εsG14 Experimental search for quantum gravity
Workshop
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SISSA/ISAS Trieste – Italy

Finsler geometry and deSitter momentum space

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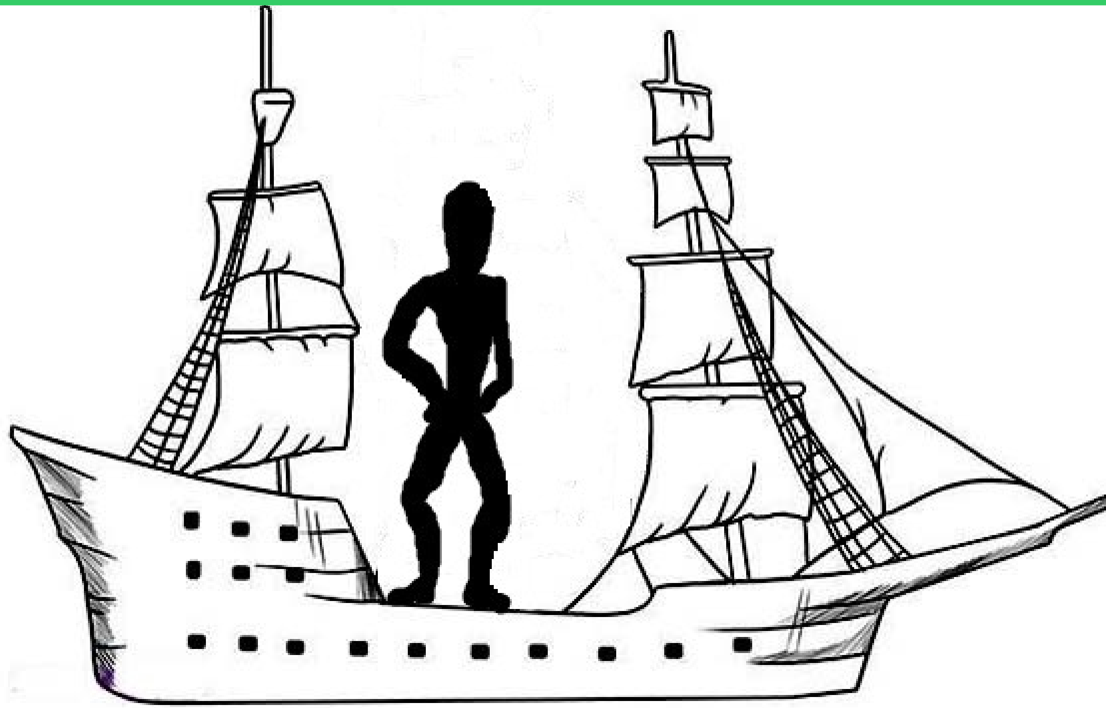
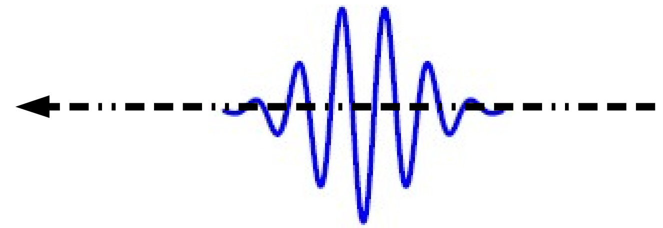
arXiv:1407.8143

With: Giovanni Amelino-Camelia, Leonardo Barcaroli, Giulia Gubitosi and Stefano Liberati

Galilean Relativity





Invariant (Casimir)

