# INTRODUCTION TO QUANTUM REGGE CALCULUS

GTC 2011

Outline:

1. Classical Regge calculus
   - formalism
   - time evolution
2. Quantum Regge calculus
   - canonical. WDW equation
   - covariant
     - analytic
     - numerical
3. Conclusions

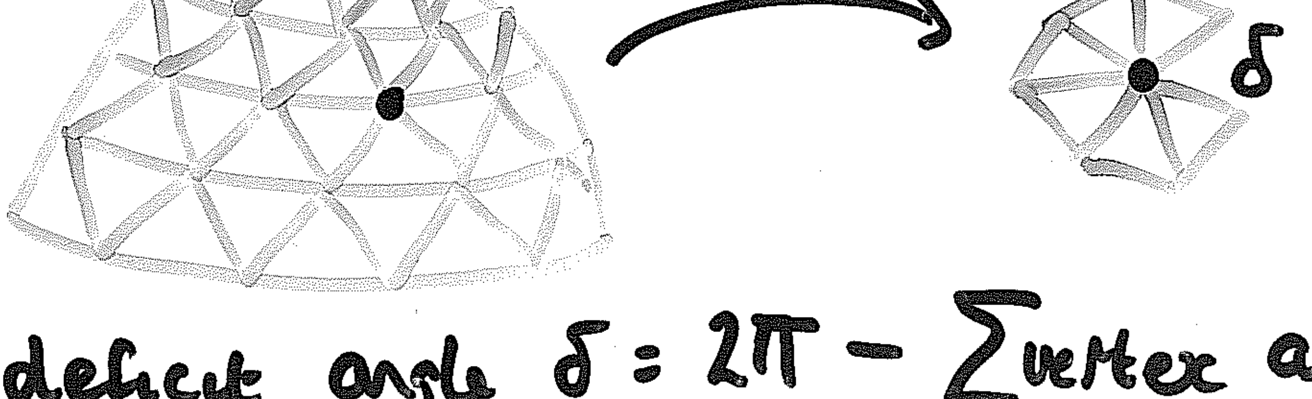
# 1. CLASSICAL REGGE CALCULUS

1961: Regge (b. 1931) – developed tool for numerical relativity.

"curved" spaces built from flat blocks.

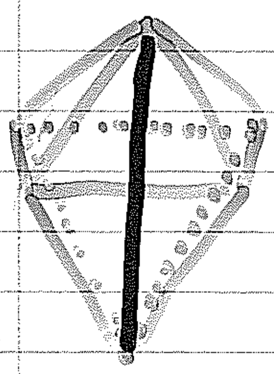
Discrete space (time) with curvature restricted to subspaces of codimension 2 – "hinges"

eg 2 dimensions: geodesic dome




deficit angle $\delta = 2\pi - \sum$ vertex angles.

3 dimensions: flat tetrahedra, curvature at edges.



deficit angle $=$ $2\pi - \sum$ dihedral angles.

4 dimensions: flat 4-simplices, curvature at triangles

and so on.

__Action:__

$$\frac{1}{2}\int d^n x \sqrt{|g|}\, R \rightarrow \sum V_h \,\delta_h$$

hinges $h$ — volume of hinge — deficit angle

Basic variables are the __edge lengths__ (cf metric)

Analogue of Einstein's equations: vary wrt edge length

$$\boxed{0 = \delta S = \sum_h \frac{\partial V_h}{\partial l_i} \delta_h} + \underbrace{\sum_h V_h \frac{\partial \delta_h}{\partial l_i}}_{=0}$$

(Schläfli identity)

(Extra term if add matter).

Use these equations to construct classical piecewise-linear Einstein spaces).

<u>Continuum limit</u> Cheeger, Müller + Schrader 1984. Regge action converges to continuum action in sense of measure provided simplices reasonably shaped.

# Time evolution

First classical applications to evolution of simple model universes. Important that each vertex in a space-like hypersurface can be evolved independently because of de-coupling of Regge equations (Sorkin algorithm).

