{r}, \tilde{z}) = - \int^\tilde{z} \partial_\tilde{r} \tilde{c} / \tilde{c}^2 d\tilde{z}' \). Then the metric (3.9) is written as

\[
ds^2 = \sqrt{\frac{\rho_s}{gm}} \chi^2 \tilde{c}^2 \left[ -dT^2 + dZ^2 + \left( \frac{\lambda_{\perp}}{\lambda \tilde{c}} \right)^2 f^2 \right] d\tilde{r}^2 + \left( \frac{\tilde{r} \lambda_{\perp}}{\lambda \tilde{c}} \right)^2 d\theta^2 - 2f \, dZ d\tilde{r}.
\]

(3.48)
Note that $f$ vanishes in the limit where we can neglect the radial derivatives of $\tilde{c}$. In that limit note that the metric takes a particularly simple diagonal form.

Relabeling $\phi := \tilde{\theta}_1$ the action for phase perturbations written in these coordinates is

$$S = -\frac{1}{2} \int d^4 x \sqrt{g} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi$$

$$= -\frac{1}{2} \int dT dZ \left[ -\Omega(T, Z)(\partial_T \phi)^2 + H(T, Z)(\partial_Z \phi)^2 \right] + \mathcal{O}(\partial_r \phi) + \mathcal{O}(\partial_\theta \phi), \quad (3.49)$$

where

$$\Omega = \int dr d\theta \sqrt{\frac{\rho_s}{gm}} r \lambda_\perp^2, \quad (3.50)$$

and

$$H = \int dr d\theta \sqrt{\frac{\rho_s}{gm}} r \lambda_\perp^2 \left[ 1 + \left( \frac{f \lambda \tilde{c}}{\lambda_\perp} \right)^2 \right] \equiv \Omega + \delta \Omega. \quad (3.51)$$

The action (3.49) has been written up to terms involving the radial and angular derivatives of the field $\phi$ which we will discard at this stage. The justification for this approximation is transparent: we consider condensates sufficiently tightly trapped in the radial direction that the infrared cutoff provided by the finite radial size and the ultraviolet cutoff provided by the minimal healing length in fact preclude the appearance of any radial or angular modes at all. In other words, in order for a mode of the field $\phi$ to satisfy the ultraviolet band-limitation it necessarily must not contain any (non-zero) radial or angular frequencies at all\(^{10}\). Or even more simply, due to the small radial size of the condensate radial and angular modes necessarily would oscillate on length scales shorter than the healing length and are hence absent. In fact, in this kind of regime the scaling solution in the radial direction is not available since it relies on the static TF approximation. The form we use above for the sound speed also is derived from the static TF approximation and is also unavailable in this tightly trapped regime. Hence above we are making an error in the numerical factors in $\Omega$ and $H$ since we are using an approximate solution outside its regime of applicability and not the exact solution for the background density profile. To be consistent with the TF approximation conditions, one should take $c$ to be a function of only time and axial $z$\(^{46}\) effectively setting the radial sound speed to zero and reducing the problem to a 1+1 dimensional one. We shall do this in what follows as well as set the scaling function $\lambda_\perp$ to unity, freezing the radial dynamics. In this approximation $\delta \Omega$ becomes zero (since it involves a radial derivative of the sound speed) so that $H = \Omega$ and the function $\Omega$ becomes explicitly

$$\Omega(t, z) = \Omega_0 \frac{1}{\sqrt{\lambda}} \left( 1 - \tilde{\omega} \frac{z^2}{\lambda^2} \right)^{1/2}, \quad (3.52)$$

\(^{10}\)Recall that evanescent waves are capable of beating a band limitation when quantum tunneling (and hence an exponentially decaying mode with an imaginary wavenumber) is present. It would be interesting to investigate possible evanescent wave contributions to the correlation structure in this context.
where
\[ \Omega_0 = \frac{2\pi A_\perp}{g} \left( \frac{\mu}{m} \right)^{3/2}. \] (3.53)

Here it is understood that \( \Omega \) is zero if the right hand side becomes negative.

Under these assumptions the 1+1 dimensional action \[^{3.49}\] gives rise to the field equation
\[ - \partial_T (\Omega \partial_T \phi) + \partial_Z (\Omega \partial_Z \phi) = 0. \] (3.54)

which we shall work with in what follows.

### 3.4.1 Conformal symmetry approximation

As a zeroth order attempt at uncovering a signature of ‘cosmological’ particle creation in the Wightman function one might be tempted to discard derivatives of the function \( \Omega \). Under such an approximation the field equation reduces to the flat 1+1 dimensional wave equation for the (massless) scaled field \( \tilde{\phi} = \Omega \phi \) whose solutions are simple exponentials. The Wightman function for the phase field \( \phi \) is then simply related to the standard Green function as
\[
\langle \phi(T, Z) \phi(T', Z') \rangle = \frac{\langle \tilde{\phi}(T, Z) \tilde{\phi}(T', Z') \rangle}{\Omega(T, Z) \Omega(T', Z')}
= -\frac{\hbar}{4\pi} \frac{\ln[\Delta x^+ \Delta x^-]}{\Omega(T, Z) \Omega(T', Z')} \] (3.55)

where \( x^\pm \) are the characteristic null coordinates. However this is immediately seen to be too strong an approximation to capture any particle production effects. In fact the above correlator can be easily plotted as in Fig. 3.10, which shows no sign of the typical transient propagating over-correlation feature normally associated with particle production (see section [3.3.3]).

This observation seems to be at odds with several results found in some of the literature concerning the correlation structure in the case of Hawking radiation. For example, in [67, 68] an analogous calculation is performed to the one here, dimensionally reducing a 3+1D BEC dynamical problem down to one in 1+1D in the case of an acoustic black hole background geometry for phase perturbations. The important point for the current discussion is that the authors of [67, 68] in fact do observe numerically a signature of created Hawking quanta in the same approximation which neglects all the derivatives of the function \( \Omega \). However, there the non-trivial ‘cross-horizon’ correlations are contained entirely in the relationship between the lab coordinates \( x \) and \( t \) and the null co-ordinates \( x^\pm \). Indeed also in [69] these correlations across the horizon are demonstrated for an explicitly 1+1D black hole geometry\(^{11}\) which, being conformally

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\(^{11}\)Note the difference between the analyses of [69] and [67, 68]: whereas [69] works exclusively with the 1+1D conformally flat geometry the minimally coupled field the 1+1D action of [67, 68] arises
3.4. Axial correlations in an expanding elongated condensate

Figure 3.10: The field (phase) correlations $G^\phi$ as a function of time $t$ for three different fixed laboratory spatial separations $0.2L_z$, $0.5L_z/2$ and $0.8L_z/2$.

flat, possesses the standard logarithmic functional form for the correlator. Again the Hawking signal is contained exclusively in the relationship between the null and lab coordinate functions.

The reason for this discrepancy is rooted in the rather different causal structure of the black hole and cosmological analogue spacetimes and in the inherently different natures of the associated particle creation processes. In the black hole spacetime the presence of the acoustic horizon is associated to an ergo-region which allows for negative energy states (w.r.t. an asymptotic observer) and these states in turn allow for the stationary Hawking flux on the black hole static spacetime. In the case of the cosmological analogue geometry there is no such ergo-region and particle production is permitted only as a result of the time dependence of the geometry. This time dependence is fully encoded in the conformal factor describing the difference from Minkowski space of the analogue geometry. It is then clear that when one introduces an approximation which induces a conformal symmetry in the field equation, no particle production can take from a dimensional reduction of a 3+1D model. This dimensional reduction is encoded in the 1+1D dynamics by the factor $\Omega$ which renders the field non-minimally coupled to the geometry.
3.4.2 Spatially Homogeneous $\Omega(Z)$ approximation

As discussed above, the approximation in which we discard both temporal and spatial derivatives of $\Omega$ is too strong to capture the physics of particle creation in the 1+1 dimensional massless case. However, we shall now argue that it is sufficient to keep the $T$ derivatives while neglecting the $Z$ derivatives of $\Omega$ to observe a particle creation signal.

Firstly, since the relationship between $Z$ and the co-moving lab coordinate $\tilde{z}$ is given by integrating (3.47) (in the approximation which neglects radial and angular derivatives, i.e. $f = 0$) to

$$Z(\tilde{z}) = \frac{\sqrt{2}}{\omega_z} \arcsin \left( \sqrt{\frac{m\omega_z^2}{2\mu}} \tilde{z} \right),$$

we see that $\partial_Z \Omega$ is naturally suppressed. One has

$$\partial_Z \Omega = \frac{\partial_z}{\partial Z} \partial_z \Omega,$$

where the first factor $\partial_z/\partial Z$ is naturally small near the boundary (but still within the region in which the TF approximation remains valid) and the second factor $\partial_z \Omega$ is naturally small in the central region where the condensate density is almost constant. Therefore, the factor $\partial_Z \Omega$ is naturally small everywhere on the condensate.

No such natural cancellation is available for $\partial_T \Omega$ since $\partial t/\partial T = \lambda_{3/2} \lambda_{\perp}$, which is not naturally a small factor. Hence in what follows we will keep only $T$ derivatives of $\Omega$ and discard the $Z$ derivatives.

Note that this approximation is not the same as assuming $\Omega$ to be constant in lab coordinates, the difference being in the extra factor of $dz/dZ$ which suppresses $\partial_Z$ at the boundary of the condensate. An important feature of our analysis here is the inclusion of the effects of non-constant density and finite condensate size.

Phase-Phase Correlations

Consider, then, (3.54) and assume $\Omega$ is a function only of the variable $T$. Then (3.54) is written

$$- \partial_T^2 \phi - \frac{\Omega_T}{\Omega} \partial_T \phi + \partial_Z^2 \phi.$$

Let $\eta$ be defined by the differential expression $dT = \Omega \, d\eta$. Then (3.58) separates to

$$\partial_\eta^2 \phi_k + \Omega^2(\eta) k^2 \phi_k = 0.$$
where \( \phi_k = (2\pi)^{-1} \int_L dZ \phi e^{-ikZ} \) is the \( k \)th Fourier component of the function \( \phi \) and \( L \) is the \( Z \)-size of the condensate.

Since we work in the compact region \( L \), inverse Fourier integrals will be replaced by discrete Fourier series in this section. Hence \( k \) in (3.59) is constrained to satisfy \( k = 2n\pi/L \) after choosing the boundary conditions \( \phi(-L/2) = \phi(L/2) = 0 \).

Such an equation of motion can be solved exactly, as we did in the previous section whenever \( \Omega^2 \) is contained in the three parameter family of functions

\[
\mathcal{S} := \left\{ \frac{a_i^2 + a_f^2}{2} + \frac{a_i^2 - a_f^2}{2} \tanh \left( \frac{\eta}{\eta_0} \right) \left| a_i, a_f, \eta_0 \in \mathbb{R} \right. \right\}
\] (3.60)

Our strategy here will be to make use of the available exact solutions and try to find a suitable element in \( \mathcal{S} \) labelled by \((a_i, a_f, \eta_0)\) which most accurately reproduces the exact function \( \Omega \) associated with free expansion of the BEC. We will see that such a class of functions is indeed appropriate for an approximation to the free expansion case with the key benefit of a description in terms of asymptotic particle states.

An alternative way to think about this approximation methodology is that one is studying the correlations in a BEC which really has this particular form for the function \( \Omega \). As will be discussed later, there is a rather direct link between \( \Omega \) and the scaling functions \( \lambda \). In fact from (3.52) we see that morally \( \Omega^2 \propto \lambda^{-1/2} \) so that \( \Omega^2 \in \mathcal{S} \) are models for “finite expansion” condensates (or “finite contraction” depending on the relative magnitudes of \( a_i \) and \( a_f \)). Such finite expansion (contraction) condensates are achievable in the lab with a suitable ramp-down or ramp-up trapping potential (see sec 3.4.3). Note that the finite expansion case qualitatively differs from the free expansion case only in the future asymptotically static region.

Recall from Fig. 3.4 that the typical frequency of produced particles during expansion is \( \dot{\lambda}(t)/\lambda(t) \) which converges to zero at late times for a linear expansion \( \lambda(t) \propto t \). For this reason we expect all the particle production in the infinite expansion case to occur only during the “accelerating” phase and that all of the interesting physics to have ceased by the time the scaling function becomes a linear function of time which occurs at late time. In this way we expect the finite expansion approximation to free expansion to capture the relevant physics (with the possible doubling of the particle creation due to a second decelerating phase relative to the free expansion) neglecting only the irrelevant linear phase of expansion at late times.

The strategy to understand particle production effects in the correlator is as follows. We choose the quantum state to be \(|\text{in}\rangle\) (and since we work in the Heisenberg picture the system remains in this state for all time) and express our results in the \( \eta \to \infty \) limit in terms of the number eigenstates of the asymptotic Hamiltonian which are the physical particle states in that region.

Let us now move back to the \( \theta \) notation for the phase perturbation field. The field
operator is written in 1+1D as the sum
\[ \theta = \sum_k b_k f_k + b_k^\dagger f_k^*, \]
(3.61)
where the functions \( f_k \) satisfy the equation of motion (3.59) and are proportional to plane waves in the past and linear combinations of plane waves in the future as
\[ f_k(\eta \to \infty, Z) \propto \sin kZ (\alpha_k e^{-i\omega_k \eta} + \beta_k e^{i\omega_k \eta}) . \]
(3.62)
The normalization for the functions \( f_k \) is not determined by (3.59) but instead by the consistency between the commutation relations for \( b_k \) with the commutation relations between the field operator \( \theta \) and \( \rho \), the density perturbation field. To fix this normalization we will use the commutator \( \left[ \theta(Z), \partial_t \theta(Z') \right] \). In 3+1D in the TF approximation with zero background velocity flow in the BEC one has \( \rho = -\partial_t \theta/g \) as well as the fundamental commutator \( \left[ \theta(x), \rho(x') \right] = -i\hbar \delta^3(x,x') \) implying \( \left[ \theta(x), \partial_t \theta(x') \right] = i\hbar g \delta^3(x,x') \).

Recall that we assume the perturbation field to be constant over the cross sectional area of the condensate (which was related to the existence of a UV cutoff at the healing length). Hence integrating this commutator we get for the 1+1D phase field an extra factor of the area from \( \rho(3+1) A_\perp = \rho^{(1+1)} \) and hence
\[ \left[ \theta(x), \partial_t \theta(x') \right] = \frac{i\hbar g}{A_\perp} \delta(x,x'). \]
(3.63)
Further, expressing this commutator in \( Z \) coordinates we recall that the conjugate momentum is a density of weight one whereas the field \( \theta \) is a scalar so that one picks up a factor of the Jacobian under a coordinate transformation
\[ \left[ \theta(Z), \partial_t \theta(Z') \right] = \left[ \theta(x), \partial_t \theta(x') \right] \times \frac{dZ}{dx} . \]
(3.64)
Therefore
\[ \left[ \theta(Z), \partial_t \theta(Z') \right] = \frac{i\hbar g \lambda_f}{c_{s,0} A_\perp} \delta(Z,Z'), \]
(3.65)
where \( c_{s,0} \) is the initial sound velocity (which appears as a result of the use of \( \bar{c}_s \) instead of \( c_s \) in the definition of the coordinate \( Z \)) and \( \lambda_f \) is the final value of the scale factor \( \lambda \) (which comes from the relationship between co-moving \( \bar{x} \) and lab \( x \)). The dispersion relation is easily computed to be \( \omega_k^2 = \Omega^2 k^2 \to a_f^2 k^2 \) using the metric in \( (\eta,Z) \) coordinates. Hence we arrive at the expression
\[ \frac{i\hbar g \lambda_f}{c_{s,0} A_\perp} \delta(Z,Z') = \left[ \theta(Z), \partial_t \theta(Z') \right] \]
\[ = \sum_{k,k'} \left[ b_k, b_k^\dagger \right] (f_k \partial_t f_k^* - f_k^* \partial_t f_k) , \]
(3.66)
which relates the commutators and the Wronskian for the mode functions allowing us to fix the normalization factor.

For a Fock space (particle) representation we require \([b_k, b^\dagger_{k'}] = \delta_{kk'}\). Hence we arrive at the Wronskian constraint on the mode functions

\[
\sum_k (f_k \partial_t f^*_k - f^*_k \partial_t f_k) = \frac{ig\hbar \lambda_f}{c_{s,0} A_\perp} \delta(Z, Z').
\] (3.67)

In order to take the laboratory time derivatives above we require the relationship between \(t\) and \(\eta\). We have

\[
d\eta = \frac{dT}{\Omega} = \frac{1}{\Omega} \frac{dt}{\lambda^{3/2}} = \frac{\Omega^2}{\Omega_0^3} dt,
\] (3.68)

where we have firstly transformed to the variable \(T\) using \(dT = \Omega d\eta\) and consequently to \(t\) using (3.47), expressing the result in terms of \(\Omega\) alone. \(\Omega\) and \(\lambda\) are related by (3.52) as \(\Omega = \tilde{\Omega}_0 / \sqrt{\lambda}\) where \(\tilde{\Omega}_0\) is the spatial part of \(\Omega\), numerically \(\tilde{\Omega}_0 \approx 1.5 \times 10^{37}\). Then

\[
\frac{dt}{\Omega_0^3} d\eta = \frac{\tilde{\Omega}_0^3}{\Omega^2} \frac{d\eta}{a_f^2 + a_i^2 + \frac{a_f^2 - a_i^2}{2} \tanh \left( \frac{\eta}{\eta_0} \right)}.
\] (3.69)

Now, we wish to approximate the free expansion scale factor shown in Fig. 3.1 for approximately the first \(t = 0.15s\). This choice of the approximation region is motivated by two constraints: firstly, after about \(t = 0.15s\), a large proportion of the condensate no longer is described accurately by a TF approximation as can be seen in Fig. 3.3; Secondly the natural shape of the exact free expansion scaling function is less accurately fit by the tanh functions we use here over longer time periods since the scaling function enters a linear regime at late times.

Over this \(t = 0.15s\) time interval the scaling function \(\lambda\) increases from \(\lambda_i = 1\) to approximately \(\lambda_f = 25\). Since \(\Omega = \tilde{\Omega}_0 / \sqrt{\lambda}\) we see that the variable \(\Omega\) must decrease from \(\tilde{\Omega}_0\) to \(\tilde{\Omega}_0 / 5\). Therefore we must have \(a_f = a_i / 5\). Hence by (3.69) we have the asymptotic relations between the time variables

\[
t(\eta) = \begin{cases} 
\lambda_i \tilde{\Omega}_0 \eta, & \text{for } t \to -\infty \\
\lambda_f \tilde{\Omega}_0 \eta, & \text{for } t \to \infty
\end{cases},
\] (3.70)

where the function \(t(\eta)\), obtained by integrating exactly the differential expression (3.69), varies smoothly between these two constant asymptotes in the intermediate region. Taking the average gradient in the intermediate region for \(t(\eta)\) we see that a \(t\) interval of 0.15s corresponds to an \(\eta\) interval of approximately \(12 \times \tilde{\Omega}_0 \approx 8.2 \times 10^{-40}\) (in units of \([\Omega^{-1}]s\) or \([\text{Energy}]\cdot[\text{Time}]^2\)). Therefore, we require the function \(\Omega(\eta)\) to vary
over this time scale between the initial and final values $a_i$ and $a_f$. That is we should choose

$$\eta_0 \approx 8.2 \times 10^{-40} \text{[Energy]} \cdot \text{[Time]}^2$$

(3.71)

With these choices one finds that $\omega_k \eta = kt/\lambda_f^{3/2}$.

For the un-normalized $f_k$ we have

$$\sum_k (f_k \partial_t f_k^* - f_k^* \partial_t f_k) = 2 \sum_k \frac{ik}{\lambda_f^{3/2}} \sin kZ \sin kZ'.$$

(3.72)

Scaling the functions $f_k$ as

$$\tilde{f}_k = \sqrt{\frac{\lambda_f^{3/2} \hbar g}{c_s A_{\perp}}} f_k, \quad \text{(3.73)}$$

where Vol is the axial $Z$-length of the condensate (which has the dimension of a time), the Wronskian provides the correct numerical and functional form for the commutator.

Hence the correctly normalized mode functions are given by

$$f_k = \frac{\sqrt{\lambda_f^{3/2} \hbar g}}{\sqrt{c_s A_{\perp} \text{Vol}}} \sin(kZ) e^{-ikt/\lambda_f^{3/2}}.$$  

(3.74)

Here we have transformed back to lab $t$ coordinates in which we take the time derivatives.

In 1+1D at late times the equal time Wightman function is written in terms of $\alpha_k$ and $\beta_k$ in a very similar way to the 3+1D example above in terms of the discrete (and inhomogeneous) Fourier series

$$G^\theta(\eta, Z, Z') = N \sum_{n=1}^\infty \frac{1}{n} \sin (k_n Z) \sin (k_n Z') \times \left(1 + 2|\beta_{k_n}|^2 + 2 \Re \alpha_{k_n} \beta_{k_n}^* e^{-2ik_n a_f \eta}\right)$$

(3.75)

where the normalization factor $N$ is given by

$$N = \frac{\lambda_f^{3/2} \hbar g}{A_{\perp} c_s \alpha_{\pi}}.$$  

(3.76)

Taking symmetrically spaced points from the center of the condensate, the transient time dependent contribution is given by

$$\mathcal{I} = N \sum_{n=1}^\infty \frac{1}{n} \sin \left[n\pi \left(1 + \frac{2Z}{L}\right)\right] \sin \left[n\pi \left(1 - \frac{2Z}{L}\right)\right] \times 2 \Re \alpha_{k_n} \beta_{k_n}^* e^{-2ik_n a_f \eta},$$

(3.77)
3.4. Axial correlations in an expanding elongated condensate

Figure 3.11: The bump of correlations propagating outwards from the center of the BEC cloud. The curves are for $t = 0s$ (red, solid), $t = 0.3s$ (blue, dotted), $t = 0.8s$ (green dashed) and $t = 1.2s$ (black, dash dots) Here we plot the truncated sum up to the first 90 terms. The horizontal axis of this plot is the fraction of the half length $L_z/2$ of the condensate after expansion. The vertical axis is the integrated contribution $\text{Ent}$ in units of the normalization factor $N$.

is easily computed as a truncated sum.

With the choices made above for the parameters $a_i$, $a_f$ and $\eta_0$ we find the entanglement structure shown in Fig. 3.11. The time scales of the propagating bump are understood in terms of the crossing time for the massless modes: after the expansion the sound speed has decreased to one fifth its initial value $c_s/5$ due to the dilution of the BEC gas ($\lambda(t \to \infty) = 25$). The size of the condensate has increased by a factor of 25 so the crossing time increases by a factor of 125. Initially with an axial length of $L_z = 5 \times 10^{-5}m$ and $c_s = 2 \times 10^{-3}m/s$ the crossing time is $t_{\text{crossing}} = 5 \times 10^{-2}s$ which increases to the order of seconds after expansion. (Note again as in the 3+1D case, that the entanglement propagation speed is twice the sound speed since it represents pairs of phonons traveling in opposite directions). This is a slight overestimation since the sound speed actually decreases from a maximum at the center of the condensate to zero at the edge. However we have neglected this slowing in the analysis. This order
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Figure 3.12: The contributions to the phase Wightman function from the ‘1’ and ‘$|\beta_k|^2$’ terms in \[(3.75)\] truncated to the first 200 term of the series. Such a truncation serves as an ultraviolet regulator on the correlation singularity at the origin of the ‘1’ background contribution.

of magnitude estimate is confirmed in the plot of Fig. 3.11

In Fig. 3.12 we show also the integrated contributions from the ‘1’ and ‘$|\beta_k|^2$’ terms in the integrand, in the same units as Fig. 3.11. We note that the magnitude of the bump structure is in general much larger than the background (‘1’) and enhancement (‘$|\beta_k|^2$’) contributions. This is expected since the produced particles propagate from small distances where there is a high intrinsic correlation to long distances where the natural background fluctuations are no longer correlated.

**Density-Density Correlations**

Experimentally, although there are proposals to directly measure the phase correlations, the density correlations are those of major observational interest. Again, in the TF approximation the density field is written in terms of the phase as

$$\rho^{(3+1)} = -\frac{1}{g} \partial_t \theta - \frac{1}{g} v_z \partial_z \theta. \quad (3.78)$$

The 1+1D density is related to the 3+1D density by $\rho^{(3+1)} = \rho^{(1+1)} A_\perp$ where $A_\perp$ is the cross sectional area of the condensate (which we assume constant). Then the density
correlator is derived according to

\[ G^\rho(z, z') = \lim_{t \to t'} D G^\theta(t, t', z, z'), \]  

(3.79)

where we define the differential operator \( D \) as

\[ D = \frac{A^2}{g^2} (\partial_t + v_z \partial_z) (\partial_{t'} + v_{z'} \partial_{z'}). \]  

(3.80)

In the 1+1D finite expansion case, \( D \) reduces to simply the time derivative since \( v_z = 0 \) after the expansion has finished. In Fig. 3.13 we plot the normalized density correlator \( \tilde{G}^\rho(t, z, z) := G^\rho(t, z, z') / (\rho^{(1+1)})^2 \) for symmetrically placed points \( z, z' \) about the middle of the condensate in units of \( N \times A^2 / g^2 \). We note again the propagating structure but of a slightly more complicated shape to the phase correlation bump.

### 3.4.3 Actual finite expansion of an elongated BEC

The availability of an asymptotically static regime in the future is a necessary ingredient for the application of the Bogoliubov formalism. As discussed above there are two ways to use the tools contained in that formalism to understand a system which does not possess such a future static regime such as the case of indefinite expansion of a released BEC. Either one considers the finite expansion as a theoretical approximation to the ‘true’ unconstrained expansion case, as shown in Fig. 3.14 or alternatively one can imagine actually performing a finite expansion experiment in the lab with a condensate which is at first released and consequently ‘caught’ after a finite expansion. Above we have essentially been discussing the first interpretation. Here we briefly discuss the second.

In order for \( \lambda(t) \) to follow the correct step up profile with a future non-dynamical region, one might expect a simple two level potential would suffice. However, the equations of motion (3.16) for the scaling function admit oscillating solutions so that in order to arrive at a static condensate in the asymptotic future a rather finely tuned potential is necessary.

Again assuming the constancy of the radial scaling functions during the axial expansion we have for the trapping frequency

\[ \omega_\parallel^2(t) = \frac{\omega_\parallel^2(0)}{\lambda^3(t)} - \frac{\ddot{\lambda}(t)}{\lambda(t)}. \]  

(3.81)

The necessary trapping frequency is obtained by inserting the desired expansion profile into this expression. One finds that in fact, the squared trapping frequency becomes negative for a short time near the end of the transition from expansion to staticity indicating a temporarily repulsive force is necessary to stop the BEC from contracting further.
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Figure 3.13: The density correlation function after expansion has taken place, for the finite expansion case, in units of $N \times A_1^2/g^2$. The curves from top to bottom correspond to $t = 0.3s$, $t = 0.8s$ and $t = 1.2s$ after the onset of expansion (which we take, as described above, to last for 0.15s).

In this way it would be possible to actually create in the lab a BEC system with particular time dependent trapping frequencies such that the scaling function $\lambda$ behaves in the way appropriate (as discussed above) to induce the kind of finite expansion or contraction analogue geometry for perturbations. It is worth pointing out again at this point that we expect the general structure of a propagating bump of correlations resulting from a finite scaling to not depend in any crucial way on the fine details of the scaling. We expect the result to be insensitive for example to the fact that a precise reproduction of the tanh shape profile is practically very difficult and most probably would only be approximated in any real experiment.
3.5. Loose ends

In this short section we collect some loose ends from the discussion above on Hawking radiation and modified dispersion.

3.5.1 Relation to Hawking radiation

It is interesting to compare the above analysis with a similar one for a black hole geometry such as in [67]. In [67] an analogous calculation is performed dimensionally reducing a 3+1 dimensional BEC dynamical problem down to one in 1+1 dimensions. The crucial difference between cosmological particle creation from a finite expansion and black hole Hawking radiation is that whereas the cosmological radiation is a transient time dependent effect the Hawking flux (after discarding the transient contribution) is a stationary radiation. Hawking radiation is a geometric effect resulting from expressing a pure global state on a reduced physical domain. The fact that such a partial trace results in a mixed thermal state is a specific case of a very general result in quantum
theory [90]. On the other hand, cosmological radiation resulting from parametric excitation of the decoupled quantum modes is more fickle and does not survive when an approximation is made which results in an emergent conformal symmetry in the action. As is discussed above, one cannot neglect the time dependence on the conformal factor \( \Omega \) if one is to capture the physics of particle creation. The fact that, as is done in [67], one can capture Hawking radiation within an approximation in which the action is conformally invariant is related to the existence of a conformal anomaly whereby the classical conformal symmetry is broken in the quantum theory.

### 3.5.2 Modified dispersion

As anticipated in the introduction, we here demonstrate that a cosmological emission or DCE is not affected by ‘mildly’ modifying the dispersion relation outside the typical physics energy scale of the problem. For simplicity we assume spatial homogeneity and hence work directly in momentum space and assume the modification takes the form

\[
\partial_y^2 \phi_k + a^4 F^2(k) \phi_k = 0, \tag{3.82}
\]

which should be compared with the non-dispersive mode equation (3.32). Here we assume

\[
F^2(k) \to k^2 \quad \text{as} \quad k \to 0^+ \tag{3.83}
\]

Solutions to (3.82) are identical to those of (3.32) with \( k \) replaced by \( F(k) \), since \( k \) plays the role of an external parameter due to the decoupling of \( k \) modes. Therefore the Bogoliubov coefficient of the modified equation at momentum \( F(k) \) is equal to that of the unmodified equation at momentum \( k \) and hence the two spectra coincide as functions of these two separate variables respectively. Recall that the typical spectrum for DCE takes the Planckian-type form [3.5], dropping to zero emission at high momenta. We conclude that if the variables \( k \) and \( F(k) \) start to become different only after the spectrum of produced particles has dropped to almost zero the two spectra will be indistinguishable as functions of \( k \). That is, if the dispersion is modified only after some high momentum value \( k_{\text{cutoff}} \) and that the dynamical process giving rise to the produced particles predicts, using the linear dispersion relation a spectrum peaked at a momentum scale \( k_{\text{peak}} \ll k_{\text{cutoff}} \) then the effects of dispersion are negligible for the process.

It could be possible, however, by including ‘un-mild’ modified dispersion such as complex, non-monotonic and multiple-valued \( F(k) \) to drastically modify the spectrum. These features of the dispersion relation would correspond respectively to absorption, additional production channels, and resonances in a physical analogue model. We note here that all these features are present in non-linear optical systems which we will discuss in Chapter 4.
3.6 Conclusion and discussion

In this chapter we have considered the correlation structure of expanding BECs both in the idealized case of isotropic and homogeneous expansion as well as in the experimentally relevant case of a finite size, cigar-shaped, BEC anisotropically expanding.

In the first case we have developed a general formalism for probing the effects of particle production on correlations as embodied in the equal-time Wightman function. Specifically, for any spatially translation invariant system we have shown that it makes sense to define the correlator as in (3.27) which in the case of particle production (using Bogoliubov coefficients) takes the form (3.37). This result is completely generic to particle production in any spatially translation invariant system, applying to standard cosmologies as much as analogue spacetimes. Specifically, the general cosmological result was shown to be directly implementable in a 3+1D BEC system with a time dependent scattering length as well as a 3+1D isotropically expanding BEC, opening two alternative windows on a possible observation of this correlation signal.

In the second case (anisotropic BEC) we have shown that the condensate dynamics reduces to an effectively 1 + 1-dimensional system and that the Wightman function can be generically cast in the form (3.75) for these systems. Using this result we have studied the correlation structure and shown how it carries the signature of analogue cosmological particle production possibly observable in future experiments. In particular, we have studied the density correlations in such BEC systems, which are of current experimental interest.

The magnitude of the normalized correlations is of pertinent experimental concern. The overall scale of the propagating bump is given in terms of the normalization factor for phase perturbation along with the additional factors which relate the 1+1D density to the phase field $\langle \hat{\rho} \hat{\rho} \rangle \rho^2 \approx \frac{N g^2}{\rho^2 (\rho(1+1))^2} \approx 10^{-7}$. Although this is a very small number, it should be noted that this magnitude is in fact much larger than the magnitude of the background correlation structure (present in flat spacetime also). The relative magnitude of the background to entanglement correlations is independent of the normalization of the mode function solutions but instead depends on the structure of the Bogoliubov coefficients which derive from the wave equation alone.

The main purpose of this chapter was to investigate the characteristic signature related to the cosmological particle emission in an expanding (or more generally time-dependent) BEC, and analyze it without specific attention to its actual measurability. However, one can of course imagine altering the parameters of the background BEC in such a way to improve the signal. For example, one parameter which can be altered by an order of magnitude is the number of atoms $N$ which affects the value of $\rho_0$ entering into the denominator of the normalized correlations. Another possibility of improving the signal would be to alter the scattering length. We have discussed above the use of the analogue FRW geometry when one induces a time dependence in the
scattering length. Here we have something different in mind. Since the background density depends on the scattering length as $a^{3/5}$ (see equation (3.4)), by decreasing and holding the scattering length constant before expansion, one might be able to improve the signal. Another possibility is to choose a longer time scale on which to approximate the eternal expansion with a finite expansion: by stopping the expansion later, the correlations will be propagating on a background density which is lower which would improve the ratio in the normalized correlations. All these ways of altering $\rho_0$ will also affect the values of $\eta_0$, $a_i$ and $a_f$ possibly giving a more pronounced signal.

A separate analysis should regard the role of a non-vanishing initial temperature of the BEC (or more generally the role of a non-vacuum initial state). In fact, as already pointed out in [68, 81, 82], the net effect of such an initial temperature (i.e. a non-“empty” initial state, but thermally populated) is to significantly increase the quantity of created excitations, therefore enhancing the signal without affecting the mechanism responsible for the particle production.
Chapter 4

Quantum Vacuum Radiation in Optical Systems

This chapter represents a partial summary of a very exciting and recent interaction between experiment and theory in the analogue gravity community. The material is centered around the articles in Refs. [91, 92] (hereafter referred to as the ‘BF’ experiment) by Belgiorno et. al which appeared in 2010 describing an experiment performed with a non-linear optical system in which radiation was observed and was claimed to be related to an analogue of Hawking radiation (see section 2.4 and 2.1.3). The articles in Refs. [92, 91] and the more detailed follow up in Ref. [93] were controversial in the community, stimulating much activity in the literature with a series of comments and replies [94, 95, 96] as well as taking up major parts of two summer schools in the past year[1]. Although at this stage the community has not reached a consensus on the claims of BF, the complex nature of the experimental system has proven to be a rich source of interesting effects [97, 98, 99, ?] which are in some way or another closely related to an analogue of the Hawking effect.

The main technical results in this chapter are based on those presented in [97] on an alternative explanation of the BF experimental findings in terms of a dynamical quantum instability (DCE) (see section 2.1.1). However, we will have time and space here to discuss the broader picture of optical analogue models in the presence of dispersion and the nature of DCE in general, in particular the issue of finite size effects.

In the next section we will describe the experimental situation, relying on material found in the recent literature immediately surrounding the BF articles, referring the reader to some excellent reviews (see for example [55]) already present in the literature and Sec. 2.4 for a detailed account of the optics theory. We will focus on those aspects which are most relevant for the model we will present for the emission.

[1]The first school in was the 2011 SIGRAV school in Como http://www.centrovolta.it/sigrav2011/ while the second “Analogue horizons in fluids and superfluids” was in ICTP, Trieste.
In the second section we present a model for the DCE in optical glass in an attempt to explain the BF observations and extending them to more general situations than those directly relevant for the BF experiment. In some related works authors have proposed an alternative computations for the BF observations and we comment on those proposals in the third section.

We will end with some comments on the future outlook for such experiments and some speculations for analogue gravity in optical systems.

4.1 Optical black holes

It has been theoretically demonstrated that electromagnetic field modes propagating in a dielectric medium with a spacetime dependent refractive index can closely resemble field modes in a black hole background geometry [100, 101, 102, 103, 104, 105, 106, 107]. The idea was further developed (and analyzed in more realistic contexts) in the sequence of papers [108, 109, 91, 110, 55, 94] as we introduced in 2.4.

Specifically, in the BF experiment an ultrashort laser pulse was used to create a traveling RIP in a transparent dielectric medium (fused silica glass) through the so-called non-linear Kerr effect (see e.g. [107]). The authors of Refs. [92, 93] report experimental evidence of unpolarised photon emission, for which they suggest the most compelling explanation is a Hawking-like radiation due to the presence of the above mentioned blocking region which simulates a black hole-white hole system.

As we discussed in Sec. 2.4, the dispersion relation in the visible electromagnetic spectrum is approximately linear, opening the door for an analogue description in the visible band. However, as discussed in Sec. 2.1.3 the Hawking process is an ‘exceptional’ emission, apparently depending on energies well outside the frequency band of ‘predicted’ (assuming linear dispersion) Hawking quanta. One novel feature of the optical dispersion (shown in Fig. 4.1) is that it is neither purely sub nor purely superluminal in character, and in fact is not even single valued over momenta nor real valued (see below for a more precise description of this dispersion relation). Thus far in the literature, the Hawking mechanism has been investigated only in the presence these simplified monotonic, real and single-valued dispersion relations, and the robustness of the spectrum predicted from the linear theory shown. Only very recently [98, 111] has a systematic study of the Hawking process in the presence of the detailed optical dispersion begun. Therefore it is of interest to consider alternative physics mechanisms capable of spontaneously producing particles to the Hawking mechanism in order to more fully understand the BF results.

It is not the purpose of this chapter to go into the details of the interesting question of whether or not an analogue of the black hole geometry pertains to these optical systems, whether any notion of a horizon is present, or indeed whether or not the concept of a horizon or Hawking radiation even makes sense when the dispersion relation
4.1. Optical black holes

Figure 4.1: The optical dispersion in the lab frame $n^2(\omega)\omega = c^2 k^2$. Note the horizontal asymptotes at the absorption resonances above which the dispersion is complex as well as the multi-valuedness and the complicated sub- and super-luminal structure.

is the optical one (however see Section 4.3). Below we describe the BF experiment in terms which are independent of the vocabulary of black hole physics, distilling the main features which any model should seek to explain.

