Modified energy-momentum conservation laws and vacuum Cherenkov radiation

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1. Introduction
2. Two-body decay kinematics beyond SR
3. Threshold analysis
4. Bounds on vacuum Cherenkov radiation
5. Some final comments

Introduction

$$\boxed{e \rightarrow e\,\gamma}$$

- Forbidden reaction in Special Relativity (SR)
- A modified dispersion relation can change this
- Effective field theory framework (EFT):
  - Standard energy-momentum conservation law:

$$(-k) + p + q = 0$$

  - Dynamical theory: computation of matrix elements
- But… EFT might be too restrictive!
  - Doubly Special Relativity (DSR) ⟶ ~~standard E-p conservation law~~
  - Relative locality: curvature of momentum space induces a modified composition law, and then, a modified conservation law:

$$\hat{k} \oplus p \oplus q = 0$$

Introduction

Ingredients of the new kinematics beyond SR:

- A (rotational-invariant) modified dispersion relation for particles (MDR)
- A (rotational-invariant) modified composition law (MCL)
- Momentum power expansion at first order in the inverse energy scale Λ^{-1}
- Non-universality

$$C^{(a)}(p) = p_0^2 - \vec{p}^2 + \frac{\alpha_1^a}{\Lambda} p_0^3 + \frac{\alpha_2^a}{\Lambda} p_0 \vec{p}^2 = m^2$$

$$[p \oplus q]_0 = p_0 + q_0 + \frac{\beta_1^{ab}}{\Lambda} p_0 q_0 + \frac{\beta_2^{ab}}{\Lambda} \vec{p} \cdot \vec{q}$$

$$[p \oplus q]_i = p_i + q_i + \frac{\gamma_1^{ab}}{\Lambda} p_0 q_i + \frac{\gamma_2^{ab}}{\Lambda} p_i q_0 + \frac{\gamma_3^{ab}}{\Lambda} \epsilon_{ijk} p_j q_k$$

Two-body decay kinematics beyond SR

$$A(k) \to C(p) + D(q)$$

Conservation law at the interaction vertex: $\boxed{\hat{k} \oplus p \oplus q = 0}$

- $\hat{k}$ is the *antipode* of k : $\hat{k} \oplus k = k \oplus \hat{k} = 0$

- The composition of three momenta is determined from the two momenta MCL:

$$[\hat{k} \oplus p \oplus q]_0 = \hat{k}_0 + p_0 + q_0 + \frac{\beta_1^{ac}}{\Lambda}\hat{k}_0 p_0 + \frac{\beta_1^{ad}}{\Lambda}\hat{k}_0 q_0 + \frac{\beta_1^{cd}}{\Lambda} p_0 q_0 + \frac{\beta_2^{ac}}{\Lambda}\vec{\hat{k}} \cdot \vec{p} + \frac{\beta_2^{ad}}{\Lambda}\vec{\hat{k}} \cdot \vec{q} + \frac{\beta_2^{cd}}{\Lambda}\vec{p} \cdot \vec{q}$$

$$[\hat{k} \oplus p \oplus q]_i = \hat{k}_i + p_i + q_i + \frac{\gamma_1^{ac}}{\Lambda}\hat{k}_0 p_i + \frac{\gamma_1^{ad}}{\Lambda}\hat{k}_0 q_i + \frac{\gamma_1^{cd}}{\Lambda} p_0 q_i + \frac{\gamma_2^{ac}}{\Lambda}\hat{k}_i p_0 + \frac{\gamma_2^{ad}}{\Lambda}\hat{k}_i q_0 + \frac{\gamma_2^{cd}}{\Lambda} p_i q_0$$
$$+ \frac{\gamma_3^{ac}}{\Lambda}\epsilon_{ijl}\hat{k}_j p_l + \frac{\gamma_3^{ad}}{\Lambda}\epsilon_{ijl}\hat{k}_j q_l + \frac{\gamma_3^{cd}}{\Lambda}\epsilon_{ijl} p_j q_l$$

- Note that the conservation law is not unique:

$$\hat{k} \oplus q \oplus p = 0 \qquad p \oplus \hat{k} \oplus q = 0 \qquad k \oplus \hat{p} \oplus \hat{q} = 0$$