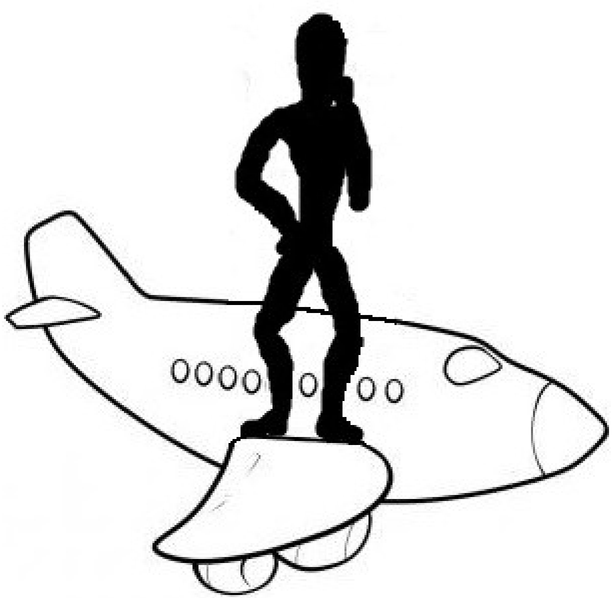
$$\mathcal{C}_G = \frac{p_1^2}{2m}$$

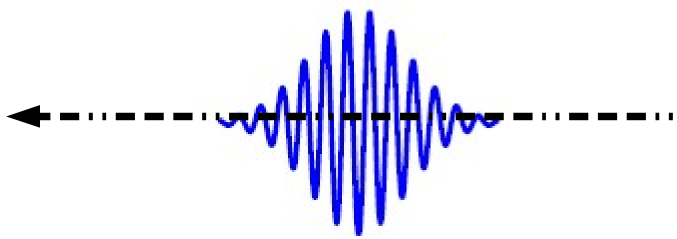
Transformation (Boost)

$$\mathcal{N}_G = x^1 m + x^0 p_1$$

Special Relativity

$$v \sim c$$





Invariant (Casimir)

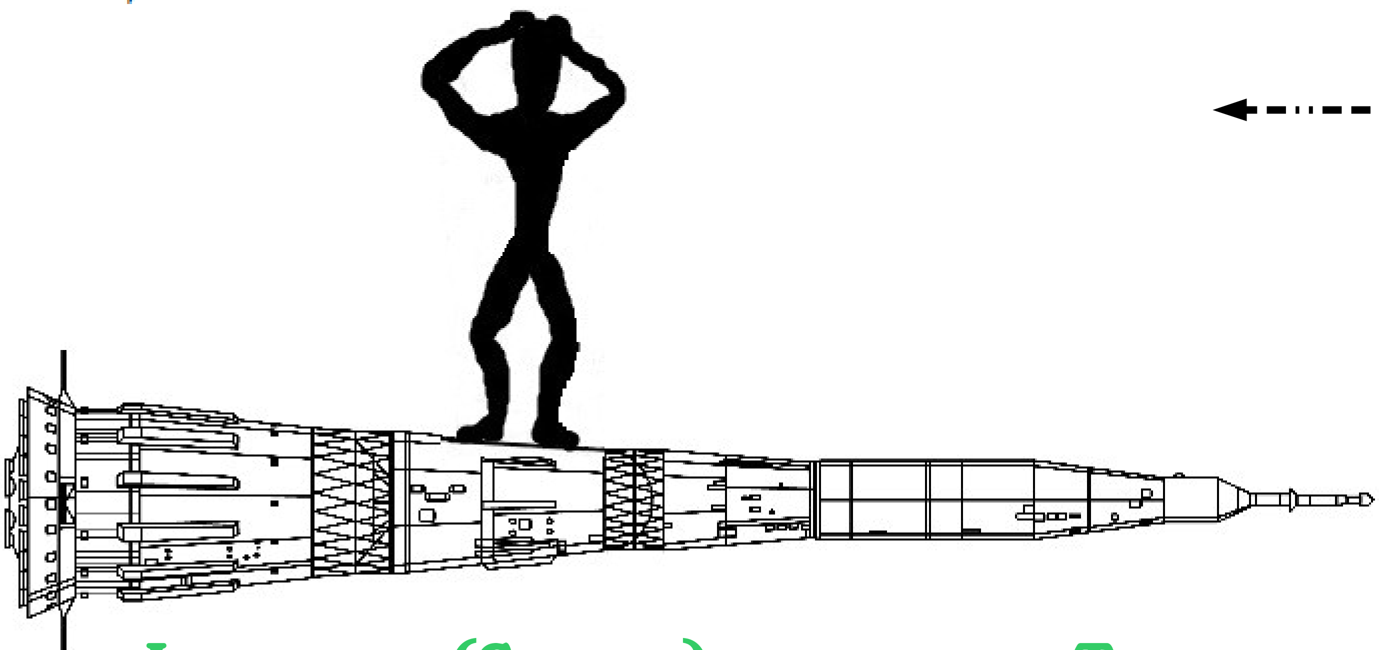
Transformation (Boost)

