Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants

Boris Dubrovin* Youjin Zhang**

* SISSA, Via Beirut 2–4, 34014 Trieste, Italy, and Steklov Math. Institute, Moscow

** Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P.R.China

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Abstract

We present a project of classification of a certain class of bihamiltonian 1+1 PDEs depending on a small parameter. Our aim is to embed the theory of Gromov - Witten invariants of all genera into the theory of integrable systems. The project is focused at describing normal forms of the PDEs and their local bihamiltonian structures satisfying certain simple axioms. A Frobenius manifold or its degeneration is associated to every bihamiltonian structure of our type. The main result is a universal loop equation on the jet space of a semisimple Frobenius manifold that can be used for perturbative reconstruction of the integrable hierarchy. We show that first few terms of the perturbative expansion correctly reproduce the universal identities between intersection numbers of Gromov - Witten classes and their descendents.

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Contents

1 Introduction

In this paper we study the structure of integrable systems of evolutionary PDEs with one spatial variable of the form

$$u^i_t = \sum_{j=1}^n A^i_j(u) u^j_x + \text{perturbation}, \quad i = 1, \dots, n. \tag{1.1}$$

The basic example of such a system is the celebrated Korteweg - de Vries equation (in this example $n = 1$)

$$u_t = u\, u_x + \frac{\epsilon^2}{12} u_{xxx}. \tag{1.2}$$

Here ϵ is the parameter of the perturbation. Our aim is to develope an approach to the problem of classification of such systems based on the deep relationship between integrable systems and quantum field theory discovered in the last decade [11, 29, 81, 26, 65, 28, 145, 27, 146, 90]. The amazing discovery of E. Witten and M. Kontsevich of the relationship between KdV and the topology of the moduli spaces of stable algebraic curves opened a new dimension in the theory of integrable systems. The surprising outcome of our classification project is that, at least at low orders of the perturbative expansion, the topology of the moduli spaces $\bar{\mathcal{M}}_{g,n}$ of stable algebraic curves is "hidden" in *every* integrable systems of our class.

Another motivation for our work was the wish to find a proper setup for the general theory of Frobenius manifolds (cf. [78]). As it has been suggested in [51, 52, 43, 44] (see also [146]) the right setting for the theory of semisimple Frobenius manifolds is the theory of hierarchies of integrable PDEs.

Frobenius manifolds were introduced by one of the authors [38] as a coordinate-free form of the WDVV associativity equations [145, 26]. We refer the reader to [41, 42, 84, 109] for the details of the theory of Frobenius manifolds. In the mathematical literature Frobenius manifolds are best known in quantum cohomologies, i.e., in the theory of the genus zero Gromov - Witten invariants of smooth projective varieties or, more generally, of compact symplectic manifolds [91, 5, 128, 113], although many ingredients of the theory of Frobenius manifolds already appeared in the singularity theory as a natural geometrical structure on the base of the universal unfolding of an isolated hypersurface singularity [129, 109, 83]. Another source of Frobenius manifolds is the geometry of the orbit spaces of finite Coxeter groups [39, 40] and their generalizations [50, 7]. The notion of flat coordinates on the orbit spaces discovered in [131, 130] was important in these constructions. The Frobenius manifolds of the singularity theory and of the theory of reflection groups are always semisimple. The origin of semisimplicity in quantum cohomology is still to be understood [139, 44, 4].

In certain cases mirror symmetry constructions, or the Arnold - Brieskorn correspondence between ADE Weyl groups and simple hypersurface singularities establish relationships between different classes of examples of Frobenius manifolds. However, the general unifying principle of the theory of Frobenius manifolds eventually covering also the theory of Gromov - Witten invariants of higher genera is still missing.

Our suggestion is that, the right framework of the theory of Frobenius manifolds of all genera is the theory of integrable PDEs along with all main ingredients of this theory, i.e., bihamiltonian structures, tau-functions, Virasoro symmetries, W-algebras etc. (an expert in the theory of integrable systems may have his own opinion about how should this list of "main ingredients" be continued).

It has already been proven in [37] that all the genus zero topological recursion relations for the so-called descendents can be correctly reproduced starting from an arbitrary Frobenius manifold. Dispersionless integrable hierarchies were crucial in this reconstruction theorem (see also [146]). Bihamiltonian structure for these hierarchies was discovered in [39, 41]. The next important step has been done in [51]. It was shown that also the genus one topological recursion relations [146] together with the E. Getzler's defining relation for elliptic Gromov - Witten invariants [72] can be reproduced starting from an arbitrary semisimple Frobenius manifold. Using this result the Virasoro conjecture of T. Eguchi *et al.* [56, 58, 59] was proved in [52] up to genus one approximation (see also [101, 99, 74] for an alternative approach to the theory of Virasoro constraints in quantum cohomology).

In [51] topology of [146, 72] was used to uniquely recover the first order integrable perturbation of the dispersionless hierarchy of [37]. In principle this approach can be extended also to higher genera (see [53] where the genus two topological recursion relations and other identities in the cohomology of $\bar{\mathcal{M}}_{g,n}$ for $g \leq 2$ [6, 73] were used in order to compute the genus 2 free energy in topological sigma models with two primaries and also [100] where a full system of equations for the genus two Gromov - Witten potential has been obtained in the general case). We now want to change completely the setting. Instead of using topology for constructing integrable hierarchies, as it was done in [37, 51] we want to develope an approach to the problem of classification of integrable hierarchies eventually reproducing *all the universal identities between Gromov - Witten invariants and their descendents of all genera* even for those integrable hierarchies that *a priori* have nothing to do with topology.

The main result of this paper is a system of four axioms of the theory of integrable hierarchies of the form (1.1) that can be used as the basis of the classification of these hierarchies. We prove that, under assumption of semisimplicity these axioms allow to uniquely reconstruct the whole structure of the hierarchy starting from the dispersionless limit $\epsilon \to 0$. We are also able to correctly reproduce, starting from our axioms, essentially all known universal identities for the Gromov - Witten cocycles and their descendents in $H^*(\bar{\mathcal{M}}_{g,n})$ written as differential constraints for the tau-function of the hierarchy. In particular, we prove the $3g-2$-conjecture of T. Eguchi and C.-S. Xiong [59] and reproduce the correct shape of the Virasoro constraints, derive the formulae for the genus 1 and genus 2 Gromov - Witten potential expressing them in terms of the genus 0 one (cf. [78, 79]) etc. Let us emphasize again that, all this has been done *for an arbitrary* semisimple Frobenius manifold. It remains an open problem to figure out what could be the meaning of our integrable hierarchies for other classes of Frobenius manifolds, e.g., in the singularity theory.

To explain our axioms let us first recall some features of KdV crucial for our classification project.

One of the starting point of the KdV theory was the discovery [118] of an infinite family of commuting evolutionary PDEs commuting with (1.2),

$$\frac{\partial u}{\partial t_j} = K_j(u, u_x, \dots, u^{(2j+1)}), \quad \frac{\partial}{\partial t_i}\frac{\partial u}{\partial t_j} = \frac{\partial}{\partial t_j}\frac{\partial u}{\partial t_i} \tag{1.3}$$

with some polynomials K_j, $j \geq 0$. For $j = 0$ one obtains the spatial translations

$$\frac{\partial u}{\partial t_0} = \frac{\partial u}{\partial x},$$

for $j = 1$ (1.3) coincides with (1.2), other equations of the so-called *KdV hierarchy* (1.3) are more complicated. E.g.,

$$\frac{\partial u}{\partial t_2} = \frac{1}{2}u^2 u' + \frac{\epsilon^2}{12}(2u'u'' + u\,u''') + \frac{\epsilon^4}{240}u^V.$$

They are obtained by a suitable recursion procedure [118, 67]. The latter has been represented [104] in the *bihamiltonian form*: the equation of the hierarchy are considered as flows on the space of functions $u(x)$ hamiltonian w.r.t. two Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$

$$\frac{\partial u}{\partial t_j} = \{u(x), H_j\}_1 = \left(j + \frac{1}{2}\right)^{-1} \{u(x), H_{j-1}\}_2, \tag{1.4}$$

$$\{u(x), u(y)\}_1 = \delta'(x-y), \quad \{u(x), u(y)\}_2 = u(x)\delta'(x-y) + \frac{1}{2}u'(x)\delta(x-y) + \frac{\epsilon^2}{8}\delta'''(x-y). \tag{1.5}$$

The crucial property in the definition of a bihamiltonian structure is *compatibility* of the pair of the Poisson brackets:

$$a_1\{\ ,\ \}_1 + a_2\{\ ,\ \}_2$$

must be a Poisson bracket for arbitrary constant coefficients a_1, a_2. The Hamiltonians are local functionals

$$H_j = \int h_j(u, u_x, \dots, u^{(2j+2)})dx$$

to be determined recursively from (1.4) starting from the Casimir

$$H_{-1} = \int u\,dx.$$

Explicitly,

$$h_0 = \frac{u^2}{2} + \epsilon^2\frac{u''}{12}, \quad h_1 = \frac{u^3}{6} + \frac{\epsilon^2}{24}(u'^2 + 2u\,u'') + \epsilon^4\frac{u^{IV}}{240},$$

$$h_2 = \frac{u^4}{24} + \frac{\epsilon^2}{24}(u\,u'^2 + u^2 u'') + \frac{\epsilon^4}{480}(3u''^2 + 4u'u''' + 2u\,u^{IV}) + \frac{\epsilon^6}{6720}u^{VI}.$$

This observation is the starting point of our study: what we are classifying is not just a single integrable PDE (1.1) but a *hierarchy* of integrable PDEs produced by a bihamiltonian recursion procedure. To say the same thing in a shorter way: we want to classify bihamiltonian structures of integrable PDEs of the form (1.1). This distinguishes our approach from the symmetry analysis technique (see [134] and references therein) or from the Painlevé test (see [95] and references therein) proved to be powerful in classification of integrable PDEs of low orders. Our approach differs also from the perturbative method of V.E. Zakharov *et al.* (see [150] and references therein) where nonlinear integrable perturbations of *linear* systems were studied.