Different kinematic channels for the process

Two-body decay kinematics beyond SR

$$\boxed{A(k) \to C(p) + D(q)}$$

Conservation law at the interaction vertex: $\boxed{\hat{k} \oplus p \oplus q = 0}$

$$\hat{k}_0 = -(p_0+q_0) + \frac{\beta_1^{ac}}{\Lambda}(p_0+q_0)p_0 + \frac{\beta_1^{ad}}{\Lambda}(p_0+q_0)q_0 - \frac{\beta_1^{cd}}{\Lambda}p_0 q_0 + \frac{\beta_2^{ac}}{\Lambda}(\vec{p}+\vec{q})\cdot\vec{p} + \frac{\beta_2^{ad}}{\Lambda}(\vec{p}+\vec{q})\cdot\vec{q} - \frac{\beta_2^{cd}}{\Lambda}\vec{p}\cdot\vec{q}$$

$$\vec{\hat{k}} = -(\vec{p}+\vec{q}) + \frac{\gamma_1^{ac}}{\Lambda}(p_0+q_0)\vec{p} + \frac{\gamma_1^{ad}}{\Lambda}(p_0+q_0)\vec{q} - \frac{\gamma_1^{cd}}{\Lambda}p_0\vec{q} + \frac{\gamma_2^{ac}}{\Lambda}(\vec{p}+\vec{q})p_0 + \frac{\gamma_2^{ad}}{\Lambda}(\vec{p}+\vec{q})q_0 - \frac{\gamma_2^{cd}}{\Lambda}\vec{p}\,q_0$$
$$- \frac{(\gamma_3^{ac} - \gamma_3^{ad} + \gamma_3^{cd})}{\Lambda}\vec{p}\wedge\vec{q}$$

- Write p_0, q_0 as a function of $\vec{p}, \vec{q}$

$$E_p \equiv \sqrt{\vec{p}^2 + m_c^2} \longrightarrow p_0 = E_p - \frac{\alpha_1^c + \alpha_2^c}{2\Lambda}E_p^2 + \frac{\alpha_2^c}{2\Lambda}m_c^2$$

- Use the MDR for $\hat{k}$

$$\hat{k}_0^2 - \vec{\hat{k}}^2 + \frac{\hat{\alpha}_1^a}{\Lambda}\hat{k}_0^3 + \frac{\hat{\alpha}_2^a}{\Lambda}\hat{k}_0\vec{\hat{k}}^2 - m_a^2 = 0$$

  - Note that, from the definition of antipode:

$$\hat{\alpha}_1^a = -\alpha_1^a - 2\beta_1^a\,, \qquad \hat{\alpha}_2^a = -\alpha_2^a - 2(\beta_2^a - \gamma_1^a - \gamma_2^a)$$

Two-body decay kinematics beyond SR

$$\boxed{A(k) \to C(p) + D(q)}$$

Conservation law at the interaction vertex: $\boxed{\hat{k} \oplus p \oplus q = 0}$

We arrive at the *modified kinematics equation* (MKE) for the 1st channel:

$$2E_pE_q - 2\vec{p}\cdot\vec{q} - m_a^2 + m_c^2 + m_d^2 = O_3$$

with

$$O_3 = \frac{E_p + E_q}{\Lambda}\{(\alpha_1^c + \alpha_2^c)E_p^2 + (\alpha_1^d + \alpha_2^d)E_q^2 + (\hat{\alpha}_1^a + \hat{\alpha}_2^a)(E_p + E_q)^2$$
$$+ 2(\beta_1^{ac} + \beta_2^{ac} - \gamma_1^{ac} - \gamma_2^{ac})(E_p + E_q)E_p + 2(\beta_1^{ad} + \beta_2^{ad} - \gamma_1^{ad} - \gamma_2^{ad})(E_p + E_q)E_q$$
$$- 2(\beta_1^{cd} + \beta_2^{cd} - \gamma_1^{cd} - \gamma_2^{cd})E_pE_q\} + \mathcal{O}\left(\frac{Em^2}{\Lambda}\right)$$