4.1.1 The Experiment Distilled

Later on in this chapter we will present a DCE model in an attempt to explain the BF observations. To this end we here collect the most salient features of the observations which should be explained by any model of the observed emission. As a preview, remarkably, our model is capable of matching almost all of the features presented below, but not all.

The central physics idea of the works cited and described above is that a sufficiently intense, localized and moving refractive index perturbation (RIP) can give rise to a region in the interior of the perturbation where electromagnetic modes cannot propagate in the direction of motion of the RIP. That is, a mode slows down while climbing the
trailing edge of the RIP, and if the RIP is sufficiently intense and fast, it is possible for
the mode’s group velocity to vanish in the frame of the RIP.

The condition for the existence of such a “blocking region”, associated with a RIP
of peak refractive index $n_{\text{max}} = n_0 + \eta$ traveling at velocity $v_{\text{RIP}}$ on top of a background
refractive index $n_0$, would be

$$\frac{c}{n_0 + \eta} < v_{\text{RIP}} < \frac{c}{n_0} \tag{4.1}$$

as represented in Fig [not done].

This can equivalently be rephrased as a rather tight constraint on the refractive
index

$$n_0 \in \left[ \frac{c}{v_{\text{RIP}} - \eta}, \frac{c}{v_{\text{RIP}}} \right] \tag{4.2}$$

The most salient features of the experimental observation are the following: The fused silica was illuminated with a laser pulse of 1 ps timescale, and the input energy was varied in the 100–1200 $\mu$J range. Radiation from the filament was then collected at 90 degrees with respect to the laser pulse propagation axis. This arrangement was chosen in order to strongly suppress, or eliminate, known spurious effects (e.g. Cherenkov radiation, fluorescence, etc.) \cite{92}. The observed emission spectra were reasonably well fitted by Gaussian profiles. These Gaussian fits showed an increasing peak wavelength of emission, and increasing overall flux, with increasing pulse intensity. Characteristic numbers for the observations were the following: the emission is centered around a peak wavelength of about $(850 \pm 25)$ nm (see Fig. 3 of \cite{92}) and has a bandwidth that appears to depend on the pulse intensity. While the determination of the actual size of such a bandwidth is fitting dependent, it is evident that the most of the photon count is concentrated in the range from 700 to 1000 nm.

The interpretation of the observed spectrum as an optical analogue of spontaneous quantum Hawking radiation is based mainly on the above mentioned window condition. In fact, while the observed radiation is clearly non-thermal, one can nevertheless claim it to be related to the blocking regions via a modification of Eq. (4.1). However, this blocking condition, which at first glance seems rather straightforward, in fact contains a number of subtleties once optical dispersion is introduced. Indeed in situations of dispersive propagation with $n_0 = n_0(\lambda)$, where $\lambda$ is now the wavelength of the specific mode under consideration, the simple refractive index blocking condition of Eq. (4.1) becomes a wavelength-dependent ‘window condition’ defining those modes for which a blocking region exists:

$$n_0(\lambda) \in \left[ \frac{c}{v_{\text{RIP}} - \eta}, \frac{c}{v_{\text{RIP}}} \right] \tag{4.3}$$

The window condition (4.3) will be satisfied by a certain set of wavelengths, a set which depends on which form for the dispersion relation one chooses as well as on
the experimental parameters such as the character of the RIP itself. However, not all dispersion relations will necessarily lead to windows — the ‘window region’ might (and often does) define an empty set.

The relevant wavelength dependence of the refractive index for fused silica is adequately approximated by the Sellmeier relation\(^2\)

\[
n(\lambda) = \sqrt{1 + \sum_{i=1}^{m} B_i \frac{\lambda^2}{\lambda_i^2 - \lambda^2}}.
\]  

(4.4)

(Except in the immediate vicinity of resonances where the unphysical poles are regulated in real physical systems by other effects such as absorption as indicated by the fact that \(n(\lambda)\) becomes complex near each pole.) In Eq. (4.4) the \(B_i\) and \(\lambda_i\) are a set of coefficients determined by matching experimental data for the dispersion. When truncating the sum to three poles, the data is best fit with the values

\[
\lambda_1 = 64.25\, \text{nm}, \quad \lambda_2 = 114.00\, \text{nm}, \quad \lambda_3 = 9938.24\, \text{nm}. \tag{4.5}
\]

which fix the location of the poles in the optical dispersion.

Typical values of the induced perturbation in the refractive index \(\eta\) are of the order of \(10^{-3}\), and higher, increasing with the intensity of the RIP. Calculating the \(v_{\text{RIP}}\) relevant for the experiment requires not only the knowledge of the group refractive index at the typical RIP frequency, but also the characteristic Bessel beam factor, which was measured to be \(\theta_B = 6.8^\circ\) [93, 112]. The factor enters the formula for the velocity as a function of refractive index as

\[
v_{\text{signal}} = \frac{c}{qn_{\text{signal}}} \quad \text{where} \quad q = \cos(\theta_B).
\]

4.1.2 Some physics questions from the experiment

In Refs. [92, 93] the window condition of Eq. (4.3) was applied with the understanding that \(v_{\text{RIP}}\) was the group velocity of the pulse, whereas the the \(n_0(\lambda)\) was taken to be the refractive index for phase velocities of modes in the fused silica. With this interpretation the window was found to be approximately (870, 940) nm. We stress that this result is very strongly sensitive to the precise numbers used for the relevant quantities in the window condition of Eq. (4.3) due to the relative flatness of the function \(n_0(\lambda)\)

\(^2\)Note there that we quote the dispersion relation, as is standard, in terms of the wavelength in vacuum (outside the glass). In practice since the modified dispersion arises from a non-instantaneous-in-time response of the polarisation to an electric field, the susceptibility and hence the refractive index are intrinsically frequency dependent objects (see section 2.4) and hence the resonance poles are in fact frequency resonances and not wavelength resonances. Our results in this chapter are also given in terms of the wavelengths as they would be outside the glass, lying in between the described resonances and hence in a region where the non-linear dispersion is approximately linear. Hence we consistently work with the wavelength representation. Any dynamical model intrinsically involving the non-linear dispersion should be carried out in the frequency representation (see Section 4.3)
over the range of wavelengths of interest here. (The Cauchy approximation provides insufficient accuracy for determining the window condition and it is essential to use the Sellmeier approximation for this particular purpose.) Such a window appears to be in good agreement with the observed spectra, hence supporting the authors’ claim of the Hawking-like nature of the observed radiation. Furthermore, the peak shift and spectral broadening with intensity are also claimed to be expected in this interpretative framework. In particular, the emission bandwidth is predicted to depend on $\eta$ which in turn is a linear function of the pulse intensity.

While these aspects of the evidence are surely supportive of a Hawking-like interpretation for the origin of the observed radiation, it was immediately noticed by several researchers in the analogue gravity community that certain other features of the experiment are problematic from this point of view (see e.g. the Comment [95], and Reply [96]).

We do not wish here to further enter into that ongoing debate, but we shall instead limit ourselves to stressing two points of main concern from our point of view:

- First of all we find odd the observation that photons are emitted at 90 degrees with respect to the ostensible horizons set by the window condition (4.1). In fact, the window condition only holds for modes aligned with the RIP direction of propagation, so one would not expect at all in this model any photon production at 90 degrees to the beam axis.

- Furthermore, we think that some concern has to be raised with the application of the window condition itself, as it rather oddly mixes the group velocity of the RIP with the phase velocities of the vacuum modes in a dispersive medium. On the contrary, we feel that there are good physics arguments for expecting the window condition to be determined by only considering group velocities.

Specifically, let us consider an observer in the rest frame of the pulse as it moves down through the glass at velocity $v_{\text{RIP}} < c$. Then a probe signal chasing the RIP from behind will be seen to be approaching with a velocity given by the relativistic formula for the combination of velocities

$$v_{\text{comoving}} = \frac{v_{\text{RIP}} \pm v_{\text{lab}}}{1 \pm v_{\text{RIP}}v_{\text{lab}}/c^2}. \quad (4.6)$$

Including the effect of a Bessel beam factor, we have $v_{\text{RIP}} = c/(q_{\text{RIP}})$. Then

$$\frac{v_{\text{signal}}}{c} = \frac{n_{\text{signal}} - q_{\text{RIP}}}{q_{\text{RIP}}n_{\text{signal}} - 1}. \quad (4.7)$$

From this we see that, in the optical band, zero approach velocity (in the RIP frame) is achieved at the location $x^*$ such that

$$n_{\text{signal}}(\lambda, x^*) = q_{\text{RIP}} = q_{\lambda_{\text{RIP}}}. \quad (4.8)$$
This defines a wavelength-dependent blocking region since the probe sees the presence of the RIP as a local increase in the ambient refractive index. These velocities are group velocities. Since \( n_{\text{signal}}(\lambda, x) \) varies between the background value \( n_0(\lambda) \) and the value at the peak of the RIP \( n_0(\lambda) + \eta \), the above condition Eq. (4.8) is equivalent to Eq. (4.3).

However, from a theoretical perspective, use of the relativistic combination of velocities formula seems to us to be well-motivated only if \( v_{\text{RIP}} \) and \( v_{\text{lab}} \) are either both group velocities or both phase velocities. Certainly \( v_{\text{RIP}} \) is a group velocity since the RIP is really a propagating wave packet. This rather strongly suggests that in determining the window region \( v_{\text{lab}} \) should also be interpreted as a group velocity.

If we now adopt a group velocity interpretation of the window condition, we are obliged to consider the group velocity refractive index which is defined as

\[
n_g(\lambda) := \frac{d[\omega n(\omega)]}{d\omega} = n + \omega \frac{dn}{d\omega} = n - \lambda \frac{dn}{d\lambda},
\]

where \( \omega \) is a frequency. Using this formula the window condition becomes

\[
n_g(\lambda) \in \left[ \frac{c q}{v_{\text{RIP}}} - \eta, \frac{c q}{v_{\text{RIP}}} \right].
\]

The problem is that using this (more physically justified) formula, and the previously given values for the characteristic quantities, there is no range of wavelengths for which the window conditions is satisfied. As a function of \( \theta_B \) the window is non-empty only for \( \theta_B \lesssim 3^\circ \) when \( \eta = 10^{-3} \) and \( \lambda_{\text{RIP}} = 1050 \text{ nm} \).

In summary, we feel that these two facts alone are quite sufficient for motivating serious consideration of alternative physical explanations for the interesting observations reported in Ref. [92]. Specifically, we shall argue here that the DCE is a competitive explanation that certainly deserves further theoretical and experimental investigation.

In particular, we conjecture that the rapidly varying refractive index in the fused silica excites vacuum modes leading to substantial photon creation. This would be naturally isotropic in the subspace orthogonal to the direction of propagation of the RIP, and would not be directly dependent on the above mentioned window condition, being independent of the presence or not of a blocking region. Furthermore the ‘non-exceptional’ nature of DCE places one in a much safer position when neglecting dispersion effects. That is, if one can show that the RIP in the lab frame can be seen as a dynamical variation of the refractive index in such a way to predict an emission lying in a band of frequencies where the dispersion is approximately constant, then one can safely assume that dispersion is negligible in the lab frame for the computation. However, this does not imply that dispersion is irrelevant for the physics in the frame of the pulse, from which one would see the optical glass as a ‘river’ of atoms flowing rapidly past. Microscopically, the dispersion relation reflects the physics of the absorption and re-emission of photons by the atoms of the glass itself, and hence it is natural...
to be able to neglect the dispersion in the lab frame while in the pulse frame have the
dispersion play a crucial role in the dynamics. We will discuss this issue of different
frames describing the same observations using vastly different physics after the next
section in which we present computations done in the lab frame neglecting dispersion.

In the following discussion we shall present an analytically solvable 1+1 model
which, in spite of its simplicity, is able to reproduce many of the salient features of the
experimentally observed photon emission.

4.2 An alternative explanation – DCE

In this section we present a model for the DCE in the lab frame in an attempt to explain
the BF experimental observations. The DCE was introduced in general in 2.1.1 while
here we apply the techniques to the particular case of electromagnetic modes traveling
in a time varying dielectric. As mentioned at the end of the previous section, we here
neglect the effects of dispersion in the wave equation for reasons also discussed there.
Essentially, since the BF experiments observe photons at around the 700 nm to 1000
nm range and the optical dispersion
\[ c^2 k^2 = n^2(\omega) \omega^2 \]  (4.11)
where \( n \) is given in terms of wavelength\(^3\) in (4.4) is approximately linear in this band
(see Figure 4.1), we can safely assume that dispersion plays no role in the production
of such photons in a DCE mechanism.

As is standard, we simplify the problem to the scalar magnitude of the electric field
component of the full electromagnetic field, with the understanding that the results
derived will apply to both photon polarisations independently, and will contribute half
of the total flux expected from a full calculation.

Below we will bootstrap our initially 1+1 dimensional results to 3+1 dimensions
using the interpretation of the equation of motion as applying to the magnitude \( k \) of
the wave vector \( k \). This is common procedure in dynamical Casimir effect calculations
due to the isotropic nature of the emitted radiation. We have in mind the idea that the
detector which observes the emitted photons actually watches only a small element of
the silica glass which, as the pulse passes through, appears to the detector to undergo a
rapid time variation in its refractive index. Within our simple model this bootstrapping
can be effectively achieved by integrating for the flux our Bogoliubov coefficient \( |\beta_k|^2 \)
over a three dimensional \( k \) space, as we will do below. The variable \( x \) in the reduced 1+1
dimensional problem represents the orthogonal coordinate to the propagation axis of
the RIP – it is the one dimensional subspace seen by the detector. In this approximation
\[ \text{Again, this is wavelength in vacuum and hence we can convert to the (conserved) frequency by } \omega^2 = 2\pi c^2/\lambda^2. \text{ Note that from glass to air, the wavelength is not conserved.} \]
we also assume a homogeneous in $x$ perturbation in the refractive index and do not consider finite size effects beyond their inclusion in the volume pre-factor for the flux. We have checked that the inclusion of finite size effects using an inhomogeneous (in $x$) refractive index perturbation does not qualitatively nor quantitatively change our results and conclusions.

One might (mistakenly) argue based on a misinterpretation of boost invariance that such an ‘artificial’ introduction of a time dependence in the field equation is unjustified since one might naively expect a moving dielectric, (for example a pair of spectacles), to not emit radiation. However, it is important to keep in mind that optical systems of the kind considered here are highly dispersive and do not, in general, enjoy exact boost invariance. Indeed, the physics as described in the frame of reference of the RIP itself is highly non-trivial and one expects particle production even in these static configurations, entirely through a dispersive mechanism alone [16]. In the laboratory frame of reference (detector frame), in the window of frequencies considered here, the refractive index is approximately constant and we can neglect the effects of dispersion in this frame. By moving to the pulse frame, one can render negligible the particle production due to time dependence$^4$ but necessarily one moves out of the regime in which dispersive effects are negligible. (See Sec. 4.3 for more comments on this point.)

The equation of motion for the 1+1 dimensional electric field $\phi$ with a time variable refractive index $n$ is

$$\frac{1}{c^2} \left( n^2(t) \phi,_{tt} \right) = \phi,_{xx}. \quad (4.12)$$

In momentum space this reads

$$\phi_{k,\tau\tau} + c^2 k^2 n^2(\tau) \phi_k = 0, \quad (4.13)$$

where we have introduced the “conformal time” variable

$$\tau(t) = \int^t dt' \frac{dt'}{n^2(t')} \quad (4.14)$$

For the refractive index we will initially choose both a sech$^2(\cdot)$ time-dependent perturbation on top of momentum-dependent background value

$$n^2(t) = n_0(k)^2 + 2\eta \ n_0(k) \ \text{sech}^2(t/t_0) + \mathcal{O}(\eta^2), \quad (4.15)$$

where $\eta$ is a small parameter and $n_0(k)$ would be a model for the refractive index such as the Sellmeier approximation. In an appendix we consider a tanh$^2(\cdot)$ profile, noting that the two profiles essentially give the same results apart from one or two caveats which we

$^4$Even this is an oversimplification: The authors have also made some preliminary investigations of a residual time dependence which remains in the pulse frame, and which might also be a viable particle production channel.
will point out. These choices are motivated not only by exact solubility of the model, but also for their close approximation to the experimental Gaussian profile reported in [92] (in the tanh case, we are motivated by the phenomena of ‘pulse steepening’ which is known to occur in optical fibres which we discuss in more depth below).

Then the equation of motion becomes

\[ \phi_{k,\tau\tau} + c^2 k^2 \left[ n_0^2(k) + 2\eta n_0(k) \operatorname{sech}^2(t(\tau)/t_0) \right] \phi_k = 0, \]  

(4.16)

which can immediately be recognized as describing a time-independent Schrödinger scattering problem

\[ \psi_{,xx} + \frac{2m}{\hbar^2} [E - V(x)] \psi = 0, \]  

(4.17)

with

\[ V(x) = V_0 \operatorname{sech}^2(x/x_0), \quad m = \frac{1}{2} c^2 k^2 \hbar^2, \]  

(4.18)

while

\[ E = n_0(k)^2, \quad \text{and} \quad V_0 = -2\eta n_0(k). \]  

(4.19)

This analogy allows us to directly write down the solutions to our equation of motion by appealing to well-known textbook results.

The Bogoliubov coefficient \( \beta_k \) for our mode equation (4.16) is related to the transmission coefficient \( T \) for the scattering problem (4.17) by \( |\beta_k|^2 = 1/T - 1 \). From the literature (see for example Ref. [113] and many references therein) we find the transmission probability

\[ T = \frac{\sinh^2 \left( \pi \sqrt{\frac{2mEx_0^2}{\hbar^2}} \right)}{\sinh^2 \left( \pi \sqrt{\frac{2mEx_0^2}{\hbar^2}} \right) + \cos^2 \left( \pi \sqrt{1 - \frac{8m V_0 x_0^2}{\hbar^2}} \right)} , \]  

(4.20)

so that

\[ |\beta_k|^2 = \frac{\cos^2 \left( \frac{\pi}{2} \sqrt{1 + 8\eta \Gamma_k^2/n_0(k)} \right)}{\sinh^2 (\pi \Gamma_k)}, \]  

(4.21)

with \( \Gamma_k = ck\tau_0 n_0(k) \). The function \( |\beta_k| \) rises from zero at \( k = 0 \) to a maximum \( \beta_{\text{max}} \) at \( k_{\text{max}} \) before again decaying to zero for large \( k \). Physically this structure reflects the transition from low \( k \) modes which experience the RIP as a sudden, approximately instantaneous influence, to high \( k \) modes for which the RIP represents an adiabatic process. There are also features at the poles (resonances) of the refractive index, but here we focus on the region between the resonances, which is the region of physical interest due to it being the only region where observations in the RIP experiments are performed.

Furthermore, as we will show below, all of the features of the function \( |\beta_k| \) deriving from the time dependence of \( n(t) \) occur for values of \( k \) for which \( n_0(k) \) is essentially
constant, whence the effects of dispersion can be ignored for the purposes of this calculation. Accordingly, we henceforth set $n_0(k) = n_0 \simeq 1.458$ for what follows.

Due to the smallness of $\eta$ we can expand the cosine about $\pi/2$ giving

$$
|\beta_k|^2 \simeq \eta^2 \frac{4\pi^2 c^4 k^4 t_0^4 n_0^2}{\sinh^2(\pi c k t_0 n_0)} - \eta^3 \frac{16\pi^2 c^6 k^6 t_0^6 n_0^3}{\sinh^2(\pi c k t_0 n_0)} + O(\eta^4),
$$

(4.22)

where we see explicitly the quadratic dependence of the spectrum on $\eta$ at lowest order.

**Bogoliubov tricks**: Transmission and reflection coefficients are related to Bogoliubov coefficients. Bogoliubov coefficients are associated with an ‘in’ frequency being converted into two out frequencies, one positive and one negative while the transmission and reflection coefficients are associated with spatial scattering problems where a positive momentum mode is converted to a positive (transmitted) and negative (reflected) momentum modes. It is known that

$$
T = \frac{1}{|\alpha|^2}
$$

(4.23)

which implies in particular since $|\alpha|^2 - |\beta|^2 = 1$ and $R + T = 1$ that

$$
R + \frac{1}{|\alpha|^2} = 1, \quad \Rightarrow \quad |\alpha|^2 R + 1 = |\alpha|^2 = |\beta|^2 + 1, \quad \Rightarrow \quad R = \frac{|\beta|^2}{|\alpha|^2}
$$

(4.24)

Alternatively we have the inverse relations such as

$$
|\beta|^2 = \frac{R}{T}
$$

(4.25)

### 4.2.1 Evaluation of $\beta_k$ directly in perturbation theory

The equation of motion (4.13) above

$$
\partial_t^2 \phi_k + c^2 k^2 n^2(t) \phi_k = 0,
$$

(4.26)

(here we have re-labeled $\tau$ by $t$) can be tackled using a perturbative method for the case of the small RIP which we are considering. This approach can be considered as a warm up problem for a later section where a perturbative solution is obtained in the more complicated case of a spatially and time dependent refractive index. Here, instead of using a sinh function to approximate the gaussian RIP shape we use a gaussian directly in order for an integral to be exactly done later on.
Define $\omega_k := n_0 c k$. Then

$$n^2(t) = n_0^2 + \eta f(t), \quad (4.27)$$

where $\eta$ is a small parameter as previously and we choose

$$f(t) := 2 c^2 k^2 n_0 e^{-(t/t_0)^2}, \quad (4.28)$$

and expand the solution to (4.26) in a power series in $\eta$ as

$$\phi_k(t) = \phi_k^{(0)}(t) + \eta \phi_k^{(1)} + O(\eta^2). \quad (4.29)$$

Collecting zeroth order terms (independent of $\eta$) we find the equation of motion

$$\partial^2_t \phi_k^{(0)}(t) + \omega_k^2 \phi_k^{(0)}(t) = 0. \quad (4.30)$$

Then the correctly normalised homogeneous solution to this equation (which is the asymptotic ‘in’ state of a traditional Bogoliubov analysis) is given by

$$\phi_k^{(0)} = \frac{e^{-i \omega_k t}}{n_0 \sqrt{2 \omega_k}}. \quad (4.31)$$

At first order in $\eta$ we find the driven harmonic oscillator problem

$$\partial^2_t \phi_k^{(1)}(t) + \omega_k^2 \phi_k^{(1)}(t) = -f(t) \phi_k^{(0)}(t). \quad (4.32)$$

where the driving force depends on the zeroth order solution $\phi_k^{(0)}$. The initial condition for $\phi_k^{(1)}$ is $\phi_k^{(1)}(t \to -\infty) = 0$ since the perturbation is absent in the infinite past. Then, using the retarded Greens function the solution can be written as

$$\phi_k^{(1)}(t) = \frac{1}{\omega_k} \int_{-\infty}^{t} dt' f(t) \phi_k^{(0)}(t) \sin \omega_k (t - t')$$

$$= \frac{\sqrt{2 c^2 k^2}}{\omega_k^{3/2}} \int_{-\infty}^{t} dt' \sin \omega_k (t - t') e^{-i \omega_k t - (t/t_0)^2}. \quad (4.33)$$

Note that the dimension of $\phi_k^{(1)}$ coincides with that of $\phi_k^{(0)}$ as it should by relation (4.29).

The integral in (4.33) can be done exactly and is given by

$$\int_{-\infty}^{t} dt' \sin \omega(t - t') e^{-i \omega_k (t - t_0)^2} = \frac{i \sqrt{\pi} t_0}{4} \left[ e^{-i \omega t} (\text{erf}(t/t_0) + 1) - e^{i \omega t} e^{-\omega^2 t_0^2} (\text{erf}(i \omega t_0 + t/t_0) + 1) \right]. \quad (4.34)$$

The error function terms ($\text{erf}$) add to a real quantity due to the property $\text{erf}(\pi) = \text{erf}(\bar{z})$. For $t \to -\infty$ we have $\phi_k^{(1)} \to 0$ whereas using the above integral and (4.33) we find

$$\phi_k^{(1)}(t) \to i \sqrt{\frac{\pi}{2 \omega_k}} \frac{c^2 k^2 t_0}{\omega_k} \left( e^{-i \omega_k t} - e^{-\omega_k^2 t_0^2 e^{i \omega_k t}} \right) \quad \text{as} \quad t \to \infty, \quad (4.35)$$
which is the time when we wish to compute the Bogoliubov coefficients, that is, after the perturbation has subsided.

From this result we see that, to first order in \( \eta \), the solution up to \( O(\eta^2) \) contains only the pure positive and pure negative norm (frequency) pieces

\[
\phi_k = \left( \frac{1}{n_0 \sqrt{2\omega_k}} + i\eta \sqrt{\frac{\pi}{2\omega_k}} \frac{e^{2k^2t_0}}{\omega_k} \right) e^{-i\omega_k t} - \left( i\eta \sqrt{\frac{\pi}{2\omega_k}} \frac{e^{-\omega_k^2 t_0}}{\omega_k} \right) e^{i\omega_k t} + O(\eta^2),
\]

(4.36)
as expected. We conclude that the Bogoliubov coefficient \( \beta_k \) is given by the coefficient of the negative norm component of the correctly normalised mode function. Given that the initial wave functions come with a denominator of \( n_0 \sqrt{2\omega_k} \), we have

\[
\beta_k = -i\eta t_0 \sqrt{\frac{\pi}{n_0}} e^{-\omega_k^2 t_0^2} + O(\eta^2),
\]

(4.37)
and hence

\[
|\beta_k|^2 = \eta^2 e^{2k^2 t_0^2 \omega_k^2 n_0^2} + O \left( \eta^3 \right).
\]

(4.38)
We see the characteristic exponential tail of the adiabatic regime for high frequencies as well as the quasi linear sudden regime for low frequencies (see Section 2.1.1 in Chapter 2 for the general structure expected of DCE spectra). It is remarkable that these features are present at this level of approximation.

For completeness we note that

\[
\alpha_k = 1 + i\eta t_0 \sqrt{\frac{\pi}{n_0^2}} \omega_k^2,
\]

(4.39)
so that

\[
|\alpha_k|^2 - |\beta_k|^2 = 1 + \eta^2 t_0^2 \frac{\omega_k^2}{n_0^2} \left( 1 - e^{2\omega_k^2 t_0^2} \right) = 1 + O(\eta^3).
\]

(4.40)
and the perturbative coefficients satisfy the Wronskian condition up to the appropriate order in \( \eta \).

### 4.2.2 Spectrum and flux

In what follows we concentrate exclusively on the result from the exact calculation given in equation (4.21) and the expansion (4.22). In the appendix we compare the results to those obtained with the perturbative method as in the previous section.

The 3 + 1 dimensional photon flux density is given by

\[
4\pi k^2 |\beta_k|^2 = \frac{dN}{dV dk}.
\]

(4.41)
Special care should be taken when converting to wavelengths (which is relevant for the comparison with the literature). The peak wavelength of emission $\lambda_{\text{peak}}$ is not simply given by $\lambda(k_{\text{peak}})$ since $\lambda_{\text{peak}}$ is the maxima of the function which enters the integral over $\lambda$, including the Jacobian factor

$$\frac{dN}{d\text{Vol} d\lambda} = \left| \frac{dk}{d\lambda} \right| \frac{dN}{d\text{Vol} dk} = \frac{2\pi}{\lambda^2} \frac{dN}{d\text{Vol} dk} \bigg|_{k=2\pi/\lambda}. \quad (4.42)$$

From the lowest order term in Eq. (4.22) we get

$$\frac{dN}{d\text{Vol}} = \int_0^\infty F(\lambda) d\lambda, \quad (4.43)$$

where

$$F(\lambda) = \frac{8}{15625} \frac{1}{\lambda^8 \sinh^2(2\pi^2 c t_0 n_0 / \lambda)}. \quad (4.44)$$

Noting that the function $1/(u^8\sinh^2(1/u))$ has its maximum at a point extremely close to $u = 1/4$ we obtain

$$\lambda_{\text{peak}} \simeq \frac{1}{2} \pi^2 c t_0 n_0. \quad (4.45)$$

Calculating the total flux we have

$$\frac{dN}{d\text{Vol}} = 4\pi \int dk \ |\beta_k|^2 = \frac{8\pi^2}{21} \frac{\eta^2}{c^3 t_0^3 n_0^5}. \quad (4.46)$$

The emitting region in this 3+1 dimensional interpretation would approximately be the cylindrical region defined by the width of the laser beam and length of the region over which the refractive index is rapidly varying. Observationally the radius of the laser beam is $W \approx 10^{-5}$ m [112], and we can estimate the length of the active region as $c t_0 n_0$, so the physical volume of the active region is $V = \pi x_0^2 (c t_0 n_0)$. The total number of emitted photons is estimated to be

$$N = \frac{8\pi^3}{21} \frac{\eta^2 W^2}{c^2 t_0^3 n_0^4}. \quad (4.47)$$

### 4.2.3 Putting numbers to the model

The parameters given explicitly in Ref. [109] are

$$n_0 = 1.458, \quad t_0 = 2.5 \times 10^{-14} \text{ s}, \quad \eta = 10^{-3}, \quad (4.48)$$

where $t_0$ is calculated from the reported initial spatial size of the pulse at its production $x_0 \approx 10^{-5}$ m and the RIP velocity. Note that, up to a small perturbation, $\tau(t) = t/n_0^2$.
which we use inside the sech\(^2(\cdot)\) function to maintain the integrability of the equation of motion. This effectively changes the above initial value of \(t_0\) to \(t_0/n_0^2 = 1.2 \times 10^{-14}\) s.

In principle these factors are all we need to derive our estimate of the peak frequency and flux. However, it is well known that the RIP shape is not invariant along its propagation (see e.g. Ref. [107]). Indeed, there are two additional effects which are certainly necessary to take into account when fitting the data. These are “pulse steepening” and “pulse cresting” which we shall discuss below.

**Pulse steepening**

**Pulse steepening** is a non-linear optical effect, akin to wave shoaling in water waves, which can be responsible for altering the size of the spatial region over which the pulse goes from its background value to its peak value and back by a factor of ten or more. This effect works oppositely on the trailing and leading edges of the RIP; while the trailing edge becomes steeper the leading edge typically becomes shallower.

An important question is what, if any, physical mechanism is responsible for setting an ultimate limit to the pulse steepening. This point is not completely clear in the extant literature, although in many experimental situations one adopts the rule of thumb that pulse steepening saturates at “about twice the carrier frequency” [107, 112].

We believe, however, that it is reasonable to expect that the limiting mechanism should be more closely related to physical properties of the silica glass. Furthermore we wish here to explore which effects can provide absolute limits to the pulse steepening and hence to the peak frequency predicted by our model. In this respect there are two obvious candidates. One is the plasma frequency \(\omega_p\) and the other is optical absorption, known to be particularly effective near poles in the dispersion relation (here characterized in terms of the Sellmeier relation).

The plasma frequency definitely provides an ultimate limit to pulse steepening, as it is fundamentally the electrons in the optical glass which communicate the propagating electromagnetic wave, and which themselves interact on a time scale given by the plasma frequency. From the Sellmeier relation, and the definition of the plasma frequency as the coefficient of the leading order deviation as the refractive index goes to 1,

\[
n^2 \rightarrow 1 - \frac{\omega_p^2}{\omega^2} = 1 - \frac{\lambda^2}{\lambda_p^2},
\]

one obtains

\[
\omega_p = \sum_i \sqrt{B_i \omega_i^2}; \quad \lambda_p = \frac{1}{\sqrt{\sum_i B_i/\lambda_i^2}}. \tag{4.50}
\]

For the suprasil glass used in the experiment one gets

\[
\omega_p = 2.6 \times 10^{16} \text{ Hz}, \quad \lambda_p = 72.7 \text{ nm}. \tag{4.51}
\]
With the above values one gets a typical timescale for the steepening of about $2\pi/\omega_p \simeq 2.4 \times 10^{-16} \text{s}$. This would be the ultimate limit to pulse steepening and we would expect the maximum steepness to closely approach, if not exactly saturate, this bound.

The function $\text{sech}^2(t/t_0)$ varies between its minimum and its maximum over a time scale of $2.4 \times 10^{-16} \text{s}$ when the parameter $t_0$ is $2 \times 10^{-16} \text{s}$.

The above estimate represents an absolute upper bound to the steepening. However, one can argue that before the plasma frequency can start to play any role, pulse steepening will be limited by dispersive effects and in particular by the pole of the Sellmeier relation at approximately 114 nm (see Eq. (4.4)). Such poles represent resonances inside the silica glass and their role in limiting the time variation or propagating signals is essentially identical to that of the plasma frequency. Nonetheless, if one takes, as we shall do, twice the time scale set by the plasma frequency $t_0 \simeq 4 \times 10^{-16} \text{s}$, the resultant steepness is just below that set by the Sellmeier pole scale, exactly in the right ballpark to model a limiting scale set by the pole’s presence. Specifically the propagating RIP varies between its maximum and minimum over a length scale of approximately 140 nm when $t_0 \simeq 4 \times 10^{-16} \text{s}$ while the Sellmeier pole sits at 114 nm.

Let us here anticipate that to match the data our model requires that the pulse steepening saturates near the above scale $t_0$ rather than, say, at twice the carrier frequency given set by the above-mentioned rule of thumb. This is a relevant point as it might be crucial in dismissing or supporting the present proposal against others.

Crest amplification

Another important effect which we expect to occur in a propagating and steepening pulse is crest amplification, whereby the maximum amplitude of the pulse increases as the RIP steepens. The cause of this effect can be understood by a conservation of energy argument. First, the integral of the intensity is constant (and given by the energy of the RIP). Second, due to the non-linear Kerr effect the intensity is proportional to the perturbation in the refractive index $\delta n(x)$. Therefore

$$\int \delta n(x) \, dx = \text{constant}, \quad (4.52)$$

so that

$$\eta \int [\text{shape of pulse}](x) \, dx = \text{constant}. \quad (4.53)$$

That is, if the pulse changes shape in such a way as to make the area under its normalized curve lower, we can expect the maximum amplitude, controlled by the parameter $\eta$ to increase. Note that the total flux of emitted quanta is (approximately) quadratically dependent on $\eta$. 

While in the current context we lack any detailed understanding of the size of this effect, we argue that one can easily expect $\eta$ to increase up to one order of magnitude. We shall hence use $\eta \simeq 10^{-2}$ in what follows.

In the closely related but distinct context of fibre-optic amplifiers, power amplifications of 20 dB, 30 dB, and 40 dB are not uncommon [114, 115, 116]. These are power amplifications of $10^2$, $10^3$, and $10^4$ respectively, corresponding to amplitude increases of 10, 32, and 100 respectively. So a cresting factor of up to 100 (corresponding to $\eta \sim 10^{-1}$) is not inconceivable.

Returning to the current context, the pulse steepening we have argued for, from $t_0 \sim 10^{-14}$ s to $t_0 \sim 10^{-16}$ s, will (provided most of the pulse is concentrated in a narrow spike with a long shallow leading edge) squeeze the pulse temporally by up to a factor of 100, thereby potentially increasing the amplitude by a factor of up to 100. Again, this implies that a cresting factor of up to 100 (corresponding to $\eta \sim 10^{-1}$) is not inconceivable. Therefore, we argue that our choice of $\eta \simeq 10^{-2}$ is rather conservative.

It is important to note that crest amplification mainly affects the total flux, at $O(\eta^2)$, and that effects on the peak wavelength and width of the spectrum are relatively subdominant, at $O(\eta^3)$.

**Peak frequency and flux matching**

We are now in shape for computing the salient features predicted by our model. From the previous calculation of the peak frequency we get (using the least amount of pulse steepening, that is the largest value of $t_0$ in the above given interval)

$$\lambda_{\text{peak}} = \frac{1}{2} \pi^2 c t_0 n_0 \simeq 858 \text{ nm}.$$  \hfill (4.54)

This is in striking agreement with the observational data.

Note again, however, that to get this nice matching for the peak frequency it is essential that the pulse steepening occur on a timescale and distance scale close to those set by the pole in the Sellmeier relation discussed above. In fact we have been generous and chosen a time scale slightly longer than the Sellmeier resonance time scale. This has the side effect of avoiding the resonance at 114 nm — which is the first resonance one encounters in the Sellmeier approximation, (moving up in frequency from the carrier frequency as the pulse steepens). Furthermore if one adopts the numerical model discussed in [93], then Fig. 10 of that reference implies a timescale of $\approx 1$ fs at the trailing edge of the RIP; this is within a factor $2 \frac{1}{2}$ of our preferred value. In contrast, if one simply uses some small multiple of the carrier frequency as an estimate of the pulse steepening, the fit is quite bad. It is therefore important to develop a clearer understanding of the amount of pulse steepening to be expected in this particular experimental setup.
Figure 4.2: Emission spectrum for the dynamical Casimir effect, as given by Eq. (4.44) and using the parameter values discussed in the text: \( n_0 = 1.45 \), \( t_0 = 4 \times 10^{-16} \) s and \( \eta = 10^{-2} \). Recall that the total flux is given by the area under this curve multiplied by the volume of the emitting region.

On the other hand the total flux becomes

\[
N = \frac{8\pi^3 \eta^2 W^2}{21 \, c^2 t_0 n_0^4} \approx 0.6 \text{ photons.} \tag{4.55}
\]

This estimate should be supplemented by a factor two to take into account the two photon polarizations and with an additional solid angle scaling \( \Omega/(4\pi) \) to account for the finite size detector. In the experiment the detector was a 5 cm diameter lens placed 10 cm from the filament \[112]. In terms of the half-angle subtended by the detector one has \( \Omega = 2\pi(1 - \cos \theta) \) whence \( \Omega/4\pi \approx 0.016 \). Unfortunately we do not have an accurate account of the observed flux.

