Quantum gravity and critical phenomena

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The general idea:
  - A brief reminder of Wilson’s RG and continuum limit ideas.
  - How could it possibly work for gravity

A concrete realization:
  - Lattice approach: (causal) dynamical triangulations

Outlook
Wilsonian renormalization group

Momentum-shell integration (or coarse graining)

\[ e^{-\Gamma_{k,\Lambda}} = \int_{k \leq p \leq \Lambda} [D\phi] \, e^{-\Gamma_{\Lambda}} \]

- It defines a flow of effective actions.
- Initial condition: \( \Gamma_{k=\Lambda,\Lambda} = \Gamma_{\Lambda} \)
- Crucial question in QFT:
  can we send \( \Lambda \to \infty \) while holding fixed the physics at scale \( k \)?
Fixed point

If the flow of effective actions has a Fixed Point
⇒ we can define a continuum limit, characterized by value of relevant couplings

[Wilson - ~'70]

[ Theory space: space of all possible initial conditions $\Gamma_\Lambda$. Arrows point towards IR. ]
Fixed points and continuum limit

\[ \Sigma_{IR} = \{ \text{irrelevant trajectories} \} : \xi = \infty \]

- \( \Sigma_{IR} \) is the locus of a 2nd order phase transition
- While sending \( \Lambda \sim \frac{1}{a} \to \infty \), tune \( \Gamma_\Lambda \) to \( \Sigma_{IR} \), where \( \xi_L \to \infty \), such that \( m = \frac{1}{a \xi_L} < \infty \).
In the continuum limit the irrelevant couplings take their FP value

- Relevant couplings at scale $k$ are fixed, as in renormalization conditions
- $\dim(\Sigma_{UV}) < \infty$ guarantees that only a finite number of couplings have to be fixed by hand (or by experiment)
Fixed points and continuum limit

\[ \Sigma_{\text{IR}} = \{ \text{irrelevant trajectories} \} : \xi = \infty \]

- In discrete approach we study \( \Gamma_{\Lambda} \) and look for phase transitions;
- in continuum approach we study \( \Gamma_k \) and look for UV-FP.
The Asymptotic Safety scenario

Gravity: $G$ (Newton’s constant) is an irrelevant coupling at the Gaussian FP (perturbative non-renormalizability)

Definition of asymptotic safety

UV controlled by a non-Gaussian fixed point (NGFP) with a finite number of relevant directions

[Weinberg - '79]
Tuning relevant parameters

- Relevant parameters are tuned to obtain phase transitions
  e.g. $m^2 > 0 \rightarrow m^2 < 0$ for paramagnetic to ferromagnetic phase transition

- Is there an analogue picture in gravity?
Phases of gravity?

- We know of a fixed point for sure, the Gaussian one
- $\Lambda$: from compact to non-compact spacetime
  - Gaussian fixed point
  - only one relevant parameter: $\Lambda$
  - only useful for defining thermodynamic limit
Phases of gravity?

- NGFP?
- Tune also $G_\Lambda$ (and more?) to obtain cont.lim. with finite $G_k$
- Old hypothesis for role of such transition:
  from extended ($\langle g_{\mu\nu} \rangle \neq 0$) to crumpled ($\langle g_{\mu\nu} \rangle = 0$) geometry
The lattice approach: Dynamical Triangulations
A statistical approach to lattice quantum gravity

The main idea is to discretize the path integral

\[ \int_{\text{geom}} \mathcal{D}[g] e^{-S[g]}, \]

directly sampling the space of (diffeo equivalent) Riemannian geometries, by means of \textit{piecewise flat geometries}:

\[ \int_{\text{geom}} \mathcal{D}[g] \rightarrow \sum_{\text{p.f.g.}}. \]
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\implies \text{A statistical model of random geometries.}
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⇒ A statistical model of random geometries.

Define a statistical ensemble

“equilateral triangulations” \( T \) of fixed topology
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- Define a statistical ensemble
- Define the Boltzmann weight

\[
A(T) = e^{-S[T]}, \quad S[T] = \kappa_D N_D - \kappa_{D-2} N_{D-2}
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- Define a statistical ensemble
- Define the Boltzmann weight
- Construct partition function and observables

\[ Z = \sum_{T} e^{-S(T)} \]
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⇒ A statistical model of random geometries.