We impose three additional constraints onto the bihamiltonian structure. The first one is existence of a *tau-function* [17]. For the example of the KdV hierarchy this means that, for an arbitrary solution $u = u(x + t_0, t_1, t_2, \ldots)$ of the hierarchy the densities of the Hamiltonians can be represented in the form

$$h_j(u, u_x, \ldots, u^{(2j+2)}) = \epsilon^2 \frac{\partial^2 \log \tau(x + t_0, t_1, t_2, \ldots)}{\partial x\, \partial t_{j+1}} \tag{1.6}$$

for some function $\tau(x + t_0, t_1, t_2, \ldots)$. In particular, for $j = -1$ one obtains the well-known formula

$$u = \epsilon^2 \frac{\partial^2 \log \tau}{\partial x^2}.$$

Existence of a *single* tau-function is a rather strong restriction onto an integrable hierarchy (cf. [115] where such a tau-function was constructed for generalized integrable hierarchies of KdV and affine Toda type). This restriction corresponds to the choice of a primitive conjugacy class [87] in the Weyl group in the setting of the theory of generalized Drinfeld - Sokolov hierarchies.

According to the conjecture by E.Witten [146] proved by M.Kontsevich [90], the logarithm of the tau-function of the particular solution of the KdV hierarchy specified by the initial data

$$u|_{t_j=0} = x,$$

$$\begin{aligned}\log\tau = \frac{1}{\epsilon^2}\Big(&\frac{t_0^3}{6} + \frac{t_0^3\,t_1}{6} + \frac{t_0^3\,t_1^2}{6} + \frac{t_0^3\,t_1^3}{6} + \frac{t_0^3\,t_1^4}{6} + \frac{t_0^4\,t_2}{24} + \frac{t_0^4\,t_1\,t_2}{8}\\
&+\frac{t_0^4\,t_1^2\,t_2}{4} + \frac{t_0^5\,t_2^2}{40} + \frac{t_0^5\,t_3}{120} + \frac{t_0^5\,t_1\,t_3}{30} + \frac{t_0^6\,t_4}{720} + \ldots\Big)\\
&+\Big(\frac{t_1}{24} + \frac{t_1^2}{48} + \frac{t_1^3}{72} + \frac{t_1^4}{96} + \frac{t_0\,t_2}{24} + \frac{t_0\,t_1\,t_2}{12} + \frac{t_0\,t_1^2\,t_2}{8} + \frac{t_0^2\,t_2^2}{24}\\
&+\frac{t_0^2\,t_3}{48} + \frac{t_0^2\,t_1\,t_3}{16} + \frac{t_0^3\,t_4}{144} + \ldots\Big)\\
&+\epsilon^2\Big(\frac{7\,t_2^3}{1440} + \frac{7\,t_1\,t_2^3}{288} + \frac{29\,t_2\,t_3}{5760} + \frac{29\,t_1\,t_2\,t_3}{1440} + \frac{29\,t_1^2\,t_2\,t_3}{576} + \frac{5\,t_0\,t_2^2\,t_3}{144}\end{aligned}$$

$$+\frac{29\,t_0\,t_3^2}{5760}+\frac{29\,t_0\,t_1\,t_3^2}{1152}+\frac{t_4}{1152}+\frac{t_1\,t_4}{384}+\frac{t_1^2\,t_4}{192}+\frac{t_1^3\,t_4}{96}+\frac{11\,t_0\,t_2\,t_4}{1440}$$
$$+\frac{11\,t_0\,t_1\,t_2\,t_4}{288}+\frac{17\,t_0^2\,t_3\,t_4}{1920}+\dots\Big)+O(\epsilon^4)$$

coincides with the generating function of the intersection numbers of the Mumford - Morita - Miller classes in $H^*(\bar{\mathcal{M}}_{g,n})$. Here $\bar{\mathcal{M}}_{g,n}$ is the moduli space of stable algebraic curves of genus g with n punctures,

$$\log\tau=\sum\epsilon^{2g-2}\mathcal{F}_g,$$

$$\mathcal{F}_g=\sum_n\frac{1}{n!}t_{p_1}\dots t_{p_n}\int_{\bar{\mathcal{M}}_{g,n}}c_1^{p_1}(\mathcal{L}_1)\wedge\dots\wedge c_1^{p_n}(\mathcal{L}_n)$$

where $\mathcal{L}_i$ is the tautological line bundle over the moduli space corresponding to the i-th puncture. In this setting different powers of the small dispersion parameter ϵ in the KdV correspond to the contributions of different genera g. In other words, the small dispersion expansion coincides with the genus expansion. In the physical literature on topological field theory the parameter ϵ is called string coupling constant.

The first two assumptions, i.e., existence of a bihamiltonian structure and of a tau-function of the integrable hierarchy imply that the *dispersionless limit* $\epsilon\to 0$ is described by a *Frobenius manifold* structure on the space of dependent variables of the hierarchy (see Section 3.5 below) or by a *degenerate Frobenius manifold* structure. The dispersionless hierarchy itself is reconstructed by the Frobenius manifold structure according to the construction of [37] (we call it Principal Hierarchy in Section 3.6 below). The Principal Hierarchy possesses all the universal properties observed in the theory of weak dispersive limits of integrable PDEs [47, 49, 35, 36, 93, 137, 10]. (We consider here only the formal geometric side of the theory of weakly dispersive integrable PDEs. We refer the reader to the papers of P. Lax, D. Levermore, S. Venakides, see [97] and references therein, for the analytic side of this theory. We hope, however, that our geometric analysis could be useful also for the analytic theory.)

The next step is the main one: we are to reconstruct the full hierarchy (1.1) together with the bihamiltonian structure starting from their dispersionless limit. (We do not consider in this paper the hierarchies corresponding to degenerate Frobenius manifolds. We plan to do it elsewhere.) The assumption of semisimplicity is to be added at this point. From the point of view of integrable systems semisimplicity ensures complete integrability, i.e., completeness of the family of commuting integrals (see Section 3.6.5 below). The last two axioms are used to provide uniqueness of the reconstruction.

The axiom 3 is the most disputable one. We call it *quasitriviality* of the hierarchy. Before explaining this axiom we are to formulate in a more precise way the classification problem. We study bihamiltonian PDEs depending on a formal small parameter ϵ represented as a (formal) small dispersion expansion

$$u_t^i=\sum_{j=1}^n A_j^i(u)u_x^j+\sum_{k>0}\epsilon^k K_{[k]}^i(u;u_x,\dots,u^{(k+1)}),\quad i=1,\dots,n \tag{1.7}$$

where $K^i_{[k]}(u; u_x, \dots, u^{(k+1)})$ is a *polynomial in the derivatives* weighted homogeneous of the degree $k+1$. It is understood that the m-th derivative $u^{i(m)}$ has degree m. We classify these PDEs and their *local* bihamiltonian structures w.r.t. the *Miura group* of transformations of the form

$$u^i \mapsto F^i_{[0]}(u) + \sum_{k>0} \epsilon^k F^i_{[k]}(u; u_x, \dots, u^{(k)}) \tag{1.8}$$

where the coefficients $F^i_{[k]}(u; u_x, \dots, u^{(k)})$ are *homogeneous polynomials in the derivatives* of the degree k and

$$\det\left(\frac{\partial F^i_{[0]}(u)}{\partial u^j}\right) \neq 0.$$

The problem of classification can be presented as the problem of description of *normal forms* of integrable PDEs w.r.t. the transformations (1.8). The Miura group acts also on local translation invariant Poisson brackets of systems of the form (1.7). We call them (0,n)-brackets (see the definition in Section 2.4.3 below). It turns out that, at least over complex numbers all (0,n) Poisson brackets are equivalent w.r.t. Miura group. This important technical step of our theory is based on the differential-geometric theory, due to S.P. Novikov and B. Dubrovin [47] of the so-called Poisson brackets of hydrodynamic type and also on the triviality of the Poisson cohomology of these brackets proved by E. Getzler [75] and also by L. Degiovanni, F. Magri, V. Sciacca [20] [1]. The main object of our study is *the problem of normal forms of bihamiltonian structures of systems (1.7) w.r.t. the Miura group.*

An integrable hierarchy (1.7) is called trivial if it can be obtained, together with the underlined bihamiltonian structure, from the dispersionless limit $\epsilon = 0$ by action of a transformation of the form (1.8). It is called *quasitrivial* if the same is true w.r.t. a transformation

$$u^i \mapsto F^i_{[0]}(u) + \sum_{k>0} \epsilon^k F^i_{[k]}(u; u_x, \dots, u^{(m_k)}) \tag{1.9}$$

where the coefficients are *rational functions* in the derivatives. The quasitriviality property seems to be unobserved even in the theory of the KdV equation (1.2). We prove it in Section 3.8. The quasitriviality transforming the hierarchy of the dispersionless KdV

$$u_t = u\, u_x$$

to the full KdV together with the bihamiltonian structure etc. reads

$$u \mapsto u + \frac{\epsilon^2}{24}(\log u_x)_{xx} + \epsilon^4\left(\frac{u^{IV}}{1152\, u'^2} - \frac{7\, u'' u'''}{1920\, u'^3} + \frac{u''^3}{360\, u'^4}\right)_{xx} + O(\epsilon^6). \tag{1.10}$$

The reader easily recognizes in (1.10) the genus expansion of the topological gravity written in the form suggested by R. Dijkgraaf and E. Witten [27] (see also [85, 55]).

