„RELATIVE LOCALITY" IN 2+1 DIMENSIONS

ESFQG – SISSA, SEPTEMBER 2014

THE PLAN

1. Curved momentum space and relative locality;
2. Alekseev-Malkin construction of effective, deformed particle lagrangian in 2+1 dimensions*;
3. Comments.

* Based on the (partially inpublished) work done with Tomasz Trześniewski.

SCALE AND GEOMETRY

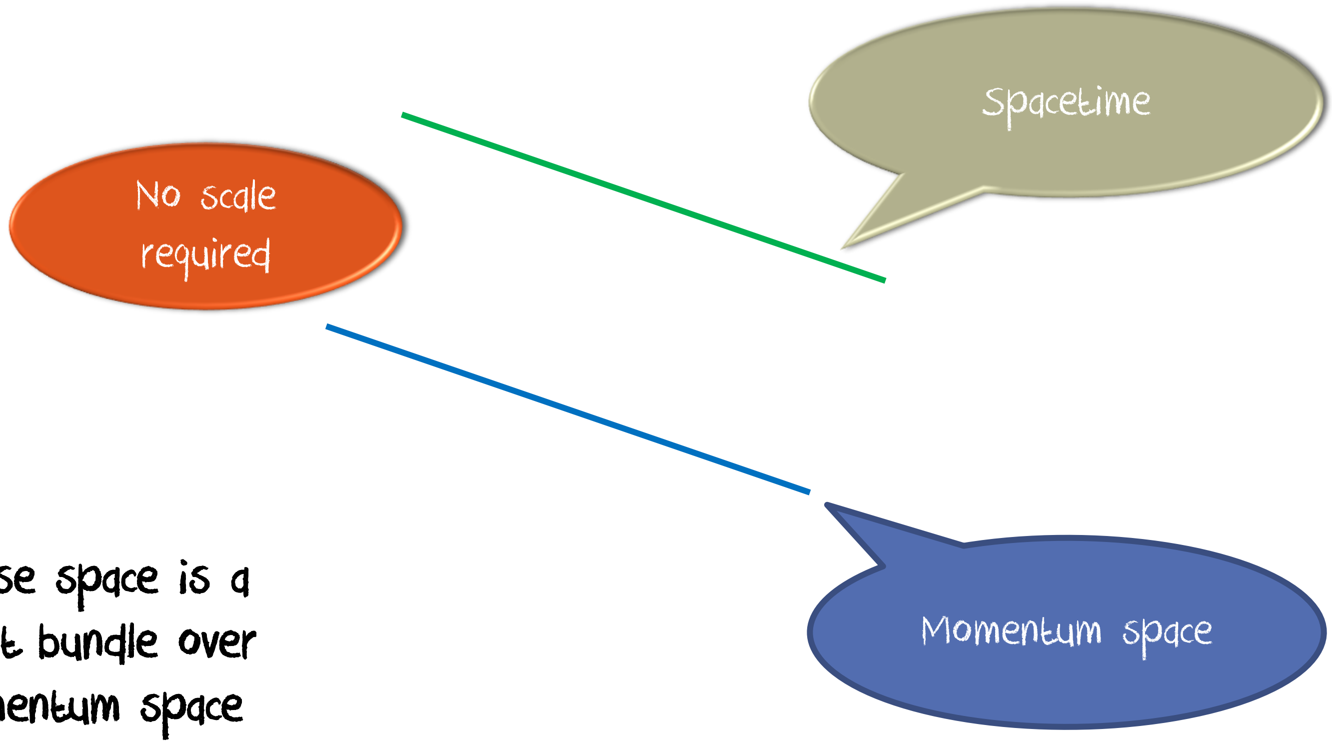
The assumption that the angle sum is less than 180° leads to a geometry quite different from Euclid's. It depends on a constant, which is not given a priori. As a joke I even wished Euclidean geometry was not true, for then we would have an absolute measure of length a priori. (Gauss, 1827)

If there is an *a priori* scale, you expect nontrivial geometry.