Two-body decay kinematics beyond SR

Ultrarelativistic (UR) limit:

$$O_3 \approx \frac{E_p + E_q}{\Lambda}\{(\alpha_1^c + \alpha_2^c)E_p^2 + (\alpha_1^d + \alpha_2^d)E_q^2 + (\hat{\alpha}_1^a + \hat{\alpha}_2^a)(E_p + E_q)^2 + 2(\beta_1^{ac} + \beta_2^{ac} - \gamma_1^{ac} - \gamma_2^{ac})(E_p + E_q)E_p$$
$$+ 2(\beta_1^{ad} + \beta_2^{ad} - \gamma_1^{ad} - \gamma_2^{ad})(E_p + E_q)E_q - 2(\beta_1^{cd} + \beta_2^{cd} - \gamma_1^{cd} - \gamma_2^{cd})E_pE_q\}$$

$$\eta^m \equiv \alpha_1^m + \alpha_2^m \qquad \hat{\eta}^m \equiv \hat{\alpha}_1^m + \hat{\alpha}_2^m \qquad \eta^{mn} \equiv \beta_1^{mn} + \beta_2^{mn} - \gamma_1^{mn} - \gamma_2^{mn}$$

$$E_p = \frac{1+x}{2}(E_p + E_q) \qquad E_q = \frac{1-x}{2}(E_p + E_q) \qquad -1 < x < 1$$

$$\boxed{O_3 \approx \frac{(E_p + E_q)^3}{\Lambda}\xi_3(x)}$$

$$\xi_3(x) = \eta^c\left(\frac{1+x}{2}\right)^2 + \eta^d\left(\frac{1-x}{2}\right)^2 + \hat{\eta}^a + 2\eta^{ac}\left(\frac{1+x}{2}\right) + 2\eta^{ad}\left(\frac{1-x}{2}\right) - 2\eta^{cd}\left(\frac{1+x}{2}\right)\left(\frac{1-x}{2}\right)$$

Threshold analysis

$$\boxed{A(k) \to C(p) + D(q)} \qquad \boxed{\hat{k} \oplus p \oplus q = 0}$$

MKE

$$2E_pE_q - 2\vec{p}\cdot\vec{q} - m_a^2 + m_c^2 + m_d^2 = O_3$$

UR limit:

$$\vec{p}\cdot\vec{q} = pq\cos\theta \approx \left(E_pE_q - \frac{m_c^2E_q}{2E_p} - \frac{m_d^2E_p}{2E_q}\right)\cos\theta$$

$$\frac{(1+x)(1-x)}{2}(E_p+E_q)^2(1-\cos\theta) + \left(\frac{1-x}{1+x}m_c^2 + \frac{1+x}{1-x}m_d^2\right)\cos\theta$$

$$= m_a^2 - m_c^2 - m_d^2 + \xi_3(x)\frac{(E_p+E_q)^3}{\Lambda}$$

$\cos\theta = 1$ at the threshold: $$\frac{2m_c^2}{1+x} + \frac{2m_d^2}{1-x} - m_a^2 = \xi_3(x)\frac{(E_p+E_q)^3}{\Lambda}$$

Threshold analysis

$$\boxed{A(k) \to C(p) + D(q)} \qquad \boxed{\hat{k} \oplus p \oplus q = 0}$$

$$\boxed{\frac{2m_c^2}{1+x} + \frac{2m_d^2}{1-x} - m_a^2 = \xi_3(x)\,\frac{(E_p+E_q)^3}{\Lambda}}$$

- To determine x at the threshold, take the derivative and make $d(E_p + E_q)/dx = 0$

$$\left(\frac{2m_c^2}{1+x} + \frac{2m_d^2}{1-x} - m_a^2\right)\frac{d\xi_3}{dx} = \left(-\frac{2m_c^2}{(1+x)^2} + \frac{2m_d^2}{(1-x)^2}\right)\xi_3(x)$$

