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On the Genus Two Free Energies for Semisimple Frobenius Manifolds

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Abstract. We represent the genus two free energy of an arbitrary semisimple Frobenius manifold as a sum of contributions associated with dual graphs of certain stable algebraic curves of genus two plus the so-called "genus two G-function". Conjecturally, the genus two G-function vanishes for a series of important examples of Frobenius manifolds associated with simple singularities, as well as for $\mathbf{P}^1$-orbifolds with positive Euler characteristics. We explain the reasons for the conjecture and prove it in particular cases.

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1. INTRODUCTION

Let $(M, \cdot, \langle\ ,\ \rangle, e, E)$ be a semisimple Frobenius manifold of dimension n. To such an object[1] one can assign (see [5]) a formal series

$$\mathcal{F} = \sum_{g\geqslant 0} \epsilon^{2g-2}\mathcal{F}_g(\mathbf{t}), \tag{1.1}$$

the so-called *free energy* of the Frobenius manifold (in the framework of the theory of Gromov–Witten invariants, its exponential is also called the *total descendent potential*). Here $\mathbf{t} = (t^{\alpha,p})$, $\alpha = 1,\dots,n$, $p = 0, 1, 2, \dots$, are coordinates on the *large phase space* which coincide with the time variables of the associated *integrable hierarchy of topological type* (see [5, 7]). The particular coordinate $x := t^{1,0}$ plays the role of the spatial variable of the integrable hierarchy. The independent parameter ϵ is referred to in the physics literature as the *string coupling constant*. Restricting the free energy to the *small phase space* $F_g(t^{1,0},\dots,t^{n,0}) := \mathcal{F}_g(\mathbf{t})|_{t^{\gamma,p}=0\ (p>0)}$, one obtains a generating function of the genus g Gromov–Witten invariants. In particular, the function $F_0(t)$, $t = (t^{1,0},\dots,t^{n,0})$, coincides with the potential of the Frobenius manifold.

Write $v_\alpha(\mathbf{t}) = \partial^2\mathcal{F}_0(\mathbf{t})/\partial t^{1,0}\partial t^{\alpha,0}$, $\alpha = 1,\dots,n$, for a particular set of the genus zero correlators. A remarkable property of the genus expansion (1.1) is that the higher genus terms can be represented in the form

$$\mathcal{F}_g(\mathbf{t}) = \hat{\mathcal{F}}_g\left(v(\mathbf{t}), v_x(\mathbf{t}), \dots, v^{(3g-2)}(\mathbf{t})\right), \quad g \geqslant 1, \tag{1.2}$$

where $v(\mathbf{t}) = \left(v^1(\mathbf{t}),\dots,v^n(\mathbf{t})\right)$ (the indices are raised with the help of the flat metric on M). The existence of such a representation, first conjectured in [8], follows from vanishing of certain intersection numbers on the moduli space of stable maps [16]; in a more general setting, it can also be derived from the bi-Hamiltonian recursion relation of the associated integrable hierarchy of topological type [5]. The functions $\hat{\mathcal{F}}_g\left(v, v_x, \dots, v^{(3g-2)}\right)$ for $g \geqslant 2$ depend rationally on the jet variables v_x, … , $v^{(3g-2)}$, while the expression for $\hat{\mathcal{F}}_1(v, v_x)$ involves also logarithms (see the formula (2.11) below). Below, the hats are omitted.

In [5], an algorithm was developed for computing $\mathcal{F}_g\left(v, v_x, \dots, v^{(3g-2)}\right)$ for $g \geqslant 1$ by recursively solving the so-called *loop equation*. In particular, an explicit formula for the genus two free energy

[1] It also depends on the choice of a so-called *calibration* of the Frobenius manifold, i.e., on the choice of a basis of horizontal sections of the deformed flat connection on M. See [5] for details.

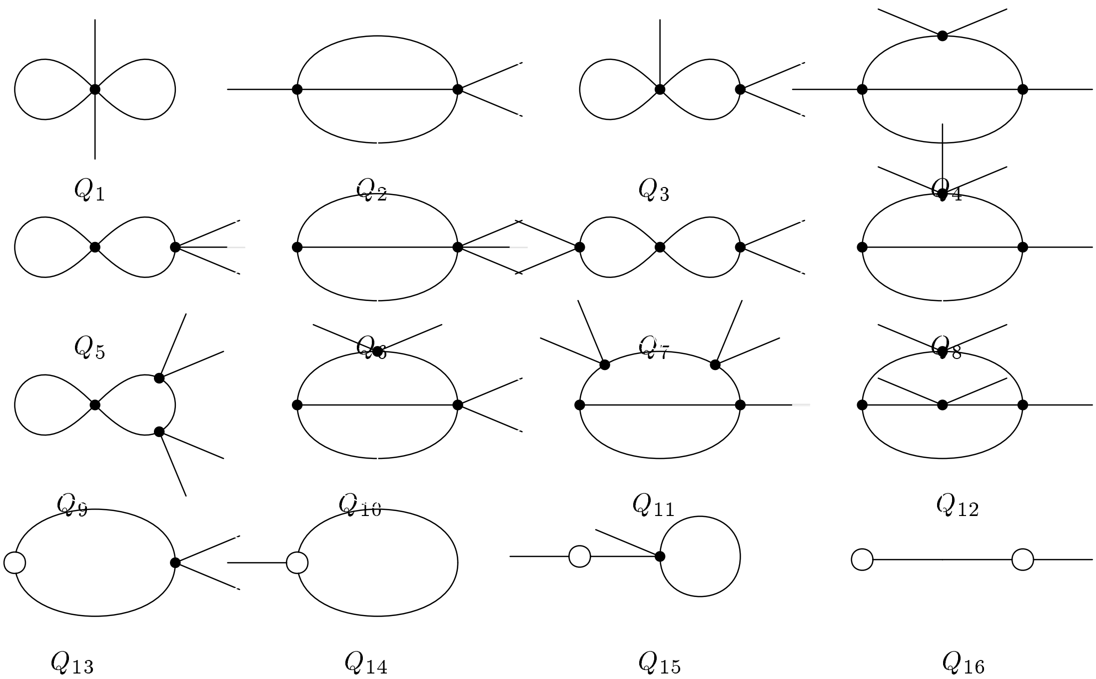
$\mathcal{F}_2 = \mathcal{F}_2(v, v_x, v^{(2)}, v^{(3)}, v^{(4)})$ is given for any semisimple Frobenius manifold. This formula (for the convenience of the reader, we reproduce it in Appendix 2 below) is represented in terms of the Lamé coefficients, rotation coefficients, and the canonical coordinates of the Frobenius manifold, which are not easy to compute for a concrete example. In this paper, we show that $\mathcal{F}_2$ can be rewritten as a sum of two parts; the first part is given by correlation functions, which is easy to compute in the flat coordinates, while the other part is still represented in terms of rotation coefficients and canonical coordinates; however, it vanishes in many examples, including the simple singularities and the $\mathbb{P}^1$-orbifolds of ADE type.

Let us proceed with formulating the main statements of the present paper.

Theorem 1.1. *Let M be a semisimple Frobenius manifold of dimension n. Denote by $\mathcal{F}_2$ the genus two free energy for M given by the formula (3.10.114) in* [5], *see the formula in Appendix B. Then*

$$\mathcal{F}_2 = \sum_{p=1}^{16} c_p\, Q_p + G^{(2)}(u, u_x, u_{xx}). \tag{1.3}$$

Here each term Q_p corresponds to one of the following sixteen graphs:



The constants c_p read as follows:

$$\begin{array}{llll}
c_1 = 0, & c_2 = -\frac{1}{960}, & c_3 = \frac{1}{5760}, & c_4 = \frac{1}{1152}, \\
c_5 = \frac{1}{2880}, & c_6 = 0, & c_7 = \frac{1}{1920}, & c_8 = -\frac{1}{2880}, \\
c_9 = -\frac{1}{1920}, & c_{10} = \frac{1}{1920}, & c_{11} = \frac{1}{1920}, & c_{12} = -\frac{1}{960}, \\
c_{13} = -\frac{1}{60}, & c_{14} = \frac{1}{48}, & c_{15} = -\frac{7}{240}, & c_{16} = \frac{7}{10}.
\end{array}$$

The function $G^{(2)}(u, u_x, u_{xx})$ is called the genus two G-function of the Frobenius manifold. An explicit expression (A.1) *of this function in the canonical coordinates $u_1, \ldots, u_n$ is given in Appendix 1.*

Before formulating the rules for computing the contributions of the sixteen graphs, let us explain their realization as *dual graphs* of stable curves of (arithmetic) genus 2. Recall (see, e.g., [20]) that dual graphs are used to encode a certain class of singular algebraic curves with marked points. The vertices of the graph correspond to the irreducible components of the curve. The genus of the normalization of such a component is called the genus of the vertex. The components of genus zero on our sixteen graphs are equipped with bullets and the components of genus 1 with circles.

All singularities of a stable curve are at most double points. The points of intersection or self-intersection of these components correspond to the edges of the dual graph, while the marked points are associated with the legs. The arithmetic genus of the stable curve is equal to the sum of genera of the vertices plus the first Betti number of the dual graph.

We are now ready to formulate the rules for computing the contribution of a dual graph. Let $\mathcal{F}_g = \mathcal{F}_g(\mathbf{t})$ be the genus $g = 0, 1$ free energy of M, and let $\partial/\partial t^{\alpha,p}$, $\alpha = 1, \dots, n$, $p \geqslant 0$, be the tangent vector fields on the large phase space. Introduce a matrix $M_{\alpha\beta} = \frac{\partial^3 \mathcal{F}_0}{\partial t^{1,0}\,\partial t^{\alpha,0}\,\partial t^{\beta,0}}$, and denote its inverse by $(M^{-1})^{\alpha\beta}$. Here and below, the summation with respect to repeated upper and lower indices is assumed. The diagram rules are formulated as follows:

i) the bullets ($\bullet$) correspond to $\mathcal{F}_0$;
ii) the circles ($\circ$) correspond to $\mathcal{F}_1$;
iii) the edges correspond to $(M^{-1})^{\alpha\alpha'}\frac{\partial}{\partial t^{\alpha,0}} \otimes \frac{\partial}{\partial t^{\alpha',0}}$;
iv) the legs correspond to $\frac{\partial}{\partial t^{1,0}}$.

It is assumed that all differential operators corresponding to edges and legs act on the vertices $\mathcal{F}_0$ or $\mathcal{F}_1$ first and all contractions with the matrix M^{-1} are to be added at the very end. Thus, for example, the terms Q_1, Q_2, Q_{15}, Q_{16} are given by

$$Q_1 = \frac{\partial^6 \mathcal{F}_0}{\partial t^{1,0}\,\partial t^{1,0}\,\partial t^{\alpha,0}\,\partial t^{\alpha',0}\,\partial t^{\beta,0}\,\partial t^{\beta',0}}\,(M^{-1})^{\alpha\alpha'}\,(M^{-1})^{\beta\beta'}, \tag{1.4}$$

$$Q_2 = \frac{\partial^4 \mathcal{F}_0}{\partial t^{1,0}\,\partial t^{\alpha,0}\,\partial t^{\beta,0}\,\partial t^{\gamma,0}}\,(M^{-1})^{\alpha\alpha'}\,(M^{-1})^{\beta\beta'}\,(M^{-1})^{\gamma\gamma'}\,\frac{\partial^5 \mathcal{F}_0}{\partial t^{1,0}\,\partial t^{1,0}\,\partial t^{\alpha',0}\,\partial t^{\beta',0}\,\partial t^{\gamma',0}}, \tag{1.5}$$

$$Q_{15} = \frac{\partial^4 \mathcal{F}_0}{\partial t^{1,0}\,\partial t^{\alpha,0}\,\partial t^{\alpha',0}\,\partial t^{\beta,0}}(M^{-1})^{\alpha\alpha'}(M^{-1})^{\beta\beta'}\,\frac{\partial^2 \mathcal{F}_1}{\partial t^{1,0}\,\partial t^{\beta',0}}, \tag{1.6}$$

$$Q_{16} = \frac{\partial^2 \mathcal{F}_1}{\partial t^{1,0}\,\partial t^{\alpha,0}}(M^{-1})^{\alpha\alpha'}\,\frac{\partial \mathcal{F}_1}{\partial t^{\alpha',0}}. \tag{1.7}$$

The other Q_ps can be computed in a similar way.

Let us now describe the characteristic properties of the above sixteen graphs distinguishing them from the other graphs. For a graph Q, denote by $N_v(Q)$, $N_e(Q)$, and $N_l(Q)$ the numbers of its vertices, edges, and legs, respectively. Let $v_1, \dots, v_m$ with $m = N_v(Q)$ be the vertices of the graph. Denote by $g(v_i)$ and $n(v_i)$ the genus and the valence of the vertex v_i. Finally, let $B_1(Q)$ be the first Betti number of the graph Q.

The above sixteen graphs are selected from the set of connected graphs by requiring that each of these graphs satisfies the following properties.

(1) Each of these graphs is the dual graph of a stable curve of arithmetic genus two. Equivalently, the graph is planar, and the valence and the genus of each of its vertices satisfy the constraints $2g(v_i) - 2 + n(v_i) > 0$ and
$$\sum_{i=1}^{m} g(v_i) + B_1(Q) = 2.$$

(2) The number of edges and the number of legs are equal to $N_v(Q) + B_1(Q) - 1$. This property is equivalent to the Euler formula for the graph, $N_e(Q) - N_v(Q) + 1 = B_1(Q)$, together with the condition that the function assigned to Q as above must have degree two with respect to the jet variables $\partial_x^p v^\alpha$, i.e.,
$$\sum_{i=1}^{m}(2g(v_i) - 2 + n(v_i)) - N_e(Q) = 2\,.$$
Recall that, according to [1, 3], such a function can be represented as a rational function of the jet variables $\partial_x^p v^\alpha$, $p \geqslant 1$, and its degree is defined by assigning the degree p to any $\partial_x^p v^\alpha$, $\alpha = 1, \dots, n$. We also note that
$$\sum_{i=1}^{m} n(v_i) = 2N_e(Q) + N_l(Q).$$

(3) Cutting off an edge connecting two genus zero vertices does not destroy the connectivity of the graph. A graph with this property is said to be *one-particle irreducible (1PI)* in the physics literature.
(4) There is at most one vertex of the valence $n(v_i) = 3 - 2g(v_i)$ in the graph. Moreover, if the graph contains only one genus one vertex, then the valence of each of its vertices v_i satisfies the inequality $n(v_i) > 3 - 2g(v_i)$.

Remark 1.2. If a graph $\tilde{Q}$ is obtained from a graph Q by adding a genus zero vertex with a leg in the middle of an edge of Q, then the functions assigned to $\tilde{Q}$ and Q are the same. This follows immediately from the above definitions. Thus, we view the new graph $\tilde{Q}$ as a graph equal to the old one, to Q.

The main point of the decomposition (1.3) of the genus two free energy into a sum of 16+1 terms is the following assertion.

Lemma 1.3. *The restrictions of the terms $Q_1, \dots, Q_{16}$ to the small phase space vanish.*

The proof of the lemma easily follows from the above explicit expressions, from the rules $v_x|_{\text{phase space}} = e$ and $v^{(k)}|_{\text{phase space}} = 0$ for $k \geqslant 2$ for restrictions of jets, and from the identity $\partial_e G = 0$ (for details, see [5]). Here e stands for the unit of the Frobenius manifold and G is the G-function of the Frobenius manifold that enters (2.11) below.

