Structure of the WDW equation

Diffeomorphism invariance and emergence of probabilistic interpretation in QC

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There is a **probabilistic interpretation** of the solutions of the Schrödinger eq., but there is **no consensus** on the interpretation of the solutions of the WDW eq.

Several options have been proposed:

- "Page-Hawking" based on the norm: $|\psi(a, \phi)|^2$
- "Vilenkin" based on the current: $\mathcal{W} = \psi^* i \partial_a \psi$
- Third quantization.
- Square root approach $i \partial_x \psi = H \psi$ (Ashtekar (2008))
- Adding "dust" (Timan (2006))
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- **Compare** (mathem.) the *structure* of the Schr. eq. to that of the WdW eq.

- Focus on **Transition Amplitudes** and use *molecular physics techniques* as a **tool** to perform the comparison.

- Establish that
  - no **exact** proba. interp. could possibly be given: 
    - **Diffeo-invar.** + **canon. quantization** → no proba. interp.
  - the **Schrod. probabilistic interpretation** is an **emergent** property of the sol. of the WDW eq.
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Why focusing on Transition Amplitudes?

- The WdW eq. also describes matter transitions, e.g. $e^+ e^-$ annihilation.
  → How to compute their amplitudes?

- Historically, Born’s statistical interpretation (1926)

  \[
  |n_0\rangle \rightarrow \sum_n c_{n,n_0} |n\rangle
  \]

  \[
  |c_{n,n_0}|^2 = \text{Proba. to find } |n\rangle \text{ at late time, (1)}
  \]
  when starting from $|n_0\rangle$.

  followed from his understanding of Transition Amplitude.

- Follow here the same logic:
  study the properties of $C_n(a)$, Trans. Amplit. in QC, then consider their interpretation.
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Why using *molecular physics techniques*?

- Appropriate for dynamical systems containing **light** and **heavy** (i.e. fast and slow) deg. of freedom.

- **Transition amplitudes** are governed by **frequency ratios** and not by **coupling constants** (called *Non-Adiabatic Transition Amplitudes NATA*).

- Certain NATA are **exponentially suppressed** w.r.t. others, thereby introducing a **hierarchy of NATA**.

- **Use this hierarchy** to
  - **Organize** solutions of the WDW eq.
  - **Get algebraic relations** between Schrod’s $c_n(t)$ and WDW’s $C_n(a)$.

- **Long tradition**: Born ’26, Heisenberg ’35, Gottfried ’66 + ’98
  All considering the statistical interpret. of QM.
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The Schroed. equation in cosmology.

- **For definiteness**, consider some matter fields governed by an hermitian $H_M$ in an expanding **compact** RW sp-time described by $a(t)$.

- The S. eq. is
  \[ i\partial_t |\psi(t)\rangle = H_M |\psi(t)\rangle. \tag{2} \]

- Because of the expansion $da/dt > 0$,
  \[ H_M = H_M(a(t)) \]
  is $t$-dependent. **This is crucial for what follows.**
To reveal its *structure*, and/or to solve it, introduce the *instantaneous eigenstates* of $H_m(a)$:

\[
H_M(a) \left| \chi_n(a) \right> = E_n(a) \left| \chi_n(a) \right>
\]
\[
\langle \chi_n(a) | \chi_m(a) \rangle = \delta_{n,m}
\] (3)

- decompose $|\psi(t)\rangle$ in this *basis*:

\[
|\psi(t)\rangle = \sum_n c_n(t) e^{-i \int^t dt' E_n(t')} \left| \chi_n(a(t)) \right>,
\] (4)

- compute the $c_n(t)$ by injecting (4) in $i \partial_t |\psi\rangle = H_M |\psi\rangle$. 
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\[ |\psi(t)\rangle = \sum_n c_n(t) e^{-i \int^t dt' E_n(t')} |\chi_n(a(t))\rangle, \] (4)

 compute the $c_n(t)$ by injecting (4) in $i\partial_t|\psi\rangle = H_M |\psi\rangle$. 