- Solve for x and substitute in the first equation, to get

$$(E_p+E_q)_* = (\Lambda m_a^2)^{1/3}\left[\frac{1}{\xi_3(x)}\left(\frac{2m_c^2/m_a^2}{1+x} + \frac{2m_d^2/m_a^2}{1-x} - 1\right)\right]^{1/3}$$

$$(\Lambda m_a^2)^{1/3} = 2.15\,\left[(\Lambda/M_P)(m_a/\mathrm{GeV})^2\right]^{1/3}\ \mathrm{PeV}$$

Bounds on vacuum Cherenkov radiation

$$\boxed{e^-(k) \to e^-(p) + \gamma(q)} \qquad \boxed{\hat{k} \oplus p \oplus q = 0}$$

$$\eta^c = \eta^e \qquad \hat{\eta}^a = \hat{\eta}^e \qquad \eta^d = \eta^\gamma \qquad \eta^{ad} = \eta^{cd} = \eta^{e\gamma}$$

$$\eta^{ac} = \eta^{ee} = \beta_1^e + \beta_2^e - \gamma_1^e - \gamma_2^e = -\frac{\hat{\eta}^e + \eta^e}{2}$$

$$\xi_3 = \eta^e \left(\frac{1+x}{2}\right)^2 + \eta^\gamma \left(\frac{1-x}{2}\right)^2 + \hat{\eta}^e + \eta^{ee}(1+x) + \eta^{e\gamma}(1-x) - \eta^{e\gamma}(1-x)\left(\frac{1+x}{2}\right)$$
$$= (1-x)\left[-\eta^{ee} - \eta^e + \frac{1}{2}\left(\eta^{e\gamma} + \frac{\eta^e + \eta^\gamma}{2}\right)(1-x)\right]$$

$$\lambda^{ee} = \eta^{ee} + \eta^e \qquad \lambda^{\gamma\gamma} = \eta^{\gamma\gamma} + \eta^\gamma$$

$$\lambda^{e\gamma} = \eta^{e\gamma} + \frac{\eta^e + \eta^\gamma}{2}$$

$$\lambda^{\gamma e} = \eta^{\gamma e} + \frac{\eta^\gamma + \eta^e}{2}$$

$$\boxed{\xi_3 = (1-x)\left[-\lambda^{ee} + \frac{\lambda^{e\gamma}}{2}(1-x)\right]}$$

Bounds on vacuum Cherenkov radiation

C_1:	$\hat{k} \oplus p \oplus q = 0$	C_2:	$\hat{k} \oplus q \oplus p = 0$	C_3:	$p \oplus \hat{k} \oplus q = 0$
C_4:	$q \oplus \hat{k} \oplus p = 0$	C_5:	$p \oplus q \oplus \hat{k} = 0$	C_6:	$q \oplus p \oplus \hat{k} = 0$
C_7:	$k \oplus \hat{p} \oplus \hat{q} = 0$	C_8:	$k \oplus \hat{q} \oplus \hat{p} = 0$	C_9:	$\hat{p} \oplus k \oplus \hat{q} = 0$
C_{10}:	$\hat{q} \oplus k \oplus \hat{p} = 0$	C_{11}:	$\hat{p} \oplus \hat{q} \oplus k = 0$	C_{12}:	$\hat{q} \oplus \hat{p} \oplus k = 0$

C_1, C_3:	$a_0 = -\lambda^{ee}$	$a_1 = \lambda^{e\gamma}$
C_2:	$a_0 = \lambda^{e\gamma} - \lambda^{\gamma e} - \lambda^{ee}$	$a_1 = \lambda^{\gamma e}$
C_4, C_6:	$a_0 = -\lambda^{ee}$	$a_1 = \lambda^{\gamma e}$
C_5:	$a_0 = \lambda^{\gamma e} - \lambda^{e\gamma} - \lambda^{ee}$	$a_1 = \lambda^{e\gamma}$
C_7, C_9:	$a_0 = -\lambda^{ee}$	$a_1 = (\lambda^{ee} + \lambda^{\gamma\gamma}) - \lambda^{e\gamma}$
C_8:	$a_0 = \lambda^{\gamma e} - \lambda^{e\gamma} - \lambda^{ee}$	$a_1 = (\lambda^{ee} + \lambda^{\gamma\gamma}) - \lambda^{\gamma e}$
C_{10}, C_{12}:	$a_0 = -\lambda^{ee}$	$a_1 = (\lambda^{ee} + \lambda^{\gamma\gamma}) - \lambda^{\gamma e}$
C_{11}:	$a_0 = \lambda^{e\gamma} - \lambda^{\gamma e} - \lambda^{ee}$	$a_1 = (\lambda^{ee} + \lambda^{\gamma\gamma}) - \lambda^{e\gamma}$