[1] The problem of normal forms of Poisson brackets of PDEs was studied also in [114]. However, a more general class of admissible transformations was considered. This simplified the solution of the classification problem.

So, with respect to the group of rational Miura transformations (1.9) the normal forms of all our hierarchies are just the dispersionless Principal Hierarchies; the reducing transformation (1.9) is the candidate for the role of the genus expansion.

Indeed, we prove that, choosing in a clever way the dependent variables of the hierarchy the reducing transformation (1.9) is expressed via second derivatives of the ϵ-expansion of the logarithm of the tau-function of the hierarchy. Moreover, we prove, using that the Poisson pencil depends polynomially in the derivatives, that the latter must have the form

$$\log\tau = \epsilon^{-2}\mathcal{F}_0 + \sum_{g\geq 1}\epsilon^{2g-2}\mathcal{F}_g(u; u_x, \dots, u^{(3g-2)}). \tag{1.11}$$

The terms with $g \geq 1$ of the expansion of the reducing transformation do not depend on the choice of solution of the hierarchy. In the setting of topological sigma-models (1.11) coincides with the so-called $3g-2$-conjecture of [59] (see also [53]).

The last axiom is used to uniquely fix the terms of the expansion (1.11). It is based on study of symmetries of the integrable PDEs. First we prove, in Section 3.10.1 that the Principal Hierarchy always admits a rich algebra of infinitesimal symmetries isomorphic to the half of the Virasoro algebra. Due to quasitriviality these symmetries can be lifted to Virasoro symmetries of the full hierarchy. We require that the generators of the action of the half of the Virasoro algebra by symmetries of the hierarchy *act linearly onto the tau-function* of the hierarchy. For the KdV example the action of the generators of the Virasoro algebra by symmetries of the hierarchy is given by the following formulae [26, 65]

$$\delta_m\tau = L_m\tau, \quad m \geq -1 \tag{1.12}$$

where

$$\begin{aligned}
L_{-1} &= \sum_{p\geq 1} t_p\partial_{p-1} + \frac{1}{2\epsilon^2}t_0^2 \\
L_0 &= \sum_{p\geq 0}\left(p+\frac{1}{2}\right)t_p\partial_p + \frac{1}{16} \\
L_1 &= \sum_{p\geq 0}\left(p+\frac{1}{2}\right)\left(p+\frac{3}{2}\right)t_p\partial_{p+1} + \frac{\epsilon^2}{8}\partial_0^2 \\
L_2 &= \sum_{p\geq 0}\left(p+\frac{1}{2}\right)\left(p+\frac{3}{2}\right)\left(p+\frac{5}{2}\right)t_p\partial_{p+2} + \frac{3\,\epsilon^2}{8}\partial_0\partial_1 \\
&\dots
\end{aligned}$$

with $\partial_k = \frac{\partial}{\partial t_k}$. The Witten - Kontsevich tau-function (1.7) is uniquely specified [88] by the following system of *Virasoro constraints*

$$\begin{aligned}
L_m\tau &= \prod_{j=1}^{m+1}\left(j+\frac{1}{2}\right)\partial_{m+1}\tau, \quad m \geq 0 \\
L_{-1}\tau &= \partial_0\tau.
\end{aligned} \tag{1.13}$$

We call this last axiom *linearization of the Virasoro symmetries.* A surprizingly looking consequence of this axioms says that, under certain assumption of monotonicity the tau functions of *all* analytic solutions to our hierarchies are annihilated by an appropriate linear combination of the Virasoro symmetries and the flows of the hierarchy.

We also use the axiom of linearization of the Virasoro symmetries to define a set of defining equations for our integrable hierarchies. Using universality of the coefficients of expansion of

$$\Delta\mathcal{F}(u; u_x, u_{xx}, \dots; \epsilon^2) = \sum_{g\geq 1} \epsilon^{2g-2}\mathcal{F}_g(u; u_x, \dots, u^{(3g-2)}) \tag{1.14}$$

we derive a kind of *loop equation* (3.10.113) for the function (1.14) on the jet space (regarding loop equations and their applications in matrix models and in topological gravity see [1, 26, 65]). Universality of $\Delta\mathcal{F}$ as a function on the jet space is used in order to develop a machinery of perturbative solution of loop equation. In particular it allows to fix ambiguities inavoidable in the standard approach to the loop equation [107] and to prove uniqueness of the reconstruction of the integrable hierarchy starting from an arbitrary semisimple Frobenius manifold.

2 Normal forms of Hamiltonian structures of evolutionary systems

2.1 Brief summary of finite-dimensional Poisson geometry

Let P be a N-dimensional smooth manifold. A *Poisson bracket* on P is a structure of a Lie algebra on the ring of functions $\mathcal{F} := \mathcal{C}^\infty(P)$

$$f, g \mapsto \{f, g\},$$

$$\{g, f\} = -\{f, g\}, \{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad a,\ b \in \mathbf{R}, f,\ g,\ h \in \mathcal{F} \tag{2.1.1}$$

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0 \tag{2.1.2}$$

satisfying the Leibnitz rule

$$\{fg, h\} = f\{g, h\} + g\{f, h\}$$

for arbitrary three functions $f,\ g,\ h \in \mathcal{F}$. In a system of local coordinates $x^1, \dots, x^N$ the Poisson bracket reads

$$\{f, g\} = h^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} \tag{2.1.3}$$

(summation over repeated indices will be assumed) where the bivector $h^{ij}(x) = -h^{ji}(x)$ $= \{x^i, x^j\}$ satisfies the following system of equations equivalent to the Jacobi identity (2.1.2)

$$\{\{x^i, x^j\}, x^k\} + \{\{x^k, x^i\}, x^j\} + \{\{x^j, x^k\}, x^i\} \equiv \frac{\partial h^{ij}}{\partial x^s}h^{sk} + \frac{\partial h^{ki}}{\partial x^s}h^{sj} + \frac{\partial h^{jk}}{\partial x^s}h^{si} = 0 \tag{2.1.4}$$

for any i, j, k. Such a bivector satisfying (2.1.4) is called *Poisson structure* on P.

Clearly any bivector constant in some coordinate system is a Poisson structure. Vice versa [98], locally all solutions to (2.1.4) of the constant rank $2n = \mathrm{r}k(h^{ij})$ can be reduced, by a change of coordinates, to the following *normal form*

$$h = \begin{pmatrix} \bar{h} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.1.5}$$

with a constant nondegenerate antisymmetric $2n \times 2n$ matrix $\bar{h} = \bar{h}^{ab}$. That means that locally there exist coordinates $y^1, \ldots, y^{2n}, c^1, \ldots, c^k$, $2n + k = N$, s.t.

$$\bar{h}^{ab} = \{y^a, y^b\}$$

and

$$\{f, c^j\} = 0, \quad j = 1, \ldots k \tag{2.1.6}$$

for an arbitrary function f.

For the case $2n = N$ the inverse matrix $(h_{ij}(x)) = (h^{ij}(x))^{-1}$ defines on P a *symplectic structure*

$$\Omega = \sum_{i<j} h_{ij}(x) dx^i \wedge dx^j, \quad \Omega^n \neq 0.$$

For $2n < N$ one obtains on P a structure of *symplectic foliation* $P = \cup_{c_0} P_{c_0}$, $c_0 = (c_0^1, \ldots c_0^k)$, of the codimension $k = N - 2n$

$$P_{c_0} := \{x \mid c^1(x) = c_0^1, \ldots, c^k(x) = c_0^k\}. \tag{2.1.7}$$

The independent functions $c^1(x), \ldots, c^k(x)$ defined in (2.1.6) are called *Casimir functions*, or simply *Casimirs* of the Poissson structure. Every leaf P_{c_0} is a symplectic manifold, and the restriction map $\mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P_{c_0})$ is a homomorphism of Lie algebras.

Example 2.1.1 *Let $\mathfrak{g}$ be n-dimensional Lie algebra. The Lie - Poisson bracket on the dual space $P = \mathfrak{g}^*$ reads*

$$\{x^i, x^j\} = c_k^{ij} x^k. \tag{2.1.8}$$

Here c_k^{ij} are the structure constants of the Lie algebra. The Casimirs of this bracket are functions on $\mathfrak{g}^*$ invariant with respect to the co-adjoint action of the associated Lie group G. The symplectic leaves coincide with the orbits of the coadjoint action with the Berezin - Kirillov - Kostant symplectic structure on them.