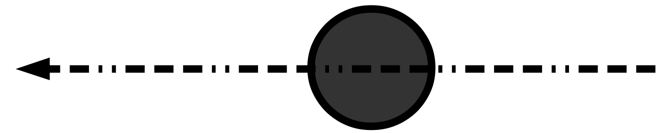
$$\mathcal{C}_{SR} = p_0^2 - p_1^2$$

$$\mathcal{N}_{SR} = x^1 p_0 + x^0 p_1$$

??? Relativity

$$E_\gamma \sim E_P$$



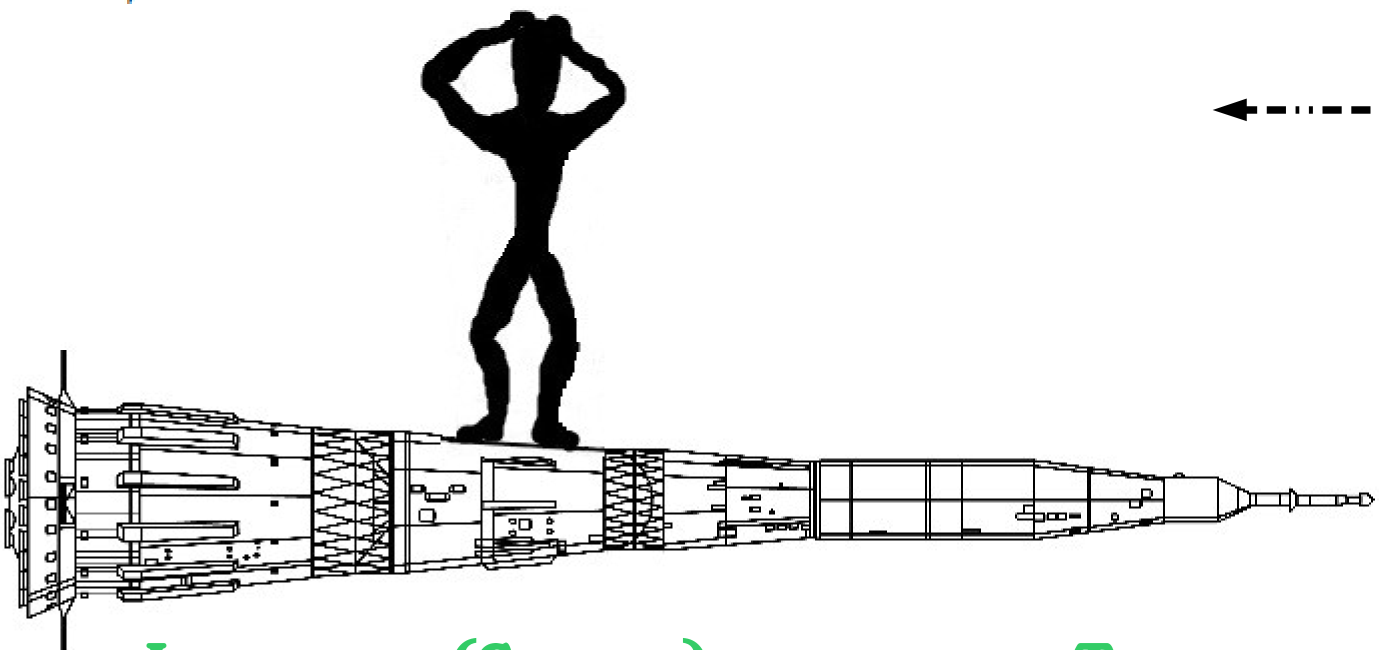


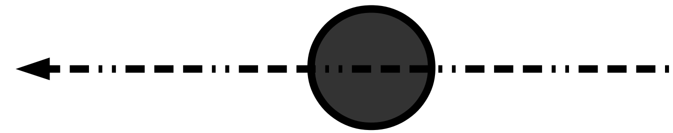
Invariant (Casimir) Transformation (Boost)

$$\mathcal{C}_\ell = p_0^2 - p_1^2 + ?$$

??? Relativity

$$E_\gamma \sim E_P$$





Invariant (Casimir)

$$\mathcal{C}_\ell = p_0^2 - p_1^2 + ?$$

Transformation (Boost)

$\boxed{\mathcal{N}_{SR}}$???

Deformed Special Relativity

κ–Poincaré algebra in bicrossproduct basis

$$\{\mathcal{N}, P_0\} = P_1 \qquad \{P_0, P_1\} = 0$$

$$\{\mathcal{N}, P_1\} = P_0 - \ell P_0^2 - \frac{\ell}{2} P_1^2$$

Deformed Special Relativity

κ–Poincaré algebra in bicrossproduct basis

$$\{\mathcal{N}, P_0\} = P_1 \qquad \{P_0, P_1\} = 0$$

$$\{\mathcal{N}, P_1\} = P_0 - \ell P_0^2 - \frac{\ell}{2} P_1^2$$

Casimir

$$\boxed{\mathcal{C}_\ell = P_0^2 - P_1^2 - \ell P_0 P_1^2}$$

Symmetry generators representation

$$P_0 = p_0, \quad P_1 = p_1,$$

$$\boxed{\mathcal{N} = x^0 p_1 + x^1 \left(p_0 - \ell p_0^2 - \frac{\ell}{2} p_1^2 \right)}$$

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Curved momentum-space

Modified symmetries → Non-trivial properties of momentum-space geometry