Thus, the part of the free energy "responsible" for the would-be genus two Gromov–Witten invariants (i.e., without descendents) is entirely contained in our genus two G-function.

Another important feature of the genus two G-function can be observed in the analysis of important examples coming from singularity theory and orbifold Gromov–Witten invariants. In the present paper we consider two classes of examples: first, the case of simple singularities and, second, the Gromov–Witten invariants of $\mathbb{P}^1$-orbifolds with positive Euler characteristic. Both classes of examples are associated with Dynkin diagrams of ADE type. The connection of the simple singularities with the ADE Weyl groups is well known. The Frobenius structure on the base of universal unfolding in this case can be constructed with the help of K. Saito theory of primitive forms [23]. The integrable hierarchies of topological type coincide with the Drinfeld–Sokolov ADE hierarchies [18, 6, 27]. The associated cohomological field theory was constructed in [26, 12, 13, 14, 15, 11].

The case of $\mathbb{P}^1$-orbifolds is relatively more recent. In this case one deals with the $\mathbb{P}^1$-orbifolds of positive Euler characteristic. Hence, there are at most three orbifold points with multiplicities p, q, and r. These positive integers must satisfy the condition $1/p + 1/q + 1/r > 1$. This inequality has only finitely many solutions, which are listed in the following table:

(p,q,r)	Dynkin diagram
$(p,q,1)$	$\tilde{A}_{p,q}$
$(2,2,r)$	$\tilde{D}_{r+2}$
$(2,3,r)$	$\tilde{E}_{r+3}$

The second column of this table refers to the so-called extended affine Weyl groups of ADE type. The Frobenius manifolds in these cases were constructed in [2]. The construction depends on the choice of a vertex of the Dynkin diagram. A connection between these Frobenius manifolds and the orbifold quantum cohomology of the $\mathbb{P}^1$-orbifolds was discovered in [21] for the $\tilde{A}_{p,q}$ case and in [22] for other Dynkin diagrams. An important connection between these Frobenius manifolds with Frobenius structures on the spaces of the so-called tri-polynomials (see below) was also established in [22] (the role of tri-polynomials in the homological mirror symmetry was revealed in [25]).

The main conjecture of the present paper is as follows.

Conjecture 1.4. *If M is a Frobenius manifold obtained from the genus zero Fan–Jarvis–Ruan–Witten ($FJRW$) invariant theory for ADE singularities or the genus zero Gromov–Witten invariant theory for $\mathbb{P}^1$-orbifolds of ADE type, then*

$$G^{(2)}(u, u_x, u_{xx}) = 0. \tag{1.8}$$

Remark 1.5. In FJRW theory includes a symmetry group G. We assume that the singularities and their symmetry groups are chosen in such a way that the corresponding Frobenius manifolds coincide with the ordinary ones constructed from the singularities of the same type [1]. In particular, when the singularities are of A and E type, or D type with even Milnor number, the group G can be chosen as the minimal one, $\langle J\rangle$. For the singularities of D type with odd Milnor number, one needs to start from the mirror of D_n, i.e., $D_n^T = x^{n-1}y + y^2$, and choose the group G to be the maximal one, G_{max}. The reason is that the FJRW theory is an A-model theory, while the construction given in [1] from singularities to Frobenius manifolds is on the B-side, and thus there are mirror symmetry phenomena between them. For more details, see [13, 15].

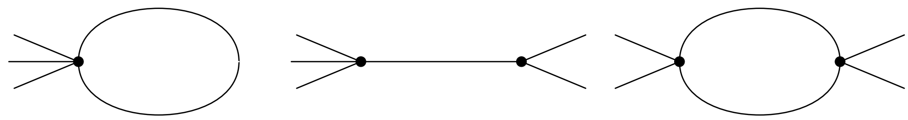
The main conjecture can also be formulated in the following way.

Conjecture 1.6. *If M is a Frobenius manifold assigned to ADE singularity or an extended affine Weyl group of ADE type, then*

$$G^{(2)}(u, u_x, u_{xx}) = 0. \tag{1.9}$$

The validity of this conjecture has been verified in many special cases; the main goal of the present paper is to explain the tools relevant for such a verification.

Remark 1.7. *Formulas for the genus two free energies for the Frobenius manifolds associated to A_2 singularity and to the extended affine Weyl group $\tilde{W}(A_1)$ are given in* [9, 10]. *They have the graph representations $\mathcal{F}_2 = \frac{1}{1152}Q_1 - \frac{1}{360}Q_2 - \frac{1}{1152}Q_3 + \frac{1}{360}Q_4$ and $\mathcal{F}_2 = \frac{1}{1152}Q_1 - \frac{1}{360}Q_2 - \frac{1}{1152}Q_3 + \frac{1}{360}Q_4 - \frac{1}{480}W_1 + \frac{7}{5760}W_2 + \frac{11}{5760}W_3$, respectively. Here W_1, W_2, and W_3 are as follows:*



W_1 W_2 W_3

When computing the coefficients c_p for our examples, we find the following interesting identity.

Theorem 1.8. *If M is the Frobenius manifold obtained from the genus zero FJRW invariant theory for ADE singularities or the genus zero Gromov–Witten invariant theory for $\mathbb{P}^1$-orbifolds of AD type, then*

$$(Q_1 - Q_6) + 2(Q_7 - Q_5) + 3(Q_8 - Q_2) + 4(Q_9 - Q_3) + 6(Q_4 + Q_{10} - Q_{11} - Q_{12}) = 0. \tag{1.10}$$

Identity (1.10) remains valid for an arbitrary two-dimensional semisimple Frobenius manifold (i.e., for a topological field theory with two primary fields in the terminology of [10]) as well as for the three-dimensional Frobenius manifolds on the orbit spaces of Coxeter groups of type B_3 or H_3. It is interesting to find necessary and sufficient conditions for the validity of this identity in the general case.

The paper is organized as follows. In Sec. 2.1 we recall first some basic properties of semisimple Frobenius manifolds and their genus zero, one, and two free energies. Then we give a proof of Theorem 1.1. In Sec. 2.2 we prove Theorem 1.8. In Secs. 2.3 and 2.4 we give some general formulas for calculating the rotation coefficients for Frobenius manifolds arising in singularity theory. In Sec. 3 we present more explicit formulas for the rotation coefficients, case by case, for simple singularities of ADE type and for $\mathbb{P}^1$ orbifolds of A and D type, and provide evidences to support the validity of the conjectures. In Appendices we give formulas for the function $G^{(2)}(u, u_x, u_{xx})$ that were presented in (1.3) and for the genus two free energy of semisimple Frobenius manifolds which was given in [5].

2. GENERAL RESULTS

2.1. Proof of Theorem 1.1

For a semisimple Frobenius manifold M^n, denote by $v^1, \ldots, v^n$ the flat coordinates, by $\langle\,,\,\rangle$ the flat metric, $\langle \partial/\partial v^\alpha, \partial/\partial v^\beta\rangle = \eta_{\alpha\beta}$, $(\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}$, and by $F(v) = F(v^1, \ldots, v^n)$ the potential. The canonical coordinates $u_1, \ldots, u_n$ are defined in such a way that the multiplication table defined on the tangent spaces is given by $\partial/\partial u_i \cdot \partial/\partial u_j = \delta_{ij}\partial/\partial u_i$. In the canonical coordinates, the flat metric takes the diagonal form $\sum_{\alpha,\beta}\eta_{\alpha\beta}dv^\alpha dv^\beta = \sum_{i=1}^n \eta_{ii}(u)du_i^2$. Write $h_i = h_i(u) = \sqrt{\eta_{ii}}$,

$i = 1, \ldots, n$ for the Lamé coefficients of the diagonal metric for some choice of the signs of the square roots. Define the rotation coefficients $\gamma_{ij} = \gamma_{ji}$ by $\gamma_{ij} = (1/h_i)(\partial h_j/\partial u_i)$ for $i \neq j$, $\gamma_{ii} = 0$. The nonzero Christoffel symbols of the Levi-Civita connection for the flat metric in the canonical coordinates are written out in the following table.

$$\Gamma_{ij}^k = \begin{cases} -\sum_{l=1}^n \gamma_{il} \frac{h_l}{h_i}, & i = j = k; \\ \gamma_{ij} \frac{h_j}{h_i}, & k = i \neq j; \\ \gamma_{ij} \frac{h_i}{h_j}, & k = j \neq i; \\ -\gamma_{ik} \frac{h_i}{h_k}, & k \neq i = j. \end{cases} \tag{2.1}$$

The canonical and the flat coordinates of the Frobenius manifold are related by the following equations:

$$\frac{\partial^2 v^\alpha}{\partial u_i \partial u_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial v^\alpha}{\partial u_k}. \tag{2.2}$$

Write $\psi_i^\alpha(u) = (1/h_i(u))(\partial v^\alpha(u)/\partial u_i)$, $\psi_{i\alpha} = \eta_{\alpha\beta}\psi_i^\beta$, where the summation with respect to the repeated upper and lower Greek indices is assumed. Assuming that the unit vector field of the Frobenius manifold is $e = \partial/\partial v^1$, we see that

$$\psi_{i1} = h_i \tag{2.3}$$

and

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F(v)}{\partial v^\alpha \partial v^\beta \partial v^\gamma} = \sum_{i=1}^n \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}. \tag{2.4}$$

The following formulas [1] will be used below to represent the correlation functions in terms of the canonical coordinates:

$$\begin{gathered} \frac{\partial v^\alpha}{\partial u_i} = \psi_{i1}\psi_i^\alpha, \qquad \frac{\partial u_i}{\partial v^\alpha} = \frac{\psi_{i\alpha}}{\psi_{i1}}; \qquad \frac{\partial \psi_{i\alpha}}{\partial u_k} = \gamma_{ik}\,\psi_{k\alpha}, \quad i \neq k, \qquad \frac{\partial \psi_{i\alpha}}{\partial u_i} = -\sum_{k=1}^n \gamma_{ik}\psi_{k\alpha}; \\ \frac{\partial \gamma_{ij}}{\partial u_k} = \gamma_{ik}\gamma_{kj}, \quad i, j, k \text{ distinct}, \quad \frac{\partial \gamma_{ij}}{\partial u_i} = \frac{\sum_{k=1}^n (u_j - u_k)\gamma_{ik}\gamma_{kj} - \gamma_{ij}}{u_i - u_j}. \end{gathered} \tag{2.5}$$

The principal hierarchy associated with the Frobenius manifold is a hierarchy of integrable Hamiltonian systems of hydrodynamic type, $\partial v^\alpha/\partial t^{\beta,q} = \eta^{\alpha\gamma}(\partial/\partial x)(\partial\theta_{\beta,q+1}/\partial v^\gamma)$, $\alpha, \beta = 1, \ldots, n$, $q \geqslant 0$. Here $\theta_\alpha(v; z) = \sum_{p\geqslant 0} \theta_{\alpha,p}(v) z^p$, $\alpha = 1, \ldots, n$, are related to the flat coordinates of the deformed flat connection of the Frobenius manifold. They satisfy the conditions

$$\theta_\alpha(v; 0) = \eta_{\alpha\gamma} v^\gamma, \qquad \langle \nabla\theta_\alpha(v, -z), \nabla\theta_\alpha(v, z)\rangle = \eta_{\alpha\beta}, \qquad \partial_\alpha\partial_\beta\theta_\gamma(v; z) = z c_{\alpha\beta}^\xi \partial_\xi \theta_\gamma(v; z),$$

$$E\left(\partial_\beta\theta_{\alpha,p}(v)\right) = p\,\partial_\beta\theta_{\alpha,p}(v) + \hat\mu_\alpha^\gamma \partial_\beta\theta_{\gamma,p}(v) + \hat\mu_\beta^\gamma \partial_\gamma\theta_{\alpha,p}(v) + (R_0)_\beta^\gamma\,\partial_\gamma\theta_{\alpha,p}(v) + \sum_{k=0}^p \partial_\beta\theta_{\gamma,p-k}(v)\ (R_k)_\alpha^\gamma,$$

where E stands for the Euler vector field of the Frobenius manifold which has the following representations in the flat coordinates and in the canonical coordinates, respectively:

$$E = \sum_{\alpha=1}^n E^\alpha(v) \frac{\partial}{\partial v^\alpha} = \sum_{i=1}^n u_i \frac{\partial}{\partial u_i},$$

and $\hat\mu$ and R_0 are the semisimple and nilpotent parts of the antisymmetric constant matrix $\mathcal{V} = (\mathcal{V}_\beta^\alpha)$ with $\mathcal{V}_\beta^\alpha = ((2-d)/2)\,\delta_\beta^\alpha - \partial E^\alpha(v)/\partial v^\beta$. The constant matrices $R_0, R_1, \ldots R_m$ (m is a certain integer depending on the Frobenius manifold) form a part of the monodromy data of the Frobenius manifold at $z = 0$ (see [1] for detail), they have the properties $(R_k)_\alpha^\gamma\,\eta_{\gamma\beta} = (-1)^{k+1}(R_k)_\beta^\gamma\,\eta_{\gamma\alpha}$, $[\hat\mu, R_k] = kR_k$, $k = 0, 1, \ldots, m$. The potential $F(v)$ can be chosen in such a way that the functions

$\theta_{\alpha,1}(v)$ have the expression $\theta_{\alpha,1}(v) = \partial_\alpha F(v)$, $\partial_\alpha = \partial/\partial v^\alpha$. Thus the first set of equations of the principal hierarchy reads $\partial v^\alpha/\partial t^{\beta,0} = \eta^{\alpha\xi} c_{\xi\beta\gamma}(v) v_x^\gamma$ with $\partial v^\alpha/\partial t^{1,0} = v_x^\alpha$, $\alpha, \beta = 1, \dots, n$. By using the above formulas, we obtain the following formula for solutions of the principal hierarchy:

$$\frac{\partial u_i}{\partial t^{\alpha,0}} \frac{\partial v^\alpha}{\partial u_j} = \frac{\partial u_i}{\partial t^{\alpha,0}} \psi_i^\alpha \psi_{i1} = \begin{cases} u_{i,x} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.6}$$

Moreover, for higher jets $u_i^{(p)} = \partial_x^p u_i$, write

$$U_j^{i,p} = \frac{\partial u_i^{(p)}}{\partial t^{\alpha,0}} \frac{\partial v^\alpha}{\partial u_j}, \qquad i, j = 1, \dots, n, \quad p \geqslant 0. \tag{2.7}$$

Then the following recursion relation holds:

$$U_j^{i,p} = \partial_x U_j^{i,p-1} - \sum_k \Gamma_{kj}^s u_{k,x} U_s^{i,p-1}, \qquad i, j = 1, \dots, n, \quad p \geqslant 1. \tag{2.8}$$

Using this recursion relation, one can represent $U_j^{i,p}$ in terms of jets $u_i^{(m)}$ with $m \geqslant 1$, the rotation coefficients γ_{ij}, and the Lamé coefficients h_i, starting from $U_j^{i,0} = \delta_j^i u_{j,x}$. Such expressions will be useful in dealing with differential operators of the form

$$\frac{\partial v^\alpha}{\partial u_i} \frac{\partial}{\partial t^{\alpha,0}} = \sum_{p \geqslant 0} U_i^{j,p} \frac{\partial}{\partial u_j^{(p)}}.$$

The topological solution $v(\mathbf{t}) = \big(v^1(\mathbf{t}), \dots, v^n(\mathbf{t})\big)$ of the principal hierarchy is found from the system of n equations $\sum \tilde{t}^{\alpha,p} \nabla \theta_{\alpha,p} = 0$, $\tilde{t}^{\alpha,p} = t^{\alpha,p} - \delta_1^\alpha \delta_1^p$. By using the topological solution $v(\mathbf{t})$, one can define the genus zero free energy $\mathcal{F}_0 = \mathcal{F}_0(\mathbf{t})$ of the Frobenius manifold [1] satisfying the equations

$$\frac{\partial^3 \mathcal{F}_0(\mathbf{t})}{\partial t^{\alpha,0} \partial t^{\beta,0} \partial t^{\gamma,0}} = c_{\alpha\beta}^\xi(v(\mathbf{t})) M_{\xi\gamma}, \quad \alpha, \beta, \gamma = 1, \dots, n, \tag{2.9}$$

where $M_{\xi\gamma} = c_{\xi\gamma\rho}(v(\mathbf{t})) v_x^\rho$.