$$\lambda^{ee} = \eta^{ee} + \eta^{e} \qquad \lambda^{\gamma\gamma} = \eta^{\gamma\gamma} + \eta^{\gamma}$$

$$\lambda^{e\gamma} = \eta^{e\gamma} + \frac{\eta^{e} + \eta^{\gamma}}{2}$$

$$\lambda^{\gamma e} = \eta^{\gamma e} + \frac{\eta^{\gamma} + \eta^{e}}{2}$$

$$\xi_3(x) = (1-x)\left[a_0 + \frac{a_1}{2}(1-x)\right]$$

Bounds on vacuum Cherenkov radiation

$$\xi_3(x) = (1-x)\left[a_0 + \frac{a_1}{2}(1-x)\right]$$

- To get the threshold, we follow the generic procedure previously explained:

$$x_* = \frac{a_0}{a_1} \qquad (E_p + E_q)_* = (\Lambda m_e^2)^{1/3}\left[\frac{2a_1}{(a_0+a_1)^2}\right]^{1/3}$$

$$-a_1 < a_0 < a_1 \qquad a_1 > 0 \qquad (E_p + E_q)_* > 0$$

- Outside the previous region and for $a_0 > 0$, the minimum (threshold) is at $x \to 1$:

$$(E_p + E_q)[x \to 1] = (\Lambda m_e^2)^{1/3}\left[\frac{1}{2a_0}\right]^{1/3}$$

$$a_0 > 0 \qquad (E_p + E_q) > 0$$

Bounds on vacuum Cherenkov radiation

- We get excluded regions in the parameter space by asking that there can be no Cherenkov emission for electron energies below 50 TeV

$$R_1: \quad a_1 > 0 \quad -a_1 < a_0 < a_1 \quad \frac{(a_0+a_1)^2}{2a_1} > 2.5\times 10^{-2}\left(\frac{\Lambda}{M_P}\right)$$

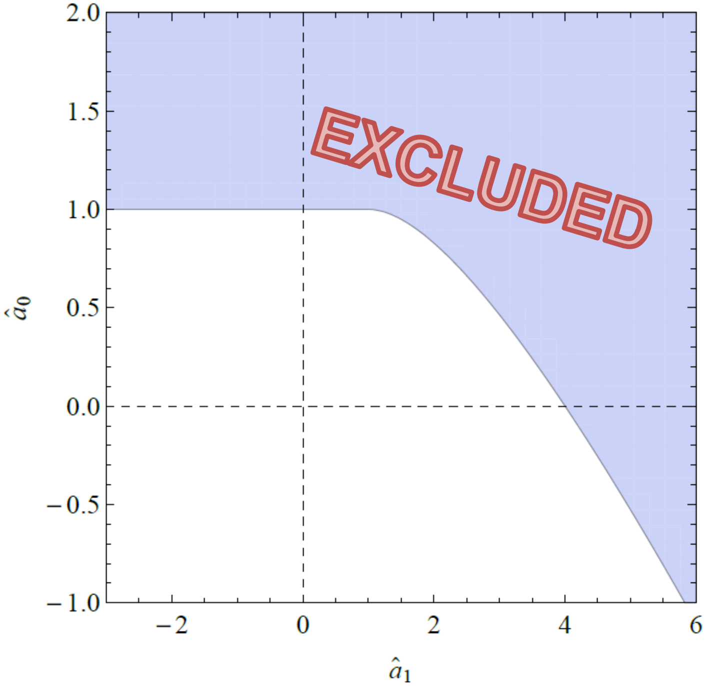
$$R_2: \quad a_1 > 0 \quad a_0 > a_1 \quad 2a_0 > 2.5\times 10^{-2}\left(\frac{\Lambda}{M_P}\right)$$

$$R_3: \quad a_1 < 0 \quad a_0 > 0 \quad 2a_0 > 2.5\times 10^{-2}\left(\frac{\Lambda}{M_P}\right)$$