New work: Dittrich + Höhn – canonical formalism using Pachner moves.

3+1 version Regge + Lund (unpublished) – more later.

# 2. QUANTUM REGGE CALCULUS



No reason not to try quantizing discrete spacetimes. May even be certain advantages eg natural cut-off (minimum length), finite no. of variables.

At the Planck scale, spacetime may be discrete anyway, with rapidly changing topologies (spacetime foam)

Approaches:

<u>Canonical:</u>

- quantize formalism of Dittrich and Höhn
- Khatsymovsky

- solve **Wheeler-DeWitt equation** 

(Hamber + RMW).

**Continuum:**

$$\left\{-(16\pi G)^2\, G_{ij,kl}(x)\, \frac{\delta^2}{\delta g_{ij}(x)\, \delta g_{kl}(x)} - \sqrt{g(x)}\left({}^3R(x) - 2\lambda\right)\right\} \Psi(g_{ij}(x)) = 0$$

with the DeWitt supermetric given by

$$G_{ij,kl} = \frac{1}{2\sqrt{g}}\left(g_{ik}g_{jl} + g_{il}g_{jk} + \alpha\, g_{ij}g_{kl}\right)$$

($\alpha = -1$ in 3-d)

Also need to satisfy diffeomorphism constraint.

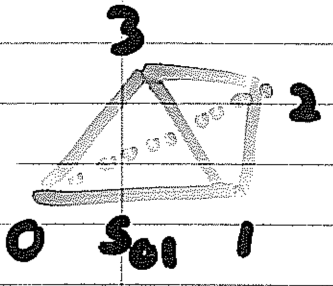
## Discrete

One WDW equation for each simplex.

Two ways of implementing "kinetic" term:

1. Use relation between metric and (edge lengths)$^2$ $\quad s_{ij} \equiv l_{ij}^2$

$$g_{ij} = \frac{1}{2}\left( s_{0i} + s_{0j} - s_{ij} \right)$$




in the simplex.

2. Use result of Regge & Lund:

$$G_{ij,kl} \frac{\delta^2}{\delta g_{ij}\, \delta g_{kl}} \longrightarrow G_{ij} \frac{\partial^2}{\partial s_i\, \partial s_j}$$

where $$G^{ij} = -\frac{d!}{V(\sigma)} \frac{\partial^2 V^2(\sigma)}{\partial s_i \, \partial s_j}$$

$V(\sigma)$ = volume of simplex $\sigma$

WDW equation is

$$\left\{ -(16\pi G)^2 \sum_{i,j \in \sigma} G_{ij}(\sigma) \frac{\partial^2}{\partial s_i \, \partial s_j} - n_d^{-1} \sum_{h \in \sigma} l_h \delta_h + 2\lambda V(\sigma) \right\} \Psi(\sigma) = 0$$

Both methods give same equation.

Results: very hard to solve for nonzero curvature. If neglect curvature, $\Psi \sim$ Bessel function of volume ((3+1)-d) or area ((2+1)-d)

# Covariant

## Ponzano-Regge model (1968)

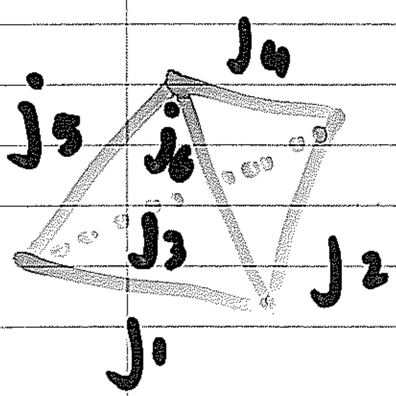
Earliest quantum Regge calculus.

Triangulated 3-manifold, edges labelled with reps of SU(2).