Everything is curved, unless it cannot be

FLAT/CURVED MOMENTUM SPACE

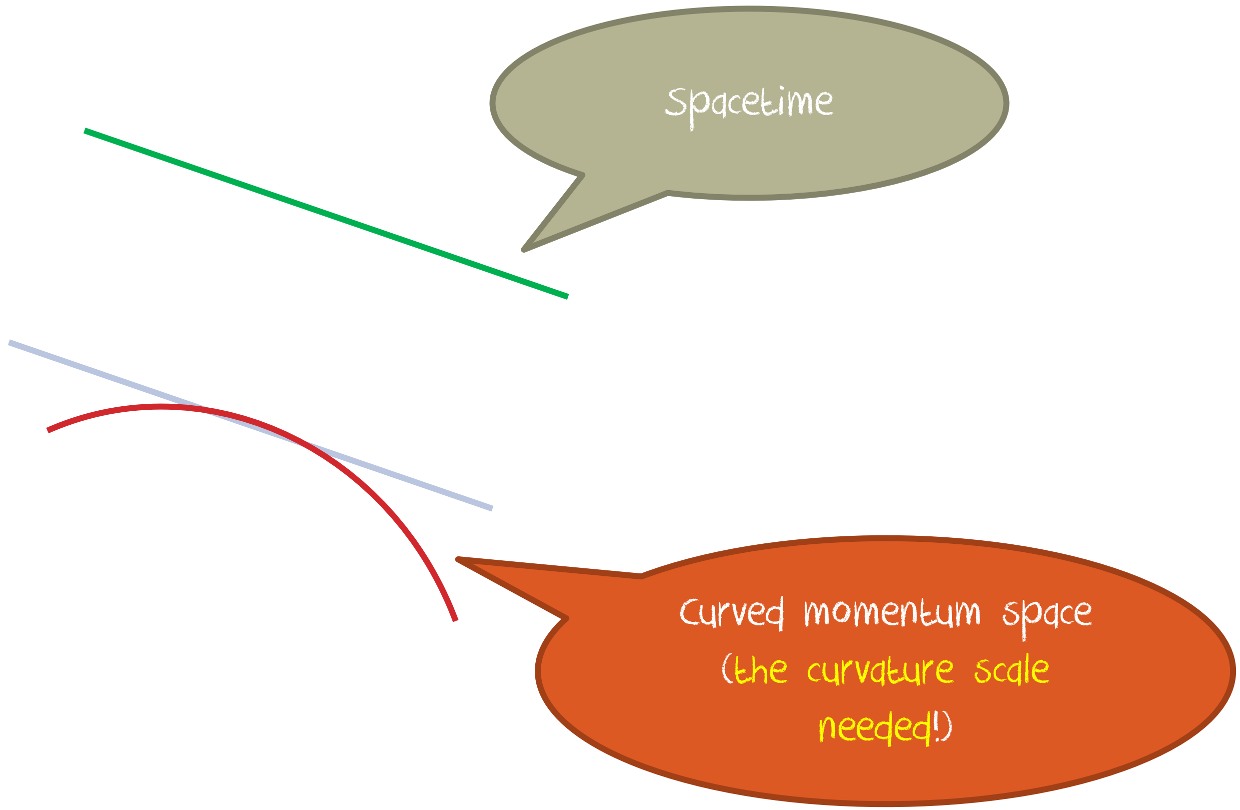
In the case of a standard relativistic particle we have flat spacetime and flat momentum space:



The phase space is a cotangent bundle over flat momentum space

FLAT/CURVED MOMENTUM SPACE

For deformed relativistic particle we have flat spacetime and curved momentum space:



The phase space is a cotangent bundle over curved momentum space

RELATIVE LOCALITY

In the RL framework the no-trivial geometry of momentum space exhibits itself in a number of ways:

1. The kinetic term for a particle has the form

$$L \sim -\dot{p}_{\alpha} E^{\alpha}_{a}(p) x^{a} + \dots,$$

$$E^{a}_{\alpha}(p) = \delta^{a}_{\alpha} + \frac{1}{\kappa} f^{a\beta}_{\alpha} p_{\beta} + \dots$$

with the nontrivial momentum space frame field.