An arbitrary foliation $P = \cup_{\phi_0} P_{\phi_0}$ of a codimension m represented locally in the form

$$P_{\phi_0} = \{x \mid \phi^1(x) = \phi_0^1, \ldots, \phi^m(x) = \phi_0^m\}$$

will be called *cosymplectic* if the $m \times m$ matrix $\{\phi^a, \phi^b\}$ does not degenerate on the leaves. In this situation a new Poisson structure $\{\ ,\ \}_D$ can be defined on P s.t. the functions $\phi^a(x)$ are Casimirs of $\{\ ,\ \}_D$. This is the *Dirac bracket* given explicitly by the formula

$$\{f,g\}_D = \{f,g\} - \sum_{a,b}\{f,\phi^a\}\{\phi^a,\phi^b\}^{-1}\{\phi^b,g\}. \tag{2.1.9}$$

It can be restricted in an obvious way to produce a Poisson structure on every leaf. The restriction map

$$(\mathcal{C}^\infty(P),\{\ ,\ \}) \to (\mathcal{C}^\infty(P_{\phi_0}),\{\ ,\ \}_D)$$

is a homomorphism of Lie algebras.

A Poisson bracket defines an (anti)homomorphism

$$\begin{gathered}\mathcal{F} \to Vect(P)\\ H \mapsto X_H := \{\cdot, H\},\\ [X_{H_1}, X_{H_2}] = -X_{\{H_1,H_2\}}.\end{gathered} \tag{2.1.10}$$

X_H is called *Hamiltonian vector field.* The corresponding dynamical system

$$\dot{x}^i = h^{ij}(x)\frac{\partial H}{\partial x^j} \tag{2.1.11}$$

is called *Hamiltonian system with the Hamiltonian* $H(x)$. It is a *symmetry of the Poisson bracket*

$$Lie_{X_H}\{\ ,\ \} = 0. \tag{2.1.12}$$

The last one is the notion of Poisson cohomology of $(P,\{\ ,\ \})$ introduced by Lichnerowicz [98]. We need to use the *Schouten - Nijenhuis bracket.* Denote

$$\Lambda^k = H^0(P, \Lambda^k TP)$$

the space of multivectors on P. The Schouten - Nijenhuis bracket is a bilinear pairing $a, b \mapsto [a,b]$,

$$\Lambda^k \times \Lambda^l \to \Lambda^{k+l-1}$$

uniquely determined by the properties of supersymmetry

$$[b,a] = (-1)^{kl}[a,b], \quad a \in \Lambda^k,\ b \in \Lambda^l \tag{2.1.13}$$

the graded Leibnitz rule

$$[c, a\wedge b] = [c,a]\wedge b + (-1)^{lk+k} a \wedge [c,b], \quad a \in \Lambda^k,\ c \in \Lambda^l \tag{2.1.14}$$

and the conditions $[f,g]=0,\ f,g \in \Lambda^0 = \mathcal{F}$,

$$[v,f] = v^i\frac{\partial f}{\partial x^i}, \quad v \in \Lambda^1 = Vect(P), \quad f \in \Lambda^0 = \mathcal{F},$$

$[v_1, v_2]$ = commutator of vector fields for $v_1, v_2 \in \Lambda^1$. In particular for a vector field v and a multivector a

$$[v,a] = Lie_v a.$$

Example 2.1.2 *For two bivectors $h = (h^{ij})$ and $f = (f^{ij})$ their Schouten - Nijenhuis bracket is the following trivector*

$$[h, f]^{ijk} = \frac{\partial h^{ij}}{\partial x^s} f^{sk} + \frac{\partial f^{ij}}{\partial x^s} h^{sk} + \frac{\partial h^{ki}}{\partial x^s} f^{sj} + \frac{\partial f^{ki}}{\partial x^s} h^{sj} + \frac{\partial h^{jk}}{\partial x^s} f^{si} + \frac{\partial f^{jk}}{\partial x^s} h^{si}. \quad (2.1.15)$$

Observe that the l.h.s. of the Jacobi identity (2.1.4) reads

$$\{\{x^i, x^j\}, x^k\} + \{\{x^k, x^i\}, x^j\} + \{\{x^j, x^k\}, x^i\} = \frac{1}{2}[h, h]^{ijk}.$$

The Schouten - Nijenhuis bracket satisfies the graded Jacobi identity [121]

$$(-1)^{km}[[a, b], c] + (-1)^{lm}[[c, a], b] + (-1)^{kl}[[b, c], a] = 0, \quad a \in \Lambda^k, \; b \in \Lambda^l, \; c \in \Lambda^m. \quad (2.1.16)$$

It follows that, for a Poisson bivector h the map

$$\partial : \Lambda^k \to \Lambda^{k+1}, \; \partial a = [h, a] \quad (2.1.17)$$

is a differential, $\partial^2 = 0$. The cohomology of the complex (Λ^*, ∂) is called *Poisson cohomology* of $(P, \{ \; , \; \})$. We will denote it

$$H^*(P, \{ \; , \}) = \oplus_{k \geq 0} H^k(P, \{ \; , \}).$$

In particular, $H^0(P, \{ \; , \})$ coincides with the ring of Casimirs of the Poisson bracket, $H^1(P, \{ \; , \})$ is the quotient of the Lie algebra of infinitesimal symmetries

$$v \in Vect(P), \; Lie_v\{ \; , \; \} = 0$$

over the subalgebra of Hamiltonian vector fields, $H^2(P, \{ \; , \})$ is the quotient of the space of infinitesimal deformations of the Poisson bracket by those obtained by infinitesimal changes of coordinates (i.e., by those of the form $Lie_v\{ \; , \; \}$ for a vector field v).

On a symplectic manifold $(P, \{ \; , \})$ Poisson cohomology coincides with the de Rham one. The isomorphism is established by "lowering the indices": for a cocycle $a = (a^{i_1 \dots i_k}) \in \Lambda^k$ the k-form

$$\sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \quad \omega_{i_1 \dots i_k} = h_{i_1 j_1} \dots h_{i_k j_k} a^{j_1 \dots j_k}$$

is closed. In particular, for $P =$ ball the Poisson cohomology is trivial. In the general case r$k(h^{ij}) < \dim P$ the Poisson cohomology does not vanish even locally (see [98]). We will prove now a simple criterion of triviality of 1- and 2-cocycles.

Lemma 2.1.3 *Let $h = (h^{ij}(x))$ be a Poisson structure of a constant rank $2n < N$ on a sufficiently small ball U. 1). A one-cocycle $v = (v^i(x)) \in H^1(U, h)$ is trivial* iff *the vector field v is tangent to the leaves of the symplectic foliation (2.1.7). 2). A 2-cocycle $f = (f^{ij}(x)) \in H^2(U, h)$ is trivial* iff

$$f(dc', dc'') = 0 \quad (2.1.18)$$

for arbitrary two Casimirs of h.

Of course, the statement of the lemma can be easily derived from the results of [98]. Nevertheless, we give a proof since we will use similar arguments also in the infinite dimensional situation.

Proof 1). For a coboundary $v = \partial f$ and for any Casimir c of h we have

$$\partial_v c = \{c, f\} = 0.$$

This means that v is tangent to the symplectic leaves (2.1.7). To prove the converse statement let us choose the canonical coordinates $x = (y^1, \dots, y^{2n}, c^1, \dots, c^k)$ reducing the bracket to the constant form (2.1.5). Here c^1, …, c^k are independent Casimirs (2.1.6). In these coordinates $v = (v^1, \dots, v^{2n}, 0, \dots, 0)$. The 1-form $\omega = (\omega_1, \dots, \omega_{2n}, 0, \dots, 0)$ given by

$$\omega_i = \sum_{j=1}^{2n} \bar{h}_{ij} v^j$$

has the property

$$d\omega|_{P_{c_0}} = 0.$$

Therefore a function g locally exists s.t.

$$dg = \sum_{i=1}^{2n} \omega_i dy^i + \sum_{a=1}^{k} \phi_a dc^a$$

for some functions ϕ_1, …, ϕ_k. This function is the Hamiltonian for the vector field v.

2). We will again use the canonical coordinates for h as in the proof of the first part. For an exact 2-cocycle $f = \partial v$ and arbitrary two functions c', c''

$$f(dc', dc'') = -\{c', v^i\}\partial_i c'' - \partial_i c'\{v^i, c''\}.$$

This is equal to zero if c' and c'' are Casimirs of the bracket h.