$$ds^2 = \zeta^{\mu\nu}(p)dp_\mu dp_\nu$$

Curved momentum-space

Modified symmetries $\longrightarrow$ Non-trivial properties of momentum-space geometry

$$ds^2 = \zeta^{\mu\nu}(p) dp_\mu dp_\nu$$

(Modified) Dispersion Relation obtained through

$$m^2 = d^2(p,0) = \int_\gamma ds\, \zeta^{\mu\nu}(p)\, \dot{\gamma}_\mu \dot{\gamma}_\nu$$

In the κ-Poincaré (bicrossproduct basis) case

$$\zeta^{\mu\nu}(p) = \begin{pmatrix} 1 & 0 \\ 0 & -(1+2\ell p_0) \end{pmatrix}$$

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Finsler Geometry

Finsler Norm

$$\boxed{F(x,\dot{x}) = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}$$

$$x \in M$$
$$\dot{x} \in T_x M$$

- Positive function in the tangent bundle
- Homogeneus of degree one in $\dot{x}$

$$\begin{cases} F(\dot{x}) \neq 0 \quad \text{if} \quad \dot{x} \neq 0 \\ F(\epsilon\dot{x}) = |\epsilon| F(\dot{x}) \end{cases}$$

Velocity-dependent generalization of Riemannian metric

$$\boxed{g_{\mu\nu}(x,\dot{x}) = \frac{1}{2}\frac{\partial^2 F^2(x,\dot{x})}{\partial \dot{x}^\mu \partial \dot{x}^\nu}}$$

Rund 1959

Finsler Geometry of a particle with MDR

Action of a free relativistic particle

$$I = \int \left(\dot{x}^{\mu} p_{\mu} - \lambda \left(\mathcal{M}(p) - m^2 \right) \right) d\tau$$