RELATIVE LOCALITY

The mass-shell relation is defined as a square of the distance from zero to the point P, with coordinates $p_\mu(P)$:

$$C(p) = D^2(p) - m^2$$

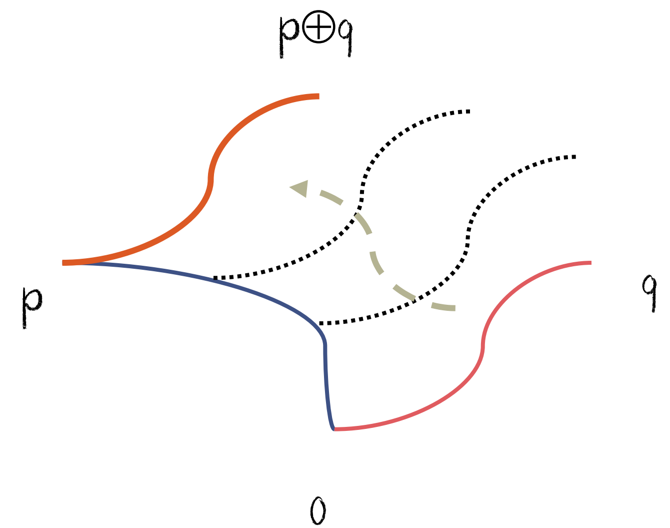
We need metric to define the mass-shell relation !

$$D^2(p) = \int_0^p ds\, g^{\mu\nu} \dot{p}_\mu \dot{p}_\nu \Big|_{geodesic}$$

If the metric is nonlinear we need the mass scale to define it.

MOMENTUM ADDITION

In order to add two momenta the notion of connection (parallel transport) is needed.



If the composition p⊕q is nonlinear we need a mass scale to define it.

$$(p \oplus q)_\alpha = p_\alpha + q_\alpha + \frac{1}{\kappa}\Gamma_\alpha^{\beta\gamma}\, p_\beta q_\gamma + \dots$$

RELATIVE LOCALITY

In theories with curved momentum space, locality might be relative:

1. The translation (and/or Lorentz transformation) of a particle wordline depends on the momentum that the particle carries;
2. Therefore the worldlines of the particles with different momenta transform differently;
3. As a result, locality of events (defined by worldlines intersections) is not absolute, and becomes relative.

It seems that relative locality is neither logically inconsistent nor does it contradict any observational data.

G. Amelino-Camelia, L. Freidel, JKG, L. Smolin *The principle of relative locality*. Phys.Rev. **D84** 084010, arXiv:1101.0931 [hep-th];
Relative locality: A deepening of the relativity principle. Gen.Rel.Grav. **43** 2547arXiv:1106.0313 [hep-th]

FUNDAMENTAL OR EMERGENT?

Is the curved momentum space fundamental or emergent?

1. If it is fundamental, what is the associated dynamics?
2. If it is emergent, how does it arise?

In the emergent case the master theory must provide a momentum scale.

In 3+1D

$$G_4 = l_{Pl} / M_{Pl}$$

One can imagine the regime, in which Planck length is small, while Planck mass stays finite (relative to the characteristic scales of the problem)

IN 2+1 D THERE IS A MASS SCALE!

In 2+1 dimensions the Newton's constant G_3 has the dimension of inverse mass $G_3 \sim 1/K$.

This suggests that the momentum space might be curved, and I will show that it indeed is.

Of course, 2+1D gravity is a toy model, and we do not have to believe that it tells you anything relevant for the real world.

However, it is interesting because:

1. This is the only example that we have;
2. There might be real physical systems, which are effectively 2+1 dimensional.