State sum:

$$Z(M) = \sum_{\substack{j_i \\ \text{interior} \\ \text{edges}}} \prod_{\substack{\text{interior} \\ \text{edges}}} (-1)^{2j_i}(2j_i+1)$$

$$\prod_{\text{tetrahedra}} \underbrace{\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}}_{\text{6j-symbol}}$$



As $j_i \to \infty$,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \sim \frac{1}{\sqrt{12\pi V}} \cos\left(\sum j_i \theta_i + \frac{\pi}{4}\right)$$

In this limit, $Z$ contains a term of the form

$$\underbrace{\int \prod_i dj_i \, (2j_i + 1) \left(\prod \frac{1}{\sqrt{V_R}}\right)}_{\text{"measure"}} e^{\overset{\uparrow}{iS}}$$

Regge action

of path integral for 3-d Regge gravity.

This leads to spin foams, group field theories etc!

Most work in 4-d uses the path integral approach. Analytic calculations very complicated

eg <u>calculation of the Regge propagator in the weak field limit</u> (Roček + RMW)

Perturbation theory about classical solution (flat space!)

- Many fluctuations decouple, leaving correct number of variables per vertex.
- Discrete and continuum propagators agree in weak field limit.

Numerical work Principal tool.

Several groups eg Tallahassee (Berg), Vienna (Beirl et al), Irvine (Hamber).

Focus on Hamber's work.

4-d Euclidean lattice (hypercubes divided into simplices).

Action: $S_R = \sum_h \left(\lambda V_h - k A_h \delta_h + a \frac{A_h^2 \delta_h^2}{V_h}\right)$

$k = \frac{1}{8\pi G}$, $V_h$ = volume per hinge

cf continuum action

$$S_c = \int d^4x \sqrt{g} \left(\lambda - \frac{k}{2} R + \frac{a}{4} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\right)$$

Higher derivative term introduced to ensure positivity of action but in practice can let $a \to 0$.

Partition function:

$$Z_R = \int \prod_s (V(s))^\sigma \prod_{ij} dl_{ij}\ \Theta(l_{ij}^2)\, e^{-S_R}$$

4-simplices

imposes triangle inequalities etc.

cf continuum

$$Z_c = \int \prod_x \left(\sqrt{g(x)}\right)^\sigma \prod_{\mu \geq \nu} dg_{\mu\nu}(x)\, e^{-S}$$

Measure is analogue of DeWitt Measure in continuum.

Method: Monte Carlo simulations

- changes made in edge lengths, rejected if increase action, accepted with certain probability if decrease it. System evolves to equilibrium state, about which it makes quantum fluctuations. Calculate expectation values and correlations

**Questions to ask:**

- stable ground state?
- second-order phase transition?
- cut-off independent quantities?

eg average curvature

$$\propto \langle l^2 \rangle \frac{\langle \sum A_h \delta_h \rangle}{\langle \sum V_h \rangle}$$

- correlation length?

correlations in scalar curvature at distance $d \underset{d \gg \xi}{\sim} e^{-d/\xi}$

correlation length.

etc.

**Results** limited by computer time

$4^4$ — $32^4$ lattices

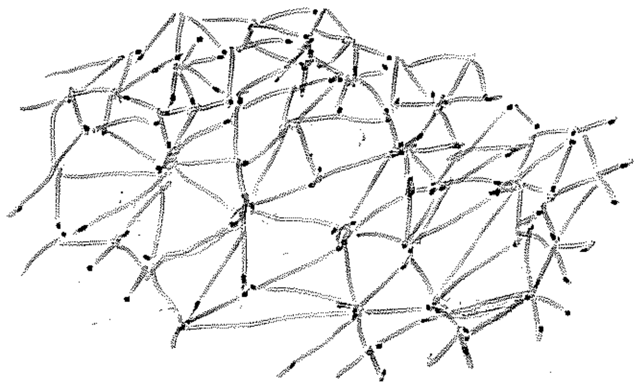
3840 edges ~ $15 \times 10^6$ edges.