Girelli, Liberati, Sindoni, PRD 2007

Finsler Geometry of a particle with MDR

Action of a free relativistic particle

$$I = \int \left(\dot{x}^\mu p_\mu - \lambda\left(\mathcal{M}(p) - m^2\right)\right) d\tau$$

By using Hamilton's equation $\dot{x}^\mu = \lambda \dfrac{\partial \mathcal{M}}{\partial p_\mu}$ **We find** $\boxed{p_\mu(\dot{x}, \lambda)}$

$$\downarrow$$

$$I = \int \mathcal{L}(\dot{x}, \lambda(\dot{x})) d\tau \qquad \mathcal{L}(\dot{x}, \lambda(\dot{x})) \equiv mF(\dot{x})$$

Girelli, Liberati, Sindoni, PRD 2007

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$$\downarrow$$

$$I = \int \mathcal{L}(\dot{x}, \lambda(\dot{x})) d\tau \qquad \mathcal{L}(\dot{x}, \lambda(\dot{x})) \equiv m F(\dot{x})$$

$$\downarrow$$

$$\boxed{I = m \int F(\dot{x}) d\tau = m \int \sqrt{g_{\mu\nu}(\dot{x}) \dot{x}^\mu \dot{x}^\nu}}$$

Girelli, Liberati, Sindoni, PRD 2007

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Finsler Geometry and κ-Poincaré symmetries

We apply this procedure to the case $\boxed{m^2 = \mathcal{C}_\ell(p)}$

$$I = \int \left(\dot{x}^\mu p_\mu - \lambda\left(p_0^2 - p_1^2 - \ell p_0 p_1^2 - m^2\right)\right) d\tau$$

Finsler Geometry and κ-Poincaré symmetries

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$$\downarrow$$

$$\begin{aligned} p_0 &= m\frac{\dot{x}^0}{\sqrt{(\dot{x}_0)^2-(\dot{x}_1)^2}} - \frac{\ell}{2}m^2\frac{(\dot{x}^1)^2((\dot{x}_0)^2+(\dot{x}_1)^2)}{((\dot{x}_0)^2-(\dot{x}_1)^2)^2} \\ p_1 &= -m\frac{\dot{x}^1}{\sqrt{(\dot{x}_0)^2-(\dot{x}_1)^2}} + \frac{\ell}{2}m^2\frac{\dot{x}^1(\dot{x}_0)^3}{((\dot{x}_0)^2-(\dot{x}_1)^2)^2} \end{aligned}$$

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Finsler Geometry and κ-Poincaré symmetries

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$$I = \int \left(\dot{x}^\mu p_\mu - \lambda\left(p_0^2 - p_1^2 - \ell p_0 p_1^2 - m^2\right)\right) d\tau$$

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$$F(\dot{x}) = \left(\sqrt{(\dot{x}^0)^2-(\dot{x}^1)^2} + \frac{\ell}{2}m\frac{\dot{x}^0(\dot{x}^1)^2}{(\dot{x}^0)^2-(\dot{x}^1)^2}\right)$$

Finsler spacetime metric

$$g_{\mu\nu}(x,\dot{x}) = \begin{pmatrix} 1+\frac{3}{2}\ell m \frac{\dot{x}^0(\dot{x}^1)^4}{((\dot{x}^0)^2-(\dot{x}^1)^2)^{5/2}} & \ell\frac{m}{2}\frac{-4(\dot{x}^0)^2(\dot{x}^1)^3+(\dot{x}^1)^5}{((\dot{x}^0)^2-(\dot{x}^1)^2)^{5/2}} \\ \ell\frac{m}{2}\frac{-4(\dot{x}^0)^2(\dot{x}^1)^3+(\dot{x}^1)^5}{((\dot{x}^0)^2-(\dot{x}^1)^2)^{5/2}} & -1+\frac{1}{2}\ell m(\dot{x}^0)^3\frac{2(\dot{x}^0)^2+(\dot{x}^1)^2}{((\dot{x}^0)^2-(\dot{x}^1)^2)^{5/2}} \end{pmatrix}$$