GRAVITY IN 2+1 D WITH PARTICLES

The Lagrangian of 2+1 gravity with one (massive) particle at the origin is

$$L = \frac{1}{2\kappa}\int d^2x\, \epsilon^{ij} \left\langle \dot{A}_i A_j \right\rangle - \int d\tau \left\langle h^{-1}\dot{h} C \right\rangle$$

$$+\int d^2x \left\langle A_0 \left(\frac{k}{2\pi}\epsilon^{ij} F_{ij} - hCh^{-1}\delta^2(\vec{x}) \right) \right\rangle$$

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$$C = mJ_0 + sP_0,$$
h *is 'translation + Lorentz'*

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$$
\begin{aligned}
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& + \int d^2x \left\langle A_0 \left(\frac{1}{\kappa}\epsilon^{ij} F_{ij} - hCh^{-1}\delta^2(\vec{x}) \right) \right\rangle
\end{aligned}
$$

Constraint: curvature is nonzero only at the position of the particle

$h \in$ Poincare group

$C = mJ_0 + sP_0$

GRAVITY IN 2+1 D WITH PARTICLES

$$L = \frac{1}{2\kappa}\int d^2x\, \epsilon^{ij}\left\langle \dot{A}_i A_j \right\rangle - \left\langle h^{-1}\dot{h}C \right\rangle$$

The idea is to solve the constraint

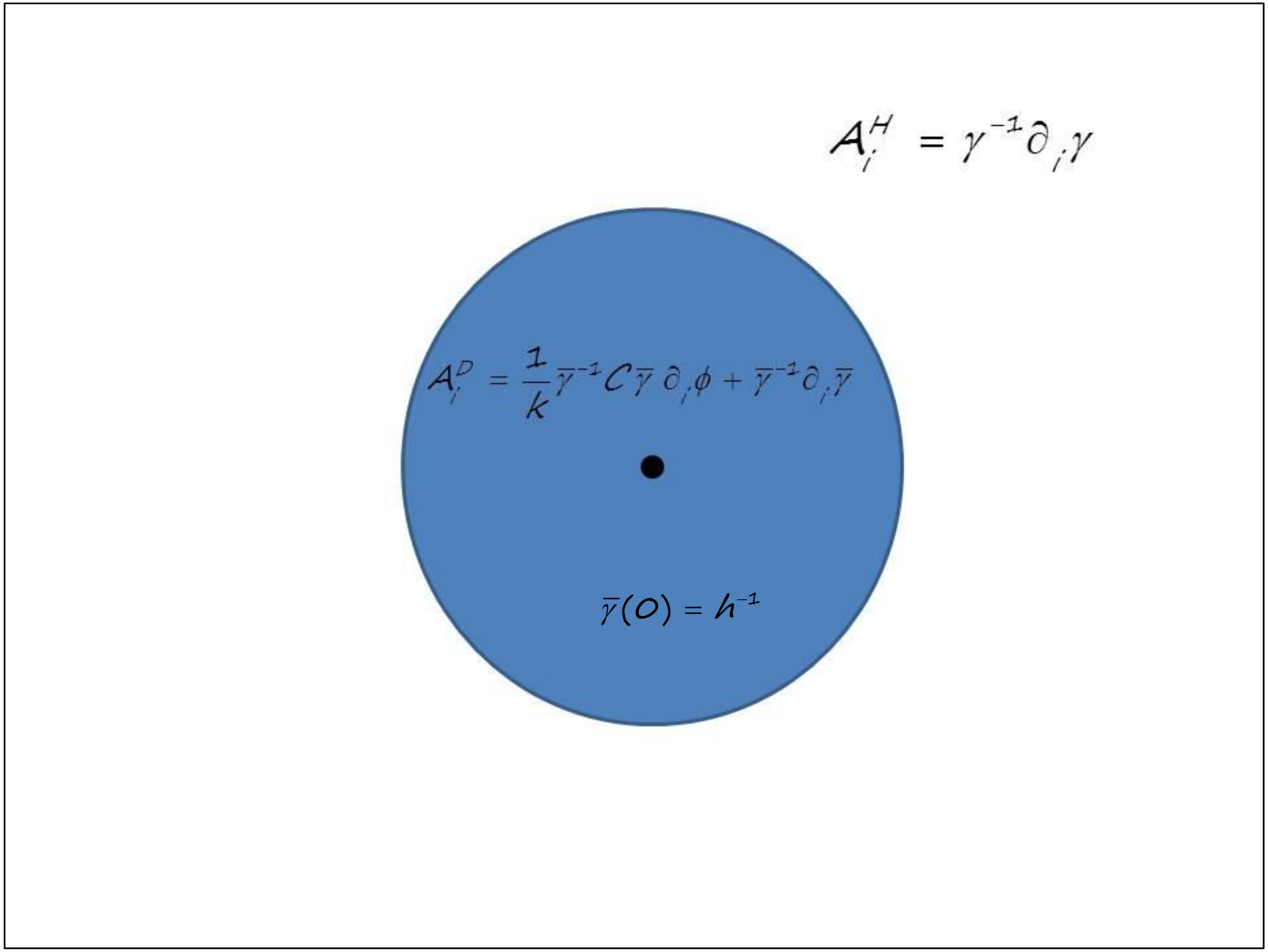
$$\frac{1}{\kappa}\epsilon^{ij}F_{ij} = hCh^{-1}\delta^2(\vec{x})$$