Periodic boundary conditions

- average curvature negative for small $k$
- critical point $k_c$ s.t. $(\xi \to \infty)$

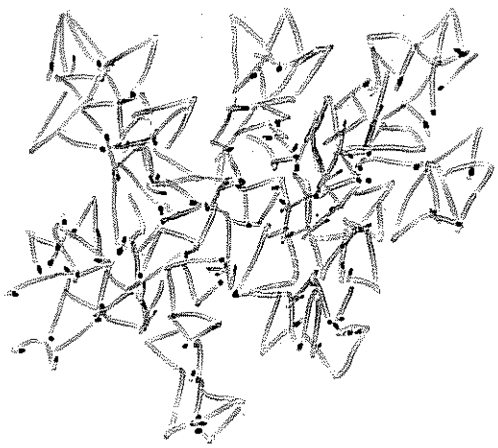
$\underline{k < k_c}$ : well-behaved ground state
$(G > G_c)$ - smooth phase

$R \simeq 0$



physical

$\underline{k > k_c}$ : branched polymer. Curvature
$(G < G_c)$ large - lattice collapses into degenerate configurations - very long thin simplices. Effective dimension 2.



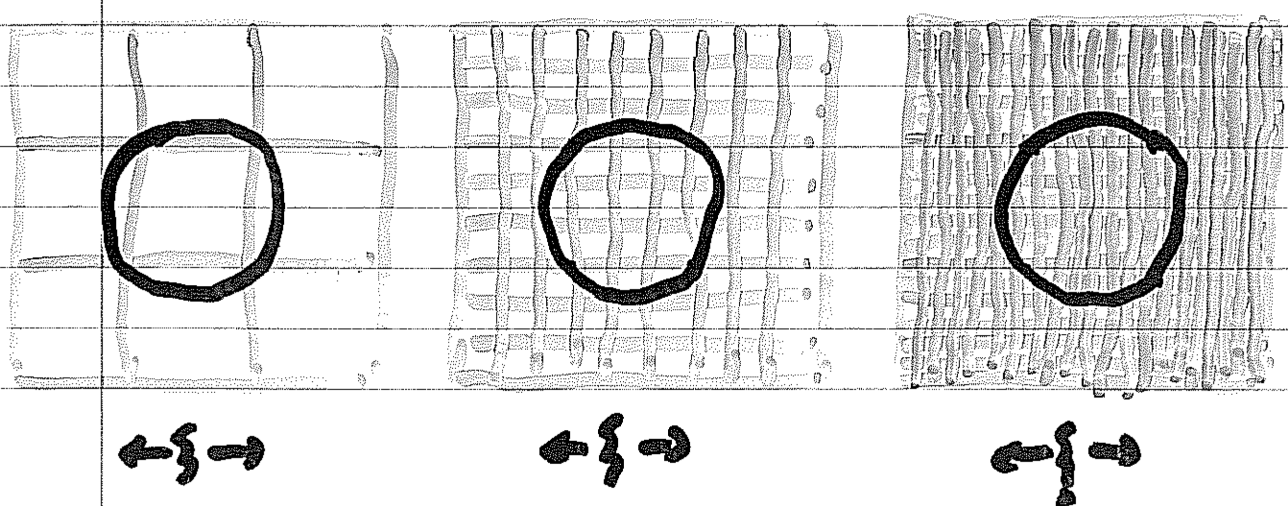
unphysical

Appears to be second-order phase transition $\Rightarrow$ continuum limit.

Evidence for continuous phase transition also found by Vienna group. Not seen in Euclidean dynamical triangulations — relation between theories needs to be understood.

What would happen in Hamber simulations in Lorentzian domain?

Lattice continuum limit gradually approached by considering sequences of lattices with increasingly large correlation lengths $\xi$ in lattice units:



# 3. CONCLUSIONS

Quantum Regge calculus has many problems (some inherited from continuum "quantization of GR") but it also has some "successes" eg PR model for 3-d, continuum limit in 4-d simulations.

Space-time is treated as fundamental, rather than being emergent, and the analogue of the Einstein equations derived. Quantization mainly uses methods of statistical field theory. (Romano paper)