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Structure of the WDW equation
This gives a $N \times N$ matricial eq.

$$
\partial_t c_n = \sum_m \langle \partial_t \chi_m | \chi_n \rangle e^{-i \int^t \! dt' (E_m - E_n)} c_m.
$$

(5)

where

$$
\langle \chi_m | \partial_t \chi_n \rangle = \frac{\langle \chi_m | \partial_t H_M | \chi_n \rangle}{E_n(a) - E_m(a)}, \quad n \neq m
$$

(6)

NB. When $da/dt = 0$, one has

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\partial_t c_n \equiv 0.
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Hence all NA transitions are induced by $da/dt$. 
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Structure of the WDW equation
The amplitudes $d_n(a)$ defined in the corresponding contracting universe, $da/dt \to -da/dt$, obey the same equation with $i \to -i$.

The Schrod. eq. can be written in terms of $a$:

$$\partial_a c_n = \sum_{m \neq n} \langle \partial_a \chi_m | \chi_n \rangle \ e^{-i \int^a da' (dt/da') (E_m - E_n)} c_m(a).$$  \hspace{1cm} (7)

The cosmic time $t$ only appears through $t(a)$. 
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The cosmic time $t$ only appears through $t(a)$.
Starting with $c_1(t = -\infty) = 1$, the amplitude to find the system in the state $n$ is $c_n(+\infty)$.

To first order in NA, it is

$$c_n(+\infty) \simeq \int_{-\infty}^{+\infty} dt \langle \chi_n | \partial_t \chi_1 \rangle \ e^{-i \int_{-\infty}^{t} dt' (E_1(t') - E_n(t'))}$$

When $c_n \ll 1$, can be evaluated by a saddle point approx.

The sp time $t^*$ is complex, and hence $c_n$ is expon. damped:

$$c_n(+\infty) \simeq C \ e^{-i \int_{-\infty}^{t^*} dt' (E_1 - E_n)}$$

NB. $C \to 1$ in the adiabatic limit, see refs.
Starting with \( c_1(t = -\infty) = 1 \), the amplitude to find the system in the state \( n \) is \( c_n(\infty) \).

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We now replace \([-i\partial_t + H_M] |\psi(t)\rangle = 0\) by the WDW eq.

\[
[ H_G + H_M ] |\psi(a)\rangle = 0
\]  \(\text{(9)}\)

where

\[
H_G = \frac{-G^2 \pi_a^2 - a^2 + \Lambda a^4}{2Ga}.
\]  \(\text{(10)}\)

\(\hat{\pi}_a = -i\partial_a\) is the momentum conjugated to \(a\).

With (9, 10) we have displaced the Heisenberg cut so as to include \((a, \pi_a)\) in the quantum description.

Eqs. (9, 10) follow from \(S = \int d^4x \sqrt{g}R + L_M\) when

- 3 geometries are compact, and
- matter distribution is sufficiently homogeneous.

N.B. \(H_G + H_M = 0\) is the Friedmann eq.
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The unique conserved quantity associated with the Schrod. eq. is the norm

\[ N = \langle \psi(t) | \psi(t) \rangle \equiv Cst. = \sum_n |c_n(t)|^2. \]  \hspace{1cm} (11)

Instead the unique one associated with the WDW eq. is the Wronskian

\[ W = \langle \psi(a) \mid i \delta_a \mid \psi(a) \rangle \equiv Cst. \]  \hspace{1cm} (12)

Question: Can \( W \) be written as \( W = \sum_n |C_n(a)|^2 \)?
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Question: Can \( W \) be written as \( W = \sum_n |C_n(a)|^2 \)?
Decompose

\[ |\psi(a)\rangle = \sum_{n} C_{n}(a) \psi_{n}(a) |\chi_{n}(a)\rangle, \]  

where \( |\chi_{n}(a)\rangle \) is the same as for the Schrod. eq. and \( \psi_{n}(a) \) is the WKB solution of

\[ [H_{G} + E_{n}(a)] \psi_{n}(a) = 0 \]  