In terms of momenta

$$g_{\mu\nu}(x,p) = \begin{pmatrix} 1+\frac{3}{2}\ell\frac{p_0 p_1^4}{m^4} & \frac{\ell}{2}\frac{4p_0^2 p_1^3-p_1^5}{m^4} \\ \frac{\ell}{2}\frac{4p_0^2 p_1^3-p_1^5}{m^4} & -1+\frac{\ell}{2}p_0^3\frac{2p_0^2+p_1^2}{m^4} \end{pmatrix}$$

Finsler spacetime metric

$$g_{\mu\nu}(x,\dot{x}) = \begin{pmatrix} 1+\frac{3}{2}\ell m \frac{\dot{x}^0(\dot{x}^1)^4}{((\dot{x}^0)^2-(\dot{x}^1)^2)^{5/2}} & \ell\frac{m}{2}\frac{-4(\dot{x}^0)^2(\dot{x}^1)^3+(\dot{x}^1)^5}{((\dot{x}^0)^2-(\dot{x}^1)^2)^{5/2}} \\ \ell\frac{m}{2}\frac{-4(\dot{x}^0)^2(\dot{x}^1)^3+(\dot{x}^1)^5}{((\dot{x}^0)^2-(\dot{x}^1)^2)^{5/2}} & -1+\frac{1}{2}\ell m(\dot{x}^0)^3\frac{2(\dot{x}^0)^2+(\dot{x}^1)^2}{((\dot{x}^0)^2-(\dot{x}^1)^2)^{5/2}} \end{pmatrix}$$

In terms of momenta

$$g_{\mu\nu}(x,p) = \begin{pmatrix} 1+\frac{3}{2}\ell\frac{p_0 p_1^4}{m^4} & \frac{\ell}{2}\frac{4p_0^2p_1^3-p_1^5}{m^4} \\ \frac{\ell}{2}\frac{4p_0^2p_1^3-p_1^5}{m^4} & -1+\frac{\ell}{2}p_0^3\frac{2p_0^2+p_1^2}{m^4} \end{pmatrix}$$

Despite its horrible aspect this metric defines some simple relations:

- Its inverse is related to the particle's MDR

$$\boxed{g^{\mu\nu}(p)p_\mu p_\nu = p_0^2 - p_1^2 - \ell p_0 p_1^2 = \mathcal{C}_\ell}$$

- In terms of g momenta are simply related to velocities

$$\boxed{p_\mu = m\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}}$$

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On the invariance of the Lagrangian

Edge terms

$$\delta_{\mathcal{N}}\mathcal{L} = \{\mathcal{N}, \mathcal{L}\} = m\frac{g_{\mu\nu}\dot{x}^{\nu}}{F}\{\mathcal{N}, \dot{x}^{\mu}\} = p_{\mu}\delta_{\mathcal{N}}\dot{x}^{\mu}$$

On the invariance of the Lagrangian

Edge terms

$$\delta_{\mathcal{N}}\mathcal{L} = \{\mathcal{N}, \mathcal{L}\} = m\frac{g_{\mu\nu}\dot{x}^{\nu}}{F}\{\mathcal{N}, \dot{x}^{\mu}\} = p_{\mu}\delta_{\mathcal{N}}\dot{x}^{\mu}$$

$$\downarrow$$

$$\delta_{\mathcal{N}}\mathcal{L} - p_{\mu}\delta_{\mathcal{N}}\dot{x}^{\mu} = \delta_{\mathcal{N}}(2m^2\lambda(\dot{x})) = 0$$

On the invariance of the Lagrangian

Edge terms

$$\delta_{\mathcal{N}}\mathcal{L} = \{\mathcal{N}, \mathcal{L}\} = m\frac{g_{\mu\nu}\dot{x}^{\nu}}{F}\{\mathcal{N}, \dot{x}^{\mu}\} = p_{\mu}\delta_{\mathcal{N}}\dot{x}^{\mu}$$

$$\downarrow$$

$$\delta_{\mathcal{N}}\mathcal{L} - p_{\mu}\delta_{\mathcal{N}}\dot{x}^{\mu} = \delta_{\mathcal{N}}(2m^2\lambda(\dot{x})) = 0$$