and plug the solution back to the lagrangian. (This can be done explicitly!)

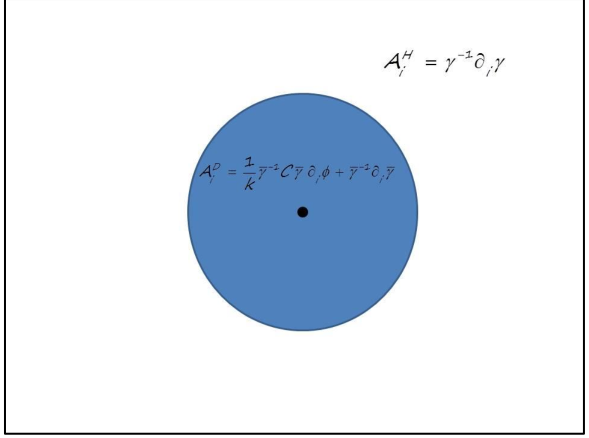
A.Y. Alekseev and A.Z. Malkin, Commun. Math. Phys. 169, 99 (1995) [arXiv:hep-th/9312004; C. Meusburger and B. J. Schroers, Class. Quant. Grav. 20 (2003) 2193 [arXiv:gr-qc/0301108.

Solving the constraint

$$\frac{1}{\kappa}\epsilon^{ij}F_{ij} = hCh^{-1}\delta^2(\vec{x})$$



CONTINUITY CONDITION



The connection must be continuous across the boundary of the disk.

Decomposing the Poincare group elements into Lorentz and translational parts

$$\gamma = \mathfrak{q}\mathfrak{l}, \quad \bar{\gamma} = \bar{\mathfrak{q}}\,\bar{\mathfrak{l}}$$

one finds that the Lorentz part of the continuity condition reads

$$\mathfrak{l}(t,\phi) = \mathfrak{n}(t)\, exp\left(\frac{m}{2\pi\kappa} J_0 \phi\right) \bar{\mathfrak{l}}(t,\phi)$$

Lorentz group element!

HOLONOMY AND MOMENTUM

Since the connection A^H is gauge trivial, its holonomy along the boundary is given by

$$Hol_{\Gamma}(A^{H}) \equiv \Xi(0)\Xi^{-1}(2\pi) = \bar{\Xi}^{-1}(0)\exp\left(\frac{2\pi}{k}C_{J}\right)\bar{\Xi}(0) = \Pi$$

Here is the group valued momentum characterizing motion of the particle.