- Define a statistical ensemble
- Define the Boltzmann weight
- Construct partition function and observables
- Study the phase diagram of the model
Thermodynamic limit

- Phase transitions only really occur in the thermodynamic limit
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- The average (discrete) volume is given by

$$\langle N_D \rangle = \frac{1}{Z} \sum_T e^{-S(T)} N_D(T) = -\frac{\partial}{\partial \kappa_D} \ln Z$$
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  \Rightarrow Thermodynamic limit, or infinite (lattice) volume limit: tune \( \kappa_D \rightarrow \kappa_D^c \)

\[
\langle N_D \rangle \sim \frac{1}{\kappa_D - \kappa_D^c} \rightarrow \infty
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- For the continuum limit we want

\[
a \rightarrow 0, \quad \text{s.t.} \quad a^D \langle N_D \rangle = V \sim \frac{1}{\Lambda_{IR}^2} < \infty
\]

We are tuning the bare \( \Lambda_{UV}(\kappa_D) \) to its critical value, keeping \( \Lambda_{IR} \) fixed!
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  We are tuning the bare \(\Lambda_{UV} (\kappa_D)\) to its critical value, keeping \(\Lambda_{IR}\) fixed!

- Usually not enough: in order to have propagating degrees of freedom we need to tune some other coupling to a 2nd order phase transition
Phases of dynamical triangulations

- Hausdorff dimension as an order parameter:

\[ \langle n(r) \rangle \sim r^{d_H - 1} \]

\[ d_H \sim \infty \]

\[ d_H \sim 2 \]

[ Agishtein, Migdal; Ambjørn, Jurkiewicz; Bialas et al.; Catteral, Kogut, et al.; de Bakker, Smit; etc. ]
Phases of dynamical triangulations

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  \[
  \langle n(r) \rangle \sim r^{d_H - 1}
  \]
  \[
  d_H \sim \infty \quad d_H \sim 2
  \]

- No classical geometry
- Phase transition is 1st order: no diverging correlation length :-(
Changing the phase diagram by non-local constraints

The “causality” condition:  

- introduce a foliation in triangulation
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- forbid “spatial” topology change $\Rightarrow$ no “baby universes”
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- introduce a foliation in triangulation
- in standard DT baby universes appear everywhere (in continuum limit)
- forbid “spatial” topology change $\Rightarrow$ no “baby universes”

[ Ambjørn, Loll ]
Phases of causal dynamical triangulations (CDT)

\[ \kappa_4 \quad \kappa_{4}^{\text{crit}} (\kappa_0, \Delta) \]

[Ambjørn, Jurkiewicz, Loll, Görlich, Jordan]
Phases of causal dynamical triangulations (CDT)

- phase A: \( d_H \sim 2 \).

- phase B: \( d_H \sim \infty \).

- phase C: \( d_H \sim 4 \); profile averages to 4-sphere.

[Ambjørn, Jurkiewicz, Loll, Görlich, Jordan]
Comparison to Hořava-Lifshitz gravity

- Hořava has proposed an anisotropic theory of gravity. In $m + 1$ dimensions he proposed an action of the type:

$$S = \int dt d^m x \sqrt{g} \left\{ \frac{2}{\kappa^2} \left[ \lambda K^2 - K_{ij} K^{ij} \right] + \mathcal{V}(g_{ij}) \right\}$$

- Pro:
  For a potential $\mathcal{V}$ containing up to $2z = 2m$ spatial derivatives of the fluctuations of the spatial metric $g_{ij}$, the theory is power-counting renormalizable, and with no higher-derivatives in time.

- Con:
  Spacetime diffeomorphisms are reduced to foliation-preserving diffeomorphisms.
  \[ \Rightarrow \text{new scalar dof} \]
Comparison to Hořava-Lifshitz gravity

The Landau free energy for a generic \((d, m)\)-Lifshitz point with scalar order parameter is

\[
    F = \frac{1}{2} \int d^d x \left[ c_\parallel (\nabla_\parallel \phi)^2 + c_\perp (\nabla_\perp \phi)^2 + D (\nabla_\perp^2 \phi)^2 + r \phi^2 + u \phi^4 \right]
\]

The point \(r = 0, c_\perp = 0\) is a Lifshitz point.

It is tempting to compare to the CDT phase diagram.
Phases of gravity?

- phase A ↔ modulated phase: Lorentzian geometries?

- phase B ↔ paramagnetic phase: 
  \[ \langle g_{\mu\nu} \rangle = 0 \] ?

- phase C: ↔ ferromagnetic phase: 
  \[ \langle g_{\mu\nu} \rangle \neq 0 \] ?
Spectral dimension

- We can say a little bit more for the comparison
  - Spectral dimension measured in $d = 4$ CDT:
    - from $4$ (IR) to $2$ (UV)
    
    [Ambjorn et al.]
  