Remark 2.1. By taking $\alpha = 1$ in (2.9), we see that the matrix $M_{\beta\gamma}$ coincides with the one occurred in the definition of the sixteen diagrams of Theorem 1.1. For this reason, we use the same notation.

Observe the following useful formula for the entries of the inverse matrix:

$$\big(M^{-1}\big)^{\alpha\beta} = \sum_{i=1}^n \frac{1}{h_i^2 u_{i,x}} \frac{\partial v^\alpha}{\partial u_i} \frac{\partial v^\beta}{\partial u_i}. \tag{2.10}$$

We also need the genus one free energy $\mathcal{F}_1(\mathbf{t})$ defined for a semisimple Frobenius manifold by the following expression:

$$\mathcal{F}_1(\mathbf{t}) = F_1(u, u_x)|_{v^\alpha = v^\alpha(\mathbf{t})} \quad \text{with} \quad F_1(u, u_x) = \frac{1}{24} \sum_{i=1}^n \log u_{i,x} + G(u), \tag{2.11}$$

where the function G is called the G-function of the Frobenius manifold. It is given by a quadrature, due to the following equations [3]:

$$\frac{\partial G(u)}{\partial u_i} = \frac{1}{2} \sum_{j \neq i} (u_i - u_j) \gamma_{ij}^2 - \frac{1}{24} \sum_{k \neq i} \gamma_{ik} \left(\frac{h_i}{h_k} - \frac{h_k}{h_i} \right). \tag{2.12}$$

To express the correlation functions in terms of canonical coordinates, we write $C_{i_1,i_2,\dots,i_m} = \dfrac{\partial^m \mathcal{F}_0(\mathbf{t})}{\partial t^{\alpha_1,0}\partial t^{\alpha_2,0}\dots\partial t^{\alpha_m,0}}\dfrac{\partial v^{\alpha_1}}{\partial u_{i_1}}\dfrac{\partial v^{\alpha_2}}{\partial u_{i_2}}\cdots\dfrac{\partial v^{\alpha_m}}{\partial u_{i_m}}$, $D_{i_1,i_2,\dots,i_m} = \dfrac{\partial^m \mathcal{F}_1(\mathbf{t})}{\partial t^{\alpha_1,0}\partial t^{\alpha_2,0}\dots\partial t^{\alpha_m,0}}\dfrac{\partial v^{\alpha_1}}{\partial u_{i_1}}\dfrac{\partial v^{\alpha_2}}{\partial u_{i_2}}\cdots\dfrac{\partial v^{\alpha_m}}{\partial u_{i_m}}$ for the indices $1 \leqslant i_1,\dots,i_m \leqslant n$. Then

$$C_{i_1,i_2,i_3} = \begin{cases} h_i^2\, u_{i_1,x} & \text{if} \quad i_1 = i_2 = i_3, \\ 0 & \text{otherwise}; \end{cases} \qquad D_i = \sum_{p=0}^{1} U_i^{j,p}\,\frac{\partial F_1(u,u_x)}{\partial u_j^{(p)}}. \tag{2.13}$$

By using the relation (2.2), we obtain the following recursive formula:

$$X_{i_1,i_2,\dots,i_{m+1}} = \sum_{k=1}^{n}\sum_{p=0}^{m-2}\frac{\partial X_{i_1,\dots,i_m}}{\partial u_k^{(p)}}\,X_{i_{m+1}}^{k,p} - \sum_{k=1}^{m} X_{i_1,\dots,i_{k-1},s,i_{k+1},\dots,i_m}\Gamma^s_{i_k i_{m+1}}u_{i_{m+1},x}, \tag{2.14}$$

which holds for $X = C$ and $X = D$.

Proof of Theorem 1.1. Since the genus two free energy F_2 given in [5] is represented as a rational function of the canonical coordinates u_i, their x-derivatives $u_i^{(p)} = \partial_x^p u_i$, the rotation coefficients γ_{ij}, and the Lamé coefficients h_i, it follows that, to prove the theorem, we are to represent the functions $Q_1,\dots,Q_{16}$ assigned to the 16 dual graphs as rational functions of the above variables. In fact, for the functions Q_1 and Q_{16} defined in (1.4), we have

$$\begin{aligned} Q_1 &= \frac{\partial^6\mathcal{F}_0}{\partial t^{1,0}\,\partial t^{1,0}\,\partial t^{\alpha,0}\,\partial t^{\alpha',0}\,\partial t^{\beta,0}\,\partial t^{\beta',0}}\sum_{j_1,j_2=1}^{n}\frac{1}{h_{j_1}^2 u_{j_1,x}}\frac{\partial v^{\alpha}}{\partial u_{j_1}}\frac{\partial v^{\alpha'}}{\partial u_{j_1}}\frac{1}{h_{j_2}^2 u_{j_2,x}}\frac{\partial v^{\beta}}{\partial u_{j_2}}\frac{\partial v^{\beta'}}{\partial u_{j_2}} \\ &= \sum_{i_1,i_2,j_1,j_2=1}^{n}\frac{\partial v^{\alpha_1}}{\partial u_{i_1}}\frac{\partial v^{\alpha_2}}{\partial u_{i_2}}\frac{\partial^6\mathcal{F}_0}{\partial t^{\alpha_1,0}\,\partial t^{\alpha_2,0}\,\partial t^{\alpha,0}\,\partial t^{\alpha',0}\,\partial t^{\beta,0}\,\partial t^{\beta',0}}\frac{1}{h_{j_1}^2 u_{j_1,x}}\frac{\partial v^{\alpha}}{\partial u_{j_1}}\frac{\partial v^{\alpha'}}{\partial u_{j_1}}\frac{1}{h_{j_2}^2 u_{j_2,x}}\frac{\partial v^{\beta}}{\partial u_{j_2}}\frac{\partial v^{\beta'}}{\partial u_{j_2}} \\ &= \sum_{i_1,i_2,j_1,j_2=1}^{n}\frac{C_{i_1,i_2,j_1,j_1,j_2,j_2}}{h_{j_1}^2 h_{j_2}^2 u_{j_1,x}u_{j_2,x}}, \end{aligned}$$

and

$$Q_{16} = \frac{\partial^2\mathcal{F}_1}{\partial t^{1,0}\partial t^{\alpha,0}}\sum_{i=1}^{n}\frac{1}{h_i^2 u_{i,x}}\frac{\partial v^{\alpha}}{\partial u_i}\frac{\partial v^{\alpha'}}{\partial u_i}\frac{\partial\mathcal{F}_1}{\partial t^{\alpha',0}} = \sum_{i,j=1}^{n}\frac{\partial^2\mathcal{F}_1}{\partial t^{\beta,0}\partial t^{\alpha,0}}\frac{\partial v^{\beta}}{\partial u_j}\frac{1}{h_i^2 u_{i,x}}\frac{\partial v^{\alpha}}{\partial u_i}\frac{\partial v^{\alpha'}}{\partial u_i}\frac{\partial\mathcal{F}_1}{\partial t^{\alpha',0}} = \sum_{i,j=1}^{n}\frac{D_i D_{i,j}}{h_i^2 u_{i,x}}.$$

Here we have used the identity $\sum_{i=1}^n \partial v^\alpha/\partial u_i = \partial v^\alpha/\partial v^1 = \delta_1^\alpha$ (since the unit vector field e of the Frobenius manifold is equal to $\partial/\partial v^1 = \sum_{i=1}^n \partial/\partial u_i$).

It follows from formulas (2.5)–(2.14) that the functions $C_{i_1,i_2,j_1,j_1,j_2,j_2}$, D_i, and $D_{i,j}$ can also be represented as rational functions of the canonical coordinates u_i, their x-derivatives $u_i^{(p)} = \partial_x^p u_i$, the rotation coefficients γ_{ij}, and the Lamé coefficients h_i. In a similar way, we can find similar expressions for other functions $Q_2, \dots Q_{15}$. Now, by subtracting the linear combination of the 16 functions $Q_1,\dots,Q_{16}$ occurring on the right-hand side of (1.3) from the linear combination given by the left-hand side of (1.3), we obtain the desired expression for $G^{(2)}(u,u_x,u_{xx})$ by a tedious but straightforward computation. This completes the proof of the theorem.

2.2. Proof of Theorem 1.8

In this section, we reduce identity (1.8) to a simpler one, (2.17).

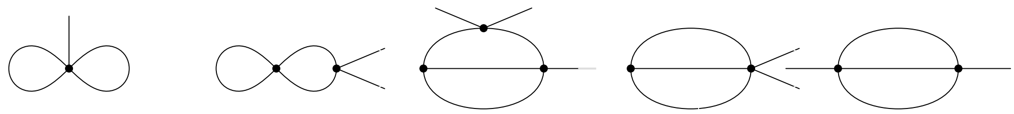
Lemma 2.2. *Let Γ be a dual graph, and let $x = t^{1,0}$. Then*

$$\partial_x \Gamma = \sum_{v:\ \text{vertex of } \Gamma} \Gamma_v - \sum_{e:\ \text{edge of } \Gamma} \Gamma_e, \tag{2.15}$$

where Γ_v is the dual graph obtained from Γ by adding a new leg at the vertex v, and Γ_e is the dual graph obtained from Γ by adding a new vertex of genus zero with two legs on the edge e.

Proof. The dual graph Γ corresponds to the product of several multi-point correlation functions and the inverse of the matrix M. According to the Leibniz rule, when the operator ∂_x acts on multi-point correlation functions, we obtain terms standing in the first summation on the right-hand side of (2.15) and, when it acts on the inverse of M, we obtain terms occurring in the other summation. This completes the proof of the lemma.

Introduce the following dual graphs:



P_1 P_2 P_3 P_4 P_5

O_1 O_2

An auxiliary assertion holds.

Lemma 2.3. *The following identities hold*:

$$\partial_x P_1 = Q_1 - 2\,Q_3, \quad \partial_x P_2 = Q_3 + Q_5 - Q_7 - 2\,Q_9, \quad \partial_x P_3 = Q_4 + Q_8 + Q_{10} - 2\,Q_{11} - 2\,Q_{12},$$

$$\partial_x P_4 = Q_6 + Q_2 - 3\,Q_{10}, \quad \partial_x P_5 = 2\,Q_2 - 3\,Q_4, \quad \partial_x O_1 = P_1 - 2\,P_2, \quad \partial_x O_2 = P_4 + P_5 - 3\,P_3,$$

and hence

$$(Q_1 - Q_6) + 2(Q_7 - Q_5) + 3(Q_8 - Q_2) + 4(Q_9 - Q_3) + 6(Q_4 + Q_{10} - Q_{11} - Q_{12}) = \partial_x^2 \left(O_1 - O_2\right). \tag{2.16}$$

Proof. These relations are easy consequences of Lemma 2.2.

Lemma 2.4. *For any semisimple Frobenius manifold, the following identity holds*:

$$O_1 - O_2 = \sum_{1 \leqslant i < j \leqslant n} \gamma_{ij} \frac{(h_i^2 + h_j^2)^2}{h_i^3\, h_j^3}. \tag{2.17}$$

Proof. The functions O_1 and O_2 have the following expressions:

$$O_1 = \sum_{1 \leqslant j_1, j_2 \leqslant n} \frac{C_{j_1, j_1, j_2, j_2}}{h_{j_1}^2 h_{j_2}^2 u_{j_1, x} u_{j_2, x}}, \quad O_2 = \sum_{1 \leqslant j_1, j_2, j_3 \leqslant n} \frac{C_{i_1, j_1, j_2, j_3} C_{j_1, j_2, j_3}}{h_{j_1}^2 h_{j_2}^2 h_{j_3}^2 u_{j_1, x} u_{j_2, x} u_{j_3, x}}.$$

By using formulas (2.13) and (2.14), one can see that

$$O_1 = \sum_{1 \leqslant i < j \leqslant n} \gamma_{ij} \frac{(h_i^2\, u_{j,x} + h_j^2\, u_{i,x})^2 - (h_i^4 + h_j^4)(u_{i,x} - u_{j,x})^2}{h_i^3 h_j^3 u_{i,x} u_{j,x}} + \sum_{i=1}^{n} \frac{u_{i,xx}}{h_i^2 u_{i,x}^2},$$

$$O_2 = \sum_{1 \leqslant i < j \leqslant n} \gamma_{ij} \frac{(h_i^4\, u_{i,x} - h_j^4\, u_{j,x})(u_{j,x} - u_{i,x})}{h_i^3 h_j^3 u_{i,x} u_{j,x}} + \sum_{i=1}^{n} \frac{u_{i,xx}}{h_i^2 u_{i,x}^2}.$$

Then it can readily be seen that the difference $O_1 - O_2$ is equal to the right-hand side of (2.17). This completes the proof of the lemma.

To prove Theorem 1.8, it suffices to prove the following lemma.

Lemma 2.5. *For a Frobenius manifold associated to ADE singularities, or $\mathbb{P}^1$-orbifolds of AD type, the difference $O_1 - O_2$ is always a constant.*

We will give the proof of the above lemma case by case in Section 3.

2.3. Rotation coefficients for simple singularities

Let f be a polynomial on $\mathbb{C}^m$ which has an isolated critical point at $0 \in \mathbb{C}^m$ of ADE type. Let n be the Milnor number of f. The coordinates in $\mathbb{C}^m$ are $z = (z^1, \dots, z^m)$. Denote by ∂_α or ∂_{z^α} the partial derivatives $\frac{\partial}{\partial z^\alpha}$.

Let $F\colon \mathbb{C}^m \times B \to \mathbb{C}$, $(z,t) \mapsto F(z,t)$, be a miniversal unfolding of f (avoid confusions with the potential of the Frobenius manifold!), where B is an open ball in $\mathbb{C}^n$. Let $C \subset B$ be the caustic. For a given point t in the complement $B \setminus C$, the function $F(z,t)$ has n Morse critical points $z^{(i)}(t) = (z^{(i),1}, \dots, z^{(i),m})$ $(i = 1, \dots, n)$, $\partial_\alpha F(z,t)|_{z=z^{(i)}(t)} = 0$, $\alpha = 1, \dots, m$. Define the canonical coordinates u_i on $B \setminus C$ as the critical values

$$u_i(t) = F(z^{(i)}(t), t), \quad i = 1, \dots, n. \tag{2.18}$$

We often use the brief notation ∂_i or ∂_{u_i} for the partial derivatives $\partial/\partial u_i$.