We consider the above numbers to be somewhat encouraging and supportive of the dynamical Casimir effect explanation. In Fig. 4.2 we show the emission spectrum for the dynamical Casimir effect as a function of wavelength as given by the function \( F(\cdot) \) in Eq. (4.44).

**Peak shift and broadening**

Of course, having shown that the relatively unsophisticated and crude model here proposed can can get anywhere near reproducing the observed peak frequency, and
Figure 4.3: Shift of the location of the maxima of the emission spectrum as a function of wavelength (in nm) for the sech^2(-) profile as the input energy is increased, here modeled by increasing the parameter \( \eta \) from \( 2.5 \times 10^{-3} \) to \( 10^{-1} \).

provide anywhere near a reasonable flux, while encouraging, is certainly not enough for claiming to have explained the experiment under investigation. In particular, we have not yet explained the above-mentioned shift of the peak frequency and broadening of the spectrum. We shall now argue that our model can account for the former, and we shall critically discuss the relevance of the latter.

Experimentally, working from Fig. 3 in Ref. [109], we can summarize the above two points with the following two empirical formulas:

\[
\lambda_{\text{peak}} \approx 850 \text{ nm} + 25 \text{ nm/mJ} \times E. \tag{4.56}
\]

\[
\sigma_\lambda \approx 100 \text{ nm/mJ} \times E. \tag{4.57}
\]

where \( \sigma \) is the full-width-at-half-maximum (FWHM) of the Gaussian fits and \( E \) is the input beam energy.

*Peak shift* – From Eq. (4.22) we see that the peak wavenumber (and hence also the peak wavelength of emission) depends on \( \eta \) when we move to higher orders in the \( \eta \) expansion of the exact result. In Fig. 4.3 we plot the maxima of the spectral function \( F(\lambda) \) for a range of values of \( \eta \) between \( 0.25 \times 10^{-2} \) and \( 10^{-1} \). Note that this shifting is consistent with the shift observed in the Belgiorno et al article [109]; increasing the input energy shifts the peak emission wavelength to a higher value. We also note that the shifting is not exactly linear once we also take into account the \( \eta^4 \) term in the expansion of \( |\beta_k|^2 \).
From Fig. 3 of Ref. [109] one might argue that the peak wavelengths of emission are approximately

\[ \lambda_{\text{peak}} \simeq 860 \text{ nm at } I = 270 \mu\text{J}, \quad (4.58) \]

\[ \lambda_{\text{peak}} \simeq 875 \text{ nm at } I = 1280 \mu\text{J}. \quad (4.59) \]

That is, the experiments observe a 15 nm shift in peak wavelength from a factor of 5 increase of the input energy. From our result shown in Fig. 4.3 we see that our model predicts a spectral shift of about 10 nm from a factor of 10 increase of the input energy (given by \( \eta \)).

We note that with an increase of the input energy by a factor of 10, our model predicts an increase in flux by a factor of (approximately) 100. In the case of a factor of 5 increase, as in the experimental data, we would expect from our model a flux increase of a factor of (approximately) 25. This predicted behaviour is qualitatively consistent with the observed spectra shown in Fig. 3 of Ref. [109] where one sees an increase in peak flux from approximately 2 photoelectron counts at the lowest input energy (270 \( \mu\text{J} \)) to approximately 50 at the highest input energy (1280 \( \mu\text{J} \)).

Peak broadening – Peak broadening, as we have mentioned above, is a feature observed in the data by matching the measured spectrum to Gaussian curves by eye. It is interesting to note that although the magnitude of the effect does not seem to be in agreement with the observational data, our oversimplified model also predicts a spectral broadening as the input energy (\( \eta \)) is increased. We use the FWHM definition for bandwidth, and find that \( \sigma_{\lambda} \) behaves as

\[ \sigma_{\lambda} \simeq 767 \text{ nm for } \eta = 10^{-3}, \]

\[ \sigma_{\lambda} \simeq 771 \text{ nm for } \eta = 10^{-2}, \]

\[ \sigma_{\lambda} \simeq 787 \text{ nm for } \eta = 10^{-1}. \quad (4.60) \]

As an increasing function of \( \eta \), we see that the predicted broadening is in qualitative (though not quantitative) agreement with the data. The broadening predicted by our simple model is shown in Fig. 4.4 where we plot the scaled spectrum:

\[ \frac{F_{\eta}(\lambda)}{[F_{\eta_0}(\lambda)]_{\text{peak}}} \times \left( \frac{\eta_0}{\eta} \right)^2. \quad (4.61) \]

That is, we plot the spectrum scaled to the maximum of the spectrum at \( \eta_0 = 10^{-3} \), multiplied by an extra factor of \((\eta_0/\eta)^2\). This is done in order for the spectral curves to be visually comparable, and to observe the non-quadratic dependence of the spectrum on \( \eta \). Note that in this figure we have used an exaggerated range of values for the parameter \( \eta \) to highlight the effect. We also note that this figure shows the shifting peak wavelength of emission, where we see the peak moving to higher wavelengths as \( \eta \) is increased. We again emphasize that the reported broadening in Fig. 3 of Ref. [109]
Figure 4.4: Normalized spectra $F_\eta(\lambda) \times (\eta_0/\eta)^2/[F_{\eta_0}(\lambda)]_{\text{peak}}$ for $\eta = 10^{-3}$ (red solid curve), $\eta = 10^{-2}$ (green dashed curve) and $\eta = 10^{-1}$ (blue dash dotted curve). Here $F(\cdot)$ is obtained from Eq. (4.44), but modified to include the higher order terms indicated in Eq. (4.22). We normalize to the peak of the spectrum for $\eta_0 = 10^{-3}$, and display the result as a function of wavelength in meters. By multiplying by $(\eta_0/\eta)^2$, we can see how the shape and maximum of the spectrum as a function of $\eta$ is not exactly quadratic in $\eta$, but in fact depends on the higher order terms in indicated in (4.22). A numerical routine to find the FWHM bandwidths of these curves (which are independent of the amplitude) give the results shown in Eq. (4.60).

arises from matching noisy data to purely phenomenologically chosen Gaussian curves (there is no physics reason underlying the choice of Gaussian curves), and that the reported magnitudes might, in a slightly more conservative analysis, be lower than those reported.

### 4.2.4 The effect of a finite pulse width

In the model presented above we have been considering the ‘homogeneous’ approximation for the physical time dependent region. That is, although we work in the (stationary with respect to the lab) orthogonal 1+1 dimensional subspace to the propagating pulse, we assume the finite time dependent perturbation to be of infinite extent in the spatial direction. This kind of approximation does not effect the particle number density in the active region, which is the number of particles divided by the ‘volume’ of that region and this is the reason why we can still get accurate total flux results in
the homogeneous approximation by simply multiplying the homogeneous result for the flux density by the (finite) size of the active region (see sec 4.2.2 for this discussion). A more accurate way of carrying out the calculation would be to assume from the start a spatially finite (and of course temporally finite) pulse, calculating the Bogoliubov coefficients directly in the inhomogeneous theory. In what follows in this short subsection we report on the results of carrying out this kind of finite size computation using a perturbative method.

The equation of motion we wish to work with here is

\[ \phi_{,tt} - c^2 n^2(t, x) \phi_{,xx} = 0, \]  

(4.62)

where \( cn \) is a time and spatially dependent wave speed. This kind of problem is interesting mathematically since the ordinary methods such a Fourier transform and the inner product need to modified in order to find, for example, the normal modes even in the time independent asymptotic regions.

One simplifying feature of the RIP system that allows us to make quick progress with minimal mathematical complications, however, is the fact that the time and space dependence of the wave speed \( cn \) is a small perturbation on top of a homogeneous background

\[ n^2(t, x) = n^2_0 + \eta f^2(t, x); \quad \eta \ll 1. \]  

(4.63)

Hence we can calculate the first terms of the power series expansion of the Bogoliubov coefficients \( \beta_k \) in \( \eta \) using perturbation theory.

Assuming the perturbative ansatz that \( \phi(t, x) = \phi^{(0)}(t, x) + \eta \phi^{(1)}(t, x) + O(\eta^2) \), inserting it into (4.62) and collecting terms in powers of \( \eta \) we find at zeroth order the equation of motion

\[ \phi^{(0)}_{,tt} - c^2 n^2_0 \phi^{(0)}_{,xx} = 0. \]  

(4.64)

The boundary condition corresponding to the assumption that the quantum state of the field is the vacuum state in the asymptotic past (only positive norm modes) demands the form of the solution\[^5\]

\[ \phi^{(0)}(t, x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{e^{ikx - i\omega_k t}}{n_0 \sqrt{2\omega_k}}, \]  

(4.65)

where \( \omega_k^2 = c^2 n^2_0 k^2 \).

Collecting the first order terms we have

\[ \partial_t^2 \left( \phi^{(0)} + \eta \phi^{(1)} \right) - c^2 \left( n_0(k)^2 + \eta f^2 \right) \partial_x^2 \left( \phi^{(0)} + \eta \phi^{(1)} \right) = 0, \]  

(4.66)

so that the equation of motion for the perturbation is

\[ \partial_t^2 \phi^{(1)} - c^2 n^2_0 \partial_x^2 \phi^{(1)} = c^2 f^2 \partial_x^2 \phi^{(0)} =: F(t, x), \]  

(4.67)

\[^5\] This integral can be done explicitly and comes out to be proportional to \( \phi^{(0)} = (1 - i)/\sqrt{8n^3_0c(x + n_0ct)} + (1 + i)/\sqrt{8n^3_0c(x - n_0ct)} \).
where we have defined the function $F$. Fourier transforming in the variable $x$ we have

$$\partial^2_t \phi_k^{(1)} + \omega_k^2 \phi_k^{(1)} = \tilde{F}(t, k). \quad (4.68)$$

Since the boundary condition for the first order term is $\phi^{(1)}(t) \to 0$ for $t \to -\infty$ (no perturbation before the RIP acts) it is appropriate to use the retarded Green function, giving the solution

$$\phi_k^{(1)}(t) = \frac{1}{\omega_k} \int_{-\infty}^{t} dt' \tilde{F}(t', k) \sin \omega_k(t-t'). \quad (4.69)$$

Modeling the RIP by the Gaussian bump function

$$f^2(t, x) = e^{-x^2/x_0^2-t^2/t_0^2}, \quad (4.70)$$

the function $F$ is given by

$$F(t, x) = c^2 f^2 \partial_x^2 \phi^{(0)} = -e^{-x^2/x_0^2-t^2/t_0^2} \int \frac{dk}{\sqrt{2\pi}} \frac{c^2 k^2 \sin \omega_k t}{n_0 \sqrt{2\omega_k^3}} e^{i k x}, \quad (4.71)$$

leading to the Fourier transformed object

$$\tilde{F}(t, k) = \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} \left[ e^{-x^2/x_0^2-t^2/t_0^2} \int \frac{dk}{\sqrt{2\pi}} c^2 k^2 \frac{e^{i k x - i \omega_k t}}{n_0 \sqrt{2\omega_k^3}} \right]$$

$$= e^{-t^2/t_0^2} \int \frac{dk}{\sqrt{2\pi}} \frac{e^{-i \omega_k t}}{n_0 \sqrt{2\omega_k^3}} c^2 k^2 \left[ \int \frac{dx}{\sqrt{2\pi}} e^{i(k-k)x} e^{-x^2/x_0^2} \right]$$

$$= e^{-t^2/t_0^2} \int \frac{dk}{\sqrt{2\pi}} \frac{e^{-i \omega_k t}}{n_0 \sqrt{2\omega_k^3}} c^2 k^2 \left[ x_0 e^{-(k-k)^2 x_0^2/4} \right], \quad (4.72)$$

upon changing the order of integration.

Equation (4.68) with $\tilde{F}$ as above in (4.72) is exactly soluble in terms of so-called ‘parabolic cylinder functions’. In the appendix ?? we compute this solution and show that the Bogoliubov coefficient $\beta_k$ up to $O(\eta^2)$ terms is

$$\beta_k = \beta_k(\xi) \quad (4.73)$$

given by a function of $k$ and the dimensionless ratio

$$\xi = \frac{x_0}{t_0 c n_0}, \quad (4.74)$$

which controls the finite size effects. When $\xi$ is large then the finite size of the RIP does not effect the spectrum whereas for small $\xi$ the size of the RIP starts to be comparable with the characteristic wavelength of emitted quanta which is determined by $t_0$. This
Chapter 4. Quantum Vacuum Radiation in Optical Systems

Figure 4.5: Here we show the deformation of the finite size spectrum, clockwise from top left, as the parameter $\xi$ is decreased from a regime in which the emission spectrum does not feel the finite size effects (large $\xi$) into one in which the spectrum is strongly distorted (small $\xi$). The respective values for $\xi$ are $5, 2, 1, \frac{1}{4}$. Note that even at the relatively small value $\xi = 5$ the spectrum is already very similar to the homogeneous result.

is shown in Figure 4.5 where we plot the exact spectrum derived from the homogeneous computation given in equation (4.21) against the first order perturbative result from the inhomogeneous computation here (4.73) for various values of the parameter $\xi$.

Using the physical parameters given above in Section 4.2.3 we find that for the experimental results one has

$$\xi \simeq 47.89, \tag{4.75}$$

putting us in the ‘large $\xi$’ regime where the spectrum is unaffected by the finite size. In Fig. 4.6 we compare the first order finite size result $4.73$ using the accurate value $\xi \simeq 47.89$ with the first order homogeneous result $4.38$ observing the almost identity of the spectra.

We conclude therefore that the finite size effects play little or no role in the interpretation of the experimental RIP results, that we are in a regime in which the time dependent system is naturally emitting particles whose wavelengths are much smaller than the spatial size of the time dependent region. This is the regime in which the
4.3. Alternative models for the observed emission

Recently in the literature several other works, apart from the one [97] which was the focus of this chapter, have appeared which seek to explain the BF experimental results. We here report three important contributions to the ongoing debate briefly to highlight the activity and interest of the community on the issues involved in the propagating RIP type scenarios.

- **Cosmological particle production by time dependent mass** — The article in Reference [117] chooses to adopt a DCE interpretation of the emitted radiation from the moving RIP, noting the difficulties of applying the Hawking mechanism to the highly dispersive optical systems. The authors present two calculations, one...
very similar to ours here but without making a detailed analysis including pulse steepening or crest amplification while in the second model a time dependent mass is used to model the particle production. Both models conclude that a particle emission is possible and that a Hawking mechanism is not necessary for the interpretation.

- **Anomalous Doppler effect** — The article of Reference [99] discusses the spontaneous emission from the vacuum of correlated pairs of photons due to a super-luminal RIP in a very similar scenario to the BF experiment in a way similar to the anomalous doppler effect where spontaneous emission occurs due to the movement of a rapidly moving source alone (see the article [99] itself and reference therein). The authors find emission in the forward and backward directions with respect to the pulse propagation. Although no direct attempt to interpret the BF experimental results is made, the particular emission spectrum predicted by such a mechanism is discussed in order to potentially distinguish the signal from others due to different mechanisms.

- **Dispersive effects and horizonless emission from stationary systems** — In the two works in References [98, 111] the authors carry out firstly a kinematic, then subsequently a dynamical study of the spontaneous emission spectrum due to the non-linear optical dispersion alone in the moving frame of the RIP. The analysis is similar in spirit to the study of Hawking radiation in general in the presence of non-linear dispersion (see for example [118]) and moving media. Here a higher order equation of motion is used to model the non-linear dispersion and general scattering processes investigated. Although the authors use a very simple step function model for the shape of the RIP they conclude that an emission is possible even in situations in which no notion of horizon can be defined naturally and hence argue against a Hawking mechanism interpretation for the experimental results of BF.

This model can be understood as somehow complimentary to the one we have discussed in the bulk of this chapter: in the pulse frame of [98, 111], the time dependence of the RIP system is negligible while the non-linear dispersion is certainly not, being an effect related to the absorption and re-emission of the photons by the atoms which are flowing past the RIP at near luminal speeds. Instead, in our model, which is carried out in the lab frame, the effects of dispersion are negligible (as we discussed, due to the separation of scales between the onset of non-linear dispersion and the peak frequency of emission due to the particular time dependence of the passing RIP) while the time dependence is certainly not, being responsible for all the particle production effects. In this way we can see that the physics result can be invariant under a change of the reference frame

---

6Note that even in the pulse frame, it is known that the shape of the RIP is not static but undergoes deformation on a timescale similar to that which occurs in the lab frame [119]
(particle emission from the vacuum, potentially with the same spectrum) while the mechanism and particularities are quite different in the different frames. It is possible that both pictures could be seen as two faces of the same underlying quantum vacuum emission (see the conclusion chapter 6.6 for more on this possibility).

4.3.1 A possibly distinctive test: correlations in the radiation

Based on the results on quantum correlations of Chapter 3 and various works on this subject in the extant literature on Hawking process [85, 67, 68, 66, 69], one could argue that the spectrum and total particle flux are not clean enough observables to distinguish the various production mechanisms and the one should instead be looking at quantum correlations in the observed photons.

It is known that the quantum state of the spontaneously emitted quanta of the DCE type emission is a squeezed state and one should expect a correlation signature similar to that predicted in Chapter 3 for the BEC emission. However for a horizon of finite lifetime it is known that the spectrum is not exactly thermal [35, 120] and it is possible that the quantum state may possess different correlation functions and higher order moments, which could lead to a divergence in the predictions for these two mechanisms for the observed spectra.

Using correlation measurements should also differentiate the observed emission from other, non-spontaneous effects (see [121] for more on this discussion) with an otherwise similar spectrum (e.g. thermal noise).

4.4 Conclusions

The model provided in this chapter is very simple, in many ways overly simple, but nevertheless provides a tolerable fit to many of the known aspects of the BF experimental situation. The dynamical Casimir effect envisaged in this model has a long and convoluted history in its own right (see for instance [122, 123, 124, 34, 125, 121, 126]). Particularly attractive features of the dynamical Casimir model are that it does not need any spectral window, that there is no preferred direction, and that in performing an experiment to observe it one could look from any angle.

In the present context, the most pressing aspect of the model is to more fully understand peak broadening, and to undertake a detailed confrontation of the model with empirical reality. (Specifically, it would be important to develop precise quantitative estimates of the amount of pulse steepening and crest amplification to be expected in the relevant experimental context, perhaps involving exact solutions to the Maxwell
Another potential line of development would be to go to a full 3+1 dimensional calculation. We have also undertaken a computation in perturbation theory which includes the finite size of the pulse in the 1+1 dimensional subspace of the detector (the width of the beam) but, again the results were essentially identical to the homogeneous results presented here – the calculated spectrum was peaked at wavelengths which were much smaller than the size of the emitting region by an order of magnitude.

From a theoretical perspective, as we have already mentioned above, one might argue that the natural frame of reference for understanding this experimental result is the RIP frame. Of course, in this frame the simple picture of a time dependent refractive index seems to be lacking, but one instead has to view the photons as being created by the optical glass rapidly flowing through the RIP; with the quantum ground state of the electromagnetic field being defined by the rest frame of the optical glass, quantum excitations are now generated by motion through an inhomogeneous medium. Modes endowed with modified dispersion relation will see a sequence of subluminal regions of different intensity – the asymptotic past to the left, the intermediate region where the refractive index is increased (the RIP) and the asymptotic future region to the right. This possibility, that modes be excited from the vacuum in a time independent system solely through dispersion and a background, subsonic flow has been investigated partly in [118, 63, 127, 128, 129] even in absence of any horizon/blocking region for the “dual” situation of modes subject to a supersonic dispersion relation in a “flow” with two supersonic regions of different speeds. Indeed there seems to be mathematical correspondence that leads to the same behavior when passing from super- to sub-sonic in the dispersion relation and in the character of the flow regions [118]. This was also the topic of the two recent articles [98, 111]. If confirmed this analysis would corroborate our claim that a particle production from the quantum vacuum could still take place in our system even in the absence of analogue horizons.

Finally, we emphasize that the dynamical Casimir effect is still quantum vacuum radiation, just not Hawking radiation. Both the dynamical Casimir effect and Hawking radiation produce squeezed state outputs. Direct tests of the difference between these two effects should focus on the different correlation structure in the emitted photons. Specifically, it would be interesting to consider 2-photon correlations, and back-to-back photons (see e.g. [121, 67]). These effects are likely to be different in detail. One might also need to do considerably more than just looking at the squeezed state properties. At a very practical level, the output radiation might not be “clean”, due for instance to interaction with silica defects.

Overall, we consider this RIP-based experimental scenario to be a very promising avenue in the ongoing search for a direct observation of some analogue of quantum vacuum radiation in condensed matter systems, and we hope that the analysis here will shed some light in this direction. In this context quantum vacuum radiation refers to any spontaneous emission from the vacuum and not specifically to the dynamical...
Casimir effect nor the analogue Hawking effect. It is our opinion that in highly disperse
sive systems such as that of a propagating RIP in optical glass many of the phenomena
which are distinct in the relativistic context can be seen as limiting idealizations of
more general quantum vacuum emissions.
Chapter 5

Superradiance in Dispersive Theories

In this chapter we will generalise the idea of superradiance to dispersive theories both sub- and super-luminal. We will develop general formulas describing scattering processes which admit our dispersive notion of superradiance based on the conservation of a generalised Wronskian for a class of higher order wave equations suitable to describe wave scattering on flowing fluids. We will also describe some specific models and their exact solutions therein as a way to get a tighter grip on the general case. We work with the motivation that such dispersive superradiant scattering might be observable in doable analogue gravity experiments in the near future. Indeed in the following chapter we will describe one such experimental effort currently in progress.

In the first section we make a general introduction and motivation for the chapter. The second section consists of a detailed analysis of the simplest case of zero background flow (this terminology will be explained below) but is to be regarded as a warmup exercise for the following section which generalises to non-zero flows. The analysis of the non-zero flow scattering represents the bulk of the interesting results of this chapter. We finish in the fourth section with some concluding remarks and possible future research directions.

5.1 Introduction

As described in Section 2.1.4, superradiance is a general phenomenon in physics expressed in gravitational systems as superradiant scattering from a Kerr black hole as well as the related particle phenomenon of the Penrose process. The Penrose process and superradiance can be understood as the two faces, particle-like and wave-like, of a quantum spontaneous superradiant emission.
Through the analogy with condensed matter systems, superradiant scattering has the possibility of being observed in the lab directly. Assuming one could produce in the lab a flow profile in a fluid system such that the acoustic wave equation supports superradiant scattering then superradiance moves from a hypothetical to a feasibly observable phenomenon.

As we have seen many times in this thesis, typically condensed matter systems exhibit a non-linear dispersion relation for the excitation spectrum. We have seen how some systems with non-linear dispersion can be manipulated into a regime in which the excitations satisfy a linear dispersion and can be described by an acoustic metric and wave equation. Of course this is only an approximation to the true equation of motion in the real system and there will always be corrections to the linear dispersive result due to this approximation. Therefore it is of interest, as far as analogue gravity is concerned, to attain whether some phenomenon one is studying through the analogy, for example superradiance, is robust against modifications to the dispersion relation. This might be seen as the practical side of such a theoretical investigation.

Alternatively, one might be intrinsically interested in the theoretical question of whether a spacetime phenomenon, such as superradiance, is robust against modifications of the dispersion, with the idea that deep in the UV spacetime might not be as smooth and Lorentz invariant as low energy physics would lead us to believe.

It is in this spirit that we move in the next section to the problem of superradiance with modified dispersion.

### 5.2 Warmup problem - zero flow scattering

In this section we will make the first generalisation of superradiant scattering to non-linear dispersion. This involves two steps, the first of which is to introduce the language in a non-dispersive context and secondly to generalise that language to non-linear dispersion. In a subsequent section we will generalise further to the case of non-zero flow velocity (equivalently non-flat background geometry). This is a slight abuse of terminology, however, since the zero flow case presented below can be understood (as we will describe) as arising from higher dimensional acoustic geometry with a non-zero flow in the extra dimension only as we shall show below. The meaning of zero flow in this context is that the flow is zero in the scattering direction. For example, this would attain to radial scattering in a cylindrically symmetric system with zero radial flow.

This section will be very detailed since the same steps essentially apply to the non-zero flows computations and we will skip them there in the interests of brevity.
5.2. Warmup problem - zero flow scattering

5.2.1 Non-dispersive preliminary

As we saw in [2.1.4] The equation of motion

\[ \frac{d^2 f}{dx^2} + (\omega - eV(x))^2 f = 0. \]  

(5.1)

is capable of modeling a superradiant scattering process when the function \( V \) has a particular asymptotic structure

\[ eV(x) = \begin{cases} 
0, & x \to -\infty, \\
eV_0, & x \to +\infty, 
\end{cases} \]  

(5.2)

The asymptotically flat (constant potential) regions at \( \pm \infty \) are necessary in order to have well defined plane waves (equivalently ‘particles’) there and hence a well defined scattering problem.

By way of motivating the form of equation (5.1) we note that it can be thought of as arising from the d’Alembert equation for the 2+1 dimensional geometry

\[ ds^2 = dt^2 - (dx - v dt)^2 \]  

(5.3)

and the assumption that only the second component \( V \) of \( \mathbf{v} = (U, V) \) be nonzero and that this non-zero component \( V \) only depend on the first spatial variable \( x \). This is written

\[ \frac{\partial^2 f}{\partial x^2} - \left( \frac{\partial}{\partial t} + ieV(x) \right)^2 f = 0 \]  

(5.4)

after neglecting the effective mass term induced from the momentum in the \( y \) direction.

At first these assumptions on the flow might seem unnatural for describing superradiance when we imagine that superradiant scattering should be occurring in the near horizon region around a rotating black hole. One might argue that ‘surely we need radial flow, and, surely there is no asymptotically flat region at the horizon’. However, we saw in Section [2.1.4] that, by a suitable change of coordinates and variables, the horizon region around a black hole can be regarded as asymptotically flat in the above sense, being pushed out to \( -\infty \) and allowing for a plane wave decomposition there. The ‘radial flow’ is zero and the \( \phi \) independent rotational flow is precisely the \( V(x) \) in (5.4). Of course, (5.1) can be studied in its own right without embedding it into an analogue framework but we find it useful to keep in mind where such an equation comes from.

For the equation of motion (5.1) the reflection coefficient is related to the transmission coefficient by

\[ |R|^2 = 1 - \frac{\omega - eV_0}{\omega} |T|^2 \]  

(5.5)
and consequently that $|R|^2 > 1$ is possible for sufficiently low frequencies $\omega < eV_0$.

The way we derived relation (5.5) was through the conserved Wronskian associated with (5.1). The action of the Wronskian functional

$$W[f] := f^* f - f^* f', \quad (5.6)$$

where prime $'$ denotes derivative with respect to $x$, is independent of $x$ whenever $f$ satisfies the equation of motion (5.1) as can be easily demonstrated. When we choose as an exact solution the ‘scattering solution’

$$f = \begin{cases} e^{ik_{in}x} + R e^{ik_{r}x} & x \to -\infty \\ T e^{ik_{t}x} & x \to +\infty \end{cases} \quad (5.7)$$

the conservation of $W[f]$ from $-\infty$ to $+\infty$ leads exactly to relation (5.5) independently of the form of the potential $V(x)$ at finite $x$.

The allowed wave-numbers and those chosen for the scattering solution can be concisely understood using a dispersion diagram as shown in Fig.5.1. We note that the scattering solution chooses only a subset of the set of all possible wave-numbers allowed by the dispersion relation

$$k^2 = \begin{cases} \omega^2 & x \to -\infty \\ (\omega - eV_0)^2 & x \to +\infty \end{cases} \quad (5.8)$$

in such a way that the solution can be interpreted as a scattering process: A localised wave packet with momentum centered around $k_{in}$ traveling from $-\infty$ converting into a reflected wavepacket with momentum centered around $k_r$ which travels back to $-\infty$ and a transmitted wavepacket with momentum centered around $k_t$ which travels on to $+\infty$. The direction of motion is given by the group velocity $d\omega/dk$ of the packet and the momenta are determined by the dispersion relation. Explicitly solving the dispersion relation we have

$$k_{in} = \omega \quad k_r = -\omega \quad k_t = \omega - eV_0. \quad (5.9)$$

**Conserved inner product, normalisation**

Associated with the 1+1 equation of motion (5.4) is the bilinear form

$$(f, g) = i \int dx \left[ f^* \left( \partial_t + i eV \right) g - g \left( \partial_t - i eV \right) f^* \right] \quad (5.10)$$
Figure 5.1: The linear dispersion relation \([5.8]\) and the effective frequency \(\Omega(x)\) shown in the two asymptotic regions, the green dash-dotted line at \(-\infty\) and the blue dashed line at \(+\infty\). We have shown the case where \(\omega < eV_0\), specifically \(\omega = 0.4\) and \(eV_0 = 0.7\). Note that the intersections of the blue dashed line (the effective frequency at \(+\infty\)) lie below the \(k\) axis, allowing for a negative norm transmitted mode and consequently superradiant scattering back to \(-\infty\).

which is conserved in time when \(f\) and \(g\) are solutions to \([5.4]\). The norm of a wavepacket \((f, f)\) is conserved in time even after it splits into two oppositely moving packets. A normalised fixed positive frequency wavepacket \(f\) at \(-\infty\) propagates to the right where we can compute the norm to be

\[
(f, f) = \frac{\omega - eV(x)}{\omega}.
\]  

From this we understand such a wave packet to possess a negative norm in regions where \(\omega < eV(x)\). Hence, when the initial wavepacket splits into two, the transmitted part possesses negative norm necessarily implying that the reflected part has norm (which is equal to \(|R|^2\)) larger than one.

We therefore understand that superradiance is tied to the existence of negative norm modes which, in this case, is tied to the sign of the effective frequency \(\omega - eV_0\) in the transmitted asymptotic region.

\(^1\text{Assuming the potential is such that a WKB approximation is valid for the packet and the notion of a spatially dependent wavenumber makes sense.}\)
We also learn from (5.11) how to correctly normalise modes so that they form an orthonormal basis. In each asymptotic region the orthonormal modes making up the Hilbert space of states are given by

\[ \frac{e^{ikx}}{\sqrt{|\Omega|}} = \begin{cases} \frac{e^{ikx}}{\sqrt{|\omega|}}, & x = -\infty \\ \frac{e^{ikx}}{\sqrt{|\omega-eV_0|}}, & x = +\infty \end{cases}, \quad (5.12) \]

with \( \Omega \) evaluated in the appropriate asymptotic region. The correctly normalised scattering solution therefore reads

\[ f = \begin{cases} \frac{e^{ik_{\text{in}}x}}{\sqrt{\omega}} + \frac{R e^{ik_{\text{r}}x}}{\sqrt{\omega}}, & x \to -\infty \\ \frac{T e^{ik_{\text{t}}x}}{\sqrt{|\omega-eV_0|}}, & x \to +\infty \end{cases}. \quad (5.13) \]

The effect of correctly normalising the modes in the scattering solution can equivalently be achieved by simply multiplying all transmission amplitudes obtained from the un-normalised scattering problem by a factor:

\[ T_{\text{normalised}} = \sqrt{\frac{|\omega-eV_0|}{\omega}} T_{\text{unnormalised}}. \quad (5.14) \]

This quantitative factor does not affect out qualitative results presented below which we carry out with the un-normalised solution, except when we discuss exact solutions and can be safely ignored until that point. We choose to work in the un-normalised variables for the qualitative treatment since it renders the formulas and results more transparent until quantitative results are required at which point we append the relevant factors.

**Exact solution**

In what follows we will occasionally be interested in some exact solutions to the scattering problem when higher derivative operators are appended to the equation of motion. In such cases exact solutions are very difficult to come by in the literature due to the relative disinterest in higher order ODEs compared with traditional second order equations of motion. Therefore we will find it convenient to make use of idealised stepfunction potentials for which the exact solution follows directly upon imposing continuity conditions at the step and solving a linear equation. However, the use of step-functions introduces unphysical poles in the scattering amplitudes, as we will see shortly in the non-dispersive case. These poles will also be found in the higher order solutions obtained with step-functions. Here in this short passage we show that these poles are indeed artifacts of the infinite slope in the non-dispersive case at the step by solving exactly the scattering problem using smooth potentials and comparing the
scattering amplitudes. We argue similarly that the poles in the higher order wave equations can be cured analogously by using smooth potentials.

Let \( eV(x) = eV_0 \Theta(x) \), where \( \Theta \) is the Heaviside function, and \( f \) solving (5.1) be given by (5.13). Then imposing the boundary conditions

\[
\begin{align*}
\lim_{x \to 0^+} f(x) &= \lim_{x \to 0^-} f(x) \\
\lim_{x \to 0^+} f'(x) &= \lim_{x \to 0^-} f'(x)
\end{align*}
\] (5.15) (5.16)

allows us to solve for (correctly normalised) \( R \) and \( T \) as

\[
R = \frac{k_{\text{in}} - k_t}{k_{\text{in}} + k_t},
\]

\[
T = \sqrt{\frac{eV_0 - \omega}{\omega} \frac{2k_{\text{in}}}{k_{\text{in}} + k_t}}
\] (5.17)

which are singular when \( k_{\text{in}} + k_t = 0 \) which implies from (5.9) \( \omega = eV_0/2 \).

The equation of motion (5.1) is solvable exactly when \( eV(x) \) takes the form of a smooth tanh interpolation

\[
eV(x) = \frac{eV_0}{2} (1 + \tanh(x/x_p)).
\] (5.18)

where \( x_p \) is a ‘thickness’ parameter of the interpolation. The squared transmission coefficient can be shown to be given exactly for this potential by

\[
|T|^2 = \frac{\sinh(\pi \omega x_p) \sinh(\pi (eV_0 - \omega)x_p)}{\sinh^2(\pi x_p(2\omega - eV_0))^2 + \cos^2(\pi/2 \sqrt{1 - eV_0^2 x_p^2})}.
\] (5.19)

In Figure 5.2 we show the resolution of the singularities of the solution (5.17) by the smooth model for various values of the ‘thickness’ parameter \( x_p \). We discuss this smooth model and its (non-trivial) construction more completely in Appendix B.1.

### 5.2.2 Introducing dispersion

In real, ‘dirty’ physical systems the dispersion relation is often highly non-trivial as we have seen in previous chapters on BEC and optical systems and in Section 2.5. In this chapter we will be concerned with polynomial modifications to the dispersion relation, which can be understood as low wavenumber approximations to a complicated physical dispersion relations as a Taylor series in \( k \) gives a low \( k \) approximation to an arbitrary continuous function of \( k \).
Figure 5.2: The squared reflection and transmission coefficients for the stepfunction model $R_{\text{lin}}$ and $T_{\text{lin}}$ compared with the squared reflection and transmission coefficients for the smooth model $R$ and $T$ for various values of the thickness parameter $x_p$ (see (5.18)) $5.2(\text{a}) x_p = 2$, $5.2(\text{b}) x_p = 1$, $5.2(\text{c}) x_p = 1/2$, $5.2(\text{d}) x_p = 1/4$ — note in 5.2(d) the smoothed result is not actually divergent, being regularised outside the plotted window. In all plots we take $eV_0 = 1$ and plot up to $\omega = eV_0$ from where the scattering is no longer superradiant.
5.2. Warmup problem - zero flow scattering

Inspired by the quartic dispersion relation $\Omega^2 = k^2 \pm k^4/\Lambda^2$, where $\Lambda$ is a dispersive momentum scale and the $\pm$ notates super- and sub-luminal dispersion respectively, we propose the following generalization of equation (5.1),

$$\mp \frac{1}{\Lambda^2} f^{''''} + f'' + (\omega - e\Phi(x))^2 f = 0,$$

(5.20)

where $f$ represents the fixed frequency mode after separation of the time dependence from the appropriate time dependent wave equation

$$\mp \frac{1}{\Lambda^2} f^{''''} + f'' - \left( \frac{d}{dt} - ie\Phi(x) \right)^2 f = 0,$$

(5.21)

and prime denotes derivative with respect to $x$. We choose the effective frequency $\Omega(x) = \omega - e\Phi(x)$ to satisfy asymptotic relations similar to that in equation (5.2),

$$\Omega^2(x) = \begin{cases} \omega^2, & \text{for } x \to -\infty, \\ (\omega - e\Phi_0)^2, & \text{for } x \to +\infty, \end{cases}$$

(5.22)

where $\omega > 0$ is understood as the conserved frequency of the mode and $e\Phi_0$ is a positive constant. In the asymptotic regions, the solutions of equation (5.20) are simple exponentials $\exp(i k x)$ whose wavenumbers $k$ satisfy the dispersion relations

$$\mp \frac{k^4}{\Lambda^2} - k^2 + \omega^2 = 0, \quad x \to -\infty,$$

(5.23)

$$\mp \frac{k^4}{\Lambda^2} - k^2 + (\omega - e\Phi_0)^2 = 0, \quad x \to +\infty.$$

(5.24)

In the subluminal case (lower sign), real solutions $(\omega, k)$ to the dispersion relation correspond to the intersections of a lemniscate (figure-eight) and a straight line (see Figure (5.3)) while in the superluminal case (upper sign) they correspond to the intersections of the quartic curve and a straight line, as shown in Figure (5.4).

In order to relate the asymptotic solutions at $+\infty$ and $-\infty$ without having to solve the differential equation for all values of $x$, one needs a conserved quantity analogous to the Wronskian (5.6) for second order wave equations. Since our model is non-dissipative, we expect such a quantity to exist [130].

Conserved Wronskian, inner product

After some trivial algebra, it is possible to show that the quantity

$$Z[f] = W_1 + W_2 \mp \Lambda^2 W_3,$$

(5.25)

where

$$W_1 = f^{''''*} f - f^* f^{''''},$$

$$W_2 = f^{*''} f'' - f^{*''*} f',$$

$$W_3 = f^{*''} f - f^* f',$$

(5.26)
Figure 5.3: The dispersion relation for the subluminal case shown in terms of the dimensionless variables $k/\Lambda$ and $\Omega/\Lambda$ where $\Omega = \omega - e\Phi_0$. The green dash-dotted line represents $\omega$ when $x \to -\infty$ while the blue dashed line corresponds to the region $x \to \infty$. Here $\omega < e\Phi_0$ is the fixed lab frequency satisfying the condition for superradiance.