In terms of:

$$\hat{a}_0 \equiv \frac{a_0}{b} \qquad \hat{a}_1 \equiv \frac{a_1}{b}$$

$$b \equiv 1.25\times 10^{-2}\left(\frac{\Lambda}{M_P}\right)$$



Bounds on vacuum Cherenkov radiation

- The excluded region in the (a_0, a_1) plane should be translated now into excluded regions in the parameter space $(\lambda^{ee}, \lambda^{\gamma\gamma}, \lambda^{e\gamma}, \lambda^{\gamma e})$ for each of the possible channels
- The true excluded region is the intersection of the above regions
- The multichannel excluded region would then translate into a excluded region in the six dimensional parameter space $(\eta^{e}, \eta^{\gamma}, \eta^{ee}, \eta^{\gamma\gamma}, \eta^{e\gamma}, \eta^{\gamma e})$

Particular cases:

$\boxed{\eta^{e\gamma} = \eta^{\gamma e}}$ With this simplification there are only two independent channels:

$$C_1 = \ldots = C_6 \qquad a_0 = -(\eta^{ee} + \eta^{e}) \qquad a_1 = \eta^{e\gamma} + \frac{\eta^{e} + \eta^{\gamma}}{2}$$

$$C_7 = \ldots = C_{12} \qquad a_0 = -(\eta^{ee} + \eta^{e}) \qquad a_1 = \eta^{ee} + \eta^{\gamma\gamma} - \eta^{e\gamma} + \frac{\eta^{e} + \eta^{\gamma}}{2}$$

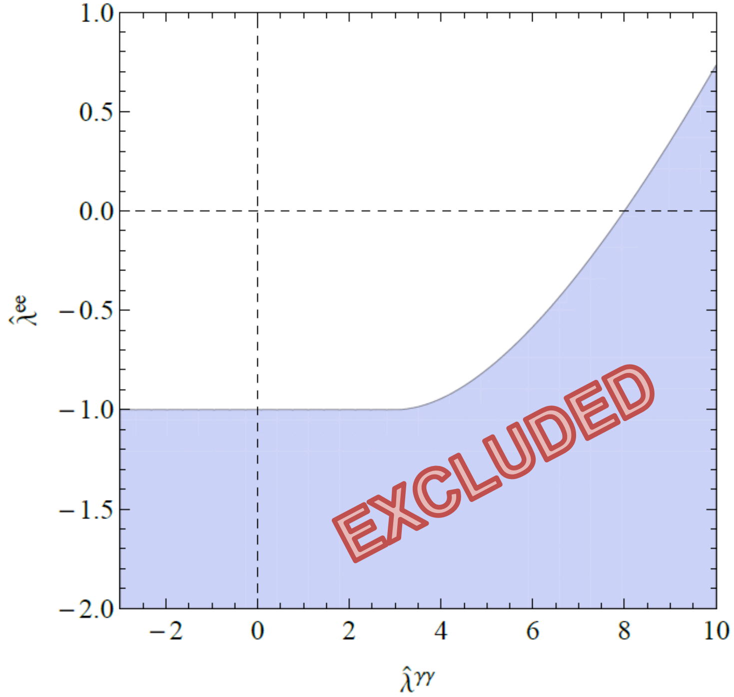
Tridimensional $(\lambda^{ee}, \lambda^{\gamma\gamma}, \lambda^{e\gamma})$ space, or pentadimensional $(\eta^{e}, \eta^{\gamma}, \eta^{ee}, \eta^{\gamma\gamma}, \eta^{e\gamma})$ space

Bounds on vacuum Cherenkov radiation

$$\boxed{\eta^{e\gamma} = \eta^{\gamma e} = (\eta^{ee} + \eta^{\gamma\gamma})/2}$$