There is a semisimple Frobenius manifold structure on the base space $B \setminus C$. The flat metric $\langle\ ,\ \rangle$ is defined by

$$\langle \partial', \partial'' \rangle_t = -\operatorname{Res}_{z=\infty} \frac{(\partial' F(z,t))(\partial'' F(z,t))\ dz^1 \wedge \dots \wedge dz^m}{\partial_{z^1} F \cdots \partial_{z^m} F} \tag{2.19}$$

for any ∂', $\partial'' \in T_t B$. Write $h_{\alpha\beta}(z,t) = \partial_\alpha \partial_\beta F(z,t)$ and $H(z,t) = \det(h_{\alpha\beta}(z,t))$. Let $(h^{\alpha\beta})$ be the inverse matrix of $(h_{\alpha\beta})$. Then the residue theorem implies that

$$\langle \partial', \partial'' \rangle_t = \sum_{k=1}^n \left. \frac{(\partial' F(z,t))(\partial'' F(z,t))}{H(z,t)} \right|_{z=z^{(k)}(t)}. \tag{2.20}$$

Write

$$\eta^{ii}(t) = H(z^{(i)}(t), t), \quad \eta_{ii}(t) = \Big(H(z^{(i)}(t), t)\Big)^{-1}. \tag{2.21}$$

By using (2.20) and the identity

$$\partial_i F(z,t)|_{z=z^{(k)}(t)} = \delta_{ik}, \tag{2.22}$$

we then obtain

$$\langle \partial_i, \partial_j \rangle_t = \sum_{k=1}^n \left. \frac{(\partial_i F(z,t))(\partial_j F(z,t))}{H(z,t)} \right|_{z=z^{(k)}(t)} = \sum_{k=1}^n \frac{\delta_{ik}\delta_{jk}}{\eta^{kk}(t)} = \delta_{ij}\, \eta_{ii}(t). \tag{2.23}$$

It follows from the definition of the critical points $z^{(k)}(t)$ that

$$\partial_i \partial_\alpha F(z,t)|_{z=z^{(k)}(t)} = -h_{\alpha\beta}(z^{(k)}(t), t)\, \partial_i z^{(k),\beta}(t). \tag{2.24}$$

$$\partial_i z^{(k),\beta}(t) = -h^{\alpha\beta}(z^{(k)}(t), t)\, \partial_i \partial_\alpha F(z,t)|_{z=z^{(k)}(t)}. \tag{2.25}$$

By using these equations and the identity $\partial_x \det A(x) = \det A(x) \operatorname{Tr}\big(A^{-1}(x) \partial_x A(x)\big)$ for any non-degenerate matrix function $A(x)$, we obtain

$$\begin{aligned} \frac{\partial_i \eta^{kk}}{\eta^{kk}} &= \big(h^{\alpha\beta}(z,t)\partial_i h_{\alpha\beta}(z,t) - h^{\alpha\beta}(z,t)\partial_\gamma h_{\alpha\beta}(z,t)\, h^{\gamma\sigma}(z,t)\, \partial_i \partial_\sigma F(z,t)\big)\,|_{z=z^{(k)}(t)} \\ &= \big(h^{\alpha\beta}(z,t)\partial_i h_{\alpha\beta}(z,t) + \partial_\alpha h^{\alpha\sigma}\, \partial_i \partial_\sigma F(z,t)\big)\,|_{z=z^{(k)}(t)} = \partial_\alpha \left(h^{\alpha\beta}(z,t)\, \partial_i \partial_\beta F(z,t)\right)|_{z=z^{(k)}(t)}. \end{aligned} \tag{2.26}$$

As above, denote by $h_i = \sqrt{\eta_{ii}}$ the Lamé coefficients and by $\gamma_{ki} = \frac{\partial_i h_k}{h_i}$ the rotation coefficients of the metric $\sum_{i=1}^n \eta_{ii}(du_i)^2$. We often use the coefficients Γ_{ki}^k of the Christoffel symbols of the metric with $k \neq i$, for this reason, we introduce a notation for these coefficients, namely,

$$\Gamma_{ki} := \Gamma_{ki}^k = \frac{\partial_i \eta_{kk}}{2\,\eta_{kk}} = -\frac{1}{2}\partial_\alpha \left(h^{\alpha\beta}(z,t)\, \partial_i \partial_\beta F(z,t)\right)|_{z=z^{(k)}}. \tag{2.27}$$

Then

$$\gamma_{ki} = (h_k/h_i)\Gamma_{ki}. \tag{2.28}$$

Remark 2.6. Equations (2.12) satisfied by the G-function of the Frobenius manifold can be represented as

$$\partial_i G(u) = \frac{1}{2}\sum_{k\neq i}(u_i - u_k)\Gamma_{ki}\Gamma_{ik} - \frac{1}{24}\sum_{k\neq i}(\Gamma_{ki} - \Gamma_{ik}). \tag{2.29}$$

The explicit expressions of Γ_{ki} given in Sec. 3 for the Frobenius manifolds associated to ADE singularities can be used to re-derive the known explicit formulas $G = 0$ [17, 24] for the G-functions of this class of Frobenius manifolds. We can also obtain the explicit formulas (3.21), (3.23) for the G-functions of the Frobenius manifolds defined on the orbit spaces of the extended affine Weyl groups of AD type. Strachan [24] proved formula (3.21) (see below) and conjectured formula (3.23).

Equations (2.27) and (2.28) give us a formula to compute the rotation coefficients of the Frobenius manifold. However, the computation of the derivatives of $F(z,t)$ with respect to the canonical coordinates is also needed. To this end, we assume below that the miniversal deformation $F(z,t)$ is given by $F(z,t) = f(z) + \sum_{j=1}^{n} t^j\,\phi_j(z)$, where $\phi_1(z),\dots,\phi_n(z)$ is a basis of the Milnor ring. Define $W\colon (\mathbb{C}^m)^n \to \mathbb{C}$ by $W(z_1,\dots,z_n) = \det(\phi_j(z_i))$.

Lemma 2.7.

$$\partial_i F(z,t) = \frac{W(z^{(1)},\dots,z^{(i-1)},z,z^{(i+1)},\dots,z^{(n)})}{W(z^{(1)},\dots,z^{(n)})}. \tag{2.30}$$

Proof. By (2.22), $\sum_{j=1}^{n}\partial t^j/\partial u_i\,\phi_j(z^{(k)}(t)) = \delta_{ik}$. Thus, $\partial u_i/\partial t^j = \phi_j(z^{(i)}(t))$. Next, consider the following system of linear equations for partial derivatives $\partial_i F(z,t) = \partial F(z,t)/\partial u_i$:

$$\phi_j(z) = \frac{\partial F(z,t)}{\partial t^j} = \partial_i F(z,t)\,\frac{\partial u_i}{\partial t^j} = \phi_j(z^{(i)})\partial_i F(z,t),\ j = 1,\dots,n.$$

The statement of the lemma now follows by using Cramer's rule.

2.4. Rotation coefficients for $\mathbb{P}^1$-orbifolds

Let p,q,r be positive integers satisfying $1/p+1/q+1/r > 1$. It is shown in [22] that the quantum cohomology of the $\mathbb{P}^1$-orbifold $\mathbb{P}^1_{p,q,r}$ is isomorphic to the Frobenius structure on the space of tri-polynomials of type (p,q,r).

We take $m = 3$ and $n = p+q+r-1$. A tri-polynomial is a function $F\colon \mathbb{C}^m \times B \to \mathbb{C},\ (z,t) \mapsto F(z,t)$ such that

$$F(z,t) = -\,z^1z^2z^3 + P_1(z_1) + P_2(z_2) + P_3(z_3), \tag{2.31}$$

$$P_1(z_1) = \sum_{i=1}^{p-1} t_i z_1^i + z_1^p, \tag{2.32}$$

$$P_2(z_2) = \sum_{i=1}^{q-1} t_{p-1+i} z_2^i + z_2^p, \tag{2.33}$$

$$P_3(z_3) = \sum_{i=0}^{r} t_{p+q-1+i} z_3^i, \tag{2.34}$$

where B is an open set in $\mathbb{C}^{n-1}\times\mathbb{C}^*$ defined by the condition $t^n \neq 0$. Let $C \subset B$ be the caustic. As in the previous section, the critical values

$$u_i(t) = F(z^{(i)}(t),t), \quad i = 1,\dots,n, \tag{2.35}$$

define the canonical coordinates u^i on $B \setminus C$.

The flat metric of the Frobenius structure on the space of tri-polynomial is also defined by (2.19). One can easily see that all lemmas from the previous section hold true also for tri-polynomials.

3. EXAMPLES

3.1. A_n singularities

In this case, $m = 1$, $f(z) = z^{n+1}$, and $\phi_j = z^{n-j}$.

Lemma 3.1.

$$\partial_i F(z,t) = \frac{1}{z - z^{(i)}} \frac{F'(z,t)}{F''(z^{(i)},t)}. \tag{3.1}$$

Proof. The lemma can readily be proved by using the identities

$$F'(z,t) = (n+1)\prod_{k=1}^{n}(z - z^{(k)}(t)), \qquad F''(z^{(i)}(t),t) = (n+1)\prod_{k\neq i}(z^{(i)}(t) - z^{(k)}(t)),$$

and Lemma 2.7.

Lemma 3.2.

$$\Gamma_{ki}(t) = \frac{1}{(z^{(k)}(t) - z^{(i)}(t))^2 F''(z^{(i)}(t),t)}. \tag{3.2}$$

Proof. This follows from (2.26) and Lemma 3.1.

Remark 3.3. By applying the residue theorem to the meromorphic functions

$$m(z) = \frac{F(z) - F(z^{(i)})}{F'(z)(z - z^{(i)})^4}, \qquad \tilde{m}(z) = \frac{F''(z) - F''(z^{(i)})}{F'(z)(z - z^{(i)})^2},$$

one can easily prove that the G-functions of the A_n singularities vanish.

Let us now use formula (3.2) to verify the validity of Conjecture 1.4 for A_n singularities. We use the critical points $z^{(1)}, \dots, z^{(n)}$ and an additional parameter $z^{(0)}$ to represent $F(z,t) = z^{n+1} + t^1 z^{n-1} + \dots + t^n$ in the form

$$F(z,t) = \lambda(z) = \int_0^z (n+1)\prod_{k=1}^{n}(\xi - z^{(k)})d\xi + z^{(0)} \tag{3.3}$$

Note that $z^{(1)}, \dots z^{(n)}$ are not independent, because they satisfy the relation

$$z^{(n)} = -\sum_{k=1}^{n-1} z^{(k)}. \tag{3.4}$$

We have

$$u_i = \lambda(z^{(i)}), \quad h_i = \psi_{i,1} = \frac{1}{\sqrt{\lambda''(z^{(i)})}}, \quad \gamma_{ij} = \frac{h_i\, h_j}{(z^{(i)} - z^{(j)})^2}. \tag{3.5}$$

By substituting these expressions into formula (1.1) for $G^{(2)}(u, u_x, u_{xx})$, we obtain a rational function of $z^{(0)}, \dots z^{(n-1)}$. For $n \leqslant 8$, one can check (with the help of a suitable symbolic computations software) that this rational function vanishes, and thus Conjecture 1.4 is valid in these cases.

Proof of Lemma 2.5 for A_n singularities. First,

$$\sum_{1\leqslant i<j\leqslant n} \gamma_{ij} \frac{(h_i^2 + h_j^2)^2}{h_i^3\, h_j^3} = \sum_{1\leqslant i<j\leqslant n} \frac{1}{(z_i - z_j)^2}\left(\frac{\lambda''(z_i)}{\lambda''(z_j)} + \frac{\lambda''(z_j)}{\lambda''(z_i)} + 2\right) = \sum_{i=1}^{n}\sum_{j\neq i} \frac{\lambda''(z_i) + \lambda''(z_j)}{(z_i - z_j)^2\, \lambda''(z_j)}.$$

For a fixed i,

$$\sum_{j\neq i}\frac{\lambda''(z_i)+\lambda''(z_j)}{(z_i-z_j)^2\,\lambda''(z_j)}=\sum_{j\neq i}\mathrm{Res}_{z=z_j}\frac{\lambda''(z)+\lambda''(z_i)}{(z-z_i)^2\,\lambda'(z)}=-\,\mathrm{Res}_{z=z_i}\frac{\lambda''(z)+\lambda''(z_i)}{(z-z_i)^2\,\lambda'(z)}=-\frac{1}{6}\frac{\lambda^{(4)}(z_i)}{\lambda''(z_i)}.$$

Thus,

$$\sum_{i=1}^{n}\sum_{j\neq i}\frac{\lambda''(z_i)+\lambda''(z_j)}{(z_i-z_j)^2\,\lambda''(z_j)}=-\frac{1}{6}\sum_{i=1}^{n}\frac{\lambda^{(4)}(z_i)}{\lambda''(z_i)}=-\frac{1}{6}\sum_{i=1}^{n}\mathrm{Res}_{z=z_i}\frac{\lambda^{(4)}(z)}{\lambda'(z)}=\frac{1}{6}\,\mathrm{Res}_{z=\infty}\frac{\lambda^{(4)}(z)}{\lambda'(z)}=0\,.$$

The lemma is proved.

3.2. D_n singularities

In this case, $m=2$. Write $x=z^1$, $y=z^2$, and $f(z)=x^{n-1}+x\,y^2$. A basis in the Milnor ring is given by $\phi_j=x^{n-j-1}$ $(j=1,\dots,n-1)$, $\phi_n=y$. The critical points are defined by the equations $F_x=(n-1)x^{n-2}+\dots+t^{n-2}+y^2=0$ and $F_y=2\,x\,y+t^n=0$, or, equivalently, $y=-\frac{t^n}{2\,x}$ and $(n-1)x^{n-2}+\dots+t^{n-2}+\frac{(t^n)^2}{4\,x^2}=0$. Introduce the function

$$\lambda(x,t)=x^{n-1}+\sum_{j=1}^{n-1}t^j\,\phi_j-\frac{(t^n)^2}{4\,x}.$$

Then the critical points and the critical values of $F(z,t)$ are given by those of $\lambda(x,t)$. Write $z^{(i)}=(x_i,y_i)$.