We have indicated in boldface those modes which are included in the scattering solution (5.36) $k_c = k_{1\text{in}}$, $k_b = k_{r_1}$, $k_d = k_{r_2}$, $k_B = k_{t_1}$ and $k_D = k_{t_2}$.

is conserved in space i.e. $dZ/dx = 0$ for any solution $f$ of equation (5.20). However, it will be more convenient to work with the scaled functional $X$ defined by

$$X[f] = \frac{Z[f]}{2i\Lambda^2}.$$  

This is because the action of $X$ on a linear combination of ‘on shell’ plane waves (wavenumbers satisfying the dispersion relation) is very simple, given by

$$X \left[ \sum_n A_n e^{ik_n x} \right] = \sum_n \Omega \frac{d\Omega}{dk_n} |A_n|^2.$$  

The inner product in the dispersive theory is identical [30] to the non-dispersive one defined in (5.10). It is conserved when acting on solutions to equation (5.21) as can be easily verified.
Figure 5.4: The dispersion relation for the superluminal case shown in terms of the dimensionless variables $k/\Lambda$ and $\Omega/\Lambda$. The blue dashed curve corresponds to the dispersion relation at $x \to +\infty$ while the green dash-dotted curve corresponds to the dispersion at $x \to -\infty$. We have chosen here a frequency in the superradiant interval $\omega < e\Phi_0$.

Again, for wave-packets $\phi$ and $\psi$ normalised at $-\infty$, respectively peaked around the single momenta $k$ and $k'$, with fixed frequency $\omega$ and localised in a region where $\Phi(x)$ is approximately constant, one can show that

$$ (\phi, \psi) = \frac{\omega - e\Phi(x)}{\omega} \delta(k - k'), \quad (5.29) $$

independently of the dispersion. Therefore, modes for which $\omega - e\Phi_0 < 0$ possess negative norm at $+\infty$.

Again we see that the correct normalisation of plane waves to render them orthonormal in their asymptotic region the effective frequency:

$$ \text{orthonormal mode} = \frac{e^{ikx}}{\sqrt{|\Omega|}} \quad (5.30) $$

As already mentioned we remain with the un-normalised modes unless otherwise stated.
Quartic polynomials – In fact, the action of $X$ in (5.28) can be shown to be

$$X \left[ \sum_i A_i e^{ik_i x} \right] = \sum_i |A_i|^2 \Omega \frac{d\Omega}{dk_i} + \text{“terms”},$$

(5.31)

where “terms” are all proportional to

$$(k_i + k_j)(\Lambda^2 - k_i^2 - k_j^2),$$

(5.32)

for pairs of roots. These terms vanish either when $k_i = -k_j$ or when they sum to $\Lambda^2$.

In general the roots of the polynomial $z^4 + a_2 z^2 + a_1 z + a_0$, satisfy the relations

$$\sum_n z_n = 0 \quad \text{and} \quad \sum_n z_n^2 = -2a_2$$

known as Vieta’s formulas.

The dispersion relation in the asymptotic regions considered above is

$$\omega^2 = k^2 - \frac{k^4}{\Lambda^2}, \quad (x \to -\infty),$$

(5.33)

$$\left(\omega - e\Phi_0\right)^2 = k^2 - \frac{k^4}{\Lambda^2}, \quad (x \to +\infty).$$

(5.34)

Hence the roots of the dispersion relation satisfy

$$\sum_n k_n^2 = 2\Lambda^2, \quad \sum_n k_n = 0,$$

(5.35)

in both asymptotic regions. In this zero flow case, the roots come in pairs of opposite sign. Either the two roots in any cross term are of opposite sign and the first factor of (5.32) vanishes or they are not and contribute half to the sum of squares i.e: they square sum to $\Lambda^2$ and the second factor of (5.32) vanishes. Hence “terms” all vanish.

5.2.3 Subluminal scattering

Consider firstly the subluminal dispersion relation. In realistic scenarios, we do not expect the dispersion relation $\omega(k)$ to have a maximum/minimum value above/below which only imaginary solutions of the dispersion relation are possible\(^2\). Therefore, in order to guarantee that at least one real mode is available for the scattering, we assume that the dispersion parameter $\Lambda$ is large compared to the interaction potential $e\Phi_0$, i.e. we assume that $e\Phi_0 < \Lambda/2$.

\(^2\)A notable exception to this expectation is the optical dispersion described in Section 2.4.
Then the scattering process is captured succinctly in Figure 5.3 where the green (Ω = ω) and blue (Ω = ω − eΦ0) lines represent the effective frequency when x → −∞ and x → +∞, respectively. Note that there exist exactly four roots, corresponding to four propagating degrees of freedom as one would expect from a fourth order differential equation. We also note that, when ω < eΦ0, the blue dashed line is located below the Ω = 0 axis and the roots in the region x → +∞ inherit a relative sign between their group and phase velocities with respect to the roots at x → −∞ (compare the intersections of the green dash-dotted line (x → −∞) and the blue dashed line (x → +∞) with the red curve).

We label the four roots associated with a frequency 0 < ω < eΦ0 as k_a < k_b < k_c < k_d when x → −∞ and as k_A < k_B < k_C < k_D when x → +∞ (they correspond to the intersections of the straight lines with the figure-eight in Figure 5.3). The following table summarises the character of these modes, where u and v notate right-movers and left-movers respectively and blue boldface indicates those modes which are included in the scattering solution.

<table>
<thead>
<tr>
<th>Roots at x → −∞</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>group velocity</td>
<td>u</td>
<td>v</td>
<td>u</td>
<td>v</td>
</tr>
<tr>
<td>phase velocity</td>
<td>v</td>
<td>v</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>norm</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Roots at x → +∞</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>group velocity</td>
<td>v</td>
<td>u</td>
<td>v</td>
<td>u</td>
</tr>
<tr>
<td>phase velocity</td>
<td>v</td>
<td>v</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>norm</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

We set up our scattering experiment in the standard way so that all incident waves originate from x → −∞, i.e. there is only one source of waves, located at x → −∞ and no incoming signal from x → +∞. Therefore, we impose the boundary condition that the modes A and C are unpopulated in the interaction region since they correspond to left-moving modes at x → +∞. Furthermore, we choose the incoming mode at x → −∞ to be entirely composed of low-momentum c modes with no high-momentum a mode component. Hence there are only 5 distinct roots which are relevant for the scattering process (indicated in boldface in Figure 5.3) and we relabel them as k_in := k_c, the reflected modes k_{r1} := k_b, k_{r2} := k_d, and the transmitted modes k_{t1} := k_B, k_{t2} := k_D.

The scattering solution of equation (5.20) in the asymptotic limits is, therefore, given by

\[
f \rightarrow \begin{cases} 
    e^{ik_{in}x} + R_1 e^{ik_{r1}x} + R_2 e^{ik_{r2}x}, & x \to -\infty, \\
    T_1 e^{ik_{t1}x} + T_2 e^{ik_{t2}x}, & x \to +\infty.
\end{cases}
\] (5.36)

In the asymptotic regions, it is possible to solve exactly the dispersion relation and
find the following explicit expressions for the roots,

\[
k_{in} = -k_{r1} = \frac{\Lambda}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \frac{4\omega^2}{\Lambda^2}}},
\]

\[
k_{r2} = \frac{\Lambda}{\sqrt{2}} \sqrt{1 + \sqrt{1 - \frac{4\omega^2}{\Lambda^2}}},
\]

\[
k_{t1} = -\frac{\Lambda}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \frac{4(\omega - e\Phi_0)^2}{\Lambda^2}}},
\]

\[
k_{t2} = \frac{\Lambda}{\sqrt{2}} \sqrt{1 + \sqrt{1 - \frac{4(\omega - e\Phi_0)^2}{\Lambda^2}}},
\]

(5.37)

Substituting equation (5.36) into the expression for the conserved generalised Wronskian, equation (5.27), we find, after straightforward algebraic manipulation, the following relation between the coefficients \(R_1, R_2, T_1\) and \(T_2\),

\[
|R_1|^2 + \frac{k_{r2}}{k_{in}} |R_2|^2 = 1 + \sqrt{\frac{\Lambda^2 - 4(\omega - e\Phi_0)^2}{\Lambda^2 - 4\omega^2}} \left( |k_{i1}| |T_1|^2 + |k_{i2}| |T_2|^2 \right) > 1. \tag{5.38}
\]

This relation should be compared to the standard result for non-dispersive 1D scattering \(|R|^2 = 1 - |T|^2\) and its generalization in the presence of an external potential \(|R|^2 = 1 - (\omega - e\Phi_0)|T|^2/\omega\), as in equation (5.5).

As discussed above, in the linear dispersion case it is possible to achieve \(|R| > 1\) for sufficiently low frequency scattering \(\omega < e\Phi_0\). For subluminal dispersion, the conclusion is similar. From expression (5.38) above, which is valid only when \(0 < \omega < e\Phi_0 < \Lambda/2\), we conclude that the total reflection coefficient (i.e. the left hand side of equation (5.38)) is always greater than one, characterizing superradiance.

Note that, in the general case without an exact solution we have no information about how this total reflection is distributed between the low and high wavenumber channels represented by \(R_1\) and \(R_2\) respectively. However, by looking at the \(\Lambda\) series expansion of the reflected and transmitted wavenumbers, we can draw some interesting conclusions about the regime \(\Lambda \gg 1\).

From equation (5.37), we obtain the following expansions for the relevant wavenum-
5.2. Warmup problem - zero flow scattering

All possible modes of the system have the same energy, which is determined by the conserved lab frequency $\omega$. The momentum, on the other hand, is determined by the wavenumber $k$ and, therefore, is not the same for every mode.

From the expansions above, we note that $k_{t_1}, k_{r_2} \sim O(\Lambda)$ and $k_{in}, k_{r_1}, k_{t_1} \sim O(\Lambda^0)$. In particular, the difference between the high momentum modes $k_{r_2}$ and $k_{t_2}$ is small $k_{r_2} - k_{t_2} \sim O(\Lambda^{-1})$. Intuitively, this means that the creation of a pair of these modes $(k_{r_2}, k_{t_2})$ requires a negligible momentum change in the system even when $\Lambda \gg 1$. Consequently, we expect mode production in these high momentum channels to be non-negligible. Moreover, regarding the low momentum channels, incident modes $k_{in}$ are being converted into $k_{r_1}$ modes and $k_{t_1}$ modes in the system, and should be unaffected when $\Lambda \gg 1$.

In the regime $\Lambda \gg 1$, we therefore anticipate that the scattering consists of two separate kinds of balance, high and low momentum, in such a way that the Wronskian relation (5.38) becomes equivalent to two different conservation relations, namely

$$|R_1|^2 \simeq 1 + \sqrt{\frac{\Lambda^2 - 4(\omega - e\Phi_0)^2}{\Lambda^2 - 4\omega^2}} \left| \frac{k_{t_1}}{k_{in}} \right| |T_1|^2 > 1,$$

for the low-momentum modes and

$$\left| \frac{k_{r_2}}{k_{in}} \right| |R_2|^2 \simeq \sqrt{\frac{\Lambda^2 - 4(\omega - e\Phi_0)^2}{\Lambda^2 - 4\omega^2}} \left| \frac{k_{t_2}}{k_{in}} \right| |T_2|^2;$$

(5.41)

for the high-momentum modes. Note that, outside the regime $\Lambda \gg 1$, the expressions can not be expected to remain valid independently. In such a case, only the original Wronskian relation (5.38) is satisfied and we expect a mixing between all available channels.

This kind of separation of the Wronskian is confirmed below by using a stepfunction model for $e\Phi(x)$ to compute the exact solution.
Exact step-function solution

We can lend weight to the argument for the decoupling of high and low momentum channels by solving exactly the idealised problem in which the potential is chosen to be a step function, i.e. \( V(x) = e \Phi_0 \Theta(x) \). The boundary condition at the step is such that the solution and its first 3 derivatives are continuous there. Then the un-normalised solution is

\[
R_1 = \frac{(k_{in} - k_{r2})(k_{in} - k_{t1})(k_{in} - k_{t2})}{(k_{r2} - k_{r1})(k_{r1} - k_{t1})(k_{r1} - k_{t2})},
\]
\[
R_2 = \frac{(k_{in} - k_{r1})(k_{in} - k_{r1})(k_{in} - k_{t2})}{(k_{r1} - k_{r1})(k_{r1} - k_{r2})(k_{r1} - k_{t2})},
\]
\[
T_1 = \frac{(k_{r1} - k_{t1})(k_{r1} - k_{t2})(k_{r2} - k_{t1})}{(k_{r1} - k_{r1})(k_{r1} - k_{r2})(k_{r2} - k_{t1})},
\]
\[
T_2 = \frac{(k_{r1} - k_{r1})(k_{r1} - k_{r2})(k_{r2} - k_{t1})}{(k_{r1} - k_{r1})(k_{r1} - k_{r2})(k_{r2} - k_{t1})}.
\]

Note the poles at exactly the same frequency as in the non-dispersive solutions \((5.17)\) which we assume are resolved in a smooth mode\(^3\)(from Figure 5.3 we see that \( k_{t2} = k_{r2} \) at the same frequency that \( k_{t1} = k_{r1} \), that is, at \( \omega = e \Phi_0 / 2 \)).

Using the exact expressions \((5.37)\), we can compute the action of the functional \( X \) on each mode and its coefficient appearing in the scattering solution\(^4\) \((5.36)\) as

\[
X \left[ e^{ik_{in}x} \right] = \omega + O \left( \Lambda^{-1} \right)
\]
\[
X \left[ R_1 \, e^{ik_{r1}x} \right] = \frac{(e \Phi)^2 \omega}{(2 \omega - e \Phi)^2} + O \left( \Lambda^{-1} \right)
\]
\[
X \left[ R_2 \, e^{ik_{r2}x} \right] = -\frac{16 \omega^2 (\Lambda - 2 \omega - 2e \Phi)}{(2 \omega - e \Phi)^2} + O \left( \Lambda^{-1} \right)
\]
\[
X \left[ T_1 \, e^{ikt_1x} \right] = \frac{4(\omega - e \Phi_0)^2 \omega^2}{(2 \omega - e \Phi)^2} + O \left( \Lambda^{-1} \right)
\]
\[
X \left[ T_2 \, e^{ikt_2x} \right] = -\frac{16 \omega^2 (\Lambda - 2 \omega - 2e \Phi_0)}{(2 \omega - e \Phi)^2} + O \left( \Lambda^{-1} \right).
\]

\(^3\)We have not shown that the poles are resolved in the dispersive theory since an exact smooth solution is only available in the non-dispersive theory but believe they should be nevertheless be resolved in this way.

\(^4\)Note that, by \((5.14)\) we have

\[
X \left[ T_{\text{normalised}} e^{-ikx} / \sqrt{\Omega} \right] = X \left[ T_{\text{normalised}} e^{ikx} / \sqrt{\omega} \right] = X \left[ T_{\text{normalised}} e^{ikx} / \sqrt{\omega} \right] = X \left[ T_{\text{normalised}} e^{ikx} \right] / \sqrt{\omega}
\]

so that we can use both the unnormalised solution and unnormalised coefficients to get the same result as the properly normalised solution and coefficients.
Figure 5.5: The squared reflection and transmission coefficients for the low momentum channels $|R_1|^2$ and $|T_1|^2$ for the subluminally dispersive solutions to the stepfunction model for $e\Phi_0 = 1$ as computed from (5.42) compared with the reflection and transmission coefficients computed from the the non-dispersive stepfunction model where these are the only scattering channels. In 5.5(a) the results are shown for $\Lambda = 5$ (significant dispersion) while in 5.5(b) we show the results for $\Lambda = 50$ (small deviation from linear dispersion).

Hence the action of the functional $X$ on the two channels $k_r^2$ and $k_t^2$ is identical up to terms of $O(1/\Lambda)$ meaning that these contributions cancel from equation (5.38) and hence equation (5.40) is satisfied up to $O(1/\Lambda)$, implying superradiance in the low momentum channel alone. This also implies that (5.41) is satisfied up to $O(1/\Lambda)$. Note that this result also shows that the non-dispersive scattering process is recovered in the non-dispersive $\Lambda \to \infty$ limit.

In Figure 5.5 we plot the low momentum channel squared reflection and transmission coefficients for two different regimes, significant (small $\Lambda$) and slight (large $\Lambda$) dispersion and compare them with the result from the linear dispersive case. In Figure 5.6 on the other hand we plot only the high momentum channel coefficients $|R_2|^2$ and $|T_2|^2$ including as a reference the low momentum $|R_1|^2$ coefficient.

These plots provide us with an interesting result since they show that, using step functions, the population of the new channels due to dispersion, $k_{r_2}$ and $k_{t_2}$, do not go to zero when the dispersion is slight, confirming the expectations we had above about a momentum balance favouring simultaneous production of $k_{r_2}$ and $k_{t_2}$ modes. In fact, each of these high momentum channels contributes a universally large amount which
Figure 5.6: The reflection and transmission coefficients \( |R_2|^2 \) and \( |T_2|^2 \) for the dispersive stepfunction model for \( e\Phi_0 = 1 \) as computed from (5.42) in 5.6(a) for \( \Lambda = 4 \) while in 5.6(b) for \( \Lambda = 50 \). In both cases we include for reference also the coefficient \( |R_1|^2 \), showing that the high momentum channels \( R_2 \) and \( T_2 \) are less occupied than the low momentum channels for low frequencies but much more populated for high frequency up to the superradiance bound \( \omega = e\Phi_0 \).

cancels out from the total reflection coefficient (5.38) in the limit of large \( \Lambda \).

Another very interesting observation is that the presence of the additional channels corresponding to the high momenta \( k_{r_2} \) and \( k_{t_2} \) in fact act to increase the total reflection received back at \( -\infty \) even though we only send in one low momentum \( k_{\text{in}} \) incident wave. This is shown in Figure 5.7 where we plot the result from linear dispersion \( |R_{\text{lin}}|^2 \) along with two objects which one might consider representative of the total reflection: the particle current given by the left-hand side of (5.38) \( |R_1|^2 + |k_{r_2}/k_{\text{in}}||R_2|^2 \) in one sub figure and the total power of reflection given sums of square amplitudes \( |R_1|^2 + |R_2|^2 \). We plot these quantities for the generic values of \( e\Phi_0 = 1 \) and \( \Lambda = 2 \), noting that the coefficient \( R_2 \) unphysically grows without bound as \( \Lambda \to +\infty \) further enhancing this result (perhaps unphysically) at higher \( \Lambda \).

### 5.2.4 Superluminal scattering

Let us now turn our attention to the case of superluminal dispersion. Once again, we can understand much of the scattering process from the dispersion diagram, shown in Figure 5.4. The main difference with the subluminal case is that now there are only
two real roots in either asymptotic region, corresponding to one left-moving and one right-moving mode. The other two solutions to the fourth order equation are imaginary roots, corresponding to exponentially decaying and exponentially growing modes.

Similarly to the subluminal case, we are interested in the scattering of an incident low-frequency ($\omega < e\Phi_0$) wave originating from $x \to -\infty$, which is converted into a reflected left-moving mode, a transmitted right-moving mode and exponentially decaying modes (as a boundary condition, we impose the fact that there can be no exponentially growing modes in the asymptotic regions). Therefore, the solution of equation (5.20) corresponding to this scattering process is given by,

$$\begin{align*}
f &\to \begin{cases} 
e^{ik_{in}x} + R e^{ik_{r}x} + E_r e^{k_{er}x}, & x \to -\infty, \\
T e^{ik_{t}x} + E_t e^{-k_{et}x}, & x \to +\infty, \end{cases} \tag{5.45}
\end{align*}$$

where the coefficients $R$ and $T$ are related, respectively, to reflection and transmission coefficients (see below), and the coefficients $E_r$ and $E_t$ are the coefficients of the exponential modes.

The wavenumbers $k$ in equation (5.45) are obtained directly from the dispersion
relations, equations (5.23) and (5.24),

\[ k_{\text{in}} = -k_r = \frac{\Lambda}{\sqrt{2}} \sqrt{-1 + \sqrt{1 + \frac{4\omega^2}{\Lambda^2}}}, \]

\[ k_{\text{er}} = \frac{\Lambda}{\sqrt{2}} \sqrt{\frac{1 + 4\omega^2}{\Lambda^2}}, \quad (5.46) \]

\[ k_t = -\frac{\Lambda}{\sqrt{2}} \sqrt{-1 + \sqrt{1 + \frac{4(\omega - e\Phi_0)^2}{\Lambda^2}}}, \]

\[ k_{\text{et}} = \frac{\Lambda}{\sqrt{2}} \sqrt{\frac{1 + 4(\omega - e\Phi_0)^2}{\Lambda^2}}. \]

A conservation equation can be obtained by plugging equation (5.45) into equation (5.28) and equating the generalised Wronskian at \( \pm \infty \). One might expect a distinction with the standard result due to the extra two exponentially decaying modes in the solution. However, it turns out that these extra contributions to the generalised Wronskian are also exponentially decaying and one obtains the following reflection coefficient (valid for \( 0 < \omega < e\Phi_0 \)),

\[ |R|^2 = 1 + \sqrt{\frac{\Lambda^2 + 4(\omega - e\Phi_0)^2}{\Lambda^2 + 4\omega^2} \left| \frac{k_t}{k_{\text{in}}} \right| |T|^2} > 1, \quad (5.47) \]

which, similarly to the non-dispersive case, is always greater than one. Note also that the expression above reduces to the usual non-dispersive reflection coefficient (5.5) in the limit \( \Lambda \rightarrow \infty \).

In Figure 5.8 we plot the exact reflection and transmission coefficients for the step-function model and the non-dispersive result (the algebraic result (5.17) applies to both scattering problems since the solution (5.17) was obtained independently of the scattering momenta) showing the agreement with the non-dispersive result.

### 5.3 Non-zero flow (or, curved geometry)

In this section we make the second generalisation anticipated in the introduction, that is, to non-zero flows in the direction of scattering. This is equivalent to a generalisation to a non-flat background geometry in the analogue language.

The equation of motion we will use can be understood to arise again from an acoustic geometry of the form

\[ ds^2 = dt^2 - (dx - v(x)dt)^2 \]

\[ (5.48) \]
5.3. Non-zero flow (or, curved geometry)

Figure 5.8: The reflection and transmission coefficients for the low momentum channels $|R|^2$ and $|T|^2$ for the superluminally dispersive solutions to the stepfunction model for $e\Phi_0 = 1$ as computed from (5.42) compared with the reflection and transmission coefficients computed from the non-dispersive stepfunction model where these are the only scattering channels. In 5.8(a) the results are shown for $\Lambda = 1$ (significant dispersion) while in 5.8(b) we show the results for $\Lambda = 5$ (‘small’ deviation from linear dispersion).

but now with the first component $W$ of the flow velocity $\mathbf{v} = (W, V)$ non-zero. We still requite that both $W$ and $V$ be only functions of one component $x$ of the spatial coordinate $\mathbf{x} = (x, y)$.

The non-dispersive d’Alembertian equation associated to (5.48) is

$$
\left[ \left( \frac{\partial}{\partial t} - ipV - \frac{\partial}{\partial x} W \right) \left( \frac{\partial}{\partial t} - ipV - W \frac{\partial}{\partial x} \right) - p^2 + \frac{\partial^2}{\partial x^2} \right] f_{\omega,p}(x) = 0 \quad (5.49)
$$

where $p$ is the momentum eigenvalue in the $y$ direction.

Below we will work with a slightly modified equation of motion without the $p^2$ term in the interests of simplicity as we did above. This additional term acts as a mass for modes propagating only in the $x$ direction, where the scattering takes place, and later on we will discuss the implications of including such a mass term which has been recently discussed in \cite{132,133}. Such a mass term is also considered in \cite{30} in the non-dispersive, zero flow context where it was shown that the window of frequencies where superradiance occurs is not only bounded from above by $e\Phi_0$ but also from below by $p^2$ as we saw in 2.1.4.
5.3.1 Equation of motion, dispersion

Specifying the modified dispersion relation in the co-moving frame of the fluid as the quartic one, the equation of motion (5.20) generalises, through (5.49) to

\[ \mp \frac{1}{\Lambda^2} \frac{d^4 f}{dx^4} + \frac{d^2 f}{dx^2} + \left( \omega - e\Phi(x) + i \frac{d}{dx} W(x) \right) \left( \omega - e\Phi(x) + i W(x) \frac{d}{dx} \right) f = 0. \tag{5.50} \]

Similarly to the zero-flow case, it is useful to write the equation above in the form

\[ f^{'''}(x) + \alpha(x) f^{''}(x) + \beta(x) f'(x) + \gamma(x) f = 0, \tag{5.51} \]

where

\[
\alpha = \pm \Lambda^2 \left( 1 - W^2(x) \right) \\
\beta = \mp \Lambda^2 \left[ 2W(x) \partial_x W(x) - 2iW(x) \left( \omega - e\Phi(x) \right) \right] \\
\gamma = \pm \Lambda^2 \left[ (\omega - e\Phi(x))^2 + i \partial_x (W(x)(\omega - e\Phi(x))) \right].
\]

Observe that these coefficients are not independent but satisfy the following relations,

\[
\gamma - \gamma^* = 2i \text{Im}(\gamma) = i \text{Im}(\partial_x \beta) = \partial_x \left( \frac{\beta - \beta^*}{2} \right), \tag{5.53}
\]

\[
\text{Re}(\beta) = \frac{1}{2} (\beta + \beta^*) = \partial_x \alpha. \tag{5.54}
\]

We choose the functions \( W(x) \) and \( \Phi(x) \) to be asymptotically constant

\[
W(x) = \begin{cases} 
0, & x \to -\infty, \\
W_0, & x \to +\infty,
\end{cases} \tag{5.55}
\]

\[
e\Phi(x) = \begin{cases} 
0, & x \to -\infty, \\
e\Phi_0, & x \to +\infty,
\end{cases} \tag{5.56}
\]

so that, at \( \pm \infty \), any solution to (5.50) can be decomposed into plane waves which satisfy the dispersion relation

\[
k^2 \pm k^4 = \begin{cases} 
\omega^2, & x \to -\infty, \\
(\omega - e\Phi_0 - kW_0)^2, & x \to +\infty,
\end{cases} \tag{5.57}
\]

where the \( \pm \) stands for super- and sub-luminal dispersion. At this stage we will not specify whether \( W_0 > 1 \) or \( < 1 \) corresponding to supersonic and subsonic asymptotic flow speeds respectively. As previously, we work under the assumption that \( \omega, e\Phi_0 \ll \Lambda \) in order for the wavenumbers to be real.
5.3. Non-zero flow (or, curved geometry)

5.3.2 The Wronskian and inner product reconsidered

The generalised Wronskian found previously, equation (5.25), is not conserved when the flow \( W \) is non-zero. However, one can show that a modification of it, namely

\[
Z[f] = W_1 + W_2 + \alpha W_3 - i \text{Im} \left( \beta \right) |f|^2,
\]

(5.58)

and the corresponding scaled quantity \( X[f] = Z[f]/(2i\Lambda^2) \) are independent of \( x \) when \( f \) satisfies the equation of motion (5.50).

The action of the functional \( X \) on a linear combination of ‘on-shell’ plane waves takes precisely the same form as in the zero-flow case,

\[
X \left[ \sum_n A_n e^{i \omega_n x} \right] = \sum_n |A_n|^2 \Omega \frac{d\omega}{dk_n},
\]

(5.59)

but now with \( \Omega = \omega - e\Phi - kW \). In particular, this relation holds in the asymptotic regions at \( \pm \infty \). In the zero-flow case, it was the quantity \( \omega - e\Phi_0 \) which entered the right hand side of equation (5.59) in place of \( \Omega \) and the existence of superradiance was related to the condition that \( \omega - e\Phi_0 < 0 \). We can anticipate that the condition of superradiance in non-zero flows will be favoured by modes for which \( \Omega < 0 \).

The generalization of the inner product (5.10) to non-zero flows is given by

\[
(\phi, \psi) = i \int dx \left[ \phi^* (\partial_t - i e\Phi + W \partial_x) \psi - \psi (\partial_t + i e\Phi + W \partial_x) \phi^* \right],
\]

which is conserved for solutions of the time-dependent version of the equation of motion (5.50) as long as we choose the quartic modification to the dispersion\footnote{It suffices that we modify by an even power of \( k \) for this conclusion to hold.} as we show in Appendix B.2. It is clear, therefore, that the norm of a localised wavepacket \( \phi \) normalised at \( -\infty \) (that is, with the free frequency, \( \omega \)) of momentum \( k \) is proportional to the factor \( \Omega \) evaluated at its position,

\[
(\phi, \phi) = \frac{\omega - e\Phi(x) - kW(x)}{\omega} \delta(0),
\]

(5.60)

in corroboration of our expectations above on the role of the sign of effective frequency \( \Omega \) being the litmus test for superradiance. That is, the negative norm modes are those for which \( \Omega < 0 \).

**Wronskian trick** – In fact, again, the action of \( X \) (5.59) on the the linear combination of plane waves also contains some “cross terms” which can be shown to be

\[
\text{“cross term”} = \left[ \Omega(k_i) \frac{d\omega}{dk_i} + \Omega(k_j) \frac{d\omega}{dk_j} + \frac{1}{\Lambda^2} (k_i - k_j)^2 (k_i + k_j) \right] \text{Re} \left( A_i A_j^* e^{i(k_i-k_j)x} \right)
\]
Chapter 5. Superradiance in Dispersive Theories

Spoiling the nice result (5.59). In the zero flow case, we were able to find algebraic relations amongst the roots and use a symmetry argument to show that the cross terms all vanish. No such algebraic result is available in this case but we know that such terms must vanish because of the independent fact that $\frac{dX[f]}{dx} = 0$.

5.3.3 Subluminal scattering

Let us consider first the case of a subluminal dispersion relation. The inclusion of a flow velocity introduces many new features to the analysis which both complicate the computations but also allow a description of new phenomena and physical processes.

As shown in the dispersion diagram in Figure 5.9 for fixed $e\Phi_0/\Lambda < W_0 < 1$, there are two distinct intervals of frequencies separated by a critical frequency $\omega_{\text{crit}} < e\Phi_0$ in which we expect superradiance: For $0 < \omega < \omega_{\text{crit}}$ (region I) two propagating degrees of freedom are admitted, both right-moving in the lab frame; for $\omega_{\text{crit}} < \omega < e\Phi_0$ (region II) there are four real roots of the dispersion relation corresponding to four propagating degrees of freedom, three right-moving and one left-moving (with respect to the lab-frame). Note that the requirement that $W_0$ is not too small, specifically $W_0 > e\Phi_0/\Lambda$, is necessary in order to guarantee that the left-most root in region I (i.e. the intersection of an $\omega = \text{constant}$ line with the red dispersion curve in Figure 5.9) has positive group velocity and hence defines a true transmitted mode $^7$. If $0 < W_0 < e\Phi_0/\Lambda$, the situation is qualitatively identical as the zero flow case discussed in section 5.2.3.

On the other hand, if $W_0 > 1$ (see the blue-dashed curve in Figure 5.9), there is only one possible regime: For all frequencies $0 < \omega < e\Phi_0$, two right-moving modes are admitted.

An interesting fact is that, due to the absence entirely of left-moving modes at $+\infty$ when $\omega < \omega_{\text{crit}}$, this system is a model for the event horizon of an analogue black hole with modified dispersion relations. However, the precise location of the horizon, besides being $\omega$-dependent, is also rather ill-defined, relying on a global solution to the equation of motion in the vicinity of a classical turning point. This region can be studied by WKB methods and Hamilton–Jacobi theory $^\text{134}$, but it is not of specific interest to us here.

According to the analysis above, the scattering of an incoming wave from $-\infty$ will result in transmission through two or three channels, depending on whether $\omega < \omega_{\text{crit}}$

$^6$In terms of the other parameters of the problem, $\omega_{\text{crit}}$ is given by

$$\omega_{\text{crit}} \simeq -\left[\frac{2}{3}(1-W_0)\right]^{3/2} \Lambda + e\Phi_0.$$

$^7$See the Appendix for further discussion on this peculiarity of the quartic dispersive model as well as a discussion on the role of the asymptotic flow $W(x \to -\infty)$. 
5.3. Non-zero flow (or, curved geometry)

Figure 5.9: The subluminal dispersion curve in the lab frame at $x \to +\infty$ for two different flow velocities $e\Phi_0/\Lambda < W_0 < 1$ (red solid line) and $W_0 > 1$ (blue dashed curve). The coloured regions described in the text are associated with the red curve: region I (green), $\omega < \omega_{\text{crit}}$, region II (blue) $\omega_{\text{crit}} < \omega < e\Phi_0$, region III (red), non-superradiantly scattering region. We have plotted the dispersion for $e\Phi_0 = 0.4$ and $\Lambda = 1$. In the non-interacting region, since $W(x) \to 0$ there, the dispersion is described by the green dash-dotted curve of Figure 5.3.

or not. An exact solution to the scattering problem can be decomposed as

$$f = \begin{cases} e^{ik_{\text{in}}x} + R_1e^{ik_{r1}x} + R_2e^{ik_{r2}x}, & x \to -\infty, \\ T_1e^{ik_{t1}x} + T_2e^{ik_{t2}x}, & x \to +\infty, \end{cases}$$

(5.62)

when $0 < \omega < \omega_{\text{crit}}$ (if $W_0 < 1$) or $0 < \omega < e\Phi_0$ (if $W_0 > 1$) and as

$$f = \begin{cases} e^{ik_{\text{in}}x} + R_1e^{ik_{r1}x} + R_2e^{ik_{r2}x}, & x \to -\infty, \\ T_1e^{ik_{t1}x} + T_2e^{ik_{t2}x} + T_3e^{ik_{t3}x}, & x \to +\infty, \end{cases}$$

(5.63)
when \( \omega_{\text{crit}} < \omega < e\Phi_0 \) (only possible if \( W_0 < 1 \)). The wavenumbers \( k_{in}, k_{r_1} \) and \( k_{r_2} \) are given by equation (5.37) and labeled in Figure 5.3, while the transmitted wavenumbers are not, in general, expressible as simple functions of the parameters. Plugging the solutions (5.62) and (5.63) into the functional \( X \) of equation (5.59), we find, after some algebraic manipulation, the following relationship between the transmission and reflection coefficients,

\[
| R_1 |^2 + \left| \frac{k_{r_2}}{k_{in}} \right| | R_2 |^2 = 1 - \frac{\Lambda}{\sqrt{\Lambda^2 - 4\omega^2}} \left( \sum_n v_{g,n} (\omega - e\Phi_0 - k_{tn} W_0) |T_n|^2 \right), \tag{5.64}
\]

where the sum is over all (2 or 3, depending on \( \omega \) and \( W_0 \)) transmission channels. Here, \( v_{g,n} = v_g(k_{tn}) \) are the group velocities of the transmitted modes at \(+\infty\). The left hand side of equation (5.64) can be interpreted as the total reflection coefficient for the incident modes.

Since the group velocities \( v_{g,n} \) of the transmitted modes are, by definition, always positive, the sign of the contribution from each channel \( k_{tn} \) to the right hand side of (5.64) is determined by the factor \( \Omega(k) = \omega - e\Phi_0 - kW_0 \) evaluated at \( k_{tn} \), as we anticipated previously. When only two transmission channels are admitted, one of the two factors \( \Omega(k) \) is strictly negative while in the case of three transmission channels, two of the three factors \( \Omega(k) \) are strictly negative. Therefore, in both situations there is one root which contributes an overall negative amount to the right hand side of equation (5.64) and thus reduces the magnitude of the total reflection. Because of these troublesome modes, the right hand side is not strictly greater than 1 and we cannot straightforwardly conclude superradiance. In general, in order to fully answer the question of superradiance, one would need to specify \( W(x) \) and \( e\Phi(x) \) in all space and solve for the coefficients \( R_n \) and \( T_n \) which we will do later on using a stepfunction model for \( W \) and \( e\Phi \).

**Large \( \Lambda \) approximation: \( W_0 < 1 \)**

To better understand the relation between equation (5.64) and superradiance, we now focus on small deviations (\( \Lambda \gg 1 \)) from the linear dispersion case. For fixed \( W_0 < 1 \), the size of region I in Figure 5.9 is zero since as \( \Lambda \) increases local minima of the dispersion curve shown drops below the horizontal axis. The value of \( \Lambda \) at which this occurs is

\[
\Lambda = \frac{e\Phi_0}{\left[ \frac{2}{3}(1 - W_0) \right]^{2/3}}; \tag{5.65}
\]

which is large when \( W_0 \) is close to 1 (see the Section 5.3.3 for more details of \( \omega_{\text{crit}} \)). We therefore assume that the frequency \( \omega \) lies in region II, where three transmission and

---

8Again the properly normalised solution involves plane waves divided by a factor of \( \sqrt{\left| \Omega \right|} \) as in the zero-flow case, the difference being that now \( \Omega \) is also a function of \( k \). Hence when we come to exact solutions we multiply the un-normalised transmission amplitudes by a factor of \( \sqrt{|\Omega|/\omega} \).
two reflection channels are available (the wave function is described by equation (5.63)). As explained above, in the most general case one would need to solve the equation of motion for all \( x \) in order to conclude superradiance or not from equation (5.64).

Similarly to the zero flow case, by looking at the series expansions of the relevant wavenumbers, we can make useful predictions about the transmission coefficients in this \( \Lambda \gg 1 \) regime.