This is a subcase of the previous case; there are 4 independent parameters of the modified kinematics $(\eta^{e}, \eta^{\gamma}, \eta^{ee}, \eta^{\gamma\gamma})$, two independent combinations $(\lambda^{ee}, \lambda^{\gamma\gamma})$ and only one channel:

$$a_0 = -(\eta^{ee} + \eta^{e}) \qquad a_1 = \frac{(\eta^{ee} + \eta^{e})}{2} + \frac{(\eta^{\gamma\gamma} + \eta^{\gamma})}{2}$$



Excluded region in the plane

$$(\hat{\lambda}^{\gamma\gamma}, \hat{\lambda}^{ee}) \equiv (\lambda^{\gamma\gamma}/b, \lambda^{ee}/b)$$

Bounds on vacuum Cherenkov radiation

$$\eta^{\gamma} = \eta^{e},\ \eta^{\gamma\gamma} = \eta^{e\gamma} = \eta^{\gamma e} = \eta^{ee}$$

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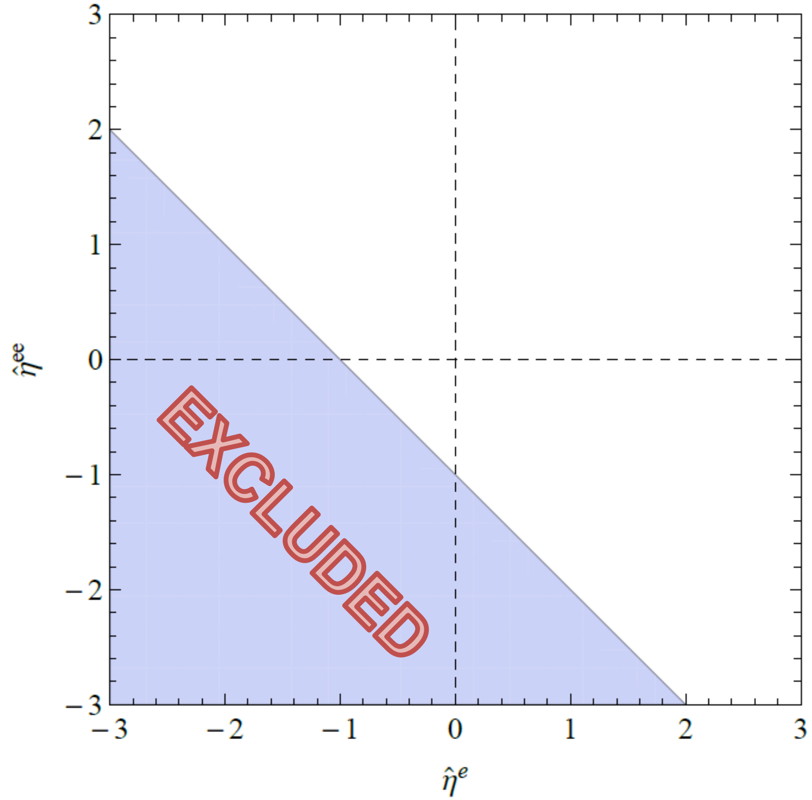
Again a subcase of the previous case, there are 2 independent parameters (η^{e}, η^{ee})

$$a_0 = -\lambda^{ee} \qquad a_1 = \lambda^{ee} \qquad \Rightarrow \qquad a_1 = -a_0$$



Excluded region in the plane

$$(\hat{\eta}^{e}, \hat{\eta}^{ee}) \equiv (\eta^{e}/b, \eta^{ee}/b)$$

Bounds on vacuum Cherenkov radiation

$\eta^e = \eta^\gamma = 0$ MCL ONLY

In this case

$$\lambda^{ee} = \eta^{ee} \qquad \lambda^{e\gamma} = \eta^{e\gamma} \qquad \lambda^{\gamma e} = \eta^{\gamma e} \qquad \lambda^{\gamma\gamma} = \eta^{\gamma\gamma}$$

- There are twelve different channels, just as in the general case
- The effect of a MDR can be reabsorbed into a MCL, as far as the process of electron decay is concerned $\longrightarrow$ degeneracy in the bounds