Lemma 3.4.

$$\partial_i F(z,t)=\frac{1}{x-x_i}\frac{x}{x_i}\frac{\lambda'(x)}{\lambda''(x_i)}+\frac{t^n(2\,x\,y+t^n)}{4\,x\,x_i^2\lambda''(x_i)}. \tag{3.6}$$

Proof. Let us compute the denominator and numerator of the right-hand side of (2.30). Since $y_i=-\frac{t^n}{2\,x_i}$, the denominator can be converted to a Vandermonde determinant

$$W(z^{(1)},\dots,z^{(n)})=-\frac{t^n}{2\,x_1\cdots x_n}\prod_{1\leqslant k<l\leqslant n}(x_k-x_l).$$

To compute the numerator, we rewrite y in the form $y=\left(-\frac{t^n}{2\,x}\right)+\left(y+\frac{t^n}{2\,x}\right)$ and then split the determinant into two parts,

$$W(z^{(1)},\dots,z^{(i-1)},z,z^{(i+1)},\dots,z^{(n)})$$
$$=\begin{vmatrix} x_1^{n-2} & x_1^{n-3} & \cdots & x_1 & 1 & -\frac{t^n}{2\,x_1}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ x^{n-2} & x^{n-3} & \cdots & x & 1 & -\frac{t^n}{2\,x}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ x_n^{n-2} & x_n^{n-3} & \cdots & x_n & 1 & -\frac{t^n}{2\,x_n}\end{vmatrix}+\begin{vmatrix} x_1^{n-2} & x_1^{n-3} & \cdots & x_1 & 1 & -\frac{t^n}{2\,x_1}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 0 & 0 & y+\frac{t^n}{2\,x}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ x_n^{n-2} & x_n^{n-3} & \cdots & x_n & 1 & -\frac{t^n}{2\,x_n}\end{vmatrix}.$$

The first determinant is similar to the denominator, while the second one, by the Laplace expansion, is again a Vandermonde determinant, and thus we have

$$\begin{aligned}W(z^{(1)},\dots,z^{(i-1)},z,z^{(i+1)},\dots,z^{(n)})=&-\frac{t^n}{2\,x_1\cdots x_n}\frac{x_i}{x}\prod_{1\leqslant k<l\leqslant n}(x_k-x_l)\prod_{k\neq i}\frac{x-x_k}{x_i-x_k}\\&-(-1)^n\left(y+\frac{t^n}{2\,x}\right)\prod_{1\leqslant k<l\leqslant n}(x_k-x_l)\prod_{k\neq i}\frac{1}{x_i-x_k}.\end{aligned}$$

Using Lemma 2.7, we see that

$$\partial_i F(z,t) = \frac{x_i}{x}\prod_{k\neq i}\frac{x-x_k}{x_i-x_k} + (-1)^n \frac{2\,x_1\cdots x_n}{t^n\,\prod_{k\neq i}(x_i-x_k)}\left(y+\frac{t^n}{2\,x}\right).$$

Applying the simple identities

$$\frac{x^2\,\lambda'(x)}{n-1} = \prod_{k=1}^n (x-x_k),\quad \frac{x_i^2\,\lambda''(x_i)}{n-1} = \prod_{k\neq i}^n (x_i-x_k),\qquad x_1\cdots x_n = (-1)^n\frac{(t^n)^2}{4(n-1)},$$

we can compete the proof of the lemma in a straightforward way.

Lemma 3.5.

$$\Gamma_{ki} = \frac{x_k+x_i}{(x_k-x_i)^2\,2\,x_i\,\lambda''(x_i)}. \tag{3.7}$$

Proof. This follows from (2.26) and Lemma 3.4.

Remark 3.6. By computing residues of the meromorphic functions

$$m(x) = \frac{(\lambda(x)-\lambda(x_i))(x+x_i)^2}{(x-x_i)^4\,x\,\lambda'(x)},\qquad \tilde m(x) = \frac{(x\,\lambda''(x)-x_i\,\lambda''(x_i))(x+x_i)}{(x-x_i)^2\,x\,\lambda'(x)},$$

one can easily prove that the G-functions of D_n singularities vanish.

To verify Conjecture 1.4 for D_n singularities, we represent $\lambda(x)=\lambda(x,t)$ in terms of $x_1, \ldots,$ x_{n-1} and x_0 in the form

$$\lambda(x) = \int_0^x (n-1)\xi^{-2}\prod_{k=1}^n(\xi-x_i)d\xi + x_0. \tag{3.8}$$

Here $\frac{1}{x_n} = -\sum_{k=1}^{n-1}\frac{1}{x_k}$. Then

$$u_i = \lambda(x_i),\quad h_i = \psi_{i,1} = \frac{1}{\sqrt{2x_i\lambda''(x_i)}},\quad \gamma_{ij} = \frac{(x_i+x_j)h_ih_j}{(x_i-x_j)^2}. \tag{3.9}$$

By using these data, one can also verify Conjecture 1.4 for small n.

Proof of Lemma 2.5 for D_n singularities. First,

$$\sum_{1\leqslant i<j\leqslant n}\gamma_{ij}\frac{(h_i^2+h_j^2)^2}{h_i^3\,h_j^3} = \sum_{i=1}^n\sum_{j\neq i}\frac{x_i+x_j}{(x_i-x_j)^2}\frac{x_i\,\lambda''(x_i)+x_j\,\lambda''(x_j)}{x_j\,\lambda''(x_j)}.$$

For a fixed i,

$$\begin{aligned}
&\sum_{j\neq i}\frac{x_i+x_j}{(x_i-x_j)^2}\frac{x_i\,\lambda''(x_i)+x_j\,\lambda''(x_j)}{x_j\,\lambda''(x_j)} = \sum_{j\neq i}\operatorname{Res}_{z=x_j}\frac{z+x_i}{(z-x_i)^2}\frac{z\,\lambda''(z)+x_i\,\lambda''(x_i)}{z\,\lambda'(z)}\\
&= -\left(\operatorname{Res}_{z=0}+\operatorname{Res}_{z=x_i}\right)\frac{z+x_i}{(z-x_i)^2}\frac{z\,\lambda''(z)+x_i\,\lambda''(x_i)}{z\,\lambda'(z)} = \frac{2}{x_i}-\left(\frac{1}{x_i}+\frac{\lambda'''(x_i)}{\lambda''(x_i)}+\frac{3\,x_i\,\lambda^{(4)}(x_i)}{\lambda''(x_i)}\right)\\
&= \frac{1}{x_i}-\frac{\lambda'''(x_i)}{\lambda''(x_i)}-\frac{3\,x_i\,\lambda^{(4)}(x_i)}{\lambda''(x_i)},
\end{aligned}$$

and thus

$$\begin{aligned}
&\sum_{i=1}^n\sum_{j\neq i}\frac{x_i+x_j}{(x_i-x_j)^2}\frac{x_i\,\lambda''(x_i)+x_j\,\lambda''(x_j)}{x_j\,\lambda''(x_j)} = \sum_{i=1}^n\left(\frac{1}{x_i}-\frac{\lambda'''(x_i)}{\lambda''(x_i)}-\frac{3\,x_i\,\lambda^{(4)}(x_i)}{\lambda''(x_i)}\right)\\
&= \sum_{i=1}^n\frac{1}{x_i}+\left(\operatorname{Res}_{z=0}+\operatorname{Res}_{z=\infty}\right)\left(\frac{\lambda'''(z)}{\lambda'(z)}+\frac{3\,z\,\lambda^{(4)}(z)}{\lambda'(z)}\right) = 0+0+0+0+0=0\,.
\end{aligned}$$

3.3. E_6 and E_8 singularities

In this case, $m = 2$ $E_6\colon f(x,y) = x^3 + y^4$, and $E_8\colon f(x,y) = x^3 + y^5$. Let $\nu = n/2$; then $f(x,y) = x^3 + y^{\nu+1}$, and the miniversal deformation F reads

$$F(z,t) = x^3 + p(y)\,x + q(y), \tag{3.10}$$

where

$$p(y) = \sum_{k=1}^{\nu} t_k\, y^{\nu-k},\ \ q(y) = y^{\nu+1} + \sum_{k=1}^{\nu} t_{\nu+k}\, y^{\nu-k}.$$

Here the indices of t's are written as subscripts for convenience. The critical points are defined by the equations $F_x = 3x^2 + p(y) = 0$ and $F_y = p'(y)\,x + q'(y) = 0$. Thus, $x = -q'(y)/p'(y)$, and $R(y) := R(F_x, F_y, x) = 3\,q'(y)^2 + p(y)p'(y)^2 = 0$. Here and below, $R(f_1(u), f_2(u), u)$ stands for the resultant of polynomials f_1 and f_2 with respect to the variable u. The $R(y)$'s roots give us the y-components of all the critical points $z^{(k)} = (x_k, y_k)$ $(k = 1, \dots, n)$. The corresponding x-components x_k's can be found from $x_k = -q'(y_k)/p'(y_k)$, $k = 1, \dots, n$.

Lemma 3.7. *Let $\Delta = R(q'(y), p'(y), y)$. Then*

$$W(z^{(1)}, \dots, z^{(n)}) = (-1)^{\nu}\,\frac{(\nu+1)^{2\nu-2}}{\Delta} \prod_{1\leqslant k<l\leqslant n} (y_k - y_l). \tag{3.11}$$

Proof. By definition,

$$W(z^{(1)}, \dots, z^{(n)}) = \begin{vmatrix} x_1\, y_1^{\nu-1} & x_1\, y_1^{\nu-2} & \cdots & x_1 & y_1^{\nu-1} & y_1^{\nu-2} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{2\nu}\, y_{2\nu}^{\nu-1} & x_{2\nu}\, y_{2\nu}^{\nu-2} & \cdots & x_{2\nu} & y_{2\nu}^{\nu-1} & y_{2\nu}^{\nu-2} & \cdots & 1 \end{vmatrix}.$$

Thus,

$$\begin{aligned} &(-1)^{\nu} \Big(\prod_{k=1}^{n} p'_k \Big)\, W(z^{(1)}, \dots, z^{(n)}) \\ &= \begin{vmatrix} q'_1\, y_1^{\nu-1} & q'_1\, y_1^{\nu-2} & \cdots & q'_1 & p'_1\, y_1^{\nu-1} & p'_1\, y_1^{m-2} & \cdots & p'_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ q'_{2\nu}\, y_{2\nu}^{\nu-1} & q'_{2\nu}\, y_{2\nu}^{\nu-2} & \cdots & q'_{2\nu} & p'_{2\nu}\, y_{2\nu}^{\nu-1} & p'_{2\nu}\, y_{2nu}^{\nu-2} & \cdots & p'_{2\nu} \end{vmatrix} = |U \cdot V| = |U| \cdot |V|. \end{aligned}$$

Here $p'_i = p'(y_i)$ and $q'_i = q'(y_i)$, and the matrices U, V read

$$U = \begin{pmatrix} y_1^{n-1} & y_1^{n-2} & \cdots & y_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ y_n^{n-1} & y_n^{n-2} & \cdots & y_n & 1 \end{pmatrix},$$

$$V = \begin{cases} \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 2t_4 & 0 & 4 & 2t_1 & 0 & 0 \\ t_5 & 2t_4 & 0 & t_2 & 2t_1 & 0 \\ 0 & t_5 & 2t_4 & 0 & t_2 & 2t_1 \\ 0 & 0 & t_5 & 0 & 0 & t_2 \end{pmatrix}, & \nu = 3, \\ \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3t_5 & 0 & 5 & 0 & 3t_1 & 0 & 0 & 0 \\ 2t_6 & 3t_5 & 0 & 5 & 2t_2 & 3t_1 & 0 & 0 \\ t_7 & 2t_6 & 3t_5 & 0 & t_3 & 2t_2 & 3t_1 & 0 \\ 0 & t_7 & 2t_6 & 3t_5 & 0 & t_3 & 2t_2 & 3t_1 \\ 0 & 0 & t_7 & 2t_6 & 0 & 0 & t_3 & 2t_2 \\ 0 & 0 & 0 & t_7 & 0 & 0 & 0 & t_3 \end{pmatrix}, & \nu = 4. \end{cases}$$

The matrix U is just the Vandermonde matrix of $y_1, \dots, y_n$, and therefore

$$|U| = \prod_{1 \leqslant k < l \leqslant n} (y_k - y_l).$$

The determinant formula for the resultant $R(q'(y), p'(y), y) = \Delta$ gives now $|V| = (\nu+1)^2\,\Delta$. On the other hand, according to the properties of the resultant, we have

$$\prod_{k=1}^{n} p'_k = \left(\Delta(\nu+1)^{-(\nu-2)}\right)^2.$$

This completes the proof of the lemma.

Lemma 3.8. *We have*

$$\partial_i F = \frac{1}{(y-y_i)\,R'(y_i)}\,\frac{p'(y_i)}{p'(y)}\left(R(y) - 3\,F_y(x,y)\,\Sigma\right) \tag{3.12}$$

where Σ *reads*

$$\Sigma = \begin{cases} F_y(x_i, y), & \nu = 3, \\ F_y(x_i, y) + \frac{t_1^2}{5}(y - y_i)p'(y), & \nu = 4. \end{cases}$$

Proof. According to Lemma 2.7, we have $\partial_i F = W_2/W_1$, where $W_1 = W(z^{(1)}, \dots, z^{(n)})$ and $W_2 = W(z^{(1)}, \dots, z^{(i-1)}, z, z^{(i+1)}, \dots, z^{(n)})$. Let us now compute W_2.

First, represent W_2 in the form $W_2 = A(x - \tilde{x}) + B$, where $\tilde{x} = -q'(y)/p'(y)$, and

$$A = \begin{vmatrix} x_1\,y_1^{\nu-1} & x_1\,y_1^{\nu-2} & \cdots & x_1 & y_1^{\nu-1} & y_1^{\nu-2} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y^{\nu-1} & y^{\nu-2} & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{2\nu}\,y_{2\nu}^{\nu-1} & x_{2\nu}\,y_{2\nu}^{\nu-2} & \cdots & x_{2\nu} & y_{2\nu}^{\nu-1} & y_{2\nu}^{\nu-2} & \cdots & 1 \end{vmatrix},$$

$$B = \begin{vmatrix} x_1\,y_1^{\nu-1} & x_1\,y_1^{\nu-2} & \cdots & x_1 & y_1^{\nu-1} & y_1^{\nu-2} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tilde{x}\,y^{\nu-1} & \tilde{x}\,y^{\nu-2} & \cdots & \tilde{x} & y^{\nu-1} & y^{\nu-2} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{2\nu}\,y_{2\nu}^{\nu-1} & x_{2\nu}\,y_{2\nu}^{\nu-2} & \cdots & x_{2\nu} & y_{2\nu}^{\nu-1} & y_{2\nu}^{\nu-2} & \cdots & 1 \end{vmatrix}.$$

The determinant B is very similar to W_1, and thus we can see that

$$B = \frac{1}{(y-y_i)\,R'(y_i)}\,\frac{p'(y_i)}{p'(y)}\,R(y)\,W_1.$$

The determinant A is less easy to compute. By using the Laplace expansion, we obtain

$$A = (-1)^{\nu+1}\frac{(\nu+1)^{2\,\nu-4}}{\Delta^2}\,p'(y_i)\,\sum_{j=1}^{\nu} C_{ij}\,y^{\nu-j},$$

where C_{ij} is the (i,j)th cofactor of the matrix $U \cdot V$.

Let U_{kl} and V_{kl} be the (k,l)th minors of the matrices U and V, respectively. Then the Binet–Cauchy formula gives

$$C_{ij} = (-1)^{i+j}\sum_{k=1}^{n} U_{ik} \cdot V_{kj}.$$

The minors U_{ik} are similar to the Vandermonde determinants,

$$U_{ik} = \frac{\prod_{1\leqslant s<t\leqslant n}(y_s - y_t)}{(-1)^{i-1}\prod_{s\neq i}(y_i - y_s)} e_{k-1}(\hat{y}_i),$$

where $e_k(\hat{y}_i)$ stands for the kth elementary symmetric polynomial in $y_1, \dots, \hat{y}_i, \dots, y_n$. Note that $y_1, \dots, y_n$ are roots of the polynomial $R(y)$, and thus these elementary symmetric polynomials can be expressed as polynomials in y_i with the coefficients of $R(y)$. It is also easy to compute the minors V_{kj}. Their explicit expressions are simple but not illuminating, and we omit them here.