To prove the converse statement we first consider, for every $a = 1$, …, k, the vector field w (depending on a)

$$w^i = f^{ia}, \quad i = 1, \dots, N. \tag{2.1.19}$$

From (2.1.18) it follows that w is tangent to the symplectic leaves of h. The cocycle condition

$$0 = [h, f]^{aij} = \partial_k f^{ai} h^{kj} + \partial_k f^{ja} h^{ki} = (\partial w)^{ij} \tag{2.1.20}$$

implies $\partial w = 0$. According to the first part of the lemma, there exists a function $q^a(x)$ s.t. $w = \partial q^a$:

$$f^{ia} = h^{ik}\partial_k q^a, \quad a = 1, \dots, k. \tag{2.1.21}$$

Let us now change the cocycle by a coboundary

$$f \mapsto f + \partial z$$

where the vector field z is given by

$$z = \sum_{a=1}^{k} q^a \frac{\partial}{\partial c^a}. \tag{2.1.22}$$

After such a change, due to (2.1.21), we obtain

$$f^{ia} = f^{ai} = 0, \quad i = 1, \ldots, N.$$

The rest of the proof repeats the arguments of the first part. The 2-form

$$\omega_{ij} = \sum_{i,\, j=1}^{2n} \bar{h}_{ik} \bar{h}_{lj} f^{kl}$$

is closed along the symplectic leaves. Hence there exists a 1-form $\phi = (\phi_i)$ s.t.

$$\omega = d\phi + \tilde{\omega}$$

where every monomial in $\tilde{\omega}$ contains at least one dc^a for some a. Therefore

$$f = \partial u$$

for the vector field

$$u^i = \sum_{k=1}^{2n} \bar{h}^{ik} \phi_k, \; i = 1, \ldots, 2n, \quad u^i = 0, \; i > 2n.$$

The lemma is proved. □

2.2 Formal loop spaces

Let M be a n-dimensional smooth manifold. Our aim is to describe an appropriate class of Poisson brackets on the loop space

$$\mathcal{L}(M) = \{S^1 \to M\}.$$

In our definitions we will treat $\mathcal{L}(M)$ formally in the spirit of formal variational calculus of [22, 19]. We define the formal loop space $\mathcal{L}(M)$ in terms of ring of functions on it. We also describe calculus of differential forms and vector fields on the formal loop space. In the next section we will also deal with multivectors on the formal loop space.

Let $U \subset M$ be a chart on M with the coordinates $u^1, \ldots, u^n$. Denote $\mathcal{A} = \mathcal{A}(U)$ the space of polynomials in the independent variables $u^{i,s}$, $i = 1, \ldots, n$, $s = 1, 2, \ldots$

$$f(x; u; u_x, u_{xx}, \ldots) := \sum_{m \geq 0} f_{i_1 s_1; \ldots; i_m s_m}(x; u) u^{i_1, s_1} \ldots u^{i_m, s_m} \tag{2.2.1}$$

with the coefficients $f_{i_1 s_1;\ldots;i_m s_m}(x;u)$ being smooth functions on $S^1 \times M$. Such an expression will be called *differential polynomial*. We will often use an alternative notation for the independent variables

$$u^i_x = u^{i,1}, \quad u^i_{xx} = u^{i,2}, \ldots$$

Observe that polynomiality w.r.t. $u = (u^1, \ldots, u^n)$ is not assumed.

The operator ∂_x is defined as follows

$$\partial_x f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u^i} u^{i,1} + \ldots + \frac{\partial f}{\partial u^{i,s}} u^{i,s+1} + \ldots . \tag{2.2.2}$$

We will often use the notation

$$f^{(k)} := \partial_x^k f. \tag{2.2.3}$$

The following identities will be useful

$$\frac{\partial}{\partial u^i}\partial_x = \partial_x \frac{\partial}{\partial u^i} \tag{2.2.4}$$

$$\frac{\partial}{\partial u^{i,s}}\partial_x = \partial_x \frac{\partial}{\partial u^{i,s}} + \frac{\partial}{\partial u^{i,s-1}}. \tag{2.2.5}$$

We define the space

$$\mathcal{A}_{0,0} = \mathcal{A}/\mathbf{R}, \quad \mathcal{A}_{0,1} = \mathcal{A}_{0,0}\, dx,$$

the operator

$$d : \mathcal{A}_{0,0} \to \mathcal{A}_{0,1}, \quad df := \partial_x f\, dx \tag{2.2.6}$$

and the quotient

$$\Lambda_0 = \mathcal{A}_{0,1}/d\mathcal{A}_{0,0}. \tag{2.2.7}$$

The elements of the space Λ_0 will be written as integrals over the circle S^1

$$\bar{f} := \int f(x; u; u_x, u_{xx}, \ldots) dx \in \Lambda_0 \tag{2.2.8}$$

We will use below the following simple statement.

Lemma 2.2.1 *If $\int f g\, dx = 0$ for an arbitrary $g \in \mathcal{A}$ then $f \in \mathcal{A}$ is equal to zero.*

The expressions (2.2.8) are also called *local functionals* with the *density* f. The space of local functionals is the main building block of the “space of functions” on the formal loop space. The full ring $\mathcal{F} = \mathcal{F}(\mathcal{L}(U))$ of functions on the formal loop space by definition coincides with the tensor algebra of Λ_0

$$\mathcal{F} = \mathbf{R} \oplus \Lambda_0 \oplus \Lambda_0 \hat{\otimes} \Lambda_0 \oplus \Lambda_0 \hat{\otimes} \Lambda_0 \hat{\otimes} \Lambda_0 \oplus \ldots \tag{2.2.9}$$

Elements of $\Lambda_0^{\hat{\otimes} k}$ will be written as multiple integrals of differential polynomials of k copies of the variables that we denote $u^i(x_1)$, ..., $u^i(x_k)$, $u^i_x(x_1)$, ..., $u^i_x(x_k)$ etc.

$$\int f(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k); u_x(x_1), \ldots, u_x(x_k), \ldots) dx_1 \ldots dx_k \in \Lambda_0^{\hat{\otimes} k}. \tag{2.2.10}$$

The short exact sequence

$$0 \to \mathcal{A}_{0,0} \xrightarrow{d} \mathcal{A}_{0,1} \xrightarrow{\pi} \Lambda_0 \to 0$$

(π is the projection) is included in the *variational bicomplex*

$$\begin{array}{ccccccccc}
 & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
0 & \to & \mathcal{A}_{2,0} & \xrightarrow{d} & \mathcal{A}_{2,1} & \xrightarrow{\pi} & \Lambda_2 & \to & 0 \\
 & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
0 & \to & \mathcal{A}_{1,0} & \xrightarrow{d} & \mathcal{A}_{1,1} & \xrightarrow{\pi} & \Lambda_1 & \to & 0 \\
 & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
0 & \to & \mathcal{A}_{0,0} & \xrightarrow{d} & \mathcal{A}_{0,1} & \xrightarrow{\pi} & \Lambda_0 & \to & 0 \\
 & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here $\mathcal{A}_{k,l}$ are elements of the total degree $k+l$ in the Grassman algebra with the generators $\delta u^{i,s}$, $i = 1, \ldots, n$, $s = 0, 1, 2, \ldots$ (observe the difference in the range of the second index of $u^{i,s}$ and $\delta u^{i,s}$) and dx with the coefficients in $\mathcal{A}$ having the degree l in dx. We will often identify

$$\delta u^{i,0} = \delta u^i.$$

For example, every k-form $\omega \in \mathcal{A}_{k,0}$ is a finite sum

$$\omega = \frac{1}{k!} \omega_{i_1 s_1; \ldots; i_k s_k} \delta u^{i_1, s_1} \wedge \ldots \wedge \delta u^{i_k s_k} \tag{2.2.11}$$

where the coefficients $\omega_{i_1 s_1; \ldots; i_k s_k} \in \mathcal{A}$ are assumed to be antisymmetric w.r.t. permutations of pairs $i_p, s_p \leftrightarrow i_q, s_q$.

The exterior differential in the Grassman algebra is decomposed into a sum $d + \delta$. The horizontal differential

$$d : \mathcal{A}_{k,0} \to \mathcal{A}_{k,1}$$

is defined by

$$d\omega = dx \wedge \partial_x \omega \tag{2.2.12}$$

where the derivation ∂_x,

$$\partial_x(\omega_1 \wedge \omega_2) = \partial_x \omega_1 \wedge \omega_2 + \omega_1 \wedge \partial_x \omega_2$$

is given by (2.2.2) on the coefficients of the differential form and by

$$\partial_x \delta u^{i,s} = \delta u^{i,s+1}.$$

The elements of the quotient

$$\Lambda_k = \mathcal{A}_{k,1} / d\mathcal{A}_{k,0}$$

will be called *(local) k-forms* on the loop space. k-forms will also be written by integrals

$$\int dx \wedge \omega \in \Lambda_k, \quad \omega \in \mathcal{A}_{k,0}.$$

Example 2.2.2 *Any one-form has a unique representative*

$$\phi = \int dx \wedge \phi_i \delta u^i \tag{2.2.13}$$

(use integration by parts).