We start the analysis by solving equation (5.57) in the asymptotic region \( x \to +\infty \) and expressing the obtained transmitted wavenumbers as power series in \( \Lambda \):

\[
k_{t1} = \frac{\omega - e\Phi_0}{1 + W} + \frac{1}{2} \frac{(\omega - e\Phi_0)^3}{(1 + W)^4} \Lambda^{-2} + \mathcal{O}(\Lambda^{-4}),
\]

\[
k_{t2,3} = \pm \Lambda \sqrt{1 - W_0^2} + \frac{W_0 \omega - e\Phi_0}{1 - W_0^2} + \mathcal{O}(\Lambda^{-1}).
\]

The series expansions of the incident wavenumber \( k_{in} \) and of the reflected wavenumbers, \( k_{r1} \) and \( k_{r2} \), are given, as previously, by equation (5.39). From these \( \Lambda \) expansions, we note that, unless \( W_0 \sim 1 \), we have \( k_{t2,3}, k_{r2} \sim \mathcal{O}(\Lambda) \) and \( k_{in}, k_{r1} \sim \mathcal{O}(\Lambda^0) \). However, unlike in the zero-flow case, the difference between the high momentum transmitted modes \( k_{t2,3} \) and the high momentum reflected modes \( k_{r2} \) is not negligible for \( \Lambda \gg 1 \), being of order \( \mathcal{O}(\Lambda) \) due to the difference of \( W_0 \) from zero. Hence, the appearance of such modes in our system requires a large momentum change. Therefore, we expect the conversion of incident modes \( k_{in} \) into transmitted modes \( k_{t2} \) and \( k_{t3} \) and into reflected modes \( k_{r2} \) to be disfavored in our system in comparison with the low momentum transmission/reflection channel involving \( k_{t1} \) and \( k_{r1} \). In other words, we expect (and will give evidence for below) that the coefficients \( R_2 \) and \( T_2 \) will be much smaller than \( R_1 \) and \( T_1 \), and that the Wronskian condition (5.64) will include only the low momentum channel, i.e.

\[
|R_1|^2 = 1 - \frac{\omega - e\Phi_0}{\omega(1 + W_0)}|T_1|^2.
\]

From this relation for the reflection coefficient, we would conclude that superradiance occurs for all frequencies in region II (\( \omega_{\text{crit}} < \omega < e\Phi_0 \)) when \( \Lambda \gg 1 \). Note that this conclusion certainly does not hold in the case of a general dispersive parameter. If the condition \( \Lambda \gg 1 \) is not satisfied, there can be a mixture between the high and low momentum channels. Consequently, as discussed before, the total reflection coefficient given by equation (5.64) is not necessarily larger than one. In such a case, a definite answer about superradiance can only be obtained by solving the differential equations at every spatial point \( x \). The results of doing exactly this for a simplified step-function model are given below in Section 5.3.5.

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\(^9\)See Appendix B.3 for the details of the expansion for these roots and all other expansions appearing below.
Large $\Lambda$ approximation: $W_0 > 1$

Having analyzed the case of $W_0 < 1$, let us now fix $W_0 > 1$ and assume $0 < \omega < e\Phi_0$. As discussed previously, there are two transmission and two reflection channels available when $W_0 > 1$ and the scattering problem is now described by equation (5.62). Since we are interested in the regime of large $\Lambda$, we calculate the wavenumber of the transmitted modes up to next-to-leading order terms,

$$k_{t_1} = \frac{\omega - e\Phi_0}{1 + W} + \frac{1}{2} \frac{(\omega - e\Phi_0)^3}{(1 + W)^4} \Lambda^{-2} + O(\Lambda^{-4}),$$

$$k_{t_2} = -\frac{\omega - e\Phi_0}{W - 1} - \frac{1}{2} \frac{(\omega - e\Phi_0)^3}{(W - 1)^4} \Lambda^{-2} + O(\Lambda^{-4}).$$

(5.69)

By direct substitution of these expressions into equation (5.64), one can straightforwardly determine the reflection coefficient in powers of $\Lambda$,

$$|R|^2 + \left| \frac{k_{r_2}}{k_{in}} \right|^2 |R_2|^2 = 1 - \frac{\omega - e\Phi_0}{\omega} \left( |T_1|^2 - |T_2|^2 \right)$$

(5.70)

plus terms of $O(\Lambda^{-2})$. Note that this second channel $T_2$ is present even in the absence of dispersion, being an upstream mode which is swept downstream by a super-sonic flow. From the equation above, it also becomes evident that the relation between the norms $|T_1|$ and $|T_2|$ of the two transmission channels determines the occurrence or not of superradiance. We will show in Section 5.3.3 how the relative magnitudes of these two transmission coefficients depend on the value of the asymptotic flow rate $W_0$.

The critical case

An interesting situation to be analyzed is the critical case $\omega = \omega_{crit}$, which corresponds to the boundary between regions I and II in Figure 5.9. In this scenario, the background flow $W_0$, when expanded in powers of $\Lambda$, relates to the critical frequency according to the following expression,

$$W_0 = 1 - \frac{3}{2} (\omega_{crit} - e\Phi_0) \frac{2}{3} \Lambda^{-\frac{3}{4}} + O \left( \Lambda^{-\frac{5}{4}} \right).$$

(5.71)

Note that the previous analysis leading to equation (5.68) relied on series expansions (see equation (5.67)) which are not valid when $W_0 - 1 \sim O \left( \Lambda^{-\frac{2}{4}} \right)$. Therefore, in order to analyze the possibility of superradiance in the critical case, we cannot use equation (5.68); instead, we have to start from the original Wronskian relation, equation (5.64).

\footnote{Again, see Appendix B.3 for a derivation of these series.}
5.3. **Non-zero flow (or, curved geometry)**

**Figure 5.10:** The dispersion relation for the subluminal case in the region $x \to +\infty$ at the critical frequency. The blue line represents the effective frequency in the co-moving frame, $\Omega = \omega - e\Phi_0 - kW_0$, and the red curve represents the dispersion relation at $x \to -\infty$ in the co-moving frame. Note that $(\omega - e\Phi_0) / \Lambda$ has to be sufficiently large in order to guarantee that $k_{t2}$ be located in the top semi-plane. Otherwise, $k_{t2}$ would correspond to a left-moving mode in the interacting region and would not be included in a scattering solution.

In the critical regime, the dispersion relation (see Figure 5.10) has three distinct solutions. Two of these solutions, denoted by $k_{t1}$ and $k_{t2}$, have positive group velocities in the lab frame and, therefore, are identified as transmitted modes. The other solution, denoted by $k_0$, is a degenerate double root and, consequently, has a vanishing group velocity in the fluid frame. In order to obtain the reflection coefficient for the scattering problem, we first expand $k_{t1}$ and $k_{t2}$ as power series in $\Lambda$,

\[
k_{t1} = \frac{\omega - e\Phi_0}{2} - \frac{3}{8} [-(\omega - e\Phi_0)]^{\frac{3}{2}} \Lambda^{-\frac{3}{2}} + O \left( \Lambda^{-\frac{3}{2}} \right),
\]

\[
k_{t2} = 2 \left[-(\omega - e\Phi_0)\right]^{\frac{1}{2}} \Lambda^{\frac{1}{2}} + O(1),
\]

and then substitute the obtained expressions into equation (5.64). The final result is given by

\[
|R_1|^2 + \left| \frac{k_{r2}}{k_{in}} \right| |R_2|^2 = 1 - \frac{\omega - e\Phi_0}{\omega} \left( |T_1|^2 - 3 |T_2|^2 \right),
\]

plus terms of order $O \left( \Lambda^{-\frac{3}{2}} \right)$.

Observe again the importance of the relative sign of the norm in the two transmission channels. Note, however, that the coefficient $|T_2|$ is related to the probability of
the initial incident mode (with wavenumber $k_{in} \approx \omega + \mathcal{O}(\Lambda^{-1})$) to be converted into a transmitted mode of large wavenumber $k_{t2} \approx \mathcal{O}(\Lambda^2)$ which can only be balanced by the high momentum reflected channel $k_{r2} \mathcal{O}(\Lambda)$. On the other hand, the coefficient $|T_1|$ is related to the probability of conversion into modes of wavenumber $k_{t1} \approx (\omega - e\Phi_0)/2$ which is comparable to the low momentum $k_{r1}$ channel. We, therefore, expect the second channel to be negligible in comparison with the first one, i.e. $|T_2| \ll |T_1|$. Since $\omega_{crit} < e\Phi_0$, we also deduce that the right hand side of equation (5.73) would always be greater than one in the limit $\Lambda \gg 1$. In other words, low-frequency waves in the critical regime would be superradiantly scattered in our toy-model if small subluminal corrections are added to the dispersion relation. Again, this dominance of $|T_1|$ over $|T_2|$ will be demonstrated for the stepfunction model in Section 5.3.5 corroborating the conclusion of superradiance in this case.

5.3.4 Superluminal scattering

We now turn to superluminal scattering in the presence of a flow which goes from sub- to super-sonic in a finite region. The superluminal case is quite different to the subluminal one since there does not exist any notion of a horizon or mode-independent blocking region for high energy incident modes (the group velocity $d\omega/dk$ is unbounded as a function of $k$ and only the low frequency modes which possess quasi-linear dispersion experience a blocking region in such flows). We will follow the standard treatment [134] of analogue black holes with superluminal dispersion and analyze the transmission of an incoming wave from $-\infty$ through to $+\infty$. In principle, one could also study the scattering of an upstream supersonically propagating mode from $\infty$ back to $\pm\infty$, but we leave this analysis for a future study.

The relevant dispersion relation in the superluminal case is depicted in Figure. 5.11. Given a supersonic flow $W_0 > 1$, there exists an interval of frequencies $0 < \omega < \omega_{crit}$ (region I in Figure. 5.11) for which only two propagating modes are admitted, one right-moving and one left-moving in the lab frame. For $\omega_{crit} < \omega < e\Phi_0$ (region II in Figure. 5.11), however, there are four propagating modes, two transmitted right-movers and two left-movers. The third possibility is a subsonic flow $W_0 < 1$ with $0 < \omega < e\Phi_0$, for which there are always two propagating modes (see the blue dashed curve in fig 5.11).

Let us discuss first the cases in which only two propagating modes are available at $x \to +\infty$. Consider, therefore, the scattering of an incident wave from $-\infty$ whose frequency $\omega$ satisfies $0 < \omega < \omega_{crit}$ (if $W_0 > 1$) or $0 < \omega < e\Phi_0$ (if $W_0 < 1$). Note that exactly one of these two propagating modes is a left-moving mode. Imposing the boundary condition that no wave generator is allowed at $+\infty$, we conclude that none of these left-moving modes can be present in the scattering solution. Consequently, the solution of equation (5.50) corresponding to the scattering problem at hand is given
5.3. Non-zero flow (or, curved geometry)

Figure 5.11: Superluminal dispersion: the lab frequency as a function of wavenumber \( \omega = \pm \sqrt{k^2 + k^2/\Lambda^2 + e\Phi_0 + kW} \) when \( x \to \infty \) for two different flows, \( W_0 > 1 \) (red, solid) and \( W_0 < 1 \) (blue, dashed). The intervals of frequency indicated refer to the red curve: region I (green, \( \omega < \omega_{\text{crit}} \)), region II (blue, \( \omega_{\text{crit}} < \omega < e\Phi_0 \)) and region III (red, non-superradiant region). The intersection of horizontal straight lines with the red curve indicate propagating modes at that frequency. In both the green and blue regions we expect superradiant scattering in our model.

by,

\[
f \to \begin{cases} 
e^{ik_{in}x} + Re^{ik_{r}x}, & x \to -\infty, \\ T_1 e^{ik_{t1}x}, & x \to +\infty, \end{cases}
\]

plus negligible exponentially decaying channels. Note that the wavenumbers \( k_{\text{in}} \) and \( k_{r} \) are the same as those which appear in equation (5.83) for the superluminal \( W = 0 \) case. In addition, the wavenumber \( k_{t1} \) represents the only available transmission channel (it corresponds to the right-moving mode at \( x \to +\infty \)).

On the other hand, if \( W_0 > 1 \) and the frequency \( \omega \) of the incident wave satisfies
\( \omega_{\text{crit}} < \omega < e\Phi_0 \) (region II in fig 5.11), there are, in principle, two extra propagating channels available (four in total, as discussed above). However, because of the boundary condition imposed at \( x \to +\infty \), only one extra transmission channel is included in a scattering solution (the other extra channel is always left-moving at \( x \to +\infty \)). The scattering solution is then given by,

\[
f \to \begin{cases} 
e^{ik_{\text{in}}x} + Re^{ik_{x}x}, & x \to -\infty, \\ T_1e^{ik_{t_1}x} + T_2e^{ik_{t_2}x}, & x \to \infty, \end{cases}
\]

(5.75)

where \( k_{\text{in}} \) and \( k_x \) are again given by equation (5.83) and \( k_{t_1} \) and \( k_{t_2} \) are the wavenumbers of the transmitted modes.

Using equation (5.59) to evaluate the functional \( X \) in both asymptotic regions, we obtain, similarly to the subluminal case, the following relation between the reflection and transmission coefficients,

\[
|R|^2 = 1 - \frac{\Lambda}{\sqrt{\Lambda^2 + 4\omega^2}} \left( \sum_n \frac{v_{gn}}{k_{\text{in}}} (\omega - e\Phi_0 - k_{tn}W_0) |T_n|^2 \right),
\]

(5.76)

where the sum is over one or two transmission channels, depending on \( \omega \) and whether \( W_0 > 1 \) or \( W_0 < 1 \). Here, \( v_{g1} \) and \( v_{g2} \) are the group velocities of the transmitted modes \( k_{t_1} \) and \( k_{t_2} \), which are always positive by definition. Furthermore, it is possible to show, for frequencies \( 0 < \omega < e\Phi_0 \), that the factor \( (\omega - e\Phi_0 - k_{tn}W) \) is always negative for modes \( k_{t_1} \) and always positive for \( k_{t_2} \) modes. Therefore, since only the \( n = 1 \) transmission channel is available for frequencies lying in region I of Figure 5.11, we conclude that the right hand side of equation (5.76) is greater than 1 and, therefore, the scattering is always superradiant. The situation for \( W_0 < 1 \) and \( 0 < \omega < \omega_{\text{crit}} \) is similar: Only the first transmission channel is available and superradiance always occurs. However, for frequencies located in region II, we cannot so easily conclude superradiance since the extra transmission channel \( k_{t_2} \) contributes an overall negative factor in equation (5.76).

To obtain a conclusive answer, one would need to know the detailed structure of \( W(x) \) and \( \Phi(x) \) in the intermediate regime and solve the equations not only in the asymptotic regions but at every point \( x \).

**Large \( \Lambda \) approximation**

In order to better understand the scattering of an incident wave whose frequency is located in region II of Figure 5.11 (which only occurs for \( W_0 > 1 \)) we shall consider small deviations from linear dispersion, i.e. \( \Lambda \gg 1 \). In such a case, we can expand the two transmission channels, \( k_{t_1} \) and \( k_{t_2} \), in powers of \( \Lambda \) by solving perturbatively the
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dispersion polynomial,

\[ k_{t_1} = \frac{\omega - e\Phi_0}{1 + W} - \frac{1}{2} \frac{(\omega - e\Phi_0)^3}{(1 + W)^4} \Lambda^{-2} + O(\Lambda^{-4}), \]
\[ k_{t_2} = \frac{\omega - e\Phi_0}{W - 1} + \frac{1}{2} \frac{(\omega - e\Phi_0)^3}{(W - 1)^4} \Lambda^{-2} + O(\Lambda^{-4}), \]

Substitute the obtained wavenumbers into equation (5.76) in order to determine the reflection coefficient for the scattering,

\[ |R|^2 = 1 - \frac{\omega - e\Phi_0}{\omega} \left( |T_1|^2 - |T_2|^2 \right) + O\left(1/\Lambda^2\right). \]

Since we assume \( \omega < e\Phi_0 \) in region II, this reflection coefficient is larger than 1 whenever \( |T_2| < |T_1| \). As predicted above, whether this condition is satisfied or not in a general model would depend on the detailed structure of \( W(x) \) and \( \Phi(x) \) in the intermediate regime \cite{134} and the asymptotic value of \( W_0 \).

From a practical point of view, in order to maximise the potential for superradiance in an experiment with a superluminally dispersive medium, one should choose the asymptotic flow \( W_0 \) as small as possible while still being supersonic as this would minimise the size of region II and the extra positive-norm transmission channels therein. Choosing the flow as such to maximise the \( T_1 \) channel is also consistent with our intuition that scattering favors the channel which most closely matches the momentum of the reflected mode; in this case the wavenumber \( k_{t_1} \) is closer to \( k_r \) than \( k_{t_2} \) is. This intuition will be confirmed below when we solve exactly the stepfunction model.

The critical case

Another interesting possibility that we now consider in detail is the critical regime \( \omega = \omega_{\text{crit}} \). This situation corresponds to the boundary between regions I and II in Figure. 5.11 and is depicted, in the fluid frame, in Figure. 5.12. As we shown in Appendix B.3 the relation between the background flow \( W_0 \) and the critical frequency \( \omega_{\text{crit}} \) is given by the following expression,

\[ W_0 = 1 + \frac{3}{2} \left[ -(\omega_{\text{crit}} - e\Phi_0) \right]^{\frac{1}{2}} \Lambda^{-\frac{7}{2}} + O\left(\Lambda^{-\frac{4}{2}}\right) \]

Note that, since \( W_0 - 1 \sim O\left(\Lambda^{-\frac{3}{2}}\right) \), the approximate reflection coefficient (5.79) obtained previously is not valid in the present case (check the denominators in equation (5.90)). Consequently, we shall need different \( \Lambda \) expansions in order to obtain an appropriate expression for the reflection coefficient.

Like in the critical subluminal case, the dispersion relation has three distinct roots: the double root \( k_0 \) (with vanishing group velocity in the lab frame), a right-moving
Figure 5.12: The dispersion relation in the region $x \to \infty$. The blue line represents the effective frequency $\Omega = \omega - e\Phi_0 - kW$ of the critical frequency $\omega_{\text{crit}}$ associated with the fixed $W_0 > 1$ at which the number of transmitted channels changes from one to two.

mode (with negative group velocity in the lab frame) and a transmitted mode (with positive group velocity in the lab frame) whose wavenumber $k_t$ is given by

$$k_t = \frac{\omega - e\Phi_0}{2} \frac{3}{8} (\omega - e\Phi_0)^{\frac{5}{3}} \Lambda^{-\frac{2}{3}} + \mathcal{O}\left(\Lambda^{-\frac{4}{3}}\right).$$

(5.81)

Applying as a boundary condition the fact that only right moving modes are allowed at $+\infty$, we obtain, after substituting the relevant quantities into equation (5.76), the reflection coefficient for the scattering process,

$$|R|^2 = 1 - \frac{\omega - e\Phi_0}{\omega} |T|^2 + \mathcal{O}\left(\Lambda^{-\frac{2}{3}}\right).$$

(5.82)

Since $\omega_{\text{crit}} < e\Phi_0$, we conclude that the right hand side of the equation above is always greater than one when $\Lambda \gg 1$. Hence, superradiance is expected to occur in the superluminal critical case for small deviations from the linear regime.
5.3. Non-zero flow (or, curved geometry)

5.3.5 Some exact solutions for non-zero flow in the large $\Lambda$ approximation

In this section we collect some exact solutions to the scattering problem in the various scenarios ($W_0 > 1$, $W_0 < 1$, sub- super-luminal dispersion, different regions and the critical case) presented above.

The purpose of this section is to try to get a handle on the actual spectrum of the superradiant modes and to try to start to study the quantitative robustness of the phenomenon to modifications to the dispersion relation as well as to address our intuition from the general cases above on the relative magnitudes of the reflection and transmission coefficients. Specifically, we wish to understand the relative magnitudes of $|T_1|$ and $|T_2|$ in equations (5.70), (5.73), and (5.79) in order to conclude not super-radiance respectively in the $W_0 > 1$, and critical cases in subluminal scattering and the $W_0 > 1$ case for superluminal scattering for cases. We also give the exact solutions in the other cases for completeness. Throughout we use exclusively the stepfunction profiles for the flow $W$ and potential $e\Phi$.

The algebraic solution to the scattering problem does not depend on the actual form of the scattering momenta, only on the number of channels and consequently on the number of undetermined coefficients. Since we only have a handle on the scattering momenta as series in the dispersion parameter $\Lambda$ the results here are to be considered accurate only for moderate to mild ‘amounts’ of dispersion such that a power series in $\Lambda$ makes sense.

Subluminal models

When $W_0 > 1$ and $\omega < e\Phi_0$ the subluminal scattering involves two reflection and two transmission channels and is described by the exact solution in (5.62) which we looked at above reaching the relation (5.70). We were not able to make a general statement of superradiance in such a case due to the relative sign between the $T_1$ and $T_2$ contributions to the right hand side of that relation. Here we show that, in the step function model, indeed, $|T_1| > |T_2|$ for ‘low’ flows, implying superradiance in general there while for higher flows $|T_2|$ can dominate leading to non-superradiant scattering.

The reflected channel momenta and the incoming momentum are expressed exactly as

$$k_{in} = -k_{r_1} = \frac{\Lambda}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \frac{4\omega^2}{\Lambda^2}}}.$$  

$$k_{r_2} = \frac{\Lambda}{\sqrt{2}} \sqrt{1 + \sqrt{1 - \frac{4\omega^2}{\Lambda^2}}}.$$  

(5.83)
while the transmitted momenta are only expressible as the perturbative series
\[
\begin{align*}
  k_{t_1} &= \frac{\omega - e\Phi_0}{1 + W} + \frac{1}{2} \frac{(\omega - e\Phi_0)^3}{(1 + W)^4} \Lambda^{-2} + \mathcal{O}(\Lambda^{-4}), \\
  k_{t_2} &= \frac{\omega - e\Phi_0}{W - 1} - \frac{1}{2} \frac{(\omega - e\Phi_0)^3}{(W - 1)^4} \Lambda^{-2} + \mathcal{O}(\Lambda^{-4}).
\end{align*}
\] (5.84)

(5.85)

Imposing continuity in the first 3 derivatives and the function itself at \(x = 0\) we can solve for the reflection and transmission coefficients exactly as in the zero flow case recovering exactly (5.42) which include the resonances at \(k_{t_1} = -k_{in}\) (both \(R_1\) and \(T_1 \to \infty\)) and \(k_{t_2} = -k_{in}\) (both \(R_1\) and \(T_2 \to \infty\)). The properly normalised transmission coefficients have an additional factor of \(\sqrt{|\Omega|}/\omega\). Considering the problem at fixed \(\omega\) and \(e\Phi_0\) these resonances occur for flows
\[
W_0 = \frac{e\Phi_0}{\omega}, \quad \text{and} \quad W_0 = \frac{e\Phi_0}{\omega} - 2.
\] (5.86)

Note that we must have \(\omega < e\Phi_0\) for superradiance so these \(W_0\) lie in the intervals \((1, +\infty)\) and \((-1, +\infty)\) respectively depending on how small \(\omega\) is. Observe that these resonances are only possible due to the fact that the transmitted modes \(k_{t_i}\) are negative, allowing for some factors in the denominators to vanish.

For generic values of \(\omega\) and \(e\Phi_0\) we plot these (properly normalised) coefficients in Fig[5.13]. Note that the perturbative expansion for \(k_{t_1,2}\) breaks down for flows too close to 1.

We see that the \(T_1\) transmission is always larger than \(T_2\) until the resonance in \(T_2\) starts to take effect. After the crossover due to the resonance the \(T_2\) coefficient is always larger. Note also the superradiant reflection coefficients \(R_1\) and \(R_2\).

**Superluminal models**

For superluminal scattering at \(W_0 > 1\) the scattering solution only has one reflection coefficient
\[
f \to \begin{cases} 
  e^{ik_{in}x} + Re^{ik_{r}x}, & x \to -\infty, \\
  T_1 e^{ik_{t_1}x} + T_2 e^{ik_{t_2}x}, & x \to \infty,
\end{cases}
\] (5.87)

and we have
\[
k_{in} = -k_{r} = \frac{\Lambda}{\sqrt{2}} \sqrt{-1 + \sqrt{1 + 4\omega^2/\Lambda^2}},
\] (5.88)

and the perturbative series
\[
\begin{align*}
  k_{t_1} &= \frac{\omega - e\Phi_0}{1 + W} - \frac{1}{2} \frac{(\omega - e\Phi_0)^3}{(1 + W)^4} \Lambda^{-2} + \mathcal{O}(\Lambda^{-4}), \\
  k_{t_2} &= \frac{\omega - e\Phi_0}{W - 1} - \frac{1}{2} \frac{(\omega - e\Phi_0)^3}{(W - 1)^4} \Lambda^{-2} + \mathcal{O}(\Lambda^{-4}).
\end{align*}
\] (5.89)
(5.90)
5.3. **Non-zero flow (or, curved geometry)**

Figure 5.13: Here we chose $\omega = 0.1$, $e\Phi_0 = 0.4$ and plot the coefficients $|R_{1,2}|$ and $|T_{1,2}|$ as a function of asymptotic flow velocity $W_0$ ($R_2$ is only barely visible along the horizontal axis). In these plots we choose $\Lambda = 1$.

Again, this case was considered above where we reached the inconclusive relation (5.79) showing that the occurrence of superradiance depends on the relative magnitudes of the coefficients $|T_1|$ and $|T_2|$. Here we show for the stepfunction solution that similarly to the subluminal exact solution above, we have $|T_1| > |T_2|$ for ow flows until a resonance at higher flows in $T_2$ leads to non-superradiant scattering.

The exact solution for the coefficients obtained by matching the first 2 derivatives and the function at $x = 0$ is

\[
R = -\frac{(-k_{t_2} + k_{in})(k_{in} - k_{t_1})}{(-k_{in} - k_{t_2})(-k_{in} - k_{t_1})} \\
T_1 = -\frac{2(-k_{t_2} + k_{in})k_{in}}{(k_{t_1} - k_{t_2})(-k_{in} - k_{t_1})} \\
T_2 = \frac{2(k_{in} - k_{t_1})k_{in}}{(k_{t_1} - k_{t_2})(-k_{in} - k_{t_2})}
\]

where we note again the poles which occur at the same place as in the subluminal case (in the term independent of $\Lambda$ in the series).

This (normalised) solution is shown in Fig. 5.14 which is almost identical to the corresponding $W_0 > 1$ subluminal plot in Figure 5.13. Notice again that $|T_1|$ is greater than $|T_2|$ until the resonance in $|T_2|$ which occurs at $W_0 = e\Phi_0/\omega$. The resonance in
Figure 5.14: The reflection and transmission coefficients $|R|$ and $|T_{1,2}|$ at fixed $\omega = 0.1$ $e\Phi_0 = 0.4$, $\Lambda = 1$ plotted as a function of asymptotic flow velocity.

$|T_1|$ can be made to be absent from this analysis if we choose $e\phi_0/\omega < 2$. Also note that after the resonance in $|T_2|$ we always have $|T_2| > |T_1|$.

### 5.3.6 Loose ends

Here we collect a few odds and ends which have been left open in the analysis above.

#### Minimal flow condition

A number of thorny issues arise when generalizing the discussion of superradiance to non-zero flows in the subluminal case. One might have noticed that our model with non-zero $W_0 < 1$ flow and three transmission channels is not continuously connected to the zero flow case in which there were only two transmission channels available. Consider, therefore, Figure 5.10 and the nature of the second transmitted mode $k_{t_2}$ in the region $x \to +\infty$. If we follow this mode backwards in time (i.e. we move towards the region $x \to -\infty$), the slope of the blue line $\Omega = \omega - e\Phi(x) - kW(x)$ in Figure 5.10 will eventually decrease. As a result, depending on how $W(x)$ behaves as we move towards negative $x$, the group velocity of this root $k_{t_2}$ might increase until it becomes divergent. This occurs when $k/\Lambda = -1$ and corresponds to a background velocity $W(x) = -(\omega - e\Phi(x))/\Lambda$. At this point it is not clear how the mode will behave anymore since, somehow, the group velocity must jump discontinuously from $+\infty$ to...
In order to avoid this peculiarity, we impose the condition that our flow should not be too small, i.e. $W(x) > -(\omega - e\Phi(x))/\Lambda$ everywhere. We point out that it is not unnatural to assume such a condition when dealing with only slight deviations from linear dispersion ($\Lambda \to \infty$). Moreover, if we wish this condition to be satisfied by all frequencies in the band of interest $0 < \omega < e\Phi_0$, then we require $W(x) > e\Phi(x)/\Lambda$. In particular, at $x \to +\infty$, this inequality becomes $W_0 > e\Phi_0/\Lambda$, which is always satisfied since we always work under the assumption that $e\Phi_0 \ll \Lambda$.

An analogous discontinuity exists in the $x \to -\infty$ region when we follow the high momentum reflected mode $k_{r_2}$ backwards in time as it propagates from $x \to -\infty$ towards the region of non-zero flow. This can be seen in Figure 5.3. As we move towards the interacting region $x \to +\infty$, the slope of the green dash-dotted line becomes negative until, possibly, the intersection labelled $k_{r_2}$ crosses the $\omega$ axis and the group velocity jumps from $-\infty$ to $+\infty$. In order to avoid this unphysical jump, we assume $W(x) > (\omega - e\Phi(x))/\Lambda$ everywhere. In particular, at $x \to -\infty$, this condition becomes $W(-\infty) > (\omega - e\Phi(-\infty))/\Lambda$. It is important to point out that we have conducted our analysis in this article without indulging this subtlety; the inequality is not satisfied in any of our models, since we always assume $e\Phi(-\infty) = 0$ and $W(-\infty) = 0$. Note, however, that choosing the flow velocity $W(x)$ to satisfy this condition would imply that the $k_{r_2}$ modes are in fact not reflected modes and, therefore, should not be present in the scattering process because of the boundary conditions imposed. The main consequence of these considerations is that, by requiring the minimal flow inequality in the non-interacting region, the second reflection coefficient $R_2$ should be absent from the left hand side of all transmission/reflection relations such as (5.64). Finally, we note that this same issue is discussed in Ref. [131] in the context of Hawking radiation. A similar inequality is needed for a Hawking signal to be present in their model.

**Inertial motion superradiance**

Throughout this paper, inspired by the usual condition for rotational superradiance $(\omega - m\Omega_h < 0)$ in the black-hole case, we have analysed only scattering problems in which the frequency of the incident mode satisfies $\omega - e\Phi_0 < 0$. However, by looking at equation (5.60), one can see that the condition for negative-norm modes is given by $\omega - e\Phi_0 - kW_0 < 0$ and, therefore, in principle an amplification of an incident wave can also occur for $\omega - e\Phi_0 > 0$ given that $\omega - e\Phi_0 - kW_0 < 0$. This kind of superradiance is indeed possible and was analysed in ref. [29], where it was dubbed inertial motion superradiance and compared with rotational superradiance. In fact, inertial motion superradiance has long been known in the literature as the anomalous Doppler effect and the condition for negative norm modes is referred to as the Ginzburg-Frank condition [135]. Several phenomena in physics, like the Vavilov-Cherenkov effect and the Mach cones (which appear in supersonic airplanes) can be understood in terms of inertial motion superradiance [29].
Inclusion of mass

In reference [30], the authors consider the electrodynamic interpretation of superradiance which in the general case includes a mass term in the superradiant equation of motion

\[
\frac{d^2 f}{dx^2} + \left[ (\omega - e\Phi_0) - (p^2 + m^2) \right] f = 0. \tag{5.94}
\]

Both a regular mass and the additional effective mass arising from the dimensional reduction procedure have been included here. The authors then proceed with the general analysis, including this mass term, concluding that superradiance indeed still occurs but the window of frequencies exhibiting superradiant scattering, which was previously \( \omega \in S := (0, e\Phi_0) \) is now modified to the so-called ‘Klein region’

\[
\sqrt{p^2 + m^2} < \omega < e\Phi_0 - \sqrt{p^2 + m^2}, \tag{5.95}
\]

which differs from \( S \) in general, particularly for low frequencies. It would be interesting to also include the effects of such a mass term in dispersive superradiance.

We have mentioned previously above and throughout this chapter that our model equation for superradiance can also be thought of as arising from a dimensional reduction in a different sense to the one of [30]. The role played by the electromagnetic potential \( e\Phi(x) \) is played by a flow velocity in an additional dimension and the reduction formally introduces a mass term modification to the remaining degrees of freedom in exactly the same way as above in (5.94) (see [132] for a discussion of this in the context of black hole radiance). This reduction could apply, for example, to the radial part of a cylindrical surface wave scattering process on a cylindrically symmetric flow profile. Therefore we can bootstrap the results of [30] to conclude that in such flowing fluid systems, the dimensionally reduced dynamics can still support superradiance in the Klein region when dispersion is neglected.

In the article [132] the authors consider the effect of such a mass term on while hole and black hole flow flow configurations in the presence of modified dispersion but without the superradiance inducing potential \( e\Phi(x) \) that we have been considering in this chapter. They study the so-called ‘undulation’, a zero-frequency but non-zero momentum solution to the dispersion relation, normally present in white hole flows [64, 136]. This is show schematically in Figure 5.15. The effect of the mass is to introduce such an undulation also in black hole flows.

Dispersion and dimensional reduction

When we introduced modified dispersion at the beginning of Section 5.2.2 we did so only after dimensionally reducing to the 1+1 dimensional problem. Alternatively, we could have introduced the modification in the higher dimensional theory, and only
then dimensionally reduced. That is, we modified the dispersion by introducing higher powers of $k$ as

$$\pm k^4 - k^2 + (\omega - kv_x - pv_y)^2 - p^2 = 0. \quad (5.96)$$

where $\pm$ refers to sub- and super-luminal dispersion respectively. On the other hand one could introduce the modified dispersion before the dimensional reduction step leading also to quartic terms in $p$ (the momentum in the direction which is integrated over) leading to

$$\pm k^4 - k^2 + (\omega - kv_x - pv_y)^2 \pm p^4 - p^2 = 0. \quad (5.97)$$

This second, more physically justified model has the interesting feature of the possibility of an imaginary effective ‘mass’ in the subluminal case represented by the term $p^2 - p^4$. For sufficiently large $p$ this term becomes negative and may lead to some interesting effects.

Figure 5.15: The quartic superluminal dispersion relation for a massive excitation without the external potential (which was responsible for superradiance), the two additional zero frequency roots are labeled as $k_u$ (for ‘undulation’).
5.4 Conclusions summary and outlook

In this chapter we have investigated the qualitative robustness of superradiance against modifications to the dispersion relation. We have studied quartic polynomial modifications both sub- and superluminal and concluded that 1D scattering of low frequency waves can be superradiantly scattered. We have derived general formulas describing such dispersive scattering in both the simple case of zero flow in the direction of scattering and the more complicated case when the flow is allowed to propagate in the scattering direction. We have seen that the inclusion of such flows introduces an unexpected complexity to the scattering process and we identified various regimes in which superradiant scattering was possible and to what extent the non-dispersive results are robust. We also began a quantitative analysis in this vein, using simplified stepfunction potentials to get a handle on the reflection and transmission spectra themselves.

In the two tables that follow we summarise the results of this chapter by stating if and when superradiant scattering occurs for the various different scenarios. As we saw and as is evident from the table, the situation is far from straightforward.

For zero flow, the analysis was quite straightforward, since we were able to conclude superradiance in the general case without the need to go to the large $\Lambda$ approximation. An interesting quantitative result we obtained by moving to exact solution to the scattering by way of using an idealised stepfunction potential, was that in fact the low momentum channel can be robust against the modifications to the dispersion for large $\Lambda$ and, furthermore, the total reflection can be enhanced by the presence of the additional degrees of freedom represented by the high momentum channels $k_r^2$ and $k_t^2$.

<table>
<thead>
<tr>
<th>Zero flow</th>
<th>General case</th>
<th>Low mom. channel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subluminal</td>
<td>Always</td>
<td>For large $\Lambda$</td>
</tr>
<tr>
<td>Superluminal</td>
<td>Always</td>
<td>Always</td>
</tr>
</tbody>
</table>

For the non-zero flow calculations, we found a much richer spectrum of scattering phenomena which we have attempted to summarise in the following table.

<table>
<thead>
<tr>
<th>Non-zero flow</th>
<th>$\omega &lt; \omega_{\text{crit}}$</th>
<th>$\omega &gt; \omega_{\text{crit}}$</th>
<th>$W_0 &gt; 1$</th>
<th>Low mom. channel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subluminal</td>
<td>For low $W_0$</td>
<td>For high $\Lambda$</td>
<td>For low $W_0$</td>
<td>For high $\Lambda$</td>
</tr>
<tr>
<td>Superluminal</td>
<td>Always</td>
<td>For low $W_0$</td>
<td>For low $W_0$</td>
<td>N/A</td>
</tr>
</tbody>
</table>

We see that, in general, the superluminal scattering problem more naturally supports superradiance, partly, it seems, due to the fact that for $W_0 < 1$ it is still a two channel scattering process as in the non-dispersive case with one transmitted and one reflected model. When $W_0 > 1$ the scattering, although not always superradiant, has no essential distinction from the non-dispersive case. The subluminal case is complex
with 4 channel scattering possible but we saw that the total reflection can be larger than the reflection in the non-dispersive case.

The present work is intimately related to recent experimental realizations of analogue black holes, some of which have even studied the Hawking emission process. In particular, Ref. [23] exhibits the first detection of the classical analogue of Hawking radiation using gravity waves in an open channel flow. One of the most important lessons to be learned from that work is that even though vorticity and viscosity effects cannot be completely removed from the experimental setup, they can be made extremely small. In fact, they can be reduced to the point that the results predicted by the irrotational and inviscid theory match the results obtained experimentally with considerable accuracy. Based on this fact, together with our present results, we expect that superradiance might occur (and be detected) in laboratory using gravity waves propagating on water. This idea is currently being investigated by the authors.