The unit positive Wronskian WKB solution is

\[ \psi_{n}(a) = \frac{e^{-i \int^{a} da \ p_{n}(a)}}{\sqrt{2p_{n}(a)}} \]  

where the momentum \( p_{n}(a) > 0 \) solves \( H_{G} + E_{n}(a) = 0 \). These solutions correspond to expanding universes. Their complex conjugated describe contracting universes.
Decompose

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These solutions correspond to expanding universes. Their complex conjugated describe contracting universes.
inserting (13) in the WDW eq. gives a second order matricial equation which mixes

- corrections to WKB corrections with
- \( n \rightarrow n' \) matter transitions.

Moreover the conserved Wronskian \( W \) reads

\[
W = \sum_n |C_n(a)|^2 + \sum_n (C_n^* i \tilde{\partial}_a C_n) |\Psi_n(a)|^2. \tag{16}
\]

The first term is OK, the second unwanted.

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The "correct" decomposition

\[ |\psi(a)\rangle = \sum_n [C_n(a) \psi_n(a) + D_n(a) \psi^*_n(a)] |\chi_n(a)\rangle, \quad (17) \]

introduces \(2N\) arbitrary functions \(C_n(a), D_n(a)\).

Eq. (17) gives an **under-constrained** system.

**Exploit** this and impose

\[ \langle \chi_n | \overrightarrow{i\partial}_a |\psi\rangle = p_n [C_n \psi_n - D_n \psi^*_n]. \quad (18) \]

Now the \(C_n \ (D_n)\) **instantaneously** weigh expanding (contracting) solutions.

Insert (17) in the WdW eq. *and* use (18) to get
The 'correct' decomposition

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Insert (17) in the WdW eq. and use (18) to get
Structure of the WDW equation, 1

\[ \partial_a C_n = \sum_{m \neq n} \tilde{M}_{nm} e^{-i \int^a (p_n - p_m) da} C_m \]
\[ + \sum_m \tilde{N}_{nm} e^{-i \int^a (p_n + p_m) da} D_m. \]  (19)

+ the same eq. with \( C_n \rightarrow D_n \) and \( i \rightarrow -i \)

where

\[ \tilde{M}_{nm} = \langle \partial_a \psi_m | \psi_n \rangle \frac{p_n + p_m}{2\sqrt{p_n p_m}} \]  (20)

\[ \tilde{N}_{nm} = \langle \partial_a \psi_m | \psi_n \rangle \frac{p_n - p_m}{2\sqrt{p_n p_m}} + \delta_{nm} \frac{\partial_a p_n}{2p_n} \]  (21)

- Eqs. (19,20,21) are **exact**

- Eq.(19) is the **WDW eq.**
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\[ \tilde{N}_{nm} = \langle \partial_a \psi_m | \psi_n \rangle \frac{p_{n} - p_{m}}{2 \sqrt{p_{n} p_{m}}} + \delta_{nm} \frac{\partial_a p_n}{2 p_n} \quad (21) \]

Eqs. (19, 20, 21) are **exact**

Eq.(19) is the WDW eq.
$$\partial_a C_n = \sum_{m \neq n} \tilde{M}_{nm} e^{-i \int^a (p_n - p_m) da} C_m$$

$$+ \sum_m \tilde{N}_{nm} e^{-i \int^a (p_n + p_m) da} D_m. \quad (19)$$

+ the same eq. with $C_n \rightarrow D_n$ and $i \rightarrow -i$

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Moreover, as an identity, one has

\[ \mathcal{W} = \langle \psi(a) | i \partial_a \psi(a) \rangle \]

\[ = \sum_n |C_n(a)|^2 - |D_n(a)|^2 \equiv \text{Const.} \]  \hspace{1cm} (22)

for any Hermitian matter hamiltonian \( H_M \), and any state.