More generally, for every k-form ω written as in (2.2.11)

$$dx \wedge \omega \sim dx \wedge \tilde{\omega} \ (\mathrm{m}od\ d(\mathcal{A}_{k,0}))$$

where

$$\tilde{\omega} = \frac{!}{(k-1)!} \tilde{\omega}_{i_1;i_2 s_2;\dots;i_k s_k} \delta u^{i_1} \wedge \delta u^{i_2,s_2} \wedge \dots \wedge \delta u^{i_k,s_k} \tag{2.2.14}$$

$$\tilde{\omega}_{i_1;i_2 s_2;\dots;i_k s_k} = \frac{1}{k} \sum_{\substack{0 \le r_l \le s_l \\ 2 \le l \le k}} \sum_{s \ge r_2+\dots+r_k} (-1)^s \begin{pmatrix} & s & \\ r_2 & \dots & r_k \end{pmatrix} \omega^{(s-r_2-\dots-r_k)}_{i_1 s; i_2, s_2-r_2;\dots;i_k,s_k-r_k}$$

here

$$\begin{pmatrix} & s & \\ r_2 & \dots & r_k \end{pmatrix} = \frac{s!}{r_2! \dots r_k! (s-r_2-\dots-r_k)!} \tag{2.2.15}$$

stands for the multinomial coefficients. The coefficients $\tilde{\omega}_{i_1;i_2 s_2;\dots;i_k s_k}$ will be called *reduced components* of ω. They are antisymmetric w.r.t. pairs $i_2, s_2, \dots, i_k, s_k$ but with the permutation of i_1 and i_2 they behave as

$$\tilde{\omega}_{i_2;i_1 s_2;\dots;i_k s_k} = \sum_{\substack{0 \le t_l \le s_l \\ 3 \le l \le k}} \sum_{t \ge s_2+t_3+\dots+t_k} (-1)^{t+1} \begin{pmatrix} & t & \\ s_2\, t_3 & \dots & t_k \end{pmatrix} \tilde{\omega}^{(t-s_2-t_3-\dots-t_k)}_{i_1;i_2 s_2; i_3, s_3-t_3;\dots;i_k,s_k-t_k} \tag{2.2.16}$$

We now define vertical arrows of the bicomplex. For a monomial

$$\omega = f\, \delta u^{i_1,s_1} \wedge \dots \wedge \delta u^{i_k,s_k}$$

put

$$\delta\omega = \sum_{s\geq 0} \frac{\partial f}{\partial u^{j,t}} \delta u^{j,t} \wedge \delta u^{i_1,s_1} \wedge \ldots \wedge \delta u^{i_k,s_k}, \tag{2.2.17}$$

where we denote

$$\frac{\partial}{\partial u^{j,0}} := \frac{\partial}{\partial u^j}.$$

This gives vertical differential $\delta : \mathcal{A}_{k,0} \to \mathcal{A}_{k+1,0}$,

$$\delta^2 = 0.$$

The map δ on $\mathcal{A}_{k,1}$ is defined by essentially same formula, $\delta dx = 0$. Anticommutativity

$$\delta d = -d\delta$$

justifies action of δ on the quotient Λ_k.

Example 2.2.3 *On Λ_0 the differential δ acts as follows*

$$\delta \int f\, dx = \int dx \wedge \left(\sum_s (-1)^s \partial_x^s \frac{\partial f}{\partial u^{i,s}} \right) \delta u^i \tag{2.2.18}$$

(the Euler - Lagrange differential). We will use the notation

$$\frac{\delta \bar{f}}{\delta u^i(x)} := \sum_s (-1)^s \partial_x^s \frac{\partial f}{\partial u^{i,s}} \tag{2.2.19}$$

for the components of the 1-form, $\bar{f} = \int f\, dx$.

Theorem 2.2.4 ([19]) *For M = ball both arrows and columns of the variational bi-complex are exact.*

Example 2.2.5 *A necessary and sufficient condition for*

$$\frac{\delta \bar{f}}{\delta u^i(x)} = 0, \quad i = 1, \ldots, n.$$

is the existence of a differential polynomial $g = g(x; u; u_x; \ldots)$ such that $f = \partial_x g$.

Let us now consider the space Λ^1 of vector fields on the formal loop space. These will be formal infinite sums

$$\xi = \xi^0 \frac{\partial}{\partial x} + \sum_{k\geq 0} \xi^{i,k} \frac{\partial}{\partial u^{i,k}}, \quad \xi^{i,k} \in \mathcal{A} \tag{2.2.20}$$

where we denote

$$\frac{\partial}{\partial u^{i,0}} := \frac{\partial}{\partial u^i}.$$

The derivative of a functional $\bar f = \int f(x; u; u_x, \dots) dx \in \Lambda_0$ along ξ reads

$$\xi\,\bar f := \int \left(\xi^0 \frac{\partial f}{\partial x} + \sum \xi^{i,k} \frac{\partial f}{\partial u^{i,k}} \right) dx.$$

The Lie bracket of two vector fields is defined by

$$\begin{aligned} [\xi, \eta] = & (\xi^0 \eta^0_x - \eta^0 \xi^0_x + \xi^{j,t} \frac{\partial \eta^0}{\partial u^{j,t}} - \eta^{j,t} \frac{\partial \xi^0}{\partial u^{j,t}}) \frac{\partial}{\partial x} \\ & + \sum_{s \geq 0} \left(\xi^0 \frac{\partial \eta^{i,s}}{\partial x} - \eta^0 \frac{\partial \xi^{i,s}}{\partial x} + \xi^{j,t} \frac{\partial \eta^{i,s}}{\partial u^{j,t}} - \eta^{j,t} \frac{\partial \xi^{i,s}}{\partial u^{j,t}} \right) \frac{\partial}{\partial u^{i,s}} \end{aligned} \tag{2.2.21}$$

Evolutionary vector fields a are defined by the conditions of vanishing of the $\partial/\partial x$-component and the commutativity

$$[\partial_x, a] = 0$$

They are parameterized by n-tuples $a^1, \dots, a^n$ of elements of $\mathcal{A}$ as follows

$$a = \sum_{s \geq 0} \partial_x^s a^i \frac{\partial}{\partial u^{i,s}}. \tag{2.2.22}$$

The corresponding system of evolutionary PDEs reads

$$u^i_t = a^i(x; u; u_x, u_{xx}, \dots), \quad i = 1, \dots, n. \tag{2.2.23}$$

In particular, an evolutionary vector field a is called *translation invariant* if the coefficients a^i do not depend explicitly on x,

$$\frac{\partial a^i}{\partial x} = 0, \quad i = 1, \dots, n.$$

The contraction $i_\xi \omega$ of a k-form $\omega \in \mathcal{A}_{k,0}$ given by (2.2.11) and a vector field ξ is a $(k-1)$-form defined by

$$i_\xi \omega = \frac{1}{(k-1)!} \xi^{j,t} \omega_{jt; i_1 s_1; \dots; i_{k-1} s_{k-1}} \delta u^{i_1, s_1} \wedge \dots \wedge \delta u^{i_{k-1}, s_{k-1}}. \tag{2.2.24}$$

As usual

$$i_\xi i_\eta = - i_\eta i_\xi$$

for two vector fields ξ, η. For a form $\omega \in \mathcal{A}_{k,1}$ the contraction $i_\xi \omega \in \mathcal{A}_{k-1,1}$ is defined by essentially same formula provided the vector field ξ contains no $\partial/\partial x$-term. It is an easy exercise to check, using (2.2.5), that for an evolutionary vector field a

$$d i_a + i_a d = 0. \tag{2.2.25}$$

It readily follows that contraction with evolutionary vector fields is well-defined on the quotient $i_a : \Lambda_k \to \Lambda_{k-1}$. A more strong statement holds true

Lemma 2.2.6 *Let $\omega \in \mathcal{A}_{k,1}$. It belongs to $d(\mathcal{A}_{k,0})$ iff $i_a\omega \in d(\mathcal{A}_{k-1,0})$ for an arbitrary evolutionary vector field a.*

Proof We use induction in k. For $k=1$ we can choose a representative of the class of $\omega \wedge dx \in \Lambda_1$ with the 1-form ω given by (2.2.13). The contraction reads

$$i_a(dx \wedge \omega) = -\int dx\, a^i \omega_i.$$

Using Lemma 2.2.1 we obtain $\omega_i = 0$ for all i.

Let us assume validity of the lemma for any $(k-1)$-form. We will prove that the condition $i_a\omega \wedge dx = 0 \in \Lambda_{k-1}$ implies vanishing of all the reduced components (2.2.14). By induction the above condition is equivalent to

$$i_{b_2} \dots i_{b_k} i_a \omega \wedge dx \in d(\mathcal{A}_{0,0})$$

for arbitrary evolutionary vector fields $b_2, \dots, b_k$. Integrating by parts we rewrite the last line in the form

$$\int a^i \phi_i \, dx = 0 \tag{2.2.26}$$

where

$$\phi_i = k\, \tilde{\omega}_{i;i_2 s_2;\dots;i_k s_k} \partial_x^{s_2} b_2^{i_2} \dots \partial_x^{s_k} b_k^{i_k}.$$

From (2.2.26) it follows that $\phi_i = 0$ for all i. Since $b_2^i, \dots, b_k^i$ are arbitrary differential polynomials, this implies $\tilde{\omega}_{i;i_2 s_2;\dots;i_k s_k} = 0$. That means that the form $\omega \wedge dx$ is equivalent to zero, modulo $d(\mathcal{A}_{k,0})$. The lemma is proved. □

Corollary 2.2.7 *A form $\omega \in \mathcal{A}_{k,1}$ belongs to $d(\mathcal{A}_{k,0})$ iff*

$$i_{a_1} \dots i_{a_k} \omega \in d(\mathcal{A})$$

for arbitrary evolutionary vector fields $a_1, \dots, a_k$.

Example 2.2.8 *For the one-form $\omega = \delta \int f\, dx$ the contraction $i_a\omega$ reads*

$$i_a \omega = \int a^i \frac{\delta \bar{f}}{\delta u^i(x)} dx \in \Lambda_0.$$

This is the time derivative of the functional $\bar{f} = \int f\, dx$ w.r.t. the evolutionary system (2.2.23).