By using the above results, we obtain

$$\partial_i F = \frac{1}{(y-y_i)\,R'(y_i)}\frac{p'(y_i)}{p'(y)}\Big(R(y) - 3\,F_y(x,y)\,\Sigma\Big),$$

where

$$\Sigma = \frac{y-y_i}{\Delta}\sum_{j=1}^{\nu}\sum_{k=1}^{n}(-1)^{j+1}\,e_{k-1}(\hat{y}_i)\,V_{kj}\,y^{\nu-j}.$$

For $\nu = 3$, it is easy to show that $\Sigma = F_y(x_i, y)$. For $\nu = 4$, after a very lengthy computation, it can be seen that $\Sigma = F_y(x_i, y) + (t_1^2/5)(y - y_i)p'(y)$. This completes the proof of the lemma.

Lemma 3.9.

$$\Gamma_{ki} = 3\,\frac{x_i + x_k}{(y_i - y_k)^2}\,\eta_{ii}, \tag{3.13}$$

where $\eta_{ii} = -p'(y_i)/R'(y_i)$.

Proof. One can prove the lemma directly, using Lemma 3.8.

Remark 3.10. *The vanishing of the G-functions of the E_6 and E_8 singularities can also be proved by the residue theorem; however, the computation procedure becomes very long.*

Although for E_6 and E_8 we obtain formula (3.13) for the rotation coefficients, we still have no simple way to relate the variables y_i to t_i, which was possible above for the A_n and D_n cases. Thus, at this moment, we can only check the validity of the conjecture for the E_6 and E_8 singularities numerically. We first randomly generate the complex values of $t_1, \dots, t_n$ and solve the equations $F_x = 0, F_y = 0$ numerically to obtain the values of the critical points $z^{(1)}, \dots, z^{(n)}$. Then one can determine the data u^i, h_i, γ_{ij}. Our computation shows that Conjecture 1.4 is valid in this numerical sense for the E_6 and E_8 cases.

Proof of Lemma 2.5 for the E_6 and E_8 singularities. First,

$$\sum_{1\leqslant i<j\leqslant n}\gamma_{ij}\frac{(h_i^2+h_j^2)^2}{h_i^3\,h_j^3} = 3\sum_{1\leqslant i<j\leqslant n}\frac{x_i+x_j}{(y_i-y_j)^2}\,\frac{(h_i^2+h_j^2)^2}{h_i^2\,h_j^2}$$
$$= -3\sum_{i=1}^{n}\sum_{j\neq i}\frac{1}{(y_i-y_j)^2}\left(\frac{q'(y_j)R'(y_i)}{p'(y_i)R'(y_j)} + \frac{p'(y_j)q'(y_i)R'(y_i)}{p'(y_i)^2R'(y_j)} + 2\,\frac{q'(y_i)}{p'(y_i)}\right).$$

For a fixed i, we then have

$$-\sum_{j\neq i}\frac{1}{(y_i-y_j)^2}\frac{q'(y_j)}{R'(y_j)} = \mathrm{Res}_{y=y_i}\,\frac{q'(y)}{(y-y_i)^2\,R(y)},$$
$$-\sum_{j\neq i}\frac{1}{(y_i-y_j)^2}\frac{p'(y_j)}{R'(y_j)} = \mathrm{Res}_{y=y_i}\,\frac{p'(y)}{(y-y_i)^2\,R(y)},$$
$$-\sum_{j\neq i}\frac{1}{(y_i-y_j)^2} = \lim_{y\to y_i}\frac{d}{dy}\left(\frac{R'(y)}{R(y)} - \frac{1}{y-y_i}\right),$$

and thus

$$\sum_{1\leqslant i<j\leqslant n}\gamma_{ij}\frac{(h_i^2+h_j^2)^2}{h_i^3\,h_j^3}=3\sum_{i=1}^{n}\frac{R'(y_i)}{p'(y_i)}\operatorname{Res}_{y=y_i}\frac{q'(y)}{(y-y_j)^2R(y)}$$
$$+3\sum_{i=1}^{n}\frac{q'(y_i)R'(y_i)}{p'(y_i)^2}\operatorname{Res}_{y=y_i}\frac{p'(y)}{(y-y_j)^2R(y)}+6\sum_{i=1}^{n}\frac{q'(y_i)}{p'(y_i)}\lim_{y\to y_i}\frac{d}{dy}\left(\frac{R'(y)}{R(y)}-\frac{1}{y-y_j}\right)$$
$$=\sum_{i=1}^{n}\operatorname{Res}_{y=y_i}g(y)=-\big(\operatorname{Res}_{y=\infty}+\operatorname{Res}_{y=\text{roots of }p'(y)}\big)\,g(y),$$

where

$$g(y)=\left(\frac{3}{2}\,\frac{p'(y)q'''(y)+p'''(y)q'(y)}{p'(y)^2}\,\frac{R'(y)}{R(y)}-\frac{3}{2}\frac{p'(y)q''(y)+p''(y)q'(y)}{p'(y)^2}\frac{R''(y)}{R(y)}+\frac{q'(y)\,R'''(y)}{p'(y)\,R(y)}\right).$$

For $n=6$, $p'(y)$ has a unique root $y=-\frac{t_2}{2\,t_1}$. One can derive that $\operatorname{Res}_{y=\infty}g(y)=12/t_1$ and $\operatorname{Res}_{y=-\frac{t_2}{2t_1}}g(y)=-12/t_1$, and thus the $n=6$ case is proved. For $n=8$, denote by a_1 and a_2 the two roots of $p'(y)$. We have $\operatorname{Res}_{y=a_1}g(y)=-\operatorname{Res}_{y=a_2}g(y)=(8(10t_2t_3+9t_1t_2t_5-9t_1^2t_6))/(9t_1^3(a_1-a_2)^3)$, and $\operatorname{Res}_{y=\infty}g(y)=0$, and thus the $n=8$ case is also proved.

3.4. E_7 singularity

In this case, $m=2$ and $f(x,y)=x^3+xy^3$. The miniversal deformation can be chosen in the form $F(x,y)=x^3+p(y)\,x^2+q(y)\,x+r(y)$, where $p(y)=t_1\,y+t_2,\ q(y)=y^3+t_3\,y+t_4,\ r(y)=t_5\,y^2+t_6\,y+t_7$. The critical points are defined by the equations $F_x=3\,x^2+2\,p(y)\,x+q(y)=0$ and $F_y=p'(y)\,x^2+q'(y)\,x+r'(y)=0$, which imply $x=\frac{Q}{P}$, and $R(y)=R(F_x,F_y,x)=Q^2-P\,S=0$, where $P=2\,p\,p'-3\,q',\ Q=3\,r'-p'\,q,\ S=q\,q'-2\,p\,r'$.

Lemma 3.11.

$$\partial_iF=\frac{1}{(y-y_i)R'(y_i)}\frac{P(y_i)}{P(y)}\Big(R(y)-(P(y)\,x_i-Q(y))(P(y)\,x-Q(y))\Big)+\frac{P(y_i)}{R'(y_i)}\Big(3(y+y_i)F_x-\frac{5\,t_1}{3}F_y\Big).\tag{3.14}$$

The proof of the above lemma is quite similar to that of Lemma 3.8, and we omit it.

By using the above lemma and (2.25), one can prove the following assertion.

Lemma 3.12. *Let $\tilde{x}_k=x_k+\frac{1}{3}\,p(y_k)$. Then*

$$\Gamma_{ki}=3\frac{\tilde{x}_i+\tilde{x}_k}{(y_i-y_k)^2}\eta_{ii}\tag{3.15}$$

where $\eta_{ii}=P(y_i)/R'(y_i)$.

The above expression of Γ_{ki} is similar to that of the E_8 case. This fact has an interesting explanation. Let us first introduce a modification of the miniversal deformation of the E_7 singularity $\tilde{F}=x^3+p(y)\,x^2+q(y)\,x+\tilde{r}(y)$, where $\tilde{r}(y)=r(y)+t_8\,y^5$. Make a coordinate transformation

$$\tilde{x}=x+\frac{1}{3}\,p(y),\ \ \tilde{y}=\tau\left(y-\frac{t_1}{15\,t_8}\right),\ \text{ where } \tau=(t_8)^{\frac{1}{5}}.$$

Then, in these new coordinates, the deformation $\tilde{F}$ reads

$$\tilde{F}=\tilde{x}^3+\tilde{y}^5+\big(\tilde{t}_1\,\tilde{y}^3+\tilde{t}_2\,\tilde{y}^2+\tilde{t}_3\,\tilde{y}+\tilde{t}_4\big)\ \tilde{x}+\tilde{t}_5\,\tilde{y}^3+\tilde{t}_6\,\tilde{y}^2+\tilde{t}_7\,\tilde{y}+\tilde{t}_8,$$

which is a miniversal deformation of an E_8-type singularity. Here $\tilde{t}_i$ $(i = 1, \ldots, 8)$ are certain rational functions of t_i $(i = 1, \ldots, 7)$ and τ, and we omit their explicit expressions here.

Let us now take the limit $\tau \to 0$. Then it is easy to see that one of the canonical coordinates, say ,u^8, goes to ∞, and the other seven ones become the canonical coordinates of the original E_7 singularity. By comparing Lemma 3.9 and 3.12, one can also prove that the Christoffel symbols $\tilde{\Gamma}_{ki}$ associated to the E_8 singularity also tend to the Christoffel symbols Γ_{ki} associated to the E_7 singularity, whenever $k, i = 1, \ldots, 7$.

By using the above observation, it is easy to see that, if the G-function of the E_8 singularity vanishes, then the same holds for the G-function of the E_7 singularity. Similarly, if Lemma 2.5 had been proved for the E_8 singularity, it also holds for the E_7 singularity.

3.5. The $\mathbb{P}^1$-orbifold of $\tilde{A}_{p,q}$ type

In this case, $m = 3$, $(p, q, r) = (p, q, 1)$, and thus $n = p + q$. The tri-polynomial $F(z, t)$ reads

$$F(z, t) = -z^1 z^2 z^3 + P_1(z^1) + P_2(z^2) + t_{n-1} + t_n z^3,$$

Its critical points are defined by the equations

$$\partial_{z^1} F = -z^2 z^3 + P_1'(z^1), \tag{3.16}$$

$$\partial_{z^2} F = -z^1 z^3 + P_2'(z^2), \tag{3.17}$$

$$\partial_{z^3} F = -z^1 z^2 + t_n. \tag{3.18}$$

We introduce an auxiliary function $\lambda(z) = P_1(z) + P_2(t^n/z) + t_{n-1}$ and denote by $z_1, \ldots, z_n$ the critical points of λ. It is easy to see that z_i coincides with the first component of the critical point $z^{(i)}$ of $F(z, t)$, and the critical values of $\lambda(z)$ also coincide with the critical values of F. Thus, $u^i = \lambda(z_i)$.

The Hessian for F reads

$$H = P_1''(z^1) P_2''(z^2) P_3''(z^3) - 2z^1 z^2 z^3 - (z^1)^2 P_1''(z^1) - (z^2)^2 P_2''(z^2) - (z^3)^2 P_3''(z^3).$$

Then, using (3.16)–(3.18), one obtains $\eta^{ii} = H(z^{(i)}(t), t) = -z_i^2 \lambda''(z_i)$.

Lemma 3.13.

$$\partial_i \lambda(z) = \frac{z \lambda'(z)}{z_i (z - z_i) \lambda''(z_i)}. \tag{3.19}$$

Proof. Write

$$R(z) = \prod_{i=1}^{n} (z - z_i) = \frac{z^{q+1}}{p} \lambda'(z).$$

By using (2.22) and the Lagrange interpolation formula, one obtains

$$\partial_i \left(z^q \lambda(z) \right) = z_i^q \frac{R(z)}{(z - z_i) R'(z_i)},$$

which implies formula (3.17) immediately.

Lemma 3.14.

$$\Gamma_{ki} = \frac{z_k}{(z_k - z_i)^2 z_i \lambda''(z_i)}. \tag{3.20}$$

Proof. The proof of this lemma is very similar to the derivation of (2.26). We omit the details here.

Lemma 3.15.

$$G(t) = -\frac{\log t_n}{24}. \tag{3.21}$$

Proof. By using the residue theorem, one obtains $\partial_i G = \eta_{ii}/24$. On the other hand, the comparison of the coefficients of z^{-q} in $\lambda(z)$ and $\partial_i\lambda(z)$ yields $\partial_i \log t_n = -\eta_{ii}$. The lemma is proved.

Lemma 3.16.

$$O_1 - O_2 = \frac{1}{6}(p^3 + q^3 - p - q). \tag{3.22}$$

Proof. Note that $h_i^{-2} = -z_i^2\lambda''(z_i)$ and $\gamma_{ij} = -h_i h_j z_i z_j (z_i - z_j)^{-2}$, and thus one can prove the lemma by using the residue theorem.

3.6. $\mathbb{P}^1$-orbifold of $\tilde{D}_{r+2}$ type

In this case, $m = 3$ and $(p, q, r) = (2, 2, r)$, and thus $n = r + 3$. The tri-polynomial $F(z, t)$ reads

$$F(z,t) = -z^1z^2z^3 + (z^1)^2 + t_1z^1 + (z^2)^2 + t_2z^2 + P_3(z^3),$$

The critical points are defined by the equations $\partial_{z^1}F = -z^2z^3 + 2z^1 + t_1$, $\partial_{z^2}F = -z^1z^3 + 2z^2 + t_2$, and $\partial_{z^3}F = -z^1z^2 + P_3'(z^3)$. Introduce an auxiliary function $\lambda(z) = P_3(z) + t_1^2 + z\,t_1\,t_2 + t_2^2/(z^2-4)$ and denote by $z_1, \dots, z_n$ its critical points. Similarly to the $\tilde{A}_{p,q}$ cases, we have $u^i = \lambda(z_i)$.

The following lemmas are similar to those for the $\tilde{A}_{p,q}$ cases, and we omit their proofs.

Lemma 3.17. $\eta^{ii} = (4 - z_i^2)\lambda''(z_i)$, $|\partial_i\lambda(z) = \dfrac{4-z^2}{4-z_i^2}\dfrac{\lambda'(z)}{(z-z_i)\lambda''(z_i)}$, *and* $\Gamma_{ki} = \frac{4-z_kz_i}{\eta^{ii}(z_k-z_i)^2}$.

Lemma 3.18.

$$G(t) = -\frac{\log t_n}{24\,r} \tag{3.23}$$

Lemma 3.19.

$$O_1 - O_2 = \frac{1}{6}(r^3 - r) + 2. \tag{3.24}$$

According to the results of Lemmas 3.15, 3.16, 3.18, and 3.19, we can state the following conjecture.

Conjecture 3.20. *For $\mathbb{P}^1$-orbifolds of ADE type,*

$$G(t) = -\frac{\log t_n}{24\,r},\ O_1 - O_2 = \frac{1}{6}(p^3 + q^3 + r^3 - p - q - r). \tag{3.25}$$

For $\mathbb{P}^1$-orbifolds of E type, we were unable to verify the validity of the conjectures, even numerically, because the numerical errors are too large in these cases.