In terms Π of the lagrangian of the particle has the form

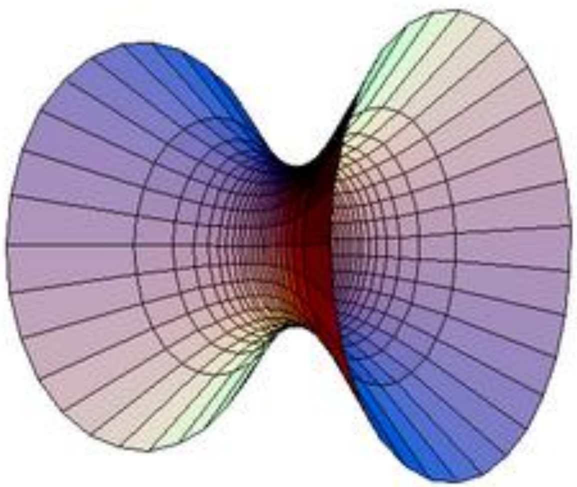
$$L = \kappa\left\langle \dot{\Pi}\Pi^{-1}\mathfrak{x}\right\rangle + (spin\ part) = -\dot{P}_{\alpha}E^{\alpha}_{a}(P)x^{a} + N(\eta^{\alpha\beta}P_{\alpha}P_{\beta} - m^{2})$$

$E^{\alpha}_{a}(P)$ is the momentum space frame field

THE MOMENTUM SPACE

The momentum of Π the particle is defined by the group element and thus the momentum space is a group manifold. In fact, the 2+1D Lorentz group, to which Π belongs is the 2+1D Anti de Sitter space

$$P_3^2 + \frac{1}{4\kappa^2}\left(P_0^2 - P_1^2 - P_2^2\right) = 1$$



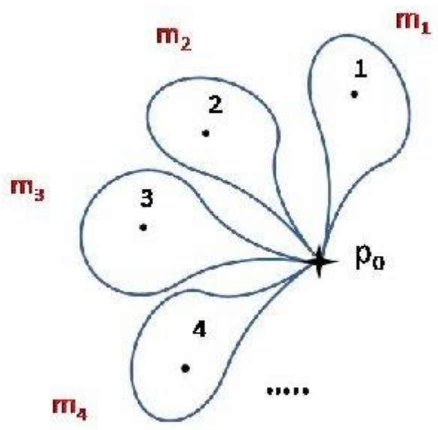
NONCOMMUTATIVE POSITIONS

By duality (Born reciprocity, Majid co-gravity) the position space is non-commutative

$$\left\{x^a, x^b\right\} = \frac{1}{\kappa}\,\epsilon^{ab}{}_c\, x^c$$

TWO PARTICLES

In the case of two (or many) particles the procedure is very similar:



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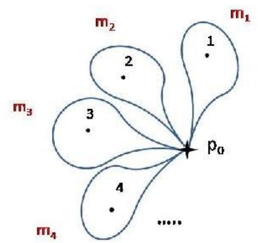