Let’s describe the consequences of Eq. 19 and 22.
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Let’s describe the consequences of Eq. 19 and 22.
Theorem 1

When, for a given matter Hamiltonian $H_M$, the Schrod. eq. gives a $N \times N$ first order matricial eq. the WDW eq. gives a $2N \times 2N$ first order matricial eq.

NB. A similar doubling is also found in non-relativistic molecular/atomic physics.

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*It is a general conseq. of QM, when moving the H-cut.*
Neglecting only the $C_n \rightarrow D_n$ transitions, one gets

$$\partial_a C_n = \sum_{m \neq n} \tilde{M}_{nm} e^{-i \int^a (p_n - p_m) da} C_m \quad (23)$$

and

$$\sum_n |C_n(a)|^2 \equiv \text{Cst.}$$

+ the same eq. with $C_n \rightarrow D_n$ and $i \rightarrow -i$.

WDW eq. thus gives two separate (and unitary) eqs: one for the expanding sector, one for the contracting.

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Theorem 3. The recovery of cosmic time

Neglecting the $C_n \to D_n$ transitions, and to first order in $E_n - E_\bar{n}$ around the mean matter state $\bar{n}$,

$$C_n(a) \equiv c_n(\bar{t}(a)),$$

(24)

where

$$\bar{t}(a) = \int^a da' \partial_E p(a') |_{E = E_\bar{n}},$$

(25)

is the HJ time to reach $a$ when matter is in the $\bar{n}$ state.

The identity (24) is Heisenberg-’35 result. It follows from

$$\tilde{M}_{nm} \equiv M^S_{mn}$$

$$e^{-i \int^a da' (\rho_n - \rho_m)} \equiv e^{-i \int^t dt' (E_n - E_m)}$$

Renaud Parentani Structure of the WDW equation
Theorem 3. The recovery of cosmic time

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For the Schrod. eq., \( c_n \) is given by the overlap

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c_n(t) = \langle \chi_n(t) | e^{i \int^t dt' E_n} | \psi(t) \rangle.
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Instead, for the WDW eq., \( C_n \) is given by

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This is the Vilenkin current, and NOT the Page-Hawking overlap \( \langle \chi_n(a) | \psi(a) \rangle \).
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Theorem 4. The $\mathcal{C}_n \rightarrow \mathcal{D}_m$ NATTransitions.

- Far from turning points (big bounce), the transitions $\mathcal{C}_n \rightarrow \mathcal{D}_m$ are exponentially suppressed w.r.t. to the $\mathcal{C}_n \rightarrow \mathcal{C}_m$.

- Because, typically

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\begin{align*}
\mathcal{C}_n \rightarrow \mathcal{C}_m & \sim e^{-(p_n - p_m) \text{Im}\Delta a^s_C} \\
\mathcal{C}_n \rightarrow \mathcal{D}_m & \sim e^{-(p_n + p_m) \text{Im}\Delta a^s_D}
\end{align*}
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$p_n + p_m$ scales with the total matter energy.

- The Lesson:
  The WDW eq. predicts a hierarchy of NAT, and governs it.
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The WDW eq. predicts a hierarchy of NAT, and governs it.
Consider a molecule of mass $M$ and CM $R$

$$H_{\text{atom}} = \frac{\mathbf{p}^2}{2M} + V(R) + H_e(R),$$

where $H_e(R)$ governs the electronic deg. of freedom.