Invariant Lagrangian

$$\lambda(\dot{x}) = \frac{\sqrt{\zeta_{\mu\nu}(\dot{x})\dot{x}^{\mu}\dot{x}^{\nu}}}{2m}$$

$$\downarrow$$

$$\boxed{\mathcal{L}_{inv} = m\sqrt{\zeta_{\mu\nu}(\dot{x})\dot{x}^{\mu}\dot{x}^{\nu}}}$$

Relative Locality and Rainbow metrics

This invariant Lagrangian suggest us to check whether

$$\{\mathcal{N}, \zeta_{\alpha\beta} x^\alpha x^\beta\} = 0$$

Invariant "rainbow" line element

$$ds^2 = \zeta_{\mu\nu}(p) dx^\mu dx^\nu$$

This would allow us to satisfy one of the key requirments of rainbow metrics which is

$$\zeta_{\mu\alpha} \zeta^{\beta\mu} = \delta^\beta_\alpha$$

Loret, arXiv:1404.5093

Geodesic equations

In terms of $\zeta(\dot{x})$:

$$\ddot{x}^{\rho} + \Gamma^{\rho}_{\mu\beta}\dot{x}^{\mu}\dot{x}^{\beta} + \Delta^{\rho}_{\mu\beta}\dot{x}^{\mu}\ddot{x}^{\beta} + E^{\rho}_{\beta\mu\nu}\dot{x}^{\beta}\dot{x}^{\mu}\dot{x}^{\nu} + Z^{\rho}_{\beta\mu\nu}\ddot{x}^{\beta}\dot{x}^{\mu}\dot{x}^{\nu} = 0$$

Geodesic equations

In terms of $\zeta(\dot{x})$:

$$\ddot{x}^{\rho} + \Gamma^{\rho}_{\mu\beta}\dot{x}^{\mu}\dot{x}^{\beta} + \Delta^{\rho}_{\mu\beta}\dot{x}^{\mu}\ddot{x}^{\beta} + E^{\rho}_{\beta\mu\nu}\dot{x}^{\beta}\dot{x}^{\mu}\dot{x}^{\nu} + Z^{\rho}_{\beta\mu\nu}\ddot{x}^{\beta}\dot{x}^{\mu}\dot{x}^{\nu} = 0$$

From the homogeneity of $\mathrm{F}(\dot{x})$ one can show that the metric $g(\dot{x})$ satisfies

$$\dot{x}^{\alpha}\frac{\partial g_{\mu\nu}}{\partial \dot{x}^{\alpha}} = \dot{x}^{\mu}\frac{\partial g_{\mu\nu}}{\partial \dot{x}^{\alpha}} = \dot{x}^{\nu}\frac{\partial g_{\mu\nu}}{\partial \dot{x}^{\alpha}} = 0$$

Geodesic equations

In terms of $\zeta(\dot{x})$:

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\beta}\dot{x}^\mu\dot{x}^\beta + \Delta^\rho_{\mu\beta}\dot{x}^\mu\ddot{x}^\beta + E^\rho_{\beta\mu\nu}\dot{x}^\beta\dot{x}^\mu\dot{x}^\nu + Z^\rho_{\beta\mu\nu}\ddot{x}^\beta\dot{x}^\mu\dot{x}^\nu = 0$$

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In terms of $g(\dot{x})$:

$$\boxed{\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}(x,\dot{x})\dot{x}^\nu\dot{x}^\rho = 0}$$

Where

$$\Gamma^\mu_{\nu\rho}(x,\dot{x}) = \frac{1}{2}g^{\mu\sigma}(x,\dot{x})\left[-\partial_\sigma g_{\nu\rho}(x,\dot{x}) + \partial_\nu g_{\rho\sigma}(x,\dot{x}) + \partial_\rho g_{\sigma\nu}(x,\dot{x})\right]$$