  - Simple argument in AS: from $d$ (IR) to $d/2$ (UV)
    
    [Lauscher, Reuter]
  
  - Simple argument in HL: from $d$ (IR) to $2$ (UV)
    
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- We found that for $d = 3$ CDT
  [DB, Henson]

\[ D_s(\infty, N_3 = 200,000) = 3.05 \pm 0.04 \]
\[ D_s(0, N_3 = 200,000) = 2.04 \pm 0.10 \]
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- The data can also be fitted with HL-gravity computation at intermediate diffusion times [Sotiriou, Visser, Weinfurtner]
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- Caveat: we are only in the thermodynamic limit
At large diffusion time, spectral curve matches that of a stretched sphere \((s \neq 1)\):

\[
ds^2_{S^3} = r^2 \left( s^2 \, d\psi^2 + \sin^2 \psi \, (d\theta^2 + \sin^2 \theta \, d\phi^2) \right)
\]

This is actually a solution of HL-gravity in \(2 + 1\) dimensions.
HL or o HL?

- Recapitulating, evidence for a CDT-HL relation comes from
  - Presence of a foliation
  - Analogy in phase diagram
  - Short-scale spectral dimension
  - Large-scale shape (stretched sphere)
HL or o HL?

- Nevertheless CDT results are not at continuum limit...

- Triple-point and end-point are different fixed points?
- Different universality classes? (Isotropic and anisotropic?)
Alternative modifications of the ensemble and/or of the amplitudes?

For example, (non-causal) DT models with modified amplitude:

\[ A(T) = e^{-\kappa_D N_D + \kappa_{D-2} N_{D-2}} \prod_i q_i^{-\beta}, \]

\[ q_i = \# \text{ of } D\text{-simplices sharing } (D - 2)\text{-simplex } i \]
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► For example, (non-causal) DT models with modified amplitude:

\[ A(T) = e^{-\kappa D N_D + \kappa D_{-2} N_{D_{-2}}} \prod_{i} q_i^{-\beta}, \]

► Numerical simulations \((D = 4)\) show signs of a new phase:

[Bilke et. al. '98; Laiho, Coumbe 2011]
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[Laiho, Coumbe 2011]

\[
\begin{align*}
D_s(\infty, N_4 = 4,000) &= 4.04 \pm 0.26 \\
D_s(0, N_4 = 4,000) &= 1.457 \pm 0.064
\end{align*}
\]
Alternative modifications of the ensemble and/or of the amplitudes?

- For example, (non-causal) DT models with modified amplitude:

  \[ A(T) = e^{-\kappa_D N_D + \kappa_D - 2 N_D - 2} \prod_i q_i^{-\beta}, \]

- Exact solution (from tensor model) at large \( \kappa_D - 2 \sim \ln N \): [DB, Gurau]

  \[
e^{N^D E} = \int d\bar{\psi} d\psi \ e^{-S(\psi, \bar{\psi})},
\]

  \[
S(\psi, \bar{\psi}) = \sum_{i=0}^{D} \sum_{\vec{p}_i, \vec{n}_i} \psi_i^{\vec{p}_i} \left( \prod_j (C^{-1})_{p_{ij} \bar{n}_{ij}} \right) \bar{\psi}_i^{\vec{n}_i} \\
+ \frac{\lambda}{N^D (D-1)/4} \sum_{\vec{n}} \prod_{i=0}^{D} \psi_i^{\vec{n}_i} + \frac{\bar{\lambda}}{N^D (D-1)/4} \sum_{\bar{n}} \prod_{i=0}^{D} \bar{\psi}_i^{\vec{n}_i},
\]

  \[
\text{Tr}[(C^T C)^q] = N q^{-\beta}
\]
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Susceptibility exponent \( E_{\text{non-analytic}} \sim (g - g_c(\beta))^{2-\gamma} \):

\[
\gamma = \begin{cases} 
  \frac{1}{2} & \text{for } \beta < \beta_c \\
  \frac{1}{2} - \beta & \text{for } \beta > \beta_c 
\end{cases}
\]

\[
\frac{d^3 g_c}{d\beta^3} \sim (\beta_c - \beta)^{2-2\beta_c} \left( \beta_c - \frac{1}{2} \right)
\]

\( \beta_c \approx 1.216 \Rightarrow 3\text{rd order phase transition} \)
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Outlook
The continuum limit?

- Analytical work in the continuum suggests existence of a NGFP – ongoing work aimed at giving more robustness to such results
  [Reuter, ...; Percacci, ...; Litim, ...; Saueressig, ...; DB, ... ]

- Numerical work in the discrete provides evidence for emergence of classical geometries and for two candidate critical points for c.l.
  [Ambjørn, Jurkiewicz, Loll, Görlich, Jordan; DB, Henson]
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   ⇒ asymptotic safety

2. Anisotropy survives in continuum limit?
   ⇒ Hořava-Lifshitz gravity
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- The (non-causal) models might also come back from their tomb:
  We proved that in DT the measure term is not inessential

[DB, Gurau; see also results of Laiho, Coumbe]