There are two levels of quantization:

The LZ way: treat $R$ classically as a given function of time $R(t)$. The residual elect. d.o.f. then obey the $t$-dep. Schrod. eq.

$$i\partial_t |\psi_e\rangle = \hat{H}_e(R(t))|\psi_e\rangle$$

The WdW way: treat both $R$ and el. d.o.f on the same footing. Then, at fixed total energy $E$, one has

$$\left[-\frac{\partial^2}{2M} + V(R) + H_{el}(R)\right] |\psi_E(R)\rangle = E |\psi_E(R)\rangle$$
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Decomposing $|\Psi_{E}(R)\rangle = \sum_{n}(C_{n}\psi_{n} + D_{n}\psi_{n}^{*})|\chi_{n}(R)\rangle$

- neglecting $C_{n} \rightarrow D_{m}$ and to first order in $E_{n}^{e} - E_{m}^{e}$, one gets

$$C_{n}(R) \equiv c_{n}(\bar{t}(R)),$$

where $c_{n}$ are the Landau-Z tr. amplitudes.

Hence the $|C_{n}|^{2}$ give transition probabilities.

- The $C_{n}(R)$ are given by $C_{n} \equiv \langle \chi_{n}|\psi_{n}^{*} i\partial_{R}^{\rightarrow} |\psi_{E}\rangle$, i.e. by the "Vilenkin" current.

- Taking into account the NAT $C_{n} \rightarrow D_{m}$, one gets, as an identity,

$$\sum_{n} |C_{n}(R)|^{2} - |D_{n}(R)|^{2} \equiv \text{Const.}$$

This is a re-expression of unitarity.

- The H-Page norm $|\langle \chi_{n}|\psi_{E}\rangle|^{2} = |C_{n}\psi_{n}(R) + D_{n}\psi_{n}^{*}(R)|^{2}$ is

  - highly interferent, and
  - answers another question.
Conclusions. 1

- The WDW eq. determines the NATA. There is NO ambiguity in **computing** these amplitudes.

- It predicts that there is a **hierarchy** of NA-regimes.
Conclusions. 2. The S. regime, the mildest one.

- The first regime, the mildest one, is obtained by
  - neglecting $C_n \rightarrow D_m$ NAT,
  - the first order in $E_n - E_{\bar{n}}$, but nothing else.

- In this regime
  
  $$C_n(a) \equiv c_n(\bar{t}(a)).$$

Therefore $|C_n(a)|^2$ is the proba. to find the state $n$ in an expand. universe at $a$ (in fact around $a$).
Conclusions. 2. The S. regime, the mildest one, 2.

- Unitarity,

\[ \sum_n |C_n(a)|^2 \equiv Cst. \]

relies on the conserved current \( W \).

- NO problem of interpretation, NO problem of time.
Conclusions. 2. The S. regime, the mildest one, 2.

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Neglecting only $C_n \rightarrow D_m$, the WDW gives two separate 1st order eqs. in $i\partial_a$.

Unitarity, i.e. $\sum_n |C_n(a)|^2 \equiv Cst.$ is obtained, as in the Schrod. regime, from the conservation of $W$, even though there is no cosmic time because no backd sp-time common to all matter states.

Lesson:
The recovery of cosmic time requires more condition than the recovery of unitarity.
Neglecting **only** \( C_n \to D_m \), the WDW gives **two separate 1st order eqs. in** \( i\partial_a \).

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**Lesson:** The **recovery of cosmic time** requires more condition than the **recovery of unitarity**.
Conclusions. 3. The second intermediate regime.

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**Lesson:**
The **recovery of cosmic time** requires more condition than the **recovery of unitarity**.
Conclusions. 4. The least Adiabatic regime.

- Taking into account $C_n \rightarrow D_m$ NAT, one faces the identity

$$\sum_n |C_n(a)|^2 - |D_n(a)|^2 \equiv \text{Cst.}$$

- How to interpret this equation?

- **Proposal:** as we just did it for cosmic time: the recovery of unitarity requires some conditions.

- Hence we conclude/conjecture:

  The *statistical interpretation* of the NATA $C_n(a)$ is *not* as a *fundamental property*, but should be conceived as an *emergent property* of QC.

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