Wordlines and symmetries

In the κ-Poincaré case

$$\Gamma^{\mu}_{\nu\rho}(x,\dot{x}) = 0 \qquad \ddot{x}^{\mu} = 0$$

We choose a parametrization

$$F(\dot{x}) = 1$$

$$\downarrow$$

$$\boxed{x^1 - \bar{x}^1 = -\frac{\sqrt{p_0^2 - m^2}}{p_0}\left(1 + \ell\frac{(2p_0^2 - m^2)}{2p_0}\right)(x^0 - \bar{x}^0)}$$

Finsler Killing equation

$$g_{\mu\rho}\,\partial_\nu\xi^\rho + g_{\nu\rho}\,\partial_\mu\xi^\rho + \frac{\partial g_{\mu\nu}}{\partial\dot{x}^\rho}\,\frac{\partial\xi^\rho}{\partial x^\sigma}\dot{x}^\sigma + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\xi_\rho = 0$$

We look for solutions and charges

$$\xi^\mu = \xi^\mu_{(0)} + \ell\,\xi^\mu_{(1)} \qquad\qquad Q_F = \xi^\mu\,p_\mu(\dot{x})$$

Finsler Killing equation

$$\boxed{g_{\mu\rho}\,\partial_\nu \xi^\rho + g_{\nu\rho}\,\partial_\mu \xi^\rho + \frac{\partial g_{\mu\nu}}{\partial \dot{x}^\rho}\,\frac{\partial \xi^\rho}{\partial x^\sigma}\dot{x}^\sigma + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\xi_\rho = 0}$$

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$$\xi^\mu = \xi^\mu_{(0)} + \ell\,\xi^\mu_{(1)} \qquad\qquad Q_F = \xi^\mu\,p_\mu(\dot{x})$$

$$Q_F^{(0)} = m\frac{d^0\dot{x}^0 - d^1\dot{x}^1 - ax^0\dot{x}^1 + ax^1\dot{x}^0}{\sqrt{(\dot{x}^0)^2 - (\dot{x}^1)^2}},$$

$$Q_F^{(1)} = m\left[\frac{A^0\dot{x}^0 - A^1\dot{x}^1}{\sqrt{(\dot{x}^0)^2 - (\dot{x}^1)^2}} + \frac{C(\dot{x}^0 x^1 - \dot{x}^1 x^0)}{\sqrt{(\dot{x}^0)^2 - (\dot{x}^1)^2}}\right]$$

Finsler Killing equation

$$g_{\mu\rho}\,\partial_\nu \xi^\rho + g_{\nu\rho}\,\partial_\mu \xi^\rho + \frac{\partial g_{\mu\nu}}{\partial \dot{x}^\rho}\,\frac{\partial \xi^\rho}{\partial x^\sigma}\dot{x}^\sigma + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\xi_\rho = 0$$

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$$Q_F^{(1)} = m\left[\frac{A^0\dot{x}^0 - A^1\dot{x}^1}{\sqrt{(\dot{x}^0)^2 - (\dot{x}^1)^2}} + \frac{C(\dot{x}^0 x^1 - \dot{x}^1 x^0)}{\sqrt{(\dot{x}^0)^2 - (\dot{x}^1)^2}}\right]$$

$$\mathcal{N} = x^0 p_1 + x^1\left(p_0 - \ell p_0^2 - \frac{\ell}{2}p_1^2\right)$$

Summary and outlook

- Finsler generalization of Riemannian geometry can be used to describe spacetime geometry seen by a particle with given (modified) dispersion relation

- This provides a consistent framework to derive physical properties of the particle: propagation, symmetries

- Equivalent to a 'rainbow' metric associated to classical particles with κ–Poincaré inspired symmetries

- Can it be used to treat more complicated cases, when gravity is introduced?

- How to introduce interactions?