3.7. Some other examples

Example 3.26. If the dimension of the Frobenius manifold is equal to 2, then it is easy to see that $O_1 - O_2 = \gamma_{12}(h_1^2 + h_2^2)^2h_1^{-3}h_2^{-3} = 0$ since $h_1^2 + h_2^2 = 0$. By using the formulas $h_1 = \sqrt{-1}\,h_2$ and $\gamma_{12} = -\frac{\sqrt{-1}\,\mu_1}{u_1-u_2}$, one can easily prove the following statement.

Lemma 3.21. *The genus two G-function vanishes if and only if* $\mu_1 = 1/2,\ 1/3,\ 1/6$, *which correspond to* $A_1 \times A_1$, A_2, *and* $\tilde{A}_{1,1}$, *respectively.*

Note that the above three cases are also the only cases for which the genus one G-function $G(t)$ is analytic on the caustics.

Example 3.26. Let M be the Frobenius manifold corresponding to the quantum cohomology of $\mathbb{P}^n$ $(n \geqslant 2)$. Then $G^{(2)}(u, u_x, u_{xx}) \neq 0$.

Indeed, the restrictions of the Q_p terms to the small phase space vanish, while the restriction of $\mathcal{F}_2$ to the small phase space does not always vanish. More generally, we obtain the following criterion.

Lemma 3.22. *The restriction of $\mathcal{F}_2$ to the small phase space vanishes if and only if $G^{(2)}|_{u_x^i=1,\ u_{xx}^i=0,\ 1\leqslant i\leqslant n}$ is equal to zero.*

Since $\mathbb{P}^n$ has nontrivial genus two Gromov–Witten invariants, we have $G^{(2)}(u, u_x, u_{xx}) \neq 0$ in this case.

4. CONCLUSION

It would be of interest to elucidate the geometric meaning of the genus two G-function $G^{(2)}$. In particular, the conditions for the vanishing of $G^{(2)} \equiv 0$ are of interest. Last but not least, finding a higher-genus $g \geqslant 3$ generalization of the decomposition (1.3) is the main challenge. We plan to address these problems in a subsequent publication.

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APPENDIX

A. THE GENUS TWO G-FUNCTION

The genus two G-function $G^{(2)}(u, u_x, u_{xx})$ depends rationally on the x-jets of the canonical coordinates

$$G^{(2)}(u, u_x, u_{xx}) = \sum_{i=1}^{n} G_i^{(2)}(u, u_x) u_{xx}^i + \sum_{i\neq j} G_{ij}^{(2)}(u) \frac{(u_x^j)^3}{u_x^i} + \frac{1}{2}\sum_{i,j} P_{ij}^{(2)}(u) u_x^i u_x^j + \sum_{i=1}^{n} Q_i^{(2)}(u) \left(u_x^i\right)^2 \tag{A.1}$$

with coefficients written in terms of the Lamé coefficients $h_i = h_i(u)$ and rotation coefficients $\gamma_{ij} = \gamma_{ij}(u)$ of the semisimple Frobenius manifold. To simplify the expressions of these coefficients, we use the function

$$H_i = \frac{1}{2}\sum_{j\neq i} u_{ij}\gamma_{ij}^2, \quad 1 \leqslant i \leqslant n$$

with $u_{ij} = u_i - u_j$, these functions are given by the gradients of the isomonodromic tau function of the Frobenius manifold [3, 4]. Then we have

$$\begin{aligned} G_i^{(2)} =& \frac{\partial_x h_i\, H_i}{60\, u_{i,x} h_i^3} - \frac{3\,\partial_i h_i H_i}{40\, h_i^3} + \frac{19\,(\partial_i h_i)^2}{2880\, h_i^4} - \frac{7\,\partial_i h_i \partial_x h_i}{5760\, u_{i,x} h_i^4} \\ &+ \sum_k \left[\frac{\gamma_{ik} H_i}{120\, h_i h_k} + \frac{\gamma_{ik} H_k}{120\, h_i h_k}\left(7 + \frac{u_{k,x}}{u_{i,x}}\right) - \frac{\gamma_{ik}}{5760\, h_i^2 h_k}\left(4\,\partial_i h_i + \frac{\partial_x h_i}{u_{i,x}}\right)\right. \\ &- \frac{\gamma_{ik}\partial_k h_k}{h_i h_k^2}\left(\frac{u_{k,x}}{1152\, u_{i,x}} + \frac{7}{2880}\right) + \frac{\gamma_{ik}\partial_k h_k}{384\, h_i^3} - \frac{\partial_k \gamma_{ik} h_k}{384\, h_i^3} + \frac{\partial_i \gamma_{ik} h_k u_{k,x}}{1920\, u_{i,x} h_i^3} \\ &+ \frac{\partial_i \gamma_{ik}}{2880\, h_i h_k} + \frac{\partial_x \gamma_{ik}}{5760\, u_{i,x} h_i h_k} + \frac{\partial_k \gamma_{ik}}{h_i h_k}\left(\frac{u_x^k}{2880\, u_x^i} + \frac{7}{2880}\right) + \frac{\gamma_{ik} h_i \partial_k h_k}{2880\, h_k^4} \end{aligned}$$

$$-\frac{\gamma_{ik}^2}{h_i^2}\left(\frac{7u_x^k}{1152\,u_x^i}+\frac{19}{720}\right)+\frac{\gamma_{ik}^2}{1440\,h_k^2}\Bigg]-\sum_{k,l}\left(\frac{h_i\gamma_{il}\gamma_{kl}}{2880\,h_kh_l^2}+\frac{u_{k,x}h_k\gamma_{il}\gamma_{kl}}{1920\,u_x^ih_ih_l^2}\right),$$

$$G_{ij}^{(2)}=-\frac{\gamma_{ij}^2H_j}{120\,h_j^2}+\frac{\gamma_{ij}^3}{480\,h_ih_j}-\frac{\gamma_{ij}}{5760}\left(\frac{\partial_i\gamma_{ij}}{h_i^2}+\frac{\partial_j\gamma_{ij}}{h_j^2}\right)+\frac{\gamma_{ij}^2}{5760}\left(\frac{\partial_ih_i}{h_i^3}+\frac{3\,\partial_jh_j}{h_j^3}\right)$$
$$+\sum_k\left(\frac{\gamma_{ij}\gamma_{ik}\gamma_{jk}}{5760\,h_k^2}+\frac{\gamma_{ij}^2}{5760\,h_k}\left(\frac{\gamma_{jk}}{h_j}-\frac{\gamma_{ik}}{h_i}\right)\right),$$

$$P_{ij}^{(2)}=-\frac{2\,\gamma_{ij}H_iH_j}{5\,h_ih_j}+\frac{\gamma_{ij}\partial_jh_jH_i}{20\,h_ih_j^2}+\frac{\gamma_{ij}h_i\partial_jh_jH_j}{20\,h_j^4}-\frac{19\,\gamma_{ij}^2H_j}{30\,h_j^2}-\frac{\partial_i\gamma_{ij}H_j}{60\,h_ih_j}$$
$$+\frac{41\,\gamma_{ij}^3}{240\,h_ih_j}-\frac{41\gamma_{ij}\partial_i\gamma_{ij}}{1440\,h_i^2}+\frac{\partial_i\gamma_{ij}\partial_jh_j}{1440\,h_ih_j^2}+\frac{79\,\gamma_{ij}^2\partial_jh_j}{1440\,h_j^3}-\frac{\gamma_{ij}\partial_ih_i\partial_jh_j}{720\,h_i^2h_j^2}-\frac{\gamma_{ij}h_i(\partial_jh_j)^2}{288\,h_j^5}$$
$$+\sum_k\Bigg(\frac{\gamma_{ij}\gamma_{ik}H_j}{60\,h_jh_k}-\frac{\gamma_{ik}\gamma_{jk}h_ih_jH_k}{30\,h_k^4}-\frac{\gamma_{ij}\gamma_{jk}h_iH_j}{60\,h_j^2h_k}+\frac{\gamma_{ik}\gamma_{jk}h_iH_j}{60\,h_jh_k^2}-\frac{7\,\gamma_{ij}\gamma_{jk}h_iH_k}{60\,h_j^2h_k}$$
$$-\frac{\gamma_{ij}\gamma_{ik}\partial_jh_j}{720\,h_j^2h_k}+\frac{\gamma_{ij}\gamma_{jk}h_i\partial_jh_j}{240\,h_j^3h_k}-\frac{\gamma_{ik}\gamma_{jk}h_i\partial_jh_j}{1440\,h_j^2h_k^2}+\frac{\gamma_{ij}\gamma_{jk}h_i\partial_kh_k}{720\,h_k^4}+\frac{\gamma_{ik}\gamma_{jk}h_ih_j\partial_kh_k}{288\,h_k^5}$$
$$+\frac{\gamma_{jk}\,\partial_i\gamma_{ij}}{1440\,h_ih_k}-\frac{h_jh_k\gamma_{ij}\partial_i\gamma_{ik}}{360\,h_i^4}-\frac{h_j(3\,\gamma_{ik}\partial_i\gamma_{ij}+2\,\gamma_{ij}\partial_i\gamma_{ik})}{1440\,h_i^2h_k}-\frac{7\,h_j\gamma_{ij}\partial_k(h_k^{-1}\gamma_{ik})}{1440\,h_i^2}$$
$$-\frac{h_ih_j\gamma_{ik}\,\partial_k\gamma_{jk}}{480\,h_k^4}+\frac{\gamma_{ij}^2\gamma_{jk}}{120\,h_jh_k}+\frac{7\,h_i\gamma_{ij}\gamma_{jk}^2}{160\,h_j^3}+\frac{11\gamma_{ij}\gamma_{ik}\gamma_{jk}}{2880\,h_k^2}+\frac{h_j\gamma_{ik}^2\gamma_{jk}}{96\,h_k^3}\Bigg)$$
$$+\sum_{k,l}\left(\frac{h_ih_j\gamma_{il}\gamma_{jl}}{720\,h_kh_l^2}\left(\frac{\gamma_{kl}}{h_l}-\frac{\gamma_{jk}}{2h_j}\right)-\frac{h_i\gamma_{ij}\gamma_{jl}\gamma_{kl}}{720\,h_kh_l^2}\right),$$

$$Q_i^{(2)}=\frac{4H_i^3}{5\,h_i^2}-\frac{7\,\partial_ih_iH_i^2}{10\,h_i^3}+\frac{7\,(\partial_ih_i)^2H_i}{48\,h_i^4}-\frac{(\partial_ih_i)^3}{120\,h_i^5}+\sum_k\Bigg(\frac{7\gamma_{ik}H_iH_k}{10\,h_ih_k}-\frac{\gamma_{ik}\partial_ih_iH_i}{120\,h_i^2h_k}$$
$$+\frac{7\,\partial_k\left(h_k^{-1}\gamma_{ik}\right)H_i}{240\,h_i}-\frac{7\gamma_{ik}\partial_ih_iH_k}{80\,h_i^2h_k}+\frac{\gamma_{ik}H_k}{576\,u_{ik}h_ih_k}+\frac{(2H_i+7H_k)\partial_i\gamma_{ik}}{240\,h_ih_k}$$
$$+\frac{\gamma_{ik}h_kH_i}{576\,u_{ik}h_i^3}-\frac{31\gamma_{ik}^2H_i}{144\,h_i^2}+\frac{\gamma_{ik}(\partial_ih_i)^2}{720\,h_i^3h_k}+\frac{253\,\gamma_{ik}^2\partial_ih_i}{5760\,h_i^3}-\frac{\partial_i\gamma_{ik}\partial_ih_i}{960\,h_i^2h_k}-\frac{\gamma_{ik}^2\partial_kh_k}{2880\,h_k^3}$$
$$-\frac{7\,\partial_k\left(h_k^{-1}\gamma_{ik}\right)\partial_ih_i}{1920\,h_i^2}-\frac{7\,\partial_i\gamma_{ik}\partial_kh_k}{5760\,h_ih_k^2}-\frac{41\,\partial_i\gamma_{ik}\partial_ih_ih_k}{5760\,h_i^4}+\frac{\partial_i(h_i\gamma_{ik})\partial_kh_k}{2880\,h_k^4}$$
$$-\frac{113\,\gamma_{ik}\partial_i\gamma_{ik}}{5760\,h_i^2}+\frac{(3\,\partial_i\gamma_{ik}+\partial_k\gamma_{ik})\,\gamma_{ik}}{1440\,h_k^2}-\frac{\partial_i\gamma_{ik}h_k}{576\,u_{ik}h_i^3}-\frac{\partial_k\gamma_{ik}}{576\,u_{ik}h_ih_k}-\frac{\gamma_{ik}^3}{240\,h_ih_k}\Bigg)$$
$$+\sum_{k,l}\Bigg(-\frac{\gamma_{kl}\partial_i(h_i\gamma_{il})}{2880\,h_kh_l^2}+\frac{\gamma_{il}^2\gamma_{kl}}{2880\,h_kh_l}-\frac{\gamma_{ik}\gamma_{il}^2}{240\,h_ih_k}-\frac{\gamma_{kl}\partial_i\gamma_{ik}}{2880\,h_ih_l}+\frac{u_{lk}\gamma_{ik}\partial_l\gamma_{kl}}{1152\,u_{il}h_ih_l}$$
$$+\frac{u_{kl}\gamma_{ik}\gamma_{kl}\partial_i\gamma_{il}}{144\,h_i^2}+\frac{h_l\gamma_{ik}\partial_i\gamma_{il}}{1440\,h_i^2h_k}+\frac{h_ku_{kl}\gamma_{kl}\partial_i\gamma_{il}}{1152\,u_{ik}h_i^3}+\frac{h_lu_{ik}\gamma_{ik}^2\partial_i\gamma_{il}}{40\,h_i^3}\Bigg).$$

In these expressions, the summations are taken over indices defining nonzero denominators.