5.4.1 What can we do next?

There are several ways in which one could generalise the theory we have presented in this chapter. Firstly one could repeat the analysis for more realistic dispersion relations and in this direction we are particularly interested in the dispersion relation for gravity waves. The article [51] studies the gravity wave dispersion in detail, but not in the context of superradiance. It is given by

\[ \omega^2(k) = \left( gk + \frac{\sigma}{\rho} k^3 \right) \tanh(kh) \]
\[ = ghk^2 - \left( \frac{gh^3}{3} - \frac{\sigma h}{\rho} \right) k^4 + \mathcal{O}(k^6), \quad (5.98) \]

where \( g \) is the acceleration due to gravity and \( \sigma \) and \( \rho \) are respectively the surface tension and density of the fluid. Hence our analysis above would apply to the case of gravity waves where we identify

\[ \frac{1}{\Lambda^2} = \left| \frac{gh^3}{3} - \frac{\sigma h}{\rho} \right| \quad (5.99) \]

When capillary effects are sufficiently small \( (\sigma/\rho < gh^2/3) \), the minus sign (subluminal case) in equation (5.50) must be chosen; otherwise, for sufficiently large capillary effects, the plus sign (superluminal) is the one to be used. Additional effects due to a mixing of sub- and superluminal dispersion might be present such a case and it would be interesting to investigate further such a system particularly in the light of the possibility of experimental detection directly of superradiant scattering in gravity wave experiments (see next chapter [6]).
Alternatively one could work with other physically interesting dispersion relations such as the optical one discussed in Chapter 4.

The generalisation of the presented theory in this chapter to higher order polynomial dispersion should be expected to be difficult based on the fact that one requires a conserved Wronskian to the higher order equation of motion. However it is possible that the form of the action of such a functional be universal in the sense that the result should always be of the form

$$X \left[ \sum_n A_n e^{ik_n x} \right] = \sum_n |A_n|^2 \Omega \frac{d\omega}{dk_n},$$

for the correct generalised conserved quantity $X$. This extension can be directly checked by looking for the appropriate generalisation for 5th or 6th degree equations of motion.

Another way in which to improve on the theory presented in this chapter would be to study the interplay between the dispersive Hawking mechanism and superradiance. In order to isolate the superradiance effect one could work with analogue geometries which have vanishing surface gravity at the ‘horizon’. This can be achieved analytically, for example, with the flow profile

$$W(x) = \begin{cases} 
W_0/2(1 + \tanh(x)) & x < 0 \\
W_0/2 \left[ 1 + \tanh \left( \frac{2x_0}{\pi} \tan \left( \frac{x}{2x_0} \right) \right) \right] & 0 < x < x_0 \\
W_0 & x > x_0
\end{cases}$$

which interpolates between 0 and $f_0$, achieving $f_0$ at finite $x$ and having zero derivative there.

A slightly more ambitious extension of this work and one in which the interplay between Hawking emission and superradiance could be addressed would a numerical investigation of the exact scattering solution for different and various flow profiles and profiles for the potential $e\Phi(x)$. Such numerical work has been carried out for the Hawking emission alone in many works in the literature the main works being [60, 118, 129, 63, 127]. Such numerical studies would also be necessary to make predictions for a real superradiant spectrum in an actual experiment with a naturally occurring (and hence not idealised) flow profile.
Chapter 6

Experimental Black Holes and a Bathtub Vortex

In this chapter we will be looking more closely at the experimental side of analogue gravity and in particular on the possibility of simulating the phenomena of superradiance known to occur for rotating black holes in a draining water vortex experiment. We refer the reader back to 2.1.4 for a background discussion on superradiance in general.

The bridge between idealised theory and the real experiment we describe can be tenuous at times. For this reason in this chapter we describe how we have adopted a mixed numerical, observational and theoretical approach to the experimental problem. One of the major lessons learned from studying analogues is that many of the phenomena formulated on smooth Lorentz invariant spacetime manifolds are surprisingly robust. It is with these lessons in mind that we find ourselves looking beyond the analogue model description of superradiance when it comes to a real experiment, often finding that an analogue description is not available or at best it is just a rough heuristic guide. Nevertheless, although the experiment we shall describe has not been fully completed at this stage, we seriously expect some form of superradiance to be manifested in our experiment and an observation of it to be feasible.

In the first section we will introduce the topic of experimental analogue gravity, and collect the material of Chapter 1. In the second we discuss the theory as it applies to the draining vortex, the modifications and complications from the simple theory discussed in Section 2.3. In the third section we will describe an experimental setup and some measurements that have been made with it on the draining vortex system while in the following section we briefly summarise the partial results obtained and lessons learned so far with the experiment. We conclude with a final section outlining the future goals and plans for the experiment.

The work presented in this chapter will described in the forthcoming article 137.
6.1 Introduction

As introduced previously in 2.1.4, superradiance was first discovered for electromagnetic wave scattering in flat spacetime and only later discovered to be a more general scattering phenomenon, including scattering from rotating Kerr black holes where it was made ‘famous’ in the gravity community. In this way, indeed, superradiance has played a special role for analogue gravity, the analogue having been discovered prior to the spacetime phenomenon. Of course the separation of ‘real’ from ‘analogy’ is only semantics as we have emphasised in previous chapters but the case of superradiance is unique in that the traditional order was reversed. Nonetheless, generating a superradiant scattering process in the lab would be a simulation of the spacetime concept of ergo-region and is therefore of great interest.

In the same way that we have separated the phenomenon of ‘Hawking radiation’ from the normal concepts of spacetime physics such as ‘horizon’, ‘Lorentz invariance’ and ‘gravitational field’ superradiance does not rely on a geometric language for its definition being simply that $|R|^2 > 1$ for a (non-dispersive) scattering process where $R$ is the reflection coefficient for reflected waves from a scattering potential (see Chapter 2 for the background discussion of superradiance in general).

A zeroth order attempt to generate superradiance experimentally would be to try to simulate directly the equatorial Kerr geometry in a flowing fluid. We discussed in Chapter 1 how this is difficult and perhaps a very unnatural thing to be trying to do to a fluid. Instead, we can draw upon our lessons from studying superradiance in general and try to find some fluid configuration which supports the necessary ingredients, namely, an acoustic regime at low wavenumbers and an ergoregion. The obvious candidate flowing fluid that comes to mind is the naturally occurring draining ‘bathtub’ vortex — the cylindrical symmetric flow from a draining tub. One huge experimental benefit of using the draining bathtub vortex is that, as mentioned, it is naturally occurring so that one would not need to enter into any complex fluid manipulation techniques in order to make the fluid flow in the correct way to generate the desired ergo-region.

6.2 The bathtub vortex theory

The physics of a draining vortex fluid is extremely complex and in fact is an active area of research even in the fluid mechanics literature [138, 139]. This section is concerned with the difficult problem of describing the background flow configuration and gravity waves on top of such a vortex and to try to discover if at least some of the complexities can be smoothed over, allowing for description simple enough to provably admit superradiance.

As we described in Section 2.3 one such complication is the ubiquitous modified dispersion relation which permeates discussions of horizons in analogue systems. In
the previous section we demonstrated that superradiance is still possible for both sub- and super-luminal modifications to the linear dispersion relation.

Another major additional technical obstacle we must face in this endeavour is the fact that the (low frequency) surface wave propagation speed is not constant, owing to the curved nature of the free surface. Thus we are confronted by the double edged sword of a non-constant flow rate and non-constant wave speed. In principle, furthermore, it is not clear that we can have anything which resembles the appropriate acoustic geometry in a region large enough to allow for superradiant scattering since the analogy has been demonstrated only for small deviations of the height of the fluid \[50\].

### 6.2.1 Vortex geometry with flat free surface

As a first approximation we might wish to model the draining vortex with a constant wave speed analogue, assuming in the process a constant depth of water above the bottom since

\[
  c_s^2 = \sqrt{gh(r)} \tag{6.1}
\]

where \( h(r) \) is the surface height as a function of \( r \) and \( g \) is the acceleration due to gravity.

Of course this is a bad approximation near the center of the vortex where the free-surface becomes almost vertical and drops to the bottom of the tank. This kind of model certainly is capable of predicting superradiance but it is almost as certainly unphysical.

For example, in the Feynman Lectures on Physics \[140\] Feynman explicitly considers the shape of the free surface of a draining vortex for ‘dry water’ (irrotational and incompressible).

The free surface \( h(r) \) can be shown very simply to be defined by

\[
  (h - h_\infty) = \frac{k}{r^2} \tag{6.2}
\]

an example shown in Figure \[6.1\]. However Feynman does not mention the flow velocities and in particular the surface flow velocities so we must look to additional theory in order to find out if there can be an ergoregion in the flow at radii large enough for the ‘slowly varying surface height’ approximation holds.

In the work \[52\] Unruh studies the problem in reverse, given a constant flow velocity and a bottom profile, what does the free surface look like? Alternatively one can provide the shape of the free surface and determine the bottom profile necessary to obtain the constant flow velocity. Surface perturbations are shown to satisfy an equation of motion similar to that which we discussed in Section \[2.3\] of Chapter \[2\].
What we shall consider in our experiment is a flat bottom tank which drains naturally. Thus the free surface is determined entirely by the internal dynamics of the fluid itself and we have observed it to be extremely curved (see below in Section 6.5 for a summary of some of our preliminary observations). Thus we would like to have a model which takes into account the curvature of the surface in such a way that we can show that the surface wave speed is surpassed by the surface flow velocity in some region where our approximations are valid.

6.2.2 Axisymmetrically draining fluid: curved free surface

The analysis of [50] and Section 2.3 can be applied to the case of a radially draining cylindrically symmetric (‘vortex’) flow [131, 137].

In the general case an analogy does not exist for surface wave propagating on a general curved surface, being valid only when surface deformations are small with respect to the wave length of the perturbations one is trying to describe and when \( h'(r)/h \ll 1 \) as we discussed in 2.3 and has been reported in [50]. However, the problem of surface wave propagation is the subject of the following subsection. Firstly here, we discuss the background flow for the draining vortex.
6.2. The bathtub vortex theory

This more physically justified approach explicitly includes the curved surface of the draining vortex in the problem and we will describe how one might make take into account the next to leading order corrections to the ‘slowly varying height’ approximation.

The relevant coordinates are cylindrical polars \((r, \phi, z)\) and we are interested in cylindrically symmetric time independent flows and in particular the height as a function of \(h(r)\) and the radial and angular surface flow velocities \(v_r(r)\) and \(v_\phi(r)\) as functions of \(r\). To get these we will need to solve the full problem as a function of \((r, \phi, z)\) as we do below.

The equations of motion, again, are the continuity

\[
\nabla \cdot \mathbf{v} = 0, \quad (6.3)
\]

and the Euler equation,

\[
\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} - g \hat{z}, \quad (6.4)
\]

where \(g\), \(\mathbf{v}\) and \(p\) are, respectively, the gravitational acceleration, the velocity and the pressure of the fluid. Note that we are assuming that the gravitational force is the only external body force acting on the fluid and incompressibility (so that the density is constant) and zero vorticity.

Besides the basic equations above, we need three boundary conditions already given in Chapter 2:

1. \(v_z|_{z=0} = 0\);
2. The rate of change in the height of the fluid must be equal to the vertical velocity of the fluid at the surface,
   \[
v_z|_{z=h} = \frac{dh}{dt} \bigg|_{z=h} = \frac{\partial h}{\partial t} + (\mathbf{v}_\parallel|_{z=h} \cdot \nabla_\parallel) h; \quad (6.5)
   \]
3. The pressure must be continuous at the interface.

We know that in the case of zero vorticity

\[
v_\phi = B/r, \quad (6.6)
\]

where \(B\) is an integration constant, on general grounds and that the velocity potential \(\psi\) (where \(\mathbf{v} = \nabla \psi\)) can be expressed in terms of a single function \(\psi_0\) of \(r\) and \(\phi\) as a

---

1 The condition \(\nabla \times \mathbf{v} = 0\) implies three separate conditions, one for each component of the vector relation. We also assume the \(\phi\) dependence of all the velocities. The vanishing of the \(z\) component implies \(\partial_z v_\phi = 0\) so that \(v_\phi\) is only a function of \(r\). Then the vanishing of the \(r\) component implies \(v_\phi = r \partial_r v_\phi\) which can be solved to find \(v_\phi = B/r\).
Chapter 6. Experimental Black Holes and a Bathtub Vortex

power series

\[ \psi = \sum_n \frac{z^n}{n!} \nabla^2_{||} \psi_0. \quad (6.7) \]

where \( \nabla_{||} \) is the gradient operator restricted to the \( r, \phi \) coordinates. The \( z \) independence of \( \partial_\phi \psi = v_\phi = B/r \) this implies that \( \psi_0 \) take the form \( \psi_0 = \xi(r) + B\phi \) in order for the infinite sum to collapse to the single \( z \)-independent term. It can be shown \[50, 137\] that the velocities \( v_r \) and \( v_z \) can be written as infinite sums of powers of the derivative operator

\[ D^2_r := \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right), \quad (6.8) \]

acting on the radial part \( \xi(r) \) of the function \( \psi_0 \):

\[ v_r = \sum_n \frac{(-1)^n z^{2n}}{(2n)!} \frac{\partial}{\partial r} D^2_r \xi(r), \quad (6.9) \]

\[ v_z = -\sum_n \frac{(-1)^n z^{2n+1}}{(2n + 1)!} D^2_r \xi(r). \quad (6.10) \]

Then the equations (6.3) and (6.4) reduce to the coupled equations

\[ \sum_n \frac{(-1)^n h^{2n}}{(2n + 1)!} \frac{\partial}{\partial r} D^2_r \xi(r) = \frac{1}{2} \int_0^h v_r(r, z) \, dz = -\frac{Ah_\infty}{rh}, \quad (6.11) \]

and

\[ \sum_n \frac{(-1)^n h^{2n}}{(2n)!} \frac{\partial}{\partial r} D^2_r \xi(r) = -\frac{1}{\sqrt{1 + h'^2}} \sqrt{2g (h_\infty - h) - \frac{B^2}{r^2}} = v_r|_{z=h}. \quad (6.12) \]

where \( h_\infty \) is the asymptotic fluid height away from the vortex core and \( A \) is an integration constant.

This pair of equations for the functions \( h(r) \) and \( \xi(r) \) is highly non-linear and in general very difficult to solve without further approximations.

One quite crude approximation consists in keeping only the \( n = 0 \) terms in the sums (neglecting the \( D^2_r \) derivatives) under which \( v_r \) reduces from (6.9) to the \( z \)-independent form \( v_r = \partial_r \xi(r) \) and the integral in (6.11) becomes trivial leaving

\[ v_r = -\frac{Ah_\infty}{rh}. \quad (6.13) \]

Combining this with (6.12) we find the single equation for \( h \)

\[ \frac{Ah_\infty}{rh} = \frac{1}{\sqrt{1 + h'^2}} \sqrt{2g (h_\infty - h) - \frac{B^2}{r^2}}, \quad (6.14) \]
which, further neglecting the derivative term \( h'(r) \), admits the solution for \( h(r) \)

\[
h(r) = \frac{1}{3} \left( h_\infty - \frac{B^2}{2gr^2} \right) \left[ 1 + 2 \cos \left( \frac{\theta}{3} \right) \right], \tag{6.15}
\]

where

\[
\theta = \cos^{-1} \left[ 1 - \frac{3^3}{g} \left( \frac{Ah_\infty}{2r} \right)^2 \left( h_\infty - \frac{B^2}{2gr^2} \right)^{-3} \right]. \tag{6.16}
\]

Therefore we have the desired solution for \( h(r) \) (6.15) and both the velocities \( v_r \) (6.13) and \( v_\phi \) (6.6), albeit under a rough approximation which neglected higher derivatives of the quantities involved.

The ergoregion for this flow and surface profile, determined by the condition \( v^2 \big|_{z=h} = v_r^2 + v_\phi^2 \geq c^2 = \tilde{g}h \) (see Section 2.3 for the definition of \( \tilde{g} \)), corresponds to the interior of the surface \( r = r_E \), where \( r_E \) is given by

\[
r_E = \frac{B}{\sqrt{gh_\infty}} \sqrt{\frac{3}{2}} \sqrt{1 + \left( \frac{3A}{2B} \right)^2}. \tag{6.17}
\]

It can be shown [142] that the horizon, where \( v_r^2 = c_s^2 = g\tilde{h} \) is located at the point

\[
r_H = \frac{B}{\sqrt{gh_\infty}} \left[ 2 \left( \frac{A}{B} \right)^{\frac{3}{2}} \sinh \left( \frac{1}{3} \cos^{-1} \left( \frac{B}{A} \right) \right) \right]^{-\frac{3}{2}}. \tag{6.18}
\]

but that, somewhat surprisingly the height function \( h \) becomes singularly steep at precisely this point. Hence the surface gravity

\[
\kappa_H = \frac{1}{2} \frac{d}{dr} (c^2 - v_r^2) \big|_{r_H} = \frac{1}{2} \left( g + 2 \frac{A^2 h_\infty^2}{h_H^3} \right) \frac{dh}{dr} \big|_{r_H} = \infty. \tag{6.19}
\]

diverges at the horizon. Therefore we conclude that this model breaks down near the predicted location of the horizon.

Another reason why the solution (6.15) might be considered unphysical is that the approximation that the radial velocity \( v_r \) be independent of height is known to be a bad approximation in draining vortices. We will see later that this is due to the presence of so-called ‘Ekman pumping’ along the bottom boundary layer which carries the bulk of the flux of radial flow to the drain hole.

For illustration, including the first \( D_r^2 \) terms in equations (6.11) and (6.12) results in the set

\[
\frac{\partial \xi}{\partial r} - \frac{h^2}{3!} \frac{\partial}{\partial r} D_r^3 \xi(r) = -\frac{Ah_\infty}{rH}, \tag{6.20}
\]

\[
\frac{\partial \xi}{\partial r} - \frac{h^2}{2!} \frac{\partial}{\partial r} D_r^2 \xi(r) = -\frac{1}{\sqrt{1 + h^2}} \sqrt{2g (h_\infty - h) - \frac{B^2}{r^2}}, \tag{6.21}
\]
A more complete analysis would include these corrections but, as we see here, these modifications result in a much more complicated set of coupled differential equations which should be solved in order to find $h(r)$ and $\xi(r)$.

### 6.2.3 Equation of motion for perturbations

Assuming that the radial flow is given by the form (6.13) and ignoring for a moment the infinite surface gravity in this approximation we can show that superradiance is predicted within this model.

With a flow profile and velocity field described by $A$, $B$ and $h(r)$ we can immediately use the results of [50], given also in Section 2.3 above to write down the metric and wave equation

$$\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \delta \psi) = 0. \quad (6.22)$$

For fixed frequency $\omega$ and angular eigenvalue $m$ the equation of motion for radial part $R(r)$ of the wave $R(r)e^{-i\omega t}e^{im\phi}$ can be written as

$$\frac{d^2 R}{dr^2} + P(r) \frac{dR}{dr} + Q(r) R = 0, \quad (6.23)$$

for some complicated functions $P$ and $Q$. We can use the results of Appendix A.1 to manipulate this into the form

$$\frac{d^2 f}{d\tilde{r}^2} + \Omega^2(\tilde{r}) f = 0 \quad (6.24)$$

similar to what we saw in Chapter 5 where $\tilde{r}$ is a modified ‘tortoise-like’ radial coordinate, $c_\infty$ and $c_H$ are the wave speeds at $+\infty$ the horizon respectively. The function $\Omega$ satisfies the asymptotic relations

$$\Omega^2 = \begin{cases} \frac{\omega^2}{c_\infty^2} & \tilde{r} \to +\infty \quad (r \to +\infty) \\ \left(\frac{\omega - \frac{mB}{r_H}}{c_H^2} \right) \frac{\omega}{c_\infty^2} & \tilde{r} \to -\infty \quad (r \to r_H) \end{cases} \quad (6.25)$$

of the form required in order to give rise to superradiance [141, 137]. Using the results presented in Section 2.1.4 we can conclude that, superradiance is present in the draining vortex system as long as the approximations leading to the metric are valid. These are: small variation of the surface profile, long wavelength surface waves (linear dispersion). Moreover the results presented in Chapter 5 strongly suggest that this superradiance is robust against modifications to the dispersion relation. Such modifications are necessarily present in the system, the full dispersion relation for surface
waves given by

\[ \omega^2(k) = \left( gk + \frac{\sigma}{\rho} k^3 \right) \tanh(kh) \]
\[ = ghk^2 - \left( \frac{gh^3}{3} - \frac{\sigma h}{\rho} \right) k^4 + \mathcal{O}(k^6) , \]  

(6.26)

### 6.2.4 Non-linear fluid dynamics of a draining vortex

Here we collect some results concerning the non-linear dynamics of a draining water vortex and compare them with our observations and the linear (and first order corrections) material developed above.

This material is supposed to be a sobering reminder of the non-linear nature of draining fluid vortices and the difficulty of simulating nice acoustic geometries with them.

The time independent and incompressible flow Navier Stokes equations are given by

\[ (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} - 2\Omega \times \mathbf{u} \; ; \quad \nabla \cdot \mathbf{u} = 0 \]  

(6.27)

where the various terms arise from competitive effects and are relevant in different parts of a draining vortex.

The draining water vortex is surprisingly complex. The flow can be roughly divided into various boundary layers, the central core and the bulk flow, indicated in Figure 6.2. These different regions are described by different approximation of the Navier Stokes equations.

**Different regions of the flow**

- **Ekman layer**

  The Ekman layer exists on the bottom of the tank and consists mainly of a radial flow pattern. It is parameterized by the thickness \( \delta = \sqrt{\nu/\Omega} \) where \( \nu \) is the kinematic viscosity, \( \Omega \) is the rotational velocity of the container and \( L \) is the size of the system.

  The Ekman number (ratio between viscous and coriolis terms in (6.27))

  \[ E_k = \frac{\nu}{2\Omega L^2} \]  

(6.28)

defines regions where this layer exists.
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Figure 6.2: The observed flow profile reported in [138, 139] (graphic taken from the paper [139]) with the Ekman upwelling region shown near the bottom of the tank near the vortex core. Note that in our experiments the central air core extended all the way down to the bottom of the tank as shown schematically in fig 6.3.

- Surface boundary layer
  It is the surface layer that we are most interested in due to the gravitational analogy for surface waves.

- Stewartson layer
  The stewartson layer exists near the boundary of the circular container and is parametrized by $LE_k^{1/4}$ where $E_k$ is the Ekman number.

Ekman Layer and pumping

In [138, 139] the flow profile in the bulk of a steady state draining vortex is investigated numerically and experimentally. The authors observe a particular boundary layer effect known as Ekman pumping whereby in certain parameter ranges the majority of the radial flow has its origin in the bottom Ekman layer of the draining fluid as shown in Fig 6.2. This Ekman layer was observed to ‘well up’ nearer the center of the vortex generating a vertical flow which consequently convets into a strong downflow near the vortex core. In the so-called linear Ekman theory valid for small Rossby number\(^2\) the upwelling (or Ekman pumping) is zero when the vorticity vanishes. Hence this upwelling is recognized as a non-linear effect due to the breakdown of the linear Ekman description and not necessarily because of the onset of vorticity near the core. The observed flow structure is diagrammatically shown in Fig. 6.2 which also labels the

\(^2\)Recall that the Rossby number is defined as the ratio between the non-linear and Coriolis terms in the Navier Stokes equations and generally scales as $R = U/(2\Omega L^2)$ where $U$ is a typical velocity, $\Omega$ is the angular velocity and $L$ is a typical scale of the system.
other boundary layers and the main bulk region which is shown to be in geostrophic balance with a $z$ independent velocity flow profile.

### 6.3 Method to experimentally determine superradiance

The traditional superradiant scattering process is in terms of the radial part of a wave form. Generating a circular symmetric wave is subtle and measuring the circular amplitudes in all angular modes a complex data analysis task. From the theory side, ideally one would like to make a clean prediction for the superradiant (or not) spectrum. This is difficult for two reasons

- The background flow and profile is not sufficiently well understood theoretically to be able to conclude absolutely that superradiance should occur for the kinds of draining vortices that naturally occur in bathtubs – for example we are not sure that there is a horizon or ergo-region at all in these vortices (see Section 6.2). Hence we are forced to make certain (sometimes drastic) approximations and simplifications in order to describe the background itself.

- More importantly (perhaps), the exact equation of motion for the propagation of perturbations on such background is difficult to obtain due to the complex nature of the excitations themselves on the highly curved free surface. Again we are forced to make approximations in order to obtain a tractable second order wave equation for the perturbations with a domain of validity which is rather unclear due to our lack of knowledge of the bulk background flow.

It is our expectation (hope!) that the approximations which lead to a tractable and superradiant predicting equation for the perturbations are valid in a region large enough to include an ergo-region for long wavelength modes.

In the initial stages of the experimental effort we have chosen to focus on the theory described in Section 6.2 with the approximations described there using the flow velocities $v_r$ and $v_\phi$ proportional to $1/r$ — experimentally match the observed flows to these functional forms by optimizing the parameters $A$ and $B$ (see (6.13) and (6.6)) to fit the data. However, due to the divergent surface gravity of the horizon in that model, we choose to abandon the derived form of the surface profile and instead measure the surface directly in the lab.

That is, in order to overcome the manifold difficulties towards a clean theoretical or clean experimental result, we use a combined numerical, observational and analytic method to determine superradiant scattering or not from the draining vortex:

---

3Recall that a rotating fluid is in geostrophic balance if the dominant balance is between Coriolis and pressure forces. In general, geostrophic balance implies a low Rossby number.
Set up stationary draining vortex
\[\Downarrow\]
Measure free surface profile \( h(r) \)
\[\Downarrow\]
Measure surface flow velocities \( A \) and \( B \)
\[\Downarrow\]
Implement \( A \), \( B \) and \( h(r) \) in a numerical routine simulating the scattering event computing \( R \) and \( T \)

In this way, the experiment combines an interesting mix of interdisciplinary physics with an interesting mix of interdisciplinary methods.

6.4 The laboratory

In this section we describe the experimental procedure we followed to determine the background free surface profile and the radial and angular surface flow velocities. We carried out experiments in three separate phases using two different experimental setups. The second and third experiments were done one year after the first.

6.4.1 Experimental Setup

For this experiment we have used a tabletop sized plastic circular bucket of radius at the base 30.1 cm and slant height (see below) 27.7 cm. The bucket walls were not vertical but instead were slightly angled outwards such that the area of the circular horizontal cross sections of the bucket increased with vertical height. Water was pumped in from a piece of plastic tubing of flow radius of approximately 1 cm and length approximately 400 cm which penetrated the surface of the water and was fixed to the bottom of the bucket with strong double sided tape such that the open end directed the water in an angular direction without a radial component giving the fluid angular momentum, see Fig. 6.3. The other end of the tube was fixed to a regular indoor house tap by forcing it over the tap, forming a watertight seal. To the open end of the tube was affixed a woven metal wire mesh to minimize turbulence at the inlet. In the bottom of the bucket a hole was cut in the centre of the bucket floor of diameter 2.9 cm.
6.5. The results so far

The results so far collected are preliminary and served mainly as a testing ground to fine-tune our experimental setup and data collection methods.
The data analysis for both the sequence of images taken from the side and the movie from above were carried out in Matlab.

The data analysis from the photographs of the free surface and movies of the test particles was carried out mainly by Silke Weinfurtner while the numerical code simulating the scattering process was written by Jason Penner.

Free surface profile

For each experimental run we recorded 10 profile images of the surface height function. In every image we applied several steps to determine the free surface profile shown in Figure 6.4.

These smooth fits were obtained from a non-linear regression optimization procedure on a three dimensional space of functions (each of the real number parameters shown were in fact given to a larger precision than two decimal places in the optimization). Note that the function space used in experimental run 4 differs from those used in the other three experiments in that a tanh function is used in place of the arctan shape. This was necessary in order to obtain a reasonable fit to the data but, as evidenced by a residuum data from that experiment (not shown here), even this space was not able to approximate the free surface to the same accuracy as was possible for the other three experiments. One possible explanation for this discrepancy is that exp. 4, being the shallowest experiment we performed, may be described by a flow dynamics in which viscosity effects are non-negligible or at least play a more significant role in the dynamics than in the other three experiments. An important remark is that the functions that best fit the experimental data are different from the theoretical profile obtained, see eq. (6.15). This accounts for the fact that the fluid flow in the experiments wasn’t completely irrotational and inviscid, as assumed in the theoretical analysis. Other contributing factor for this difference is that the slope of the height profile is not always negligible.

Our observations of the bulk flow

As a preliminary investigation we have performed a series of experiments observing the surface flow velocities and free surface (described below) while also making some observations of the bulk fluid dynamics by inserting colored dyes into the fluid at various radii. We have observed a stagnant attractor region internally surrounding the vortex.

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4The water circuit in our experiments was not recycled, being allowed to drain into the sink after flowing out the bottom of the tank, such that standard methods, i.e. illumination of free surface in water with uniformly distributed fluorescent dye, could not be used. Instead, we worked with emerged Kaliroscope flakes, and a more involved data analysis procedure to measure the water profile. Any further analysis, such as studying the actual free surface excitations, the setup needs considerable improved with this respect.
6.5. The results so far

Figure 6.4: The free surface data (in colored dots) superimposed with the result of a non-linear regression curve fitting procedure to determine an analytic form for the free surface. The discrete data is fit here by a smooth functions used in our numerical code to calculate superradiant scattering amplitudes.

and beneath the surface; the dyes coalesced in a torus shaped internal region and did not dilute even over time scales of minutes while the fluid was in a stationary phase with draining and inflow. This observation is schematically represented in fig 6.5. Such a region is reminiscent of a similar stagnant region shown to exist in accretion disks around rotating black holes [143]. This observation also shows that the approximations leading to (6.15) are not accurate specifically the approximation that $v_r(r,z)$ is independent of height $z$.

One particularly interesting feature of this non-draining region is the fact that there appeared to be no ‘rotational’ component to the toroidal shaped flow. We observed the dye to spiral inwards radially to a minimal radius leaving well separated flow lines visible from above; the spiral flow lines did not mix and remained stable over time scales of 30 seconds and more.

In a future work we hope to investigate the manipulation of the size and character of such a region by modifying the fluid flow. For example, it is our hypothesis that the intensity of the stagnant region is related to the relative radial flow due to Ek-
man pumping and the radial flow present on the surface boundary layer. Hence by optimizing the flow parameters to a region which disfavors a strong Ekman pumping, we hope to minimize the effects of the stagnant region. Note that in the theoretical analysis presented in this article above we have assumed a simplified laminar, vorticity free internal flow structure neglecting viscosity.

**Numerical simulations**

We obtained the values of $A$ and $B$ (see equations (6.13) and (6.6) above) from measurements of the flow velocity by tracking test particles and using a particle velocimetry method. Subsequently we put these data into a code which simulates the scattering process. The output from the code is a value of $T$ and $R$ as a function of frequency $\omega$. For low $\omega$ superradiant scattering is expected to occur in line with the general understanding of superradiance (see 2.1.4).

A sample of the results from simulating the scattering process according to the data collected from the first experiment is given in Figure 6.6 where we clearly see superradiant reflection coefficients. We note that these exact spectra are found using a non-dispersive theory described in Section 6.2. We include in Figure 6.7 the true water dispersion relation for water as well as the group velocity of surface modes showing the regime in which the non-dispersive result can be trusted. In Figure 6.8 we show the results zoomed in on the non-dispersive window.
6.6 Conclusion and future directions

The series of experiments described above are ongoing at present. There have been two additional separate rounds of experimental testing performed on two different prototypes for the full scale experiment.

Figure 6.6: The numerical output (the squared reflection coefficient $|R|^2$) from a simulation of the scattering process on the vortex neglecting dispersive effects. These simulations were carried out with data collected from the experiment which included three different vortices differing in their asymptotic depths at large radii. These different vortices are labelled in this plot as exp 2 - 4 and we have also included the results from simulating the scattering of the first excited angular mode ($m = 2$) showing a smaller amount of superradiance as can be expected from (6.25). Note that the data from experiment 1 was of a lower quality (as can be seen in Figure 6.4 where the free surface data seems to incorporate some systematic error (we fund that at this radius during experiment 1 there was a reflection of the light sheet from the bottom of the tank contaminating the image of the free surface).
Figure 6.7: The true dispersion relation for water (given in (6.26)) compared to the linear dispersion used in the numerical simulations in 6.7(a) while in 6.7(b) we show the corresponding group velocities for surface waves. Note that the region in which the linear dispersion is an accurate approximation is for approximately \( \omega < 25 \).

Of course, as we saw in Figure 6.6 our experiment can be said to have succeeded in showing superradiance. However, there are good reasons to believe that both the theory used and the experimental data obtained were too rough to completely trust the result. At the very least, we can see in Figure 6.7 that the results certainly should not be trusted beyond about \( \omega = 25 \text{s}^{-1} \) and that even within this regime, the model predicts up to about 20% amplification which should be observable by the naked eye and which we did not observe. Ultimately this is why we consider the results so far only as point of principle that our methodology can be carried out. It is the goal now to improve on the details of each step of the process in order to reach a conclusive result which can be shown to be trustable.

There are a number of different scientific projects which present themselves at this stage of the experiment and we plan on pursuing as many as possible in the near future.

**Experimental** – After working with the circular bucket tank described above, we have concluded that a different design is necessary in order to produce a smoother flow. One major problem with the circular bucket was that the inflow pipe penetrated the free surface, generating turbulence and surface disturbances. Another issue was that the bucket was not completely transparent, making imaging of the free surface a difficult geometric problem. Finally, our data collection methods were crude: we used
6.6. Conclusion and future directions

These difficulties have prompted us to come up with an improved design, shown in Figure 6.9, made of transparent material with long, isolates inflow channels allowing better flow characteristics and visualisation. We have also invested in some appropriate hardware, such as a dedicated computer for data analysis which streams data realtime from a high resolution black and white mounted camera. In the more distant future we plan on constructing a large-scale version of the improved experimental setup and begin taking data from it (expected to be running early in 2013).

**Theory** – It would be interesting to investigate the theoretical problem of waves propagating on a highly curved surface such as the free surface profiles we found above and have observed in the lab. So far all theoretical analyses involve the assumption that the restoring force is the gravitational force. One can imagine that this approximation is poor when surface waves propagate into the vertical neck of the vortex core. In

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**Figure 6.8:** A zoomed in plot of some of the results of Figure 6.6 showing the behaviour in the region where we can trust the approximation which neglects dispersion.
the neck, the waves propagate in fact vertically, fighting against the downflow rather than the radial flow we have been assuming. This is partly the source of some of the divergences we have seen in the linear theory. Furthermore, in the neck region, the gravitational force is not anymore the restoring force responsible for wave-like behaviour which most likely would be dominated by centrifugal forces and surface tension likely giving rise to a highly modified dispersion relation. This at the very least would give a vastly different propagation velocity for long wavelength excitations and hence a different value for $r_H$ and hence profile of superradiance. The idea is that, the further the horizon site into the $1/r$ rotational velocity field, the higher the effective angular momentum of the horizon is and hence the larger the frequency below which superradiance should occur is. The very definition of long wavelength excitation should also change character in the neck region where the typical length scale is associated with the rotation rate and capillary effects. These two additional forces have been completely ignored in the theory above. It should be very interesting to calculate the dispersion relation for such waves and study the effect it has on superradiance.

Numerical – On the numerical side, the problem of including modified dispersion relations comes hand in hand with solving difficult higher order ODEs. It has been discussed in the literature some of the numerical difficulties of working numerically with such problems which include exponentially growing modes and the reduction of noise. We saw in Chapter that, one cannot expect, in general, for the contributions from the new, dispersive channels be small.
Conclusions and Outlook

This thesis has been about the (sometimes tortuous) path between theory and experiment in analogue gravity. The relative sophistication of the theoretical developments and experimental techniques and technologies varies markedly between the diverse analogue systems and we are often led to solve quite different problems when working on the physics of different analogue models.

In Chapter 3 we looked at analogue models in BEC which have been widely studied in the literature. The theories of dispersive Hawking radiation and DCE in BEC are well developed and the most pressing issues are in practically observing the quantum emissions. In this light the idea that quantum correlations are the right observable has arisen and several authors have looked at the structure of the quantum correlations present in a Hawking emission. We have looked at the related problem for DCE and shown that there exists a very particular signal buried in the Wightman function due to the DCE emission. We showed how BECs can be used to simulate various DCE scenarios including emissions from inhomogeneous and finite size condensates which are most similar to real BEC being currently produced experimentally.

In Chapter 4 we entered the discussion surrounding a series of optics experiments where a claim was made in the literature [91, 92] of an direct observation of Hawking radiation. We modeled the experimental situation described in those articles with a very simple construction and showed how most of the pertinent features of the observed emission were reproduced in our model. In doing so we also highlighted the difficulty of making clean claims of the observation of QFT phenomena in dispersive analogue systems since multiple and distinct physical mechanisms can simultaneously play a role.

In Chapter 5 we opened the discussion of the robustness of the phenomenon of superradiance against quartic modifications to the dispersion relation, both sub- and superluminal showing the qualitative result that superradiance is possible in both static and moving media. We also began the study of the quantitative robustness for low momentum scattering processes, showing that the spectrum is robust when the dispersive momentum scale is large. Finally, we also showed that the additional scattering channels can in fact enhance the effects of superradiance making its detection more feasible in real experiments.
In Chapter 6, we described an ongoing experimental project which aims to directly or indirectly observe superradiant scattering from a common ‘bathtub’ draining water vortex. We discussed the difficulties faced when trying to confront the real physical system with idealised theoretical work. In particular we observed that the analogy certainly breaks down near the center of the vortex and highlighted the main experimental task: to generate and measure a draining vortex which supports an analogue description (or modified dispersive version thereof) of an ergo-region sufficiently far away from the highly non-linear region of the flow. The central core region is poorly understood both in terms of the existence of an analogue there as well as the basic fluid dynamics of that region.

**Generalization vs Simulation** — In the analogue gravity literature, the original goal of simulating Lorentz invariant physics seems to have been absorbed into the wider, more far reaching goal of placing the Lorentz invariant results into a more general context including violations of the linear dispersion relation. Traditionally the philosophy of analogues has been that when we reproduce some phenomenon in an analogue we are *simulating* the true effect of curved spacetime QFT. What has come out more recently, under the guidance of Parentani (summarised in [118]) and his collaborators but also stimulated by the extremely interesting experiments done by the group of Faccio et. al. (reported in [91, 92]), is the philosophy that the most important thing we are doing is *generalising* some Lorentz invariant curved spacetime physics to the Lorentz non-invariant, non-spacetime context. We are discovering that the plethora of phenomena are really much more general processes in physics. What we are also discovering, and this particularly applies to the lessons from the optics experiments, is that some of the phenomena might even be different aspects or different ways of looking at the same underlying dispersive process. When dispersion is switched off, the effects appear to decouple, becoming quite distinct in their description and formulation. Here we cite two cases which point to this possibility: The Hawking process and superradiance both are represented as multi channel scattering processes in the presence of dispersion as discussed in Chapter 5; the vacuum radiation associated with propagating optical pulses as discussed in Chapter 4 connects the DCE and Hawking process, where moving to a different frame of reference was seen to isolate one or the other aspect (Hawking or DCE) of the vacuum radiative process.