Example 2.2.9 *For a one-form $\omega = \omega_i \delta u^i \wedge dx \in \Lambda_1$ the condition of closedness*

$$\delta\omega = 0 \in \Lambda_2$$

reads

$$\frac{\partial \omega_i}{\partial u^{j,s}} = \sum_{t\geq s} (-1)^t \left(\begin{array}{c} t \\ s \end{array} \right) \partial_x^{t-s} \frac{\partial \omega_j}{\partial u^{i,t}} \tag{2.2.27}$$

for any i, $j = 1, \ldots, n$, $s = 0, 1, \ldots$. This is the classical Volterra's criterion [143] for the system of ODEs

$$\omega_1(x; u; u_x, u_{xx}, \ldots) = 0, \ldots, \omega_n(x; u; u_x, u_{xx}, \ldots) = 0$$

to be locally representable in the Euler - Lagrange form

$$\omega_i = \frac{\delta \bar{f}}{\delta u^i(x)}, \quad i = 1, \ldots, n$$

(use exactness of the variational bicomplex).

Example 2.2.10 *For a 2-form*

$$\omega = \frac{1}{2} \omega_{is;jt} \delta u^{i,s} \wedge \delta u^{j,t},$$

the contraction with two evolutionary vector fields a and b can be represented in the form

$$i_a i_b \omega = -2 \int a^i \tilde{\omega}_{i;js} \partial_x^s b^j \, dx \tag{2.2.28}$$

where

$$\tilde{\omega}_{i;js} = \frac{1}{2} \sum_{r=0}^{s} \sum_{t\geq s-r} (-1)^t \left(\begin{array}{c} t \\ s-r \end{array} \right) \partial_x^{t-s+r} \omega_{is;jr}$$

are the reduced components (2.2.14). The tilde will be omitted in the subsequent formulae. According to this we will often represent 2-forms in the reduced form

$$dx \wedge \omega = \omega_{i;js} dx \wedge \delta u^i \wedge \delta u^{j,s}. \tag{2.2.29}$$

The reduced coefficients must satisfy the antisymmetry conditions (2.2.16). They are spelled out as follows

$$\omega_{i;js} = \sum_{t\geq s} (-1)^{t+1} \left(\begin{array}{c} t \\ s \end{array} \right) \partial_x^{t-s} \omega_{j;it} \tag{2.2.30}$$

(integrate by parts in (2.2.28) and use arbitraryness of a and b).

Example 2.2.11 *A 2-form $dx \wedge \omega = \delta dx \wedge \phi$ for*

$$\phi = \phi_i \delta u^i$$

has the reduced representative (2.2.29) with

$$\omega_{i;js} = \frac{1}{2}\left(\frac{\partial \phi_i}{\partial u^{j,s}} + \sum_{t \geq s} (-1)^{t+1} \binom{t}{s} \partial_x^{t-s} \frac{\partial \phi_j}{\partial u^{i,t}}\right). \tag{2.2.31}$$

Example 2.2.12 *A 2-form (2.2.29) is closed, $\delta\omega = 0$,* iff

$$\left(\sum_{m=s}^{t+s}\sum_{r=0}^{m-s} + \sum_{m \geq t+s+1}\sum_{r=0}^{t}\right)(-1)^m \left(\begin{array}{c} m \\ r\, s \end{array}\right) \partial_x^{m-r-s} \frac{\partial \omega_{j;k,t-r}}{\partial u^{i,m}} + \frac{\partial \omega_{i;j,s}}{\partial u^{k,t}} - \frac{\partial \omega_{i;k,t}}{\partial u^{j,s}} = 0 \tag{2.2.32}$$

for any $i,\ j,\ k = 1, \ldots n,\ s = 0,\ 1,\ 2, \ldots$.

Proof By definition

$$\delta(\omega) = \sum \frac{\partial \omega_{i;js}}{\partial u^{k,l}} \delta u^i \wedge \delta u^{j,s} \wedge \delta u^{k,l} \wedge dx.$$

So $\delta\omega = 0$ means that for any three evolutionary vector fields

$$a = \sum (a^i)^{(s)} \frac{\partial}{\partial u^{i,s}}, \quad b = \sum (b^i)^{(s)} \frac{\partial}{\partial u^{i,s}}, \quad c = \sum (c^i)^{(s)} \frac{\partial}{\partial u^{i,s}}$$

the contraction $i_a i_b i_c \delta(dx \wedge \omega) \in d(\mathcal{A}_{0,0})$, i,e,

$$\begin{aligned} &\int \frac{\partial \omega_{i;j,s}}{\partial u^{k,l}} \left[a^i\, (b^k)^{(l)}\, (c^j)^{(s)} - a^i\, (b^j)^{(s)}\, (c^k)^{(l)} - (a^k)^{(l)}\, b^i\, (c^j)^{(s)} + (a^k)^{(l)}\, (b^j)^{(s)}\, c^i \right. \\ &\left. + (a^j)^{(s)}\, b^i\, (c^k)^{(l)} - (a^j)^{(s)}\, (b^k)^{(l)}\, c^i\right] dx = 0 \end{aligned} \tag{2.2.33}$$

Using integration by parts we get

$$\begin{aligned} &\int \frac{\partial \omega_{i;j,s}}{\partial u^{k,l}} \left[a^i\, (b^k)^{(l)}\, (c^j)^{(s)} - a^i\, (b^j)^{(s)}\, (c^k)^{(l)}\right] \\ &+ \sum (-1)^{m+1} \left(\begin{array}{c} m \\ l\, r \end{array}\right) \partial_x^{m-l-r} \left(\frac{\partial \omega_{i;j,s}}{\partial u^{k,m}}\right) \left(a^k\, (b^i)^{(l)}\, (c^j)^{(s+r)} - a^k\, (b^j)^{(s+l)}\, (c^i)^{(r)}\right) \\ &+ \sum (-1)^m \left(\begin{array}{c} m \\ s\, r \end{array}\right) \partial_x^{m-s-r} \left(\frac{\partial \omega_{i;j,m}}{\partial u^{k,l}}\right) \left(a^j\, (b^i)^{(s)}\, (c^k)^{(l+r)} - a^j\, (b^k)^{(l+s)}\, (c^i)^{(r)}\right) dx \\ &= 0 \end{aligned}$$

The above identity is equivalent to

$$
\begin{aligned}
&\frac{\partial \omega_{i;j,s}}{\partial u^{k,l}} - \frac{\partial \omega_{i;k,l}}{\partial u^{j,s}} \\
&+ \left(\sum_{m=l}^{l+s} \sum_{r=0}^{m-l} + \sum_{m\geq l+s+1} \sum_{r=0}^{s} \right) (-1)^{m+1} \binom{m}{l\,r} \partial_x^{m-l-r} \left(\frac{\partial \omega_{k;j,s-r}}{\partial u^{i,m}} \right) \\
&+ \left(\sum_{m=s}^{l+s} \sum_{r=0}^{m-s} + \sum_{m\geq l+s+1} \sum_{r=0}^{l} \right) (-1)^{m} \binom{m}{s\,r} \partial_x^{m-s-r} \left(\frac{\partial \omega_{j;k,l-r}}{\partial u^{i,m}} \right) \\
&+ \left(\sum_{m=l}^{l+s} \sum_{r=0}^{m-l} + \sum_{m\geq l+s+1} \sum_{r=0}^{s} \right) (-1)^{m} \binom{m}{l\,r} \partial_x^{m-l-r} \left(\frac{\partial \omega_{k;i,m}}{\partial u^{j,s-r}} \right) \\
&+ \left(\sum_{m=s}^{l+s} \sum_{r=0}^{m-s} + \sum_{m\geq l+s+1} \sum_{r=0}^{l} \right) (-1)^{m+1} \binom{m}{r\,s} \partial_x^{m-s-r} \left(\frac{\partial \omega_{j;i,m}}{\partial u^{k,l-r}} \right) = 0
\end{aligned}
$$

Now by using the antisymmetry condition (2.2.30) we see that the third term in the above sum is equal to the forth term, and by using the identity (2.2.5) and the antisymmetry condition (2.2.30) again we see that the last two terms equal to the second and first term respectively. Thus we arrive at the proof of (2.2.32). □

Remark 2.2.13 *The equations (2.2.32) were derived by Dorfman in the theory of the so-called local symplectic structures [30], see also the book [31].*

Corollary 2.2.14 *Any solution to (2.2.32) satisfying (2.2.30) can be locally represented in the form (2.2.31).*

This follows from exactness of the variational bicomplex.