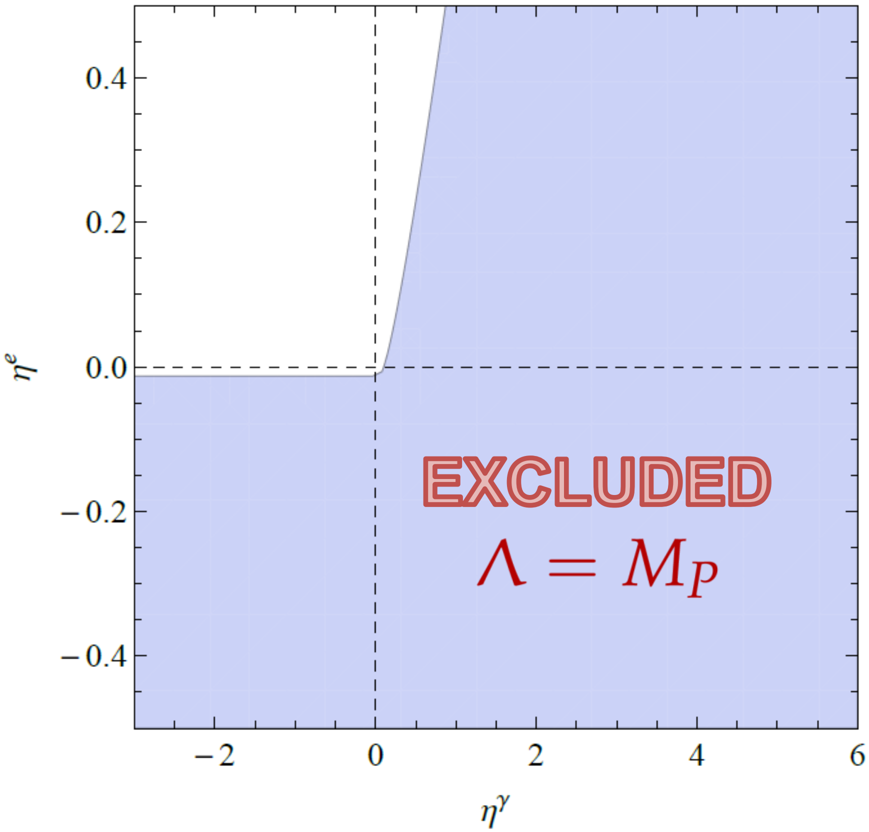
Bounds on vacuum Cherenkov radiation

$\boxed{\eta^{ee} = \eta^{e\gamma} = \eta^{\gamma e} = \eta^{\gamma\gamma} = 0}$

MDR ONLY

In this case

$$a_0 = -\eta^e \qquad a_1 = \frac{\eta^e + \eta^\gamma}{2}$$



- This is the hypothesis assumed in the EFT framework
- $(\eta^e = 0, \eta^\gamma = 1)$ is excluded when $\Lambda = M_P$: Planckian sensitivity
- However, the point $(\eta^e = 0,\ \eta^\gamma = 1,\ \eta^{ee} = \sigma/2,\ \eta^{\gamma\gamma} = \sigma/2,\ \eta^{e\gamma} = \sigma/2,\ \eta^{\gamma e} = \sigma/2)$ is not excluded for $\sigma > 9$
- Bounds within the EFT may be too naive

Some final comments

- We have explored a kinematics beyond special relativity with modified conservation laws (as well as modified dispersion relations), in a non-universal scenario

- We have used vacuum Cherenkov radiation as an example to illustrate the method to study kinematics effects due to departures from special relativity. The method we have presented can be used for any two body decay and it can also be extended to other processes. It is a first step of a systematic phenomenological analysis to establish bounds on (or identify) departures from SR in a framework going beyond the EFT framework

Some final comments

- The analysis of thresholds and bounds is much more involved in the general case owing to the presence of multi-channels
- There are interesting particular cases (universal kinematics, symmetry in the composition of different types of particles) that can be analyzed more easily
- The is a degeneracy in the bounds: different values of parameters produce identical phenomenological effects with respect to Cherenkov radiation
- This degeneracy could be removed by considering different experimental tests of the modified kinematics, including experiments to disentangle the MDR from the MCL (for example, particle propagation studies)
- Bounds extracted within the EFT framework may be too naive (ie, invalid) if the modified kinematics contains modified conservation laws