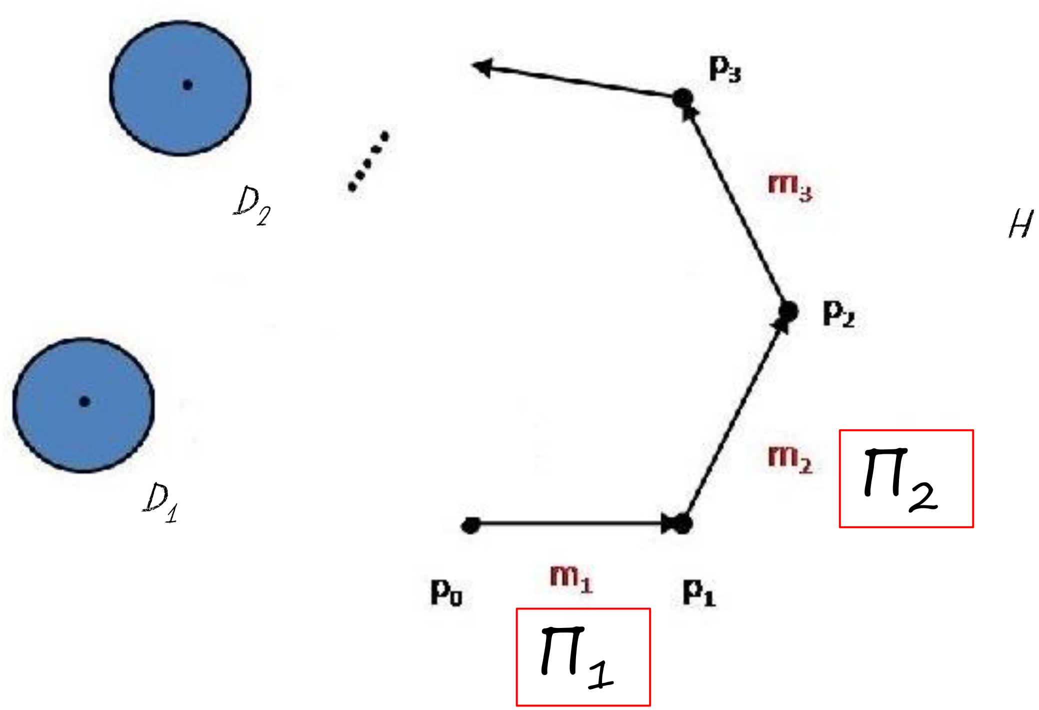

TWO PARTICLES

The deformed lagrangian has the form (no spin)

$$\mathcal{L} = \kappa\left\langle \dot{\Pi}_1 \Pi_1^{-1} \mathfrak{x}_1 \right\rangle + \kappa\left\langle \dot{\Pi}_2 \Pi_2^{-1} \mathfrak{x}_2 \right\rangle + \kappa \left\langle \left(\Pi_2 \dot{\Pi}_1 \Pi_1^{-1} \Pi_2^{-1} - \dot{\Pi}_1 \Pi_1^{-1} \right) \mathfrak{x}_2 \right\rangle$$

and there is a nontrivial coupling between particles. This is this coupling that makes the total momentum of the system equal to the deformed composition of the individual particles momenta, given by group product.

To see this define center and the relative positions

$$\mathfrak{x} = \frac{1}{2}\left(\mathfrak{x}_1 + \mathfrak{x}_2\right), \quad \mathfrak{d} = \frac{1}{2}\left(\mathfrak{x}_1 - \mathfrak{x}_2\right)$$

$$\mathcal{L} = \kappa\left\langle \dot{\Pi}\Pi^{-1} \mathfrak{x} \right\rangle + \mathfrak{d}\ terms, \quad \Pi \equiv \Pi_2 \Pi_1$$

So that the lagrangian is invariant under rigid translations which do not change the particles relative position, with the associated total momentum being Π.

COMMENTS

No relative locality! In this model there is no sign of relative locality: the relative position of the particles is invariant under rigid translations.

BTW. (for experts) this makes the construction of the vertex quite natural and unique (no sudoku needed!).

This effective particle model works only in 2+1D and cannot be extended to higher dimensions. There is a model with (kind of) the κ-Poincare structure, which can be extended to higher dimension (talk in Rome).

COMMENTS: AND WHAT ABOUT 3+1D?

There are only circumstantial evidences:

1. Gravity in 3+1D can be defined as a constrained topological field theory, when the constraint is forced to vanish one has a TFT;
2. One can couple this theory to particle(s) and even argue that such a system can be described by CS theory in 2+1D, like 2+1 gravity, but with more complicated gauge group;

THE MESSAGE

1. In 2+1D we can honesty derive the deformed single- and multi-particle actions with momentum space being a group manifold; and momenta of the particle(s) represented by group elements;
2. The total momentum of the multi-particle system is given by the group product of the group elements representing momenta of the particles;
3. The multiparticle lagrangian contains the „topological interaction terms" and the form of these terms is such that locality turns out to be absolute (not relative).
4. It is unclear if this results can be applied beyond the 2+1D setup.