This vision for the unification of all the various quantum vacuum radiations in dispersive theories is schematically shown in Figure 6.10. Such a unified perspective of the various quantum vacuum radiation effects is also presented in the recent article by Davies [144] but in the context of Lorentz invariant physics.

**Outlook** — Here we briefly summarise the status of the theoretical work as it pertains to analogue models for the three quantum vacuum radiations discussed in this thesis:

- **Dynamical Casimir effect** — Here the theory is well developed and largely free of complications. There exists a number of (perhaps) distinct versions of what
6.6. Conclusion and future directions

Figure 6.10: The various quantum vacuum radiations seen as different corners of a unified phenomenon.

is referred to as DCE with the ‘moving or time dependent boundary condition’ version currently receiving the most experimental attention. The spectrum is well known in many scenarios and some exact analytical results available. One difficult theoretical challenge is the incorporation of finite size effects which we addressed in Chapter ?? from within the confines of perturbation theory. The DCE theory in dispersive systems is made straightforward by the fact that it can be a ‘regular’ emission (as we have defined it) in some systems which sidesteps many (but not all) of the transplanckian problems plaguing the Hawking emission for example.

- **Superradiance** – Non-dispersive superradiance, its field theoretical formulation and the quantum spontaneous version are well understood theoretically being mature disciplines and playing a major role in the formulation of the black hole thermodynamics. The generalisation to dispersive wave scattering is very new, initiated in this thesis and motivated by a desire to experimentally observe superradiant scattering using (dispersive) gravity waves on a draining bathtub vortex. The connection with dispersive Hawking radiation will be interesting to investigate.

- **Hawking radiation** – Being one of the only predictions from semi-classical gravity, the non-dispersive Hawking emission has received wide interest from many communities interested in gravitational physics. Motivated by analogue models, the
study of dispersive Hawking radiation is also well developed for both sub- and superluminal modifications to the dispersion relation. Numerical simulations have shown a remarkable robustness of the spectrum of produced particles and certain more exotic configurations: multiple horizons, ‘time’ warps [145] and black hole lasers [146] have also been thoroughly investigated.

On the experimental side, important progress has been made towards an observation, direct or indirect of one or more of the quantum vacuum radiations. We summarise here the status of some of the current, ongoing, and planned experiments in some way related to analogue gravity:

- **BEC** — There are many groups around the world working on the experimental side of BEC. Recently Steinhauser and collaborators have made a BEC black hole. [147].

- **Gravity waves** — Already there have been important experimental results obtained concerning the dispersive Hawking effect by Weinfurtner et. al [23] as well as Rousseaux [48]. The so called ‘hydrodynamical jump’ white hole has also received experimental attention [49]. Superradiant scattering is currently under investigation. It is fair to say that the experimental situation for gravity waves on water is very positive, being very simple to work with but, however, are not suitable for an observation of the quantum spontaneous emissions.

- **Nonlinear Optics** — Experimental non-linear optical systems [93] perhaps currently provide the best chance to directly observe the quantum spontaneous emissions and in fact have already yielded observational results in this direction [91 92]. Much experimental work needs to be done on these complex and highly dispersive systems, the quantum processes within them and their ability to support analogs of the various spacetime concepts such as horizon and ergo-region. Due to the high complexity and sophistication of the technologies necessary to undertake these optics experiments, the groups doing experiments are usually interested in many problems of which analogue gravity simulations are just one.

In summary, it is fair to say that there is much work to be done, both theoretically and experimentally, but the current and recently generated interest in analogue gravity problems by a diverse range of experimental groups, allows the community to start to delete the question mark from the title of the founding analogue gravity article [1] ‘Experimental Black Hole Evaporation?’ as well as the related question marks for DCE and superradiant scattering.
Appendix A

Miscellaneous useful results

A.1 General method for getting rid of the friction term

Frequently we wish to solve some differential equation with a friction term, a term involving the first derivative of the function. In this small appendix we demonstrate how one can in general reduce such an equation to one involving only 2nd derivatives and the function. Such a reduction often results in an equation of motion which can be interpreted as a Schrodinger scattering problem in a particular potential — a class of problems with many known solutions or as an equation of motion such as (2.25) amenable to a Bogoliubov analysis. In this way one can select an appropriate solved Schrodinger potential which maps to the potential of the original frictional equation, even approximately. The idea here is that, the original equation is usually a toy model, or phenoomenological model for some phenomenon for which we have a choice of the potentials which best match the physical system we are studying.

Consider the equation of motion

$$\lambda f'' + \mu f' + \nu f = 0.$$  \hspace{1cm} (A.1)

Let the function $g$ be defined as

$$f = ge^{-\frac{1}{2} \int \mu/\lambda}.$$  \hspace{1cm} (A.2)

Then,

$$\left[ \lambda g'' + \left( \nu - \lambda \mu^2/4 - \lambda \mu'/2 \right) g \right] e^{-\frac{1}{2} \int \mu/\lambda} = 0.$$  \hspace{1cm} (A.3)

which is of the desired frictionless form.
A.2 Integrals

Here we collect some proper integrals which we use in the text

\[ \int_{0}^{\infty} \frac{u^n}{\sinh^2(u)} du = \frac{\pi^n}{n(n+1)}, \quad (A.4) \]
\[ \int_{0}^{\infty} u^2 e^{-u^2} du = \frac{\sqrt{\pi}}{4}, \quad (A.5) \]
\[ \int_{0}^{\infty} u^4 e^{-u^2} du = \frac{3\sqrt{\pi}}{8}, \quad (A.6) \]
\[ \int_{0}^{\infty} u^{3/2} e^{-\beta u^2 - \gamma u} du = \frac{3}{4} \sqrt{\pi} (2\beta)^{-5/4} e^{\gamma^2/(8\beta)} D_{-5/2} \left( \frac{\gamma}{\sqrt{2\beta}} \right), \quad (A.7) \]

where \( D \) is a parabolic cylinder function.

A.3 Gamma functions

Here we collect some facts about Gamma functions useful in the evaluation of certain Bogoliubov coefficients. Firstly

\[ \Gamma(1 + z) = z\Gamma(z), \quad (A.8) \]

coupled with \( \overline{\Gamma(z)} = \Gamma(\overline{z}) \) and Euler’s reflection formula

\[ \Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin \pi z}, \quad (A.9) \]

gives rise to

\[ |\Gamma(iA)|^2 = \frac{\pi}{A \sinh \pi A}. \quad (A.10) \]

Here we also used the definition of the hyperbolic function

\[ i \sinh z = \sin iz. \quad (A.11) \]

Another important feature of the Gamma function is that it is nowhere vanishing. This can be seen from the relation \( [A.9] \) \( \pi/\sin \pi z \) is nowhere vanishing. Finally, some identities involving \( \Gamma \) functions are given below.

\[ \Gamma(nz - n) = \frac{\Gamma(nz)}{\prod_{p=1}^{n}(nz - p)}, \quad (A.12) \]
\[ \Gamma \left( nz - n/2 \right) = \Gamma(nz - 1/2) \times \text{stuff}. \quad (A.13) \]
Appendix B

Superradiance Appendices

B.1 A pole in the scattering solution and its resolution

Here we present a smooth model for an ergo-region which is exactly soluble and for which we write down exact expressions for the reflection and transmission coefficients. This model was discussed in Chapter 5. The purpose of this section is to demonstrate that the pole uncovered in the reflection and transmission amplitudes for a non-dispersive superradiant scattering process in 1 dimension with a step function potential is an unphysical artifact of choosing the discontinuous step function.

B.1.1 The exact solution for modified and linear dispersion

The equation of motion we considered was

\[ f'' + (\omega - e\Phi(x))^2 f = 0, \tag{B.1} \]

where the potential \( e\Phi(x) \) was taken to be step function

\[ e\Phi(x) = \begin{cases} 0 & x < 0 \\ e\Phi_0 & x > 0. \end{cases} \tag{B.2} \]

Solving for the momenta in the two regions we have we have

\[ k_{\text{in}} \to \omega, \]
\[ k_r \to -\omega, \]
\[ k_t \to -|\omega - e\Phi_0|. \tag{B.3} \]
Appendix B. Superradiance Appendices

Matching the solution across the step function up to third derivatives gives a set of linear equations for the coefficients which has solution

\[
R = \frac{k_{in} - k_{t1}}{k_{in} + k_{t1}} = \frac{e\Phi_0}{2\omega - e\Phi_0},
\]
\[
T = \frac{2k_{in}}{k_{in} + k_{t1}} = \frac{2\omega}{2\omega - e\Phi_0},
\]

from which we confirm the standard and general result for the statement of superradiance

\[
|R_1|^2 = 1 - \frac{\omega - e\Phi_0}{\omega}|T_1|^2.
\]

We see that the general solution possesses a pole in both the reflected and transmitted amplitudes when the reflected momentum coincides with the transmitted one. This occurs when \( \omega = e\Phi_0/2 \). In the next section we will show that such a pole is avoided by choosing a smooth potential \( \Phi \) to replace the discontinuous step function.

### B.1.2 Smooth model

Consider the potential

\[
\Phi(x) = \begin{cases} 
\omega - \sqrt{\omega^2 - V(x)}, & x < x_m \\
\omega + \sqrt{\omega^2 - V(x)}, & x > x_m 
\end{cases}
\]

where

\[
V(x) = \frac{V_f}{2} [1 + \tanh(x/x_0)] + \frac{\Gamma}{\cosh^2(x/x_0)},
\]

and \( x_m \) is a matching point, coinciding with the local maxima \( V(x_m) \) of the function \( V \). Assume \( \omega < e\Phi_0 \). Choosing

\[
V_f = \omega^2 - (\omega - e\Phi_0)^2,
\]

and

\[
\Gamma = \frac{(e\Phi_0)^2}{4},
\]

we find that \( V(x_m) = \omega^2 \) and the function \( \Phi \) is continuous and interpolates smoothly monotonically between 0 and \( e\Phi_0 \) over a scale of the order of \( x_0 \). This corresponds to the promised smooth transition to an ergo-region. This behaviour is shown in Fig. [B.1] where we also display the auxiliary potential \( V \). In fact, it can be shown graphically (although algebraically it is somewhat of a mystery) that the function \( \Phi \) exactly corresponds to a tanh interpolation between 0 and \( e\Phi_0 \),

\[
\Phi(x) = \frac{e\Phi_0}{2} [1 + \tanh(x/x_0)].
\]
B.1. A pole in the scattering solution and its resolution

Figure B.1: The behaviour of the potential $\Phi(x)$ and the auxiliary potential $V(x)$ used in the smooth ergo-region model described in the text. Here we have used the values $\omega = 2$, $x_0 = 1$ and $e\Phi_0 = 3$. The

Crucially, the mode equation

$$\frac{d^2 f}{dx^2} + (\omega - \Phi(x))^2 f = 0,$$

is written in terms of the potential $V$ as

$$\frac{d^2 f}{dx^2} + (\omega^2 - V(x)) f = 0,$$

and it can be shown [148] that Eq. (B.12) with the potential $V$ given by the so-called Rosen-Morse form (B.7) is exactly soluble in terms of Hypergeometric functions. The transmission coefficient is given exactly by

$$|T|^2 = \frac{\sinh(\pi \omega x_0) \sinh(\pi (e\Phi_0 - \omega)x_0)}{\sinh^2(\pi e\Phi_0 x_0 / 2) + \cos^2(\pi/2 \sqrt{1 - (e\Phi_0)^2 x_0^2})}$$

(B.13)

By the general result [B.5] we have that the reflection coefficient $R$ is given by

$$|R|^2 = 1 - \frac{\omega - e\Phi_0}{\omega} |T|^2$$

(B.14)
and we see that the special role played by $\omega = e\Phi_0/2$ in the step function model now no longer applies and we are left with a bounded (and superradiant) reflection coefficient for all $\omega < e\Phi_0$. This was shown in Chapter 5 in more detail.

This result shows that the pathological pole in the reflection and transmission amplitudes for the step function model are an artifact of the unphysical discontinuous jump in the potential $\Phi$.

An interesting observation is that when $2\omega = e\Phi_0$ (exactly where the pole was present in the step function model) the Rosen-Morse potential B.7 reduces to a simple sech$^2$ potential and the transmission coefficient becomes that for the sech$^2$ model accordingly.

We note that this construction does not work for $\omega > e\Phi_0$.

### B.2 Conserved inner product with flow velocities

For a higher order equation of motion such as the one considered in Chapter 5 there still exists a conserved inner product. This was given in Chapter 5 by

$$
(\phi_1, \phi_2) = i \int_{\Sigma_t} \left[ \phi_1^* (\partial_t + iE + W\partial_x) \phi_2 - \phi_2 (\partial_t + iE + W\partial_x) \phi_1^* \right] dx.
$$

(B.15)

Here we show that this is conserved when the dispersion function $F^2$ is an even function (so that its Taylor expansion is in powers of $k^2$).

Firstly, it is clear that the potential $e\Phi(x)$ plays no role in the conservation since it cancels directly from the innerproduct.

Indeed we can write the inner product as being only over the dimension of interest in terms of the metric restricted to the $(t, x)$ plane as

$$
(\phi_1, \phi_2) = i \int dx \ n_\alpha g^{\alpha\beta} (\phi_1^* \partial_\beta \phi_2 - \phi_2^* \partial_\beta \phi_1^*) := i \int dx \ n_\alpha J^\alpha,
$$

(B.16)

so that the difference of this quantity at two times is given by the directed integral over the spacetime volume contained between the two time slices. By Stoke’s theorem this is equal to the integral over the spacetime volume

$$
(\phi_1, \phi_2)_{t_2} - (\phi_1, \phi_2)_{t_1} = \int_V \nabla_\alpha J^\alpha = \int_V \partial_\alpha J^\alpha,
$$

(B.17)

since the determinant of the metric is trivial. The integrand is

$$
\partial_\alpha J^\alpha = \partial_\alpha \left( g^{\alpha\beta} (\phi_1^* \partial_\beta \phi_2 - \phi_2^* \partial_\beta \phi_1^*) \right)
= \phi_1^* \left[ (\partial_t + \partial_x v) (\partial_t + \partial_x v) - \partial_x^2 \right] \phi_2 - \phi_2 \left[ (\partial_t + \partial_x v) (\partial_t + \partial_x v) - \partial_x^2 \right] \phi_1^*
= \phi_1^* \partial_x^2 \phi_2 - \phi_2 \partial_x^2 \phi_1^*.
$$

(B.18)
B.3. Perturbation theory for the roots of the modified dispersion relations

Note that this last line would be zero in the non-dispersive case. However, under the integral this term does not contribute upon integration twice by parts whence

$$\int \phi_1^* \partial_x^4 \phi_2 - \phi_2 \partial_x^4 \phi_1^* = \int (\partial_x^2 \phi_1^*)(\partial_x^2 \phi_2) - (\partial_x^2 \phi_2)(\partial_x^2 \phi_1^*) = 0$$  \hspace{1cm} (B.19)

This argument does not suffice when the dispersion function (squared) cannot be written as an expansion in even powers of the derivative operator since this last cancellation would then not be possible with an even splitting of the derivatives across the two fields.

B.3 Perturbation theory for the roots of the modified dispersion relations

The dispersion relation for the non-zero flow model in the asymptotic regions is given by

$$k^2 \mp k^4 \Lambda^2 = \begin{cases} \omega^2 & x \to -\infty \\ (\omega - e\Phi_0 - kW)^2 & x \to \infty \end{cases}$$  \hspace{1cm} (B.20)

To solve these relations for $k$ in terms of the other parameters of the problem it is best to move to the dimensionless variable $z = k/\Lambda$. Furthermore we make use of the small parameters $\epsilon = \omega/\Lambda$ and $\tilde{\epsilon} = (\omega - e\Phi_0)/\Lambda$ which are small by the assumption that we are well outside of the region where the dispersion relation has no real roots (near the top of the figure 8) or alternatively in the physical regime of low frequencies with respect to the dispersive scale. In terms of these variables the dispersion relation becomes

$$z^2 \mp z^4 = \begin{cases} \epsilon^2 & x \to -\infty \\ (\tilde{\epsilon} - zW)^2 & x \to \infty \end{cases}.$$  \hspace{1cm} (B.21)

Restricting to the subluminal case we have in the ‘free’ region (where the potentials and velocities vanish, the $x \to -\infty$ region) four roots expressed perturbatively up to $O(\epsilon^3)$ as

$$z_- = -1 + \frac{1}{2} \epsilon^2$$  \hspace{1cm} (B.22)
$$z_{-s} = -\epsilon$$  \hspace{1cm} (B.23)
$$z_{+s} = \epsilon$$  \hspace{1cm} (B.24)
$$z_+ = 1 - \frac{1}{2} \epsilon^2$$  \hspace{1cm} (B.25)

where we have labeled the roots as either positive + or negative - with the extra subscript ‘s’ to indicate the smaller (in absolute value) roots. It s the ‘s’ roots which
exist in a non-dispersive treatment, the other two roots appearing due to the modified quartic equation of motion and resulting quartic polynomial dispersion relation.

In terms of the problem parameters in the non-interacting region these roots are

\[ k_- = -\Lambda + \frac{1}{2\Lambda}\omega^2 \quad (B.26) \]
\[ k_{-s} = -\omega \quad (B.27) \]
\[ k_{+s} = \omega \quad (B.28) \]
\[ k_+ = \Lambda - \frac{1}{2\Lambda}\omega^2 \quad (B.29) \]

which agrees, as it should, with the expansion of the exact solutions in the free region

\[ k_- = -\frac{\Lambda}{\sqrt{2}}\sqrt{1 + \sqrt{1 - \frac{4\omega^2}{\Lambda^2}}} \quad (B.30) \]
\[ k_{-s} = -\frac{\Lambda}{\sqrt{2}}\sqrt{1 - \sqrt{1 - \frac{4\omega^2}{\Lambda^2}}} \quad (B.31) \]
\[ k_{+s} = \frac{\Lambda}{\sqrt{2}}\sqrt{1 - \sqrt{1 - \frac{4\omega^2}{\Lambda^2}}} \quad (B.32) \]
\[ k_+ = \frac{\Lambda}{\sqrt{2}}\sqrt{1 + \sqrt{1 - \frac{4\omega^2}{\Lambda^2}}} \quad (B.33) \]

when expanding in powers of \( \epsilon = \omega/\Lambda \).

In the interacting region, for \( W < 1 \) one finds

\[ z_- = -\sqrt{1 - W^2} + \tilde{\epsilon} \frac{W}{1 - W^2} + \frac{\tilde{\epsilon}^2}{2} \frac{2W^2 + 1}{(1 - W^2)^{5/2}} \quad (B.34) \]
\[ z_{-s} = -\tilde{\epsilon} \frac{1}{1 - W} \quad (B.35) \]
\[ z_{+s} = \tilde{\epsilon} \frac{1}{1 + W} \quad (B.36) \]
\[ z_+ = \sqrt{1 - W^2} + \tilde{\epsilon} \frac{W}{1 - W^2} - \frac{\tilde{\epsilon}^2}{2} \frac{2W^2 + 1}{(1 - W^2)^{5/2}} \quad (B.37) \]

It is rather illuminating to re-write this expansion in terms of the variable \( \lambda = \)
B.3. Perturbation theory for the roots of the modified dispersion relations

\[ \tilde{c}/(1 - W^2)^{3/2} \text{ and } q = z/\sqrt{1 - W^2} \text{ as} \]

\[ q_- = -1 + W\lambda + (2W^2 + 1)\frac{\lambda^2}{2} \quad (B.38) \]

\[ q_{-s} = -(1 + W)\lambda \quad (B.39) \]

\[ q_{+s} = (1 - W)\lambda \quad (B.40) \]

\[ q_+ = 1 + W\lambda - (2W^2 + 1)\frac{\lambda^2}{2} \quad (B.41) \]

In terms of these variables the dispersion relation reads

\[ q^2 - q^4 = (1 - W^2)\lambda^2 - 2Wq\lambda \quad (B.42) \]

where now \( \lambda \) is the small parameter. This representation shows us that the relevant small parameter governing the asymptotics is \( \lambda \) and not \( \epsilon \). This is also what is found in the article [118] where a cubic dispersion relation is used.

In any case, in the physical variables the roots are approximated by the series

\[ k_- = -\Lambda\sqrt{1 - W^2} + \frac{W(\omega - e\Phi_0)}{1 - W^2} + \frac{(2W^2 + 1)(\omega - e\Phi_0)^2}{2\Lambda(1 - W^2)^{5/2}} \quad (B.43) \]

\[ k_{-s} = -\frac{(\omega - e\Phi_0)}{1 - W} \quad (B.44) \]

\[ k_{+s} = \frac{(\omega - e\Phi_0)}{1 + W} \quad (B.45) \]

\[ k_+ = \Lambda\sqrt{1 - W^2} + \frac{W(\omega - e\Phi_0)}{1 - W^2} - \frac{(2W^2 + 1)(\omega - e\Phi_0)^2}{2\Lambda(1 - W^2)^{5/2}} \quad (B.46) \]

### B.3.1 Critical case

Let us consider the case of a subluminal dispersion. By taking the derivative of the expression \([B.20]\) with respect to \( k \), we obtain the group velocity of the modes in the lab frame as a function of the wavenumber \( k \),

\[ v_g(k) = \frac{\partial \omega}{\partial k} = W_0 \pm \frac{1 - 2\frac{k^2}{\Lambda^2}}{\sqrt{1 - \frac{k^2}{\Lambda^2}}} = W_0 + \frac{k - 2\frac{k^3}{\Lambda^2}}{\omega - e\Phi_0 - kW_0}. \quad (B.47) \]

In principle, the two equations \( v_g(k) = 0 \) and the dispersion relation \([B.20]\) are sufficient to obtain \( k_0 \) and \( W_0 \), the critical momentum of blocked upstream modes and the critical flow velocity respectively, as functions of \( \Lambda, \epsilon, \text{ and } e\Phi_0 \). However, using the quartic dispersion relation we choose here, the algebra does not allow for a simple solution. On
the other had it is easy to obtain $k_0$ as a function of $W_0$ by straightforward substitution. We have

$$k_0 = (\omega - e\Phi_0) \frac{3W_0 + \sqrt{W_0^2 + 8}}{2(W_0^2 - 1)}. \tag{B.48}$$

Plugging in the expression above into the condition $v_g(k_0) = 0$ we find

$$W_0 = \frac{\omega - e\Phi_0}{2k_0} + \frac{1}{2} \sqrt{\frac{\omega - e\Phi_0}{k_0^2} - \frac{8k_0^2}{\Lambda^2} + 4}. \tag{B.49}$$

The wavenumber $k_0$ also corresponds to a stationary (non-propagating) double root of eq. (B.20); the other two solutions correspond to the propagating modes. They can be obtained explicitly in terms of $k_0$ and $W_0$ (see fig. 5.10),

$$k_{12} = -k_0 \pm \Lambda \sqrt{1 - \frac{2k_0^2}{\Lambda^2} - W_0^2}. \tag{B.50}$$

One can obtain explicit expressions for $k_0$ and $W_0$ by making the approximation

$$\pm (k - \frac{1}{2\Lambda^2} k^3) \simeq \pm \sqrt{k^2 - k^4/\Lambda^2} = (\omega - e\Phi - kW). \tag{B.51}$$

which is valid when $k/\Lambda \ll 1$, which is accurate for the turning point when $\omega \ll \Lambda$. This approximation is accurate even for the largest roots up to a maximum error of a factor of $\sqrt{2}$.

Using this approximation we find the results

$$k_0 = \left[-\Lambda^2(\omega - e\Phi_0)\right]^{1/3} \tag{B.52}$$

$$1 - W_0 = \frac{3}{2} \left[-\frac{(\omega - e\Phi_0)}{\Lambda}\right]^{2/3} \tag{B.53}$$

which are valid, as stated above, when $\omega, \Phi_0 \ll \Lambda$ (note that, as usual, $\omega < e\Phi_0$ in these computations). Here we see that $1 - W_0$ and $1/k_0$ scale in the same way with $\Lambda$. This result was also found in [118] in the context of superluminal dispersion and Hawking radiation where a turning point existed for the negative norm modes coming from within the supersonic flow region. Here the turning point exists for positive norm modes in the subsonic region.

### B.4 Cylindrically symmetric dimensional reduction

The most general analogue metric for perturbations in a moving fluid written in cylindrical coordinates is given by

$$g \propto \begin{pmatrix}
    c^2 - v^2 & v_r & r^2 v_\phi \\
    v_r & -1 & 0 \\
    r^2 v_\phi & 0 & -r^2
  \end{pmatrix} \tag{B.54}$$
where we have chosen the \((+---)\) signature. Assuming homogeneity and a constant wave speed \(c\) we may take the constant of proportionality to be unity.

Hence the action is given by

\[
S = \frac{1}{2} \int dt d\theta d\phi r \sqrt{g} \mu \nu \partial_{\mu}\phi \star \partial_{\nu}\phi
\]

\[
= \frac{1}{2} \int dt d\theta d\phi r \left[ |(\partial_t + v_r \partial_r + v_\theta \partial_\theta) \phi|^2 - |\partial_\phi|^2 - \left| \frac{1}{r} \partial_\theta \phi \right|^2 \right]
\]  

(B.55)

Integration by parts is best done in the more abstract notation

\[
S = \frac{1}{2} \int d^3x \sqrt{g} \left[ \partial_{\mu} \phi \star (g^{\mu\nu} \partial_{\nu} \phi) \right]
\]

(B.56)

\[
= -\frac{1}{2} \int d^3x \sqrt{g} \left[ \phi \star \nabla_{\mu} (g^{\mu\nu} \partial_{\nu} \phi) \right]
\]

(B.57)

which gives rise to the familiar field equation

\[
\partial_{\mu} (\sqrt{g} g^{\mu\nu} \partial_{\nu} \phi) = 0
\]

(B.58)

Dividing through by \(\sqrt{g}\) we find

\[
(\partial_t + \frac{1}{r} \partial_r (rv_r) + v_\theta \partial_\theta) (\partial_t + v_r \partial_r + v_\theta \partial_\theta) \phi
\]

\[
- \frac{1}{r} \partial_r (r \partial_r \phi) - \frac{1}{r^2} \partial_\theta^2 \phi = 0
\]

(B.59)

(B.60)

which can also be found by direct integration by parts of the explicit form of the action (B.55).

The radial version of the Fourier transform is the order 0 Hankel transform

\[
\mathcal{H}(f)(k) =: f_k = \int_0^\infty dr f(r) r J_0(kr)
\]

(B.61)

where \(J_0\) is the order 0 Bessel J function. Using this we can show that, for example, the purely radial derivative term becomes

\[
\mathcal{H} \left( \frac{1}{r} \partial_r (r \partial_r \phi) \right) = -k^2 H(\phi) = -k^2 \phi_k
\]

(B.62)

**Proof:** Using \(J'_\nu(x) = J_{\nu-1}(x) - \nu J_{\nu}(x)/x\) and \(J_{-\nu} = (-1)\nu J_\nu\) we have

\[
\mathcal{H} \left( \frac{1}{r} \partial_r (r \partial_r \phi) \right) = \int_0^\infty dr J_0(kr) \partial_r (r \partial_r \phi)
\]

\[
= \text{[boundary term]} - \int_0^\infty dr (kr J_1(kr)) \partial_r \phi
\]

\[
= \text{[boundary term]} + k \int_0^\infty dr \partial_r (r J_1(kr)) \phi
\]

\[
= -k^2 \int_0^\infty dr r J_0(kr) \phi = -k^2 \phi_k
\]

(B.63)
However, in general the first order radial derivative terms do not enjoy such a nice analogy with the plane wave case where first derivatives become $-ik$. In this radial case, the “second derivative” operator (that operator which has as $-k^2$ the eigenvalue when acting on the Bessel basis, which arises naturally in cylindrical coordinates as above) does not simply factorise into two first order derivatives as in the plane geometric case where $-d^2/dx^2 = (-id/dx)(-id/dx)$. For this reason, under the radial Fourier transform (zeroth order Hankel transform) the wave equation does not become an algebraic equation for the coefficients of the zeroth order transform. Instead the coefficients become coupled to higher order Hankel transform (the Hankel transform involving higher order Bessel functions instead of the zeroth order ones) coefficients. We have, in regions where both $v_r$ and $v_\theta$ are constant (where we usually read off the dispersion relation),

$$\mathcal{H}^{(0)}(\partial_t + \frac{1}{r} \partial_r (rv_r) + v_\theta \partial_\theta) (\partial_t + v_r \partial_r + v_\theta \partial_\theta) \phi = -k^2 \mathcal{H}^{(0)} \phi + (\partial_t + v_\theta \partial_\theta) (-kv_r \mathcal{H}^{(1)}) \phi + (\partial_t + v_\theta \partial_\theta) (k \mathcal{H}^{(1)} - \mathcal{H}^{(0)} \frac{1}{r}) \phi + (\partial_t + v_\theta \partial_\theta)^2 \mathcal{H}^{(0)} \phi.$$  \hspace{1cm} (B.64)

In the planar case, each of these two extra factors (proportional to the first order Hankel transform $\mathcal{H}^{(1)}$) would be $-ik$ which could be brought back into the squared factor to recover the perfect square form.

Another new feature of this field equation is that the angular velocity contribution does not factorise in the way we anticipated above in our toy model. Expanding in the standard plane wave angular basis $\exp(im\theta)$, as well as in a plane wave frequency basis $\exp(-i\omega t)$ we do obtain a nice algebraic equation for the coefficients but it is of a slightly difficult form involving a mass term in addition to the anticipated terms

$$\frac{1}{r} \partial_r r \partial_r f(r) + \left[\left(\omega + \frac{1}{r} \partial_r rv_r - mv_\theta\right)\left(\omega + iv_r \partial_r - mv_\theta\right) + \frac{m^2}{r^2}\right] f(r) = 0$$

Note that this equation is valid also in regions where $v_r$ and $v_\theta$ have non-trivial $r$ dependence. It is possible to show, however, that the presence of this mass term does not exclude the existence of superradiance. We will discuss this below in the context of a simpler planar model.

This general equation of motion may also be written as

$$f''(r) + P(r)f'(r) + Q(r)f(r) = 0 \hspace{1cm} (B.65)$$
where

\[ P(r) = \frac{1}{r} + 2iv_r(\omega - mv_\theta) - \frac{1}{r}\partial_r(rv_r^2) \quad \text{(B.66)} \]
\[ Q(r) = (\omega - mv_\theta) \left[ (\omega - mv_\theta) + i \left( \frac{v_r}{r} + \partial_r(v_r) \right) \right] - imv_r\partial_r(v_\theta) + \frac{m^2}{r^2} \quad \text{(B.67)} \]

which can be put in the frictionless form by a suitable change of variables.

**Appendix C**

**Optics Appendices**

**C.1 Exact homogeneous model for DCE - \text{tanh}(\cdot) profile.**

The equation of motion for the 1 + 1 dimensional electric field \( \Phi \) with a time variable and momentum dependent refractive index \( n \) is

\[ \frac{1}{c^2} \left( n^2(t, k)\phi_{,t} \right)_{,t} = \phi_{,xx}, \quad \text{(C.1)} \]

which can be written as

\[ \partial^2_\tau \phi_k + c^2 k^2 n^2(\tau, k)\phi_k = 0, \quad \text{(C.2)} \]

upon Fourier transformation and coordinate transformation

\[ \tau(t) = \int_0^t \frac{dt'}{n^2(t')} \quad \text{(C.3)} \]

We choose for the refractive index a \text{tanh} time variable perturbation on top of momentum dependent background value

\[ n^2(t, k) = n_0(k)^2 + 2\eta n_0(k) \text{tanh}(t/t_0) + \mathcal{O}(\eta^2), \quad \text{(C.4)} \]
where $\eta$ is a small parameter and $n_0(k)$ would be a model for the refractive index such as the Sellmeier approximation. This choice is motivated by exact solubility of the model but also for its close approximation to the experimental Gaussian profile reported in [?] supplemented with the effects of pulse steepening whereby an initially gaussian-like RIP develops a steep trailing edge and a shallow leading edge (mentioned again below).

Then the equation of motion becomes

$$\phi_{k,rr} + c^2 k^2 \left( n_0^2 + 2\eta n_0 \tanh(t(\tau)/t_0) \right) \phi_k = 0, \quad (C.5)$$

which describes a time independent Schrödinger scattering problem

$$\psi_{xx} + \frac{2m}{\hbar^2} (E - V(x)) \psi = 0, \quad (C.6)$$

with $V(x) = (V_+ + V_-)/2 + (V_+ - V_-)/2 \tanh(x/x_0)$ and

$$m = \frac{1}{2} c^2 k^2 \hbar^2, \quad E = n_0^2, \quad V_- = 2\eta n_0 \quad \text{and} \quad V_+ = -2\eta n_0. \quad (C.7)$$

The Bogoliubov coefficient $\beta_k$ for our mode equation (C.5) is related to the transmission coefficient $T$ for the scattering problem (C.6) by $|\beta_k|^2 = 1/T - 1$. From the literature [148] we find the transmission

$$T = \frac{\sinh (\pi a_+ x_0) \sinh(\pi a_- x_0)}{\sinh^2 (\pi x_0 (a_+ + a_-)/2)}; \quad a_\pm := \sqrt{2m(E - V_\pm)} \hbar \quad (C.8)$$

so that after some manipulation we find

$$|\beta_k|^2 = \frac{\sinh^2 (\pi x_0 (a_+ - a_-)/2)}{\sinh(\pi x_0 a_+) \sinh(\pi x_0 a_-)}. \quad (C.9)$$

In terms of the variables of our problem one has

$$a_- = ck\sqrt{n_0^2 - 2\eta n_0} \simeq ck(n_0 - \eta), \quad a_+ = ck\sqrt{n_0^2 + 2\eta n_0} \simeq ck(n_0 + \eta), \quad (C.10)$$

so that

$$|\beta_k|^2 = \frac{\sinh^2 (\eta\pi ck t_0)}{\sinh(\pi ck t_0(n_0 + \eta)) \sinh(\pi ck t_0(n_0 - \eta))}. \quad (C.11)$$

By the smallness of $\eta$ we may expand the $\sinh^2$ and identify the factors in the denominator arriving at

$$|\beta_k|^2 \simeq \frac{\eta^2 \pi^2 c^2 k^2 t_0^2}{\sinh^2 (\pi ck t_0 n_0)} \quad (C.12)$$
C.2 Homogeneous perturbation expansion for $\beta_k$

To higher orders in $\eta$ we have

$$|\beta_k|^2 \simeq \eta^2 \frac{\pi^2 c^2 k^2 t_0^2}{\sinh^2(\pi c k t_0 n_0)} + \eta^4 \frac{\pi^4 c^2 k^4 t_0^4}{\sinh^2(\pi c k t_0 n_0)} + O(\eta^5) \quad (C.13)$$

$$|\beta_k|^4 \simeq \eta^2 \frac{4\pi^2 c^4 k^4 t_0^4 n_0^2}{\sinh^2(\pi c k t_0 n_0)} - \eta^4 \frac{16\pi^2 c^6 k^6 t_0^6 n_0^3}{\sinh^2(\pi c k t_0 n_0)} + \eta^4 \frac{16(15 - \pi^2)\pi^5 c^8 k^8 t_0^8 n_0^4}{3 \sinh^2(\pi c k t_0 n_0)} + O(\eta^5) \quad (C.14)$$

where, interestingly, we see that there is no $O(\eta^3)$ term in the expansion of the tanh spectrum.

where we have also included the result we wrote down in Chapter 4 for the sech$^2$ model for comparison.

C.2 Homogeneous perturbation expansion for $\beta_k$

In Chapter 4 we have computed the Bogoliubov $\beta_k$ coefficient in two separate ways, one exactly using known results for some specific potentials and one using a perturbative method. In the text we computed the various relevant features of the spectrum such as particle flux and peak wavenumber of emission for the exact results. Here we compute the same quantities from the perturbative solution, comparing them with the exact computations. We will see that they agree closely justifying the perturbative method as it applies to the RIP problem.

The peak wavenumber is

$$k_{\text{peak}} = \frac{1}{c t_0 n_0} \simeq 4.79 \times 10^6 \text{m}^{-1} \quad (C.15)$$

and fluxes are

$$N^{1+1} = W \times \frac{\eta^2 \pi^2 c^2 t_0^2 e^2}{2^{1/2} c^2 t_0^2 n_0^3} \int_0^\infty k^2 e^{-2c^2 t_0^2 n_0^2 k^2} dk \quad (C.16)$$

$$= \frac{\eta^2 \pi^2 c^2 t_0^2 W}{2\sqrt{2} c^2 t_0^2 n_0^3} \int_0^\infty u^2 e^{-u^2} du \quad (C.17)$$

$$= \frac{\eta^2 \pi \sqrt{\pi} W}{8\sqrt{2} c t_0 n_0^3} \quad (C.18)$$

$$\simeq 1.12 \times 10^{-5} \text{ particles} \quad (C.19)$$
Figure C.1: The exact result for the sech\(^2\) model is given in green (lower curve); this is flux against wavenumber where flux is calculated as \(k^2|\beta|^2\) with \(\beta\) from 4.21. This should be compared to the homogeneous perturbative calculation for a gaussian profile in red (upper curve) based on 4.37. In these figures we use \(t_0 = 4 \times 10^{-16}\)s for the Exact sech\(^2\) result and \(t_0 = 4.8 \times 10^{-16}\)s for the perturbative result derived from a Gaussian profile.

and

\[
N^{3+1} = 4\pi \text{Vol} \times \eta^2 \pi t_0^2 c^2 \int_0^\infty k^4 e^{-2c^2 t_0^2 n_0^2 k^2} dk \\
= \frac{4\eta^2 \pi^2 t_0^2 c^2 \text{Vol}}{4\sqrt{2}c^3 t_0^5 n_0^5} \int_0^\infty u^4 e^{-u^2} du \\
= \frac{3\eta^2 \pi^2 \sqrt{\pi} \text{Vol}}{8\sqrt{2}c^3 t_0^5 n_0^5} \\
\simeq 0.006 \text{ (shape A), } 0.275 \text{ (shape B), particles}
\]

which closely agree with the exact results derived for the exact model presented in Chapter 4 (see that chapter for definitions of shape A and shape B).