B. GENERAL FORMULA FOR THE GENUS TWO FREE ENERGY

In this formula, derived in [5], we use the notation $V_{ij}=(u_j-u_i)\gamma_{ij}$ and $u_{ij}=u_i-u_j$. A summation over repeated indices is assumed in each term (of the formula) producing nonzero denominators.

$$\mathcal{F}_2=\frac{1}{1152}\frac{u_i^{IV}}{u_i'^2\,h_i^2}-\frac{7}{1920}\frac{u_i''\,u_i'''}{u_i'^3\,h_i^2}+\frac{1}{360}\frac{u_i''^3}{u_i'^4\,h_i^2}+\frac{1}{40}\frac{V_{ij}^2\,u_i'''}{u_{ij}\,u_i'\,h_i^2}$$

$$
\begin{aligned}
&+\frac{1}{640}\frac{V_{ij}\,h_j\,u_j'\,u_i'''}{u_{ij}\,{u_i'}^2\,h_i^3}-\frac{19}{2880}\frac{V_{ij}\,u_i'''\,h_j}{u_{ij}\,u_i'\,h_i^3}+\frac{1}{1152}\frac{V_{ij}\,u_i'''\,h_i}{u_{ij}\,u_j'\,h_j^3}+\frac{7}{40}\frac{V_{ij}^2\,V_{ik}^2\,u_i''}{u_{ij}\,u_{ik}\,h_i^2}\\
&-\frac{1}{240}\frac{V_{ij}^2\,V_{ik}\,u_i''\,h_k\,(32\,u_i'-7\,u_k')}{u_{ij}\,u_{ik}\,u_i'\,h_i^3}+\frac{1}{40}\frac{V_{ij}\,V_{jk}^2\,u_i''\,h_i}{u_{ij}\,u_{jk}\,h_j^3}-\frac{1}{48}\frac{V_{ij}\,V_{jk}^2\,u_j'\,u_i''}{u_{ij}\,u_{jk}\,u_i'\,h_i\,h_j}\\
&-\frac{3}{64}\frac{V_{ij}^2\,u_i''}{u_{ij}^2\,h_i^2}-\frac{11}{480}\frac{V_{ij}^2\,{u_i''}^2}{u_{ij}\,{u_i'}^2\,h_i^2}+\frac{29}{5760}\frac{V_{ij}\,V_{jk}\,u_i''\,h_i\,h_k\,\left(u_k'-2\,u_j'\right)}{u_{ij}\,u_{jk}\,u_j'\,h_j^4}\\
&+\frac{1}{384}\frac{V_{ij}\,V_{ik}\,u_i''\,h_k\,(u_i'-u_k')}{u_{ij}\,u_{ik}\,u_j'\,h_j^3}+\frac{1}{1920}\frac{V_{ij}\,V_{ik}\,u_i''\,h_j\,h_k\,\left(54\,{u_i'}^2-25\,u_i'\,u_j'-u_j'\,u_k'\right)}{u_{ij}\,u_{ik}\,{u_i'}^2\,h_i^4}\\
&+\frac{1}{576}\frac{V_{ij}\,V_{jk}\,u_i''\,h_k\,\left(2\,u_j'-u_k'\right)}{u_{ij}\,u_{jk}\,u_i'\,h_i\,h_j^2}-\frac{1}{5760}\frac{V_{ij}\,V_{jk}\,u_k'\,u_i''\,h_k\,(27\,u_i'+u_k')}{u_{jk}\,u_{ik}\,{u_i'}^2\,h_i^3}\\
&-\frac{19}{1920}\frac{V_{ij}\,V_{jk}\,u_i''\,h_k}{u_{ij}\,u_{ik}\,h_i^3}+\frac{1}{5760}\frac{V_{ij}\,V_{jk}\,h_k\,\left(27\,u_i'\,u_k'-{u_j'}^2+2\,u_j'\,u_k'\right)\,u_i''}{u_{ij}\,u_{jk}\,{u_i'}^2\,h_i^3}\\
&+\frac{1}{288}\frac{V_{ij}\,V_{jk}\,u_i''\,h_i}{u_{jk}\,u_{ik}\,h_k^3}+\frac{1}{384}\frac{V_{ij}\,V_{jk}\,u_i'\,u_i''\,h_i}{u_{ij}\,u_{ik}\,u_k'\,h_k^3}-\frac{1}{576}\frac{V_{ij}\,V_{jk}\,u_k'\,u_i''}{u_{jk}\,u_{ik}\,u_i'\,h_i\,h_k}\\
&-\frac{1}{384}\frac{V_{ik}\,V_{jk}\,u_k'\,u_i''\,h_i}{u_{ik}\,u_{jk}\,u_j'\,h_j^3}+\frac{1}{1920}\frac{V_{ij}\,{u_i''}^2\,h_j\,\left(11\,u_i'-5\,u_j'\right)}{u_{ij}\,{u_i'}^3\,h_i^3}-\frac{1}{5760}\frac{V_{ij}\,u_i''\,u_j''\,h_j}{u_{ij}\,{u_i'}^2\,h_i^3}\\
&+\frac{1}{5760}\frac{V_{ij}\,u_i''\,h_j\,\left(57\,{u_i'}^2-27\,u_i'\,u_j'-{u_j'}^2\right)}{u_{ij}^2\,{u_i'}^2\,h_i^3}+\frac{1}{1152}\frac{V_{ij}\,u_i''\,h_i\,\left(4\,u_j'-3\,u_i'\right)}{u_{ij}^2\,u_j'\,h_j^3}\\
&-\frac{1}{576}\frac{V_{ij}\,u_j'\,u_i''}{u_{ij}^2\,u_i'\,h_i\,h_j}-\frac{1}{1152}\frac{V_{ij}\,u_i''\,u_j''}{u_{ij}\,u_i'\,u_j'\,h_i\,h_j}+\frac{1}{10}\frac{V_{ij}^2\,V_{ik}^2\,V_{il}^2\,{u_i'}^2}{u_{ij}\,u_{ik}\,u_{il}\,h_i^2}\\
&-\frac{7}{20}\frac{V_{ij}^2\,V_{ik}^2\,V_{il}\,h_l\,{u_i'}^2}{u_{ij}u_{ik}u_{il}\,h_i^3}+\frac{7}{40}\frac{V_{ij}^2\,V_{ik}^2\,V_{il}\,h_l\,u_i'\,u_l'}{u_{ij}\,u_{ik}\,u_{il}\,h_i^3}-\frac{1}{8}\frac{V_{ij}^2\,V_{ik}\,V_{kl}^2\,u_i'\,u_k'}{u_{ij}\,u_{ik}\,u_{kl}\,h_i\,h_k}\\
&+\frac{1}{40}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,h_l\,\left({u_k'}^2-3\,{u_i'}^2-2\,u_k'\,u_l'\right)}{u_{ij}\,u_{ik}\,u_{kl}\,h_i^3}+\frac{3}{40}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,u_i'\,u_l'\,h_l}{u_{ij}\,u_{ik}\,u_{il}\,h_i^3}\\
&+\frac{1}{40}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,h_l\,\left(3\,{u_i'}^2+{u_l'}^2\right)}{u_{ij}\,u_{kl}\,u_{il}\,h_i^3}+\frac{1}{48}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,h_l\,u_i'\,(2\,u_k'-u_l')}{u_{ij}\,u_{ik}\,u_{kl}\,h_i\,h_k^2}\\
&+\frac{5}{96}\frac{V_{ij}^2\,V_{ik}\,V_{il}\,h_k\,h_l\,\left(4\,{u_i'}^2-4\,u_i'\,u_k'+u_k'u_l'\right)}{u_{ij}u_{ik}u_{il}\,h_i^4}-\frac{83}{480}\frac{V_{ij}^2\,V_{ik}^2\,{u_i'}^2}{u_{ij}\,u_{ik}^2\,h_i^2}\\
&+\frac{1}{144}\frac{V_{ij}\,V_{ik}\,V_{jl}\,V_{kl}\,{u_i'}^2}{u_{ik}\,u_{jl}\,u_{il}\,h_i^2}-\frac{1}{144}\frac{V_{ij}\,V_{ik}\,V_{jl}\,V_{kl}\,{u_i'}^2}{u_{ij}\,u_{ik}\,u_{kl}\,h_i^2}-\frac{1}{48}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,u_i'\,u_l'}{u_{ij}\,u_{kl}\,u_{il}\,h_i\,h_l}\\
&+\frac{29}{1920}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\,\left(u_k'\,u_l'-u_i'\,u_k'+2\,{u_i'}^2-2\,u_i'\,u_l'\right)}{u_{ij}\,u_{ik}\,u_{il}\,h_i^4}\\
&-\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\,u_j'\,\left(2\,u_k'\,u_l'+2\,u_i'\,u_j'-u_j'\,u_k'-4\,u_i'\,u_l'\right)}{u_{ij}\,u_{ik}\,u_{jl}\,h_i^4\,u_i'}\\
&-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\,\left(4\,u_i'\,u_j'-4\,u_i'\,u_l'+u_k'\,u_l'\right)}{u_{ij}\,u_{ik}\,u_{jl}\,h_i^2\,h_j^2}\\
&-\frac{1}{384}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,\left(u_i'\,{u_j'}^2-2\,u_j'\,u_i'\,u_l'\right)}{u_{ij}\,u_{ik}\,u_{jl}\,u_k'\,h_k^3}-\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\,{u_l'}^2\,(2\,u_i'-u_k')}{u_{ik}\,u_{jl}\,u_{il}\,h_i^4\,u_i'}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u_i'}^2\,(u_i'-3\,u_l')}{u_{ij}\,u_{ik}\,u_{il}\,u_k'\,h_k^3}-\frac{1}{384}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,u_i'\,{u_l'}^2}{u_{ik}\,u_{jl}\,u_{il}\,u_k'\,h_k^3}\\
&-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u_j'}^2\left(3\,u_l'-2\,u_j'\right)}{u_{ij}\,u_{jl}\,u_{jk}\,u_k'\,h_k^3}-\frac{1}{288}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,u_j'\left(u_j'-2\,u_l'\right)}{u_{ik}\,u_{jl}\,u_{jk}\,h_k^3}\\
&+\frac{1}{576}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,u_k'\,(2\,u_k'-3\,u_l')}{u_{ik}\,u_{jk}\,u_{kl}\,h_k^3}-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u_l'}^3}{u_{jl}\,u_{kl}\,u_{il}\,u_k'\,h_k^3}\\
&+\frac{1}{288}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u_l'}^2}{u_{ik}\,u_{jl}\,u_{kl}\,h_k^3}-\frac{1}{576}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,u_l'\,(u_k'-2\,u_i')}{u_{ik}\,u_{jl}\,u_{il}\,h_i^2\,h_l}\\
&-\frac{7}{1440}\frac{V_{ij}\,V_{ik}\,V_{il}\,h_j\,h_k\,h_l\left(8\,{u_i'}^3-12\,{u_i'}^2\,u_j'-u_j'\,u_k'\,u_l'+6\,u_i'\,u_j'\,u_k'\right)}{u_{ij}\,u_{ik}\,u_{il}\,h_i^5\,u_i'}\\
&-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,u_k'\,u_l'}{u_{ik}\,u_{jl}\,u_{kl}\,h_k\,h_l}-\frac{29}{1152}\frac{V_{ij}\,V_{ik}\,V_{jk}\,{u_i'}^2}{u_{ij}\,u_{ik}^2\,h_i^2}-\frac{53}{1920}\frac{V_{ij}^2\,V_{ik}\,h_k\,u_i'\,u_k'}{u_{ij}\,u_{ik}\,u_{jk}\,h_i^3}\\
&-\frac{1}{320}\frac{V_{ij}^2\,V_{ik}\,h_k\left(3\,{u_i'}^2-8\,{u_k'}^2\right)}{u_{ij}\,u_{ik}^2\,h_i^3}-\frac{V_{ij}^2\,V_{ik}\,u_i'\,h_k}{u_{ij}^2\,u_{jk}\,h_i^3}\left(\frac{27}{640}\,u_k'-\frac{233}{2880}\,u_i'\right)\\
&-\frac{V_{ij}^2\,V_{ik}\,u_i'\,h_k}{u_{ik}^2\,u_{jk}\,h_i^3}\left(\frac{233}{2880}\,u_i'-\frac{67}{960}\,u_k'\right)+\frac{1}{1152}\frac{V_{ij}^2\,V_{ik}\,h_i\,{u_i'}^3}{u_{ij}\,u_{ik}^2\,u_k'\,h_k^3}-\frac{1}{576}\frac{V_{ij}^2\,V_{ik}\,h_i\,{u_i'}^3}{u_{ij}^2\,u_{ik}\,u_k'\,h_k^3}\\
&-\frac{1}{48}\frac{V_{ij}^2\,V_{ik}\,u_i'\,u_k'}{u_{ij}\,u_{ik}^2\,h_i\,h_k}+\frac{233}{1440}\frac{V_{ij}^3\,h_j\,{u_i'}^2}{u_{ij}^3\,h_i^3}-\frac{43}{384}\frac{V_{ij}^3\,h_j\,u_i'\,u_j'}{u_{ij}^3\,h_i^3}-\frac{1}{12}\frac{V_{ij}^3\,u_i'\,u_j'}{u_{ij}^3\,h_i\,h_j}\\
&+\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,h_j\,h_k\left(3\,u_i'\,u_k'+3\,u_j'\,u_k'+6\,u_i'\,u_j'-6\,{u_i'}^2-2\,{u_j'}^2\right)}{u_{ij}^2\,u_{ik}\,h_i^4}\\
&+\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,u_j'\,u_k'\,h_j\,h_k\,(u_k'-6\,u_i')}{u_{ij}\,u_{ik}^2\,u_i'\,h_i^4}+\frac{1}{576}\frac{V_{ij}\,V_{ik}\,u_j'\,h_k\,(2\,u_i'-u_k')}{u_{ij}^2\,u_{ik}\,h_i^2\,h_j}\\
&+\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,u_{ij}\,h_k\left(3\,{u_i'}^2\,u_k'-3\,u_i'\,{u_k'}^2+{u_k'}^3-{u_i'}^3\right)}{u_{ik}^2\,u_{jk}^2\,u_j'\,h_j^3}\\
&+\frac{1}{576}\frac{V_{ij}\,V_{ik}\,u_{ik}\,h_k\left(-{u_i'}^3+3\,{u_j'}^2\,u_k'-4\,u_i'\,u_j'\,u_k'+2\,{u_i'}^2\,u_j'-2\,{u_j'}^3\right)}{u_{ij}^2\,u_{jk}^2\,u_j'\,h_j^3}\\
&+\frac{1}{384}\frac{V_{ij}\,V_{ik}\,h_k\left(-u_i'\,{u_k'}^2+{u_i'}^3-6\,{u_j'}^2\,u_k'\right)}{u_{ij}\,u_{jk}^2\,u_j'\,h_j^3}+\frac{1}{384}\frac{V_{ij}\,V_{ik}\,h_k\,{u_i'}^2\,u_k'}{u_{ij}^2\,u_{jk}\,u_j'\,h_j^3}\\
&+\frac{1}{288}\frac{V_{ij}\,V_{ik}\,h_k\left(4\,u_i'\,u_k'+{u_k'}^2-2\,{u_i'}^2+3\,{u_j'}^2\right)}{u_{ij}\,u_{jk}^2\,h_j^3}-\frac{1}{576}\frac{V_{ij}\,V_{ik}\,u_j'\,u_k'}{u_{ik}\,u_{jk}^2\,h_j\,h_k}\\
&+\frac{1}{384}\frac{V_{ij}\,V_{ik}\,h_k\left(2\,u_i'\,{u_k'}^2-{u_i'}^2\,u_k'-{u_k'}^3\right)}{u_{ik}\,u_{jk}^2\,u_j'\,h_j^3}+\frac{1}{288}\frac{V_{ij}\,V_{ik}\,h_k\left({u_k'}^2-2\,u_i'\,u_k'+{u_i'}^2\right)}{u_{ik}\,u_{jk}^2\,h_j^3}\\
&+\frac{1}{1152}\frac{V_{ij}^2\,u_i'\left(37\,u_i'\,u_j'\,h_j^2+10\,u_i'\,u_j'\,h_i^2-3\,{u_i'}^2\,h_i^2+11\,{u_j'}^2\,h_j^2\right)}{u_{ij}^3\,u_j'\,h_i^2\,h_j^2}\\
&-\frac{1}{576}\frac{V_{ij}\,h_j\left(4\,{u_i'}^3+4\,u_i'\,{u_j'}^2-6\,{u_i'}^2\,u_j'-{u_j'}^3\right)}{u_{ij}^3\,u_i'\,h_i^3}+\frac{1}{576}\frac{V_{ij}\,u_i'\,u_j'}{u_{ij}^3\,h_i\,h_j}.
\end{aligned}
$$

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