In Fig. C.1 we compare the exact number spectrum as given by the exact analysis in 4.21 with the result obtained above using perturbation theory.

### C.3 Perturbation theory with finite size effects

As described in Chapter 4 we wish to solve the equation of motion

\[
\phi_{,tt} - c^2 n^2(t,x) \phi_{,xx} = 0
\]
when the function $n^2$ is given by a background plus a small perturbation

$$n^2(t, x) = n_0^2 + \eta f^2(t, x); \quad \eta \ll 1$$

under a perturbative assumption that $\phi(t, x) = \phi^{(0)}(t, x) + \eta \phi^{(1)}(t, x) + O(\eta^2)$.

We found that with a Gaussian RIP function

$$f^2(t, x) = e^{-x^2/\omega_0^2 - t^2/\omega_0^2},$$

that

$$\phi^{(0)}(t, x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{e^{ikx - i\omega_k t}}{n_0 \sqrt{2\omega_k}}$$

and

$$\phi_k^{(1)}(t) = \frac{1}{\omega_k} \int_{-\infty}^{t} dt' \tilde{F}(t', k) \sin \omega_k(t - t').$$

where

$$\tilde{F}(t, k) = e^{-t^2/\omega_0^2} \int \frac{d\tilde{k}}{\sqrt{2\pi}} \frac{e^{-i\omega_k t}}{n_0 \sqrt{2\omega_k}} c^2 \tilde{k}^2 \left[ x_0 e^{-(\tilde{k} - k)^2 x_0^2/4} \right]$$

and $\omega_k^2 = c^2 n_0^2 k^2$.

The first order solution is therefore written as

$$\phi_k^{(1)} = \frac{1}{\omega_k} \int_{-\infty}^{t} dt' \tilde{F}(t', k) \sin \omega_k(t - t')$$

$$= \frac{1}{\omega_k} \int_{-\infty}^{t} dt' \left[ e^{-t'^2/\omega_0^2} \int \frac{d\tilde{k}}{\sqrt{2\pi}} \frac{c^2 \tilde{k}^2 x_0}{n_0 \sqrt{2\omega_k}} e^{-i\omega_k t'} e^{-(\tilde{k} - k)^2 x_0^2/4} \right] \sin \omega_k(t - t')$$

$$= \frac{1}{\omega_k} \int \frac{d\tilde{k}}{\sqrt{2\pi}} \frac{c^2 \tilde{k}^2 x_0}{n_0 \sqrt{2\omega_k}} e^{-(\tilde{k} - k)^2 x_0^2/4} \left[ \int_{-\infty}^{t} dt' e^{-i\omega_k t'} e^{-t'^2/\omega_0^2} \sin \omega_k(t - t') \right].$$

The $t'$ integral is doable in terms of error functions and results in the asymptotic behaviour for $t \rightarrow +\infty$ as a sum of a pure positive and pure negative frequency piece

$$\phi_k^{(1)}(t) \rightarrow \frac{1}{\omega_k} \int \frac{d\tilde{k}}{\sqrt{2\pi}} \frac{c^2 \tilde{k}^2 x_0}{n_0 \sqrt{2\omega_k}} e^{-(\tilde{k} - k)^2 x_0^2/4} \left[ \frac{i\sqrt{\pi} t_0}{2} e^{-t_0^2(\omega_k - \omega_k)^2/2} e^{-i\omega_k t} \right.$$

$$+ \frac{i\sqrt{\pi} t_0}{2} e^{-t_0^2(\omega_k + \omega_k)^2/2} e^{i\omega_k t} \left. \right].$$

Rearranging we have,

$$\phi_k^{(1)} \rightarrow \frac{i c^2 x_0 t_0}{2} \left[ \frac{I_k^1}{n_0 \sqrt{2\omega_k}} + \frac{I_k^2}{n_0 \sqrt{2\omega_k}} \right].$$
where

\[ I^1_k = \frac{1}{\sqrt{\omega_k}} \int \frac{d\tilde{k}}{\sqrt{\omega_{\tilde{k}}}} \tilde{k}^2 e^{-\frac{\tilde{k}^2}{4}((\tilde{k}-k)^2 + e^{-\frac{\tilde{k}^2}{4}((\tilde{k}+k)^2) \right)} \tag{C.33} \]

\[ I^2_k = \frac{1}{\sqrt{\omega_k}} \int \frac{d\tilde{k}}{\sqrt{\omega_{\tilde{k}}}} \tilde{k}^2 e^{-\frac{\tilde{k}^2}{4}((\tilde{k}-k)^2 + e^{-\frac{\tilde{k}^2}{4}((\tilde{k}+k)^2) \right)} \tag{C.34} \]

These \( k \) integrals \( I^i_k \) may be performed exactly using the following trick to eliminate the cumbersome absolute values

\[ \int_{-\infty}^{\infty} dk f(k) = \int_{0}^{\infty} dk \left( f(k) + f(-k) \right). \tag{C.36} \]

We have

\[ I^1_k = \frac{1}{\sqrt{\omega_k}} \int_{0}^{\infty} \frac{d\tilde{k}}{\sqrt{\omega_{\tilde{k}}}} \tilde{k}^2 e^{-\frac{\tilde{k}^2}{4}(-\xi^2\tilde{k}+\tilde{k}^2) \right) \left( e^{-\frac{\xi^2}{4}((\tilde{k}-k)^2 + e^{-\frac{\xi^2}{4}((\tilde{k}+k)^2) \right)} \tag{C.37} \]

\[ I^2_k = \frac{1}{\sqrt{\omega_k}} \int_{0}^{\infty} \frac{d\tilde{k}}{\sqrt{\omega_{\tilde{k}}}} \tilde{k}^2 e^{-\frac{\tilde{k}^2}{4}(-\xi^2\tilde{k}+\tilde{k}^2) \right) \left( e^{-\frac{\xi^2}{4}((\tilde{k}+k)^2 + e^{-\frac{\xi^2}{4}((\tilde{k}+k)^2) \right)} \tag{C.38} \]

Introducing dimensionless variables

\[ K = t_0cn_0k/2, \quad \tilde{K} = t_0cn_0\tilde{k}/2 \tag{C.39} \]

and the dimensionless ratio

\[ \xi = \frac{x_0}{t_0cn_0} \tag{C.40} \]

these integrals may be expressed as

\[ I^{(1)}_k = \frac{4}{c^2n_0^2t_0x_0 \sqrt{K}} \int_{0}^{\infty} d\tilde{K} \tilde{K}^{3/2} e^{-(K-\tilde{K})^2} \left[ e^{-\xi^2(K-\tilde{K})^2} + e^{-\xi^2(K+\tilde{K})^2} \right] \tag{C.41} \]

\[ I^{(2)}_k = \frac{4}{c^2n_0^2t_0x_0 \sqrt{K}} \int_{0}^{\infty} d\tilde{K} \tilde{K}^{3/2} e^{-(K+\tilde{K})^2} \left[ e^{-\xi^2(K-\tilde{K})^2} + e^{-\xi^2(K+\tilde{K})^2} \right]. \tag{C.42} \]

in terms of which the Bogoliubov \( \beta_k \) are

\[ \beta_k = \frac{ic^2x_0t_0}{2} I^{(2)}_k \tag{C.43} \]

Let us concentrate on the integrals. Using the experimentally reported values for the parameters \( x_0 \) and \( t_0 \) (derived from the carrier wavelength of the RIP and the Sellmeier dispersion relation) we find that

\[ \xi \approx 47.89 \tag{C.44} \]
so we are in a parameter regime in which $\xi^2 \gg 1$. We will exploit this limit. In the true $\xi^2 \to \infty$ limit we have

$$\xi \left[ e^{-\xi^2(K-\tilde{K})^2} + e^{-\xi^2(K+\tilde{K})^2} \right] \to \sqrt{\pi} \delta(K - \tilde{K}) + \sqrt{\pi} \delta(K + \tilde{K})$$

(C.45)

which collapse the integrals giving the homogeneous result

$$\beta_k \to i\eta t_0 \sqrt{\frac{\omega_k}{n_0^2}} e^{\frac{-\omega_k^2 t_0^2}{2}} \text{ as } \xi \to \infty \quad (\text{equivalently } x_0 \to \infty).$$

(C.46)

This gave a spectrum which was peaked (in wavelength space) around 850nm when $t_0 = 4 \times 10^{-16}$s. Hence we expect this modified integral to be also peaked around this value which in $k$ space is at around $7.4 \times 10^6$ m$^{-1}$.

We will see below that this limit is in fact the physically relevant one for describing the real RIP experiment and that the limit is essentially realised by the parameter choices we have made to accurately describe the RIP experiment.

**Parabolic cylindrical function solution**

We need to manipulate the integrals $I^i$ a bit in order to use the result for the parabolic cylindrical functions:

$$\int_0^\infty x^{\nu-1} e^{-\beta x^2 - \gamma x} dx = \Gamma(\nu) \left(2\beta\right)^{-\nu/2} \exp\left(\frac{\gamma^2}{8\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2}\beta}\right).$$

(C.47)

We are mostly concerned with $\beta_k$ so we will focus on the integral $I_k^{(2)}$. Expanding the quadratics inside the integrals

$$I_k^{(2)} = \frac{4}{c^2 n_0^2 t_0 x_0} \frac{\xi}{\sqrt{K}} e^{-(1+\xi^2)K^2} \left[ \int_0^\infty \tilde{K}^{-3/2} e^{-(1+\xi^2)\tilde{K}^2 - 2(1+\xi^2)K\tilde{K}} \right.]$$

so that

$$I_k^{(2)} = \frac{4}{c^2 n_0^2 t_0 x_0} \frac{\xi}{\sqrt{K}} e^{-(1+\xi^2)K^2} \Gamma(5/2) \left(2(1 + \xi^2)\right)^{-5/4} \times$$

$$\left[ e^{(1+\xi^2)K^2/2} D_{-5/2}\left(\sqrt{2(1 + \xi^2)}K\right) + \exp\left(\frac{(1 - \xi^2)^2 K^2}{2(1 + \xi^2)}\right) D_{-5/2}\left(\frac{2(1 - \xi^2)K}{\sqrt{2(1 + \xi^2)}}\right) \right]$$

(C.48)

This solution has the ‘disadvantage’ that it diverges for small $k$ as $k^{-1/2}$. This is a peculiarity of the finite size computation and 1+1 dimensions. However when one moves
to the 3+1 dimensional interpretation of the flux using $k^2|\beta_k|^2$ this divergence is cured. Note that this pole is integrable and hence for 1+1 dimensional flux computations does not represent a physical divergence of created particles.

In Fig. 4.6 we compare the first order finite size result C.43 with the first order homogeneous result 4.38.

**Trying the second order term**

Above and in Chapter 4 we saw that finite size effects are practically negligible at first order in perturbation theory in the specific RIP consired. Hence it would seem unnecessary to look for the next higher order term for $\phi$ in perturbation theory directly and instead more reasonable to assume that the next higher order term $\phi^{(2)}$ is also very accurately represented by the homogeneous approximation. Nevertheless here briefly we write down the next higher order term in $\eta$ for the perturbative expansion of the finite size solution.

From C.24 and C.25 and the expansion $\phi = \phi^{(0)} + \eta\phi^{(1)} + \eta^2\phi^{(2)}$ we have

$$\phi^{(2)}_{tt} - c^2n_0^2\phi^{(2)}_{xx} = c^2f^2\phi^{(1)}_{xx} =: G(t, x)$$

(C.50)

where again we see an an equation of the forced harmonic oscillator type which is soluble in terms of a simple retarded Green function. We have

$$\phi^{(2)}(t, x) = \mathcal{F}^{-1}\left[\frac{1}{\omega_k} \int_{-\infty}^{t} \mathcal{F}(G)(t', k) \sin \omega_k(t - t') dt'\right](t, x)$$

(C.51)

where $\mathcal{F}$ is the Fourier operator. Now, the term $\mathcal{F}(G)$ is written in terms of the exact solution we obtained above in terms of Cylindrical functions for $\phi_k^{(1)}$ as

$$\mathcal{F}(G)(t, k) = \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} e^{-x^2/x_0^2 - t^2/t_0^2} \frac{d}{dx^2} \mathcal{F}^{-1}(\phi_k^{(1)}(t))(x)$$

(C.52)

(where the $\tilde{k}$ is merely a labeling device and not a variable). Integrating by parts twice we have

$$\mathcal{F}(G)(t, k) = \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} \left(\frac{4x^2}{x_0^4} + \frac{2ixk - 2}{x_0^2} - k^2\right) e^{-x^2/x_0^2 - t^2/t_0^2} \mathcal{F}^{-1}(\phi_k^{(1)}(t))(x)$$

(C.53)

$$= \mathcal{F}\left(g(t) \cdot \mathcal{F}^{-1}(\phi_k^{(1)}(t))\right)(k)$$

(C.54)

$$= \left(\mathcal{F}(g(t)) \ast \phi_k^{(1)}(t)\right)(k)$$

(C.55)

$$= \int_{-\infty}^{\infty} \mathcal{F}(g(t))(k') \cdot \phi_{k-k'}^{(1)}(t) \, dk'$$

(C.56)
where we have defined

\[ g(t, x) := \left( \frac{4x^2}{x_0^4} + \frac{2ixk - 2}{x_0^2} \right) e^{-x^2/x_0^2 - t^2/t_0^2} \tag{C.57} \]

and used the convolution theorem for Fourier transforms of products of functions. Now, the first factor in the integrand is

\[ \mathcal{F}(g(t))(k') = \int \frac{dx}{\sqrt{2\pi}} \left( \frac{4x^2}{x_0^4} + \frac{2ixk - 2}{x_0^2} \right) e^{-x^2/x_0^2 - t^2/t_0^2} \tag{C.58} \]

\[ = e^{-t^2/t_0^2} \left( -\frac{4}{x_0^4} \frac{d^2}{dk^2} + \frac{2ik}{x_0^2} \frac{d}{dk} \right) \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} e^{-x^2/x_0^2} \tag{C.59} \]

\[ = e^{-t^2/t_0^2} (-k^2) \frac{x_0}{\sqrt{2}} e^{-x_0^2k^2/2}. \tag{C.60} \]

Hence we see that the second order solution involves is a convolution of the first order solution over a Gaussian distribution.

This is rather messy but somewhere along the perturbative series one should start to see a non-isotropic Bogoliubov coefficient – that is, the second order solution (or perhaps only a higher order contribution) should be expressed not as a simple linear combination of a positive and a negative frequency piece but as a function with a non-trivial Fourier spectrum (single particle decomposition). This intuition comes from the fact that the \( \delta(k + k') \) functional behaviour of the isotropic Bogoliubov coefficients is a direct result of the use of plane waves for the initial and final particle bases and the fact that the initial plane wave does not distort in space but only in time as the time dependent perturbation acts. Here, in the finite size situation, the initial plane wave will distort spatially and temporally and we will be left with a kind of integral like

\[ \int dx \, \phi_{in}(t, x) e^{-ikx} \tag{C.61} \]

which will not in general evaluate to the delta function due to spatial distortion of the initial condition (and ‘in’ state particle wave function) \( \phi_{in} \).

Beyond perturbation theory one would need to solve exactly the equation of motion to find such non-isotropic particle production. Intuitively, from Figure 4.5 one would expect that in a 3+1 dimensional computation with one spatial dimension of the RIP small and one large with respect to the peak wavelength of emission, a non-isotropic
emission would produced.

Appendix D

BEC Appendices

D.1 Calculating the exact form of a Wightman function by contour inteation

In this appendix we calculate an integral given in the text (3.39), exactly. The integral represents the entanglement contribution to a two-point correlation function which we investigated in detail.

The integral we wish to calculate is

$$
\int_0^\infty I + T = 2\text{Re} \int_0^\infty I. \quad (D.1)
$$

where

$$
I = \frac{\sin kx}{x} \alpha_k \beta_k^* e^{-2ika_\eta} \quad (D.2)
$$

and

$$
\alpha_k = \frac{2\sqrt{AB}}{A + B} \frac{\Gamma(-iA)\Gamma(-iB)}{\Gamma^2(-i(A + B)/2)}, \quad (D.3)
$$

$$
\beta_k = \frac{-2\sqrt{AB}}{B - A} \frac{\Gamma(-iA)\Gamma(iB)}{\Gamma^2(i(B - A)/2)} \quad (D.4)
$$

and finally

$$
A = k\eta_0 a_i^2, \quad B = k\eta_0 a_f^2. \quad (D.5)
$$

$a_i$, $a_f$ and $\eta_0$ being constants.

We would like to write the integral as an integral over the entire real line and not only on $(0, \infty)$ in order to make use to the residue theorem and contour integration.
Calculating the exact form of a Wightman function by contour integration

Figure D.1: The contour $\gamma \subset \mathbb{C}$ in red used in the calculation of the integral $\int I$ with the first few poles $k_0^i, k_0^f$ and $k_1^f$ indicated.

However, upon inspection the integrand is in fact an odd function of real $k$ and the contour we wish to use will receive equal contributions from the negative real axis as the positive, evaluating to zero for an odd function. Thus we make use of the signum function $\sigma(z) = z/|z|$ which, crucially, changes the integrand into an even function of $k$ reducing to the identity on positive real numbers and minus the identity on negative real numbers. This technique has the effect of isolating only the real part of the contribution to the integral coming from $I$ since the imaginary part is rendered odd by the signum function. Thus it will not be necessary to calculate the contribution from $\overline{I}$ but we will do so anyway as a check. Our plan is to use the semi-circle contour in the complex plane shown in figure [D.1] and the residue theorem and show that the contribution from the arc section of the contour converges to zero for large arcs, leaving only the contribution from the real line. We have

$$\int_{0}^{\infty} I(k) + \overline{I}(k) = 2\text{Re} \int_{0}^{\infty} I(k) = \int_{-\infty}^{\infty} \sigma(k)I(k). \quad (D.6)$$

Below we will refer to $\sigma I$ simply as $I$.

Explicitly we have

$$I = \sigma (k)e^{-2ik\eta} \frac{\Gamma^2(-iB)\Gamma(iA)\Gamma(-iA)}{\left[\Gamma\left(-\frac{i}{2}(A + B)\right)\Gamma\left(-\frac{i}{2}(B - A)\right)\right]^2} \frac{\sin(kx)}{x}. \quad (D.7)$$

This expression possesses the following two sets of poles in the lower half plane arising
from the simple poles of the $\Gamma$ function on the negative integers:

$$-iB = -n \quad \Rightarrow \quad k = k_n^i := - \frac{in}{\eta_0 a_i^2}, \quad n \in \mathbb{N} \quad \text{(Double poles)} \quad (D.8)$$

$$-iA = -n \quad \Rightarrow \quad k = k_n^f := - \frac{in}{\eta_0 a_f^2}, \quad n \in \mathbb{N}. \quad \text{(Simple poles)} \quad (D.9)$$

These poles are in the numerator since $\Gamma$ is nonvanishing and hence does not contribute a pole from the denominator. The simple poles of the $\Gamma(iA)$ factor lie outside the contour so we do not consider them further here. Below we make use of the following important observations regarding $\Gamma$ functions and the signum function:

$$\sigma(ir) = i, \quad \sigma(-ir) = -i, \quad r > 0 \quad (D.10)$$

$$\text{Res} \left( \Gamma(z), -n \right) = \frac{(-1)^n}{n!}, \quad \text{Res}(\Gamma(\lambda z), -\frac{n}{\lambda}) = \frac{1}{\lambda} \frac{(-1)^n}{n!}, \quad n \in \mathbb{N} \quad (D.11)$$

$$\text{Res} \left( \Gamma^2(z), -n \right) =: \gamma_n \quad \text{Res} \left( \Gamma^2(\lambda z), -\frac{n}{\lambda} \right) = \frac{\gamma_n}{\lambda}, \quad n \in \mathbb{N}. \quad (D.12)$$

Here $\gamma_n$ is sum of a pure fraction and a fraction multiplying the Euler constant which converges rather rapidly to 0 for large $n$. To calculate the residues we use the well known formula for an $n$th order pole

$$\text{Res}(f, z = a) := \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} ((z-a)^n f(z)), \quad (D.13)$$

so that for the simple poles at $k_n^i$, we merely need to take a limit whereas the calculation of the residues of the double poles $k_n^f$ require differentiation of the entire expression $I(k)$. Then,

$$\text{Res}(I, k_n^i) = \sigma(k) e^{-2ika^2_0 \eta} \frac{\Gamma(-iB)^2 \Gamma(iA)}{\left[ \Gamma\left( -\frac{i}{2}(A + B) \right) \Gamma\left( -\frac{i}{2}(B - A) \right) \right]^2} \frac{\sin(kx)}{x} \text{Res}(\Gamma(-iA), k_n^i)$$

$$= -e^{-2nT \frac{a_0}{\eta_0}} \frac{\Gamma(n) \Gamma^2(-nT)}{\Gamma^2\left( -\frac{n}{2}(T + 1) \right) \Gamma^2\left( -\frac{n}{2}(T - 1) \right)} \frac{1}{x} \sinh\left( \frac{nx}{\eta_0 a_i^2} \right) \times \frac{i}{\eta_0 a_i^2 n!} \left( -\frac{1}{n} \right). \quad (D.14)$$

For the double poles the calculation is slightly longer. We use the product rule to isolate the residue of $\Gamma^2$,

$$\text{Res}(I, k_n^f) = \lim_{k \to k_n^f} \frac{d}{dk} \left[ (k - k_n^f)^2 I(k) \right]$$

$$= \Theta(k_n^f, x, \eta) \times \text{Res} \left( \Gamma(-iB)^2, k_n^f \right) + \lim_{k \to k_n^f} \left[ (k - k_n^f)^2 \Gamma(-iB)^2 \times \frac{d}{dk} \Theta(k, x, \eta) \right], \quad (D.15)$$
where
\[ \Theta(k, x, \eta) := \sigma(k) e^{-2ika^2\eta} \frac{\Gamma(iA)\Gamma(-iA)}{\left[\Gamma \left(-\frac{i}{2}(A + B)\right) \Gamma \left(-\frac{i}{2}(B - A)\right)\right]^2} \frac{\sin (kx)}{x}. \] (D.16)

For reference note that
\[ \Theta(k^I_n, x, \eta) = \Theta_n e^{-2n\frac{x}{m_0}} x^{-\frac{n}{2}} \sinh \left( \frac{nx}{\eta_0 a_f^2} \right), \] (D.17)
\[ = -e^{-2n\frac{x}{m_0}} \frac{\Gamma(nT^{-1})\Gamma(-nT^{-1})}{\Gamma^2 \left(-\frac{n}{2}(T^{-1} + 1)\right) \Gamma^2 \left(-\frac{n}{2}(1 - T^{-1})\right)} \frac{1}{x} \sinh \left( \frac{nx}{\eta_0 a_f^2} \right). \] (D.18)

It is possible to expand the Γ functions in the numerical coefficient Θn in terms of elementary functions as
\[ \Theta_n = \begin{cases} \frac{n}{8\pi T} \tan \left( \frac{\pi n}{2T} \right) \prod_{p=1}^{n/2} \left( p^2 - \frac{n^2}{4T^2} \right)^2 & \text{for } n \text{ even} \\ -\frac{T}{2\pi} \left( \frac{1}{4} - \frac{n^2}{4T^2} \right)^2 \cot \left( \frac{\pi n}{2T} \right) \prod_{p=1}^{n/2+1/2} \left( (p + \frac{1}{2})^2 - \frac{n^2}{4T^2} \right)^2 & \text{for } n \text{ odd} \end{cases}. \] (D.19)

Note that since \( k^I_n \) is a double pole of both \( I \) and \( \Gamma^2(-iB) \), the second term in \( \text{D.15} \) is finite as it should be. The first k derivative is
\[ \frac{d}{dk} \Theta(k, x, \eta) \bigg|_{k=k^I_n} = i\Theta(k^I_n, x, \eta) \left[ \eta_0 a_f^2 \left( \Psi(nT^{-1}) - \Psi(-nT^{-1}) \right) - 2a_f^2 \eta + x \frac{\cosh \left( \frac{nx}{\eta_0 a_f^2} \right)}{\sinh \left( \frac{nx}{\eta_0 a_f^2} \right)} \right] \left( \eta_0 a_f^2 + a_i^2 \right) \Psi \left( -\frac{n}{2}(T^{-1} + 1) \right) + \eta_0 a_f^2 - a_i^2 \Psi \left( -\frac{n}{2}(1 - T^{-1}) \right) \right], \] (D.20)
where \( \Psi \) is the digamma function defined as \( \Gamma(z)' = \Psi(z)\Gamma(z) \). Note that \( d\sigma(k)/dk = 2\delta(k) \) which evaluates to zero on the poles; \( \delta(k^I_n) = 0 \). Hence we obtain the residue
\[ \text{Res}(I, k^I_n) = i\Theta_n e^{-2n\frac{x}{m_0}} \frac{1}{x} \sinh \left( \frac{nx}{\eta_0 a_f^2} \right) \left[ \frac{\gamma_n}{\eta_0 a_f^2} - \frac{1}{\eta_0 a_f^2 n!^2} \left( K_n - 2a_f^2 \eta + x \frac{\cosh \left( \frac{nx}{\eta_0 a_f^2} \right)}{\sinh \left( \frac{nx}{\eta_0 a_f^2} \right)} \right) \right], \] (D.21)
where \( K_n \) stands for all the \( x \) and \( \eta \) independent terms inside the square bracket in \( \text{D.20} \). The final result for the integral is therefore given by
\[ \int_\gamma I = 2\pi \sum_{n>0} A_n e^{-2nT_{\frac{\pi}{m_0}}} x^{-\frac{n}{2}} \sinh \left( \frac{nx}{\eta_0 a_f^2} \right) + (B_n + \eta C_n) e^{-2n\frac{x}{m_0}} x^{-\frac{n}{2}} \sinh \left( \frac{nx}{\eta_0 a_f^2} \right) \]
\[ + D_n e^{-2n\frac{x}{m_0}} \cosh \left( \frac{nx}{\eta_0 a_f^2} \right), \] (D.22)
where

\[ A_n := -\frac{1}{\eta_0 a_i^2} \frac{(-1)^n}{n} \frac{\Gamma^2(-nT)}{\Gamma^2\left(-\frac{n}{2}(T+1)\right) \Gamma^2\left(-\frac{n}{2}(T-1)\right)}, \]  
\[ B_n := \Theta_n \left( \frac{\gamma_n}{\eta_0 a_f^2} - \frac{K_n}{\eta_0 a_f^4 (n!)^2} \right), \]  
\[ C_n := -\frac{2}{\eta_0 a_f^2 (n!)^2}, \]  
\[ D_n = \frac{1}{\eta_0 a_f^4 (n!)^2}. \]

Note that \( K_n \) has dimensions of time so that the dimensions of each term match.

It is also worthwhile to point out that the effect of the signum function is to contribute a factor of \( i \) in the \( I \) integral and a factor of \(-i\) in the \( \overline{I} \) integral without which the two terms would not only be imaginary but in fact would contribute equally with the opposite sign yielding the zero result anticipated above.

There remains only one last loose end and that is the fact that the integrand is not analytic on the contour \( \gamma \) due to the non-differentiability of \( \sigma(z) \) at \( z = 0 \). We get around this by modifying the contour around the origin into a small semi circle. Here we will show that the lin integral over the semi circle vanished in the limit of infinitesimal radius. We temporarily return to the old notation in which \( \tilde{I}(k) = I(k)\sigma(k) \). We have

\[ \int_\gamma I(z)\sigma(z) = \int_{-\pi}^{\pi} d\theta I(e^{i\theta}) \sigma(e^{i\theta}) \]  
\[ = \frac{-i}{\epsilon} \int_{-\epsilon}^{\epsilon} dw f(w) \]  
\[ = \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} dw \text{Im} f((w) \]  
\[ \leq \sup (\text{Im} f, [-\epsilon, \epsilon]) \frac{2\epsilon}{\epsilon} \]  
\[ = \sup (\text{Im} f, [-\epsilon, \epsilon]) \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0, \]

since \( \text{Im} f(z) \) smoothly goes to zero for \( z \) going to 0. Hence we may use the large semi-circle contour with impunity.
D.2. Adiabatic limit of the entanglement contribution to the Wightman function

D.2.1 Preliminaries

The Wightman function under consideration is written

\[ \langle \phi(\eta, x) \phi(\eta, x') \rangle = \frac{4\pi}{2a^2} \int_0^\infty dk \left[ \frac{\sin kx}{x} (1 + 2|\beta_k|^2) + I + I^* \right]. \quad (D.32) \]

It is the contribution \( I \) that we are interested in. \( I \) is written in terms of the Bogoliubov coefficients as

\[ I = \alpha_k \beta_k^* \frac{\sin kx}{x} e^{-2ika^2\eta}. \quad (D.33) \]

where

\[ A = i\eta_0 a^2, \quad B = i\eta_0 a^2. \quad (D.35) \]

The adiabatic limit is the large \( \eta_0 k \). Relabelling \( \tilde{k} = k\eta_0 \) and dropping the tilde this will be the large \( k \) limit.

It will be necessary to make use of the following asymptotic behaviour for \( \Gamma \) function

\[ \Gamma(z) \simeq \sqrt{\frac{2\pi}{z}} z^z e^{-z}, \quad \text{for} \quad |z| \to \infty. \quad (D.36) \]

We will insert the expansion [D.36] into the Gamma functions in [D.34] in steps: we calculate the contributions from the \( z^{-1/2} \), \( z^z \) and \( e^{-z} \) separately and take their product at the end as well as the other external factors not entering the asymptotic expansion. Firstly notice that the \( 2\pi s \) cancel on the top and bottom. The contribution from \( z^{-1/2} \) is

\[ \left[ \frac{1}{\sqrt{1/(A+B)}} \frac{1}{\sqrt{1/(B-A)}} \right]^2 = \frac{1}{4} \frac{(A+B)(B-A)}{-iBA} = \frac{iB^2 - A^2}{4BA}. \quad (D.37) \]

From the exponential we have

\[ \frac{e^{-(B)^2(-B)-A^2(-A)}}{[e^{-(1/2(A+B))}-(1/2(B-A))]^2} = \frac{e^{2B}}{e^{A+B+B-A}} = e^0 = 1. \quad (D.38) \]
Finally the $z^2$ terms

\[
(-B)^{-B}(-B)^{-B}A^A(-A)^{-A} \left[ \frac{1}{2(A+B)} \right]^{1/2} \frac{1}{(A-B)^{1/2}} \]

\[
= (-B)^{-2B}(-1)^{-A} \]

\[
= \frac{(2B)^{-2B}(-1)^{-A}}{(A+B)(A-B)} .
\] (D.39) (D.40) (D.41)

Hence, the large $k$ limit for the ratio of $\Gamma$ functions is

\[
\text{Gammas} \approx \frac{i B^2 - A^2}{4 BA} \frac{(2B)^{-2B}(-1)^{-A}}{(A+B)(A-B)} .
\] (D.42)

\[
= \frac{i a_f^4 - a_i^4}{4 a_f^2 a_f} \frac{(2ika_f^2)^{-2ika_f^2}(-1)^{-ika_f^2}}{a_f^2 + a_f^2} \frac{ik(a_f^2 + a_f^2)^{-ik(a_f^2 + a_f^2)}(a_f^2 - a_i^2)^{-ik(a_f^2 - a_i^2)}}{(a_f^2 + a_f^2)^{-ik(a_f^2 + a_f^2)}(a_f^2 - a_i^2)^{-ik(a_f^2 - a_i^2)}} .
\] (D.43)

\[
= \frac{i a_f^4 - a_i^4}{4 a_f^2 a_f} \frac{(2ika_f^2)^{-2ika_f^2}(-1)^{-ika_f^2}}{a_f^2 + a_f^2} \frac{(a_f^2 + a_f^2)^{-ik(a_f^2 + a_f^2)}(a_f^2 - a_i^2)^{-ik(a_f^2 - a_i^2)}}{(a_f^2 + a_f^2)^{-ik(a_f^2 + a_f^2)}(a_f^2 - a_i^2)^{-ik(a_f^2 - a_i^2)}} .
\] (D.44)

\[
= \frac{i a_f^4 - a_i^4}{4 a_f^2 a_f} \left( \frac{a_f^2 + a_f^2}{a_f^2 - a_i^2} \right)^{ika_f^2} \left( \frac{a_f^4 - a_i^4}{4a_f^2} \right)^{ika_f^2} e^{-\pi a_i^2 k} .
\] (D.45)

where we have used the fact that $\ln(-1) = i\pi$. The full function $I$ becomes

\[
I \approx -i \left( \frac{a_f^2 + a_f^2}{a_f^2 - a_i^2} \right)^{ika_f^2} \left( \frac{a_f^4 - a_i^4}{4a_f^2} \right)^{ika_f^2} \frac{\sin(kx)}{x} e^{-2ika_f^2 \eta} e^{-\pi a_i^2 k} ,
\] (D.46) (D.47) (D.48)

where we see the exponential damping for large $k$ as expected physically. Taking the real part we finally arrive at the desired result

\[
I + I^* \approx \frac{2}{x} \sin(kx) \sin k \left( \ln \left( \frac{a_f^2 + a_f^2}{a_f^2 - a_i^2} \right) a_f^2 + \ln \left( \frac{a_f^4 - a_i^4}{4a_f^2} \right) a_f^2 - 2a_i^2 \eta \right) e^{-\pi a_i^2 k} .
\] (D.49)

This result was quoted in Chapter 3.
D.3 Glossary of basic concepts and approximations in the theory of BEC

Below is a bullet pointed list of the various key concepts in the theory of BEC as it pertains to analogue gravity-ists.

- **Dilute gas limit**
  
  This is defined as the regime in which three body interactions are negligible. Algebraically we write
  
  \[ |a| < n^{-\frac{3}{2}} \]  
  
  (D.50)

  for the dilute gas limit. This is a fundamental assumption necessary for the whole picture of BEC. For example only in the dilute gas limit can we say that the mean field function is the density. We are able to ignore both the quantum and thermal depletion of the condensate.

- **Hydrodynamic Limit**
  
  This is the limit in which the quantum pressure term is negligible. This term is of quantum mechanical origin.

- **Scattering length** - the distance within which the atoms interact.

  The scattering length is defined only for low energy collisions and depends on the physical intuition that due to de Broglie wave properties of particles, the very high frequency features of a scattering potential are not visible to a slowly moving particle. One can approximate the potential by a hard sphere in the low energy collisions the radius of which is the scattering length. Essentially it is the distance a particle needs to be from another in order to scatter.

- **Healing length**

  Defined as

  \[ \xi = \frac{1}{\sqrt{8\pi n a}} \]  
  
  (D.51)

  it characterises the length scale beyond which atoms can be thought of as not existing independently? It is a characteristic length scale associated with anisotropies in the density. A small healing length is necessary for the hydrodynamic approximation.

  Another interesting characterization of the healing length is in the typical length scale that the ground state wave function goes from being
uniform to zero at the boundary. Essentially it is the solution to a constrained Laplacian problem. (However, it should be noted that, in the TF approximation the ground state wavefunction for a harmonic potential is not of this flattened blob type but of inverted parabola type with sharp edges.)

- Eikonal approximation - a high momentum approximation
- Hartree Fock ansatz

This is essentially the same as the dilute gas approximation. In the HF approximation we assume that the full wavefunction of the many body system is merely the tensor product of solutions to the GP equation. That is, if we solve the GP equation in a particular physical scenario as \( \Psi(x) \) then the full many body wavefunction, in the HF approximation is

\[
\Phi(x_1, \ldots, x_N) = \Psi(x_1) \ldots \Psi(x_N)
\] (D.52)

This approximation ignores the interactions among the elementary excitations and is not relevant to the case of perturbations around a solution where elementary excitations interact.

- Mean field theory

The mean field theory is applicable when the interatomic distance is smaller than the healing length. The mean field picture allows us to treat the condensate as a continuous field and not as a series of atoms.

- Thomas Fermi limit:

The thomas Fermi limit is that limit of the GP equation (that is, including interactions among the atoms) which neglects the quantum pressure (kinetic energy) term. The resulting TF solution is simply

\[
\Psi(r) = \sqrt{n_{TF}(r)} ; \quad n_{TF}(r) = \frac{1}{g} (\mu - V_{ext}(r)) ,
\] (D.53)

since the full equation of motion is

\[
\left( \frac{-\hbar^2}{2m} \nabla^2 + V_{ext} + g\Psi^2 \right) \Psi = \mu \Psi .
\] (D.54)

There is a smooth transition between the non-interacting solution and the TF solution given by the limits in the GP equation of the parameter

\[
\lambda_{TF} = \frac{N a_0}{a}
\] (D.55)
where $a_o$ is the length scale associated with the geometric mean of the oscillator frequencies.

An important feature of the TF approximation is that it allows us to express the chemical potential in terms of the other variables as

$$\mu = \frac{1}{2} \hbar \omega \left(15N a \sqrt{\frac{m \omega}{\hbar}}\right)^{2/5}.$$  \hspace{1cm} (D.56)

It is interesting to note that for a harmonic potential, the Schrödinger equation would most naturally give the gaussian ground state for the wave function but we see that the TF ground state is an inverted parabola. We expect the TF approximation to require corrections near the sharp edge of the parabola. There exists solution matching procedures that show exactly this sort of smoothing of the edge in the region where the potential can be approximated by a linear shape.
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