We will briefly outline necessary points of the global picture of functionals, differential forms and vector fields on the formal loop space $\mathcal{L}(M)$ for a general smooth manifold M (i.e., not only for a ball). The corresponding objects must be defined for any chart $U \subset M$ as it was explained above. On the intersections $U \cap V$ they must satisfy certain consistency conditions. For example, functionals in the charts U, V with the coordinates $u^1, \ldots, u^n$ and $v^1, \ldots, v^n$ are defined by densities $f_U(x; u; u_x, \ldots)$ and $f_V(x; v; v_x, \ldots)$ s.t.

$$f_V(x; v(u); \frac{\partial v}{\partial u} u_x, \ldots)dx = f_U(x; u; u_x, \ldots)dx \ (\mathrm{mod}\ \mathrm{Im}\, d).$$

Such objects comprise the space $\Lambda_0(M)$. As above, we obtain the ring of functions on the formal loop space taking the tensor algebra of $\Lambda_0(M)$. One-forms in the charts U,

V are described by their reduced components ω_i^U and ω_a^V s.t., on $U \cap V$ transform as components of a covector

$$\omega_a^V(x; v(u); \frac{\partial v}{\partial u} u_x, \ldots) = \omega_i^U(x; u; u_x, \ldots) \frac{\partial u^i}{\partial v^a}$$

etc. The components of evolutionary vector fields transform like vectors

$$a_V^k(x; v(u); \frac{\partial v}{\partial u} u_x, \ldots) = \frac{\partial v^k}{\partial u^i} a_U^i(x; u; u_x, \ldots). \tag{2.2.34}$$

The contraction $i_a dx \wedge \omega$ of a 1-form with an evolutionary vector field is well-defined as an element of $\Lambda_0(M)$.

The global theory of the variational bicomplex was developed in [138], [142].

2.3 Local multivectors and local Poisson brackets

We first define more general, i.e. non-local k-vectors as elements of $(\Lambda^1)^{\wedge k}$. They will be written as infinite sums of expressions of the form

$$\begin{aligned} \alpha = \frac{1}{k!}\, & \alpha^{i_1 s_1; \ldots i_k s_k}(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k); u_x(x_1), \ldots, u_x(x_k), \ldots) \\ & \times \frac{\partial}{\partial u^{i_1, s_1}(x_1)} \wedge \ldots \wedge \frac{\partial}{\partial u^{i_k, s_k}(x_k)} \end{aligned} \tag{2.3.1}$$

(in this subsection we will consider only multivectors not containing $\partial / \partial x$). The coefficients must satisfy the antisymmetry condition w.r.t. simultaneous permutations

$$i_p, s_p, x_p \leftrightarrow i_q, s_q, x_q.$$

The exterior algebra structure on multivectors is introduced in a usual way: the product of a k-vector α by a l-vector β is a $(k+l)$-vector

$$\begin{aligned} & (\alpha \wedge \beta)^{i_1 s_1; \ldots; i_k s_k; i_{k+1} s_{k+1}; \ldots; i_{k+l} s_{k+l}}(x_1, \ldots, x_{k+l}; u(x_1), \ldots, u(x_{k+l}); \ldots) \\ & = \frac{1}{k! l!} \sum_{\sigma \in S_{k+l}} (-1)^{\mathrm{sgn} \sigma}\, \alpha^{i_{\sigma(1)} s_{\sigma(1)}; \ldots; i_{\sigma(k)} s_{\sigma(k)}}(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; u(x_{\sigma(1)}), \ldots, u(x_{\sigma(k)}); \ldots) \\ & \times \beta^{i_{\sigma(k+1)} s_{\sigma(k+1)}; \ldots; i_{\sigma(k+l)} s_{\sigma(k+l)}}(x_{\sigma(k+1)}, \ldots, x_{\sigma(k+l)}; u(x_{\sigma(k+1)}), \ldots, u(x_{\sigma(k+l)}); \ldots) \end{aligned} \tag{2.3.2}$$

Example 2.3.1 *Lie derivative of a k-vector α (2.3.1) along a vector field (2.2.20) reads*

$$\begin{aligned} & Lie_\xi \alpha^{i_1 s_1; \ldots; i_k s_k} = \\ & \sum_{p=1}^k \left[\xi^0(x_p; u(x_p); \ldots) \frac{\partial}{\partial x_p} \alpha^{i_1 s_1; \ldots; i_k s_k} + \xi^{j_p, t_p}(x_p; \ldots) \frac{\partial}{\partial u^{j_p, t_p}(x_p)} \alpha^{i_1 s_1; \ldots; i_k s_k} \right] \\ & - \sum_{p=1}^k \frac{\partial \xi^{i_p, s_p}(x_p; \ldots)}{\partial u^{j_p, t_p}(x_p)} \alpha^{i_1 s_1; \ldots; i_{p-1} s_{p-1}; j_p t_p; \ldots; i_k s_k}. \end{aligned} \tag{2.3.3}$$

here we assume that ξ^0 *does not depend on* $u^{j,t}$.

Definition. A k-vector α is called *translation invariant* if

$$Lie_{\partial_x}\alpha = 0$$

and

$$\left(\frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_k}\right)\alpha = 0.$$

Lemma 2.3.2 *Every translation invariant* k*-vector* α *has coefficients of the form*

$$\begin{aligned}&\alpha^{i_1 s_1;\ldots;i_k s_k}(x_1,\ldots,x_k;u(x_1),\ldots,u(x_k);\ldots)\\ &\quad = \partial_{x_1}^{s_1}\ldots\partial_{x_k}^{s_k} A^{i_1\ldots i_k}(x_1,\ldots,x_k;u(x_1),\ldots u(x_k);\ldots)\end{aligned} \tag{2.3.4}$$

where the differential polynomials $A^{i_1\ldots i_k}(x_1,\ldots,x_k;u(x_1),\ldots,u(x_k);\ldots)$ *are antisymmetric w.r.t. simultaneous permutations*

$$i_p, x_p \leftrightarrow i_q, x_q$$

and also they satisfy

$$A^{i_1\ldots i_k}(x_1+t,\ldots,x_k+t;u(x_1),\ldots,u(x_k);\ldots) = A^{i_1\ldots i_k}(x_1,\ldots,x_k;u(x_1),\ldots,u(x_k);\ldots)$$

for any t.

The functions $A^{i_1\ldots i_k}(x_1,\ldots;x_k;u(x_1),\ldots,u(x_k);\ldots)$ will be called *components* of the translation invariant k-vector α.

Translation invariant multivectors form a graded Lie subalgebra of the full graded Lie algebra of multivectors closed w.r.t. Schouten - Nijenhuis bracket.

Example 2.3.3 *The Lie derivative of a bivector* α *with the components* $A^{ij}(x-y;u(x),u(y);\ldots)$ *along a translation invariant vector field* a *with the components* $a^i(u;u_x,\ldots)$ *has the components*

$$\begin{aligned}Lie_a\alpha^{ij} &= \partial_x^t a^k(u(x);\ldots)\frac{\partial A^{ij}}{\partial u^{k,t}(x)} + \partial_y^t a^k(u(y);\ldots)\frac{\partial A^{ij}}{\partial u^{k,t}(y)}\\ &-\frac{\partial a^i(u(x);\ldots)}{\partial u^{k,t}(x)}\partial_x^t A^{kj} - \frac{\partial a^j(u(y);\ldots)}{\partial u^{k,t}(y)}\partial_y^t A^{ik}.\end{aligned} \tag{2.3.5}$$

Example 2.3.4 *Let α, β be two translation invariant bivectors with the components $A^{ij}(x-y;u(x),u(y);u_x(x),u_x(y),\ldots)$ and $B^{ij}(x-y;u(x),u(y);u_x(x),u_x(y);\ldots)$ that we redenote resp. $A^{ij}_{x,y}$ and $B^{ij}_{x,y}$ for brevity. The Schouten - Nijenhuis bracket $[\alpha,\beta]$ is a translation invariant trivector with the components*

$$[\alpha,\beta]^{ijk}_{x,y,z} = \frac{\partial A^{ij}_{x,y}}{\partial u^{l,s}(x)}\partial^s_x B^{lk}_{x,z} + \frac{\partial B^{ij}_{x,y}}{\partial u^{l,s}(x)}\partial^s_x A^{lk}_{x,z} + \frac{\partial A^{ij}_{x,y}}{\partial u^{l,s}(y)}\partial^s_y B^{lk}_{y,z} + \frac{\partial B^{ij}_{x,y}}{\partial u^{l,s}(y)}\partial^s_y A^{lk}_{y,z}$$
$$+\frac{\partial A^{ki}_{z,x}}{\partial u^{l,s}(z)}\partial^s_z B^{lj}_{z,y} + \frac{\partial B^{ki}_{z,x}}{\partial u^{l,s}(z)}\partial^s_z A^{lj}_{z,y} + \frac{\partial A^{ki}_{z,x}}{\partial u^{l,s}(x)}\partial^s_x B^{lj}_{x,y} + \frac{\partial B^{ki}_{z,x}}{\partial u^{l,s}(x)}\partial^s_x A^{lj}_{x,y}$$
$$+\frac{\partial A^{jk}_{y,z}}{\partial u^{l,s}(y)}\partial^s_y B^{li}_{y,x} + \frac{\partial B^{jk}_{y,z}}{\partial u^{l,s}(y)}\partial^s_y A^{li}_{y,x} + \frac{\partial A^{jk}_{y,z}}{\partial u^{l,s}(z)}\partial^s_z B^{li}_{z,x} + \frac{\partial B^{jk}_{y,z}}{\partial u^{l,s}(z)}\partial^s_z A^{li}_{z,x}. \qquad (2.3.6)$$

For a translation invariant k-vector α and k 1-forms $\omega^1, \ldots, \omega^k$,

$$\omega^j = \omega^j_{is}\delta u^{i,s}\wedge dx \in \mathcal{A}_{1,1}, \quad j=1,\ldots,k$$

the contraction

$$< \alpha, \omega^1\wedge\ldots\wedge\omega^k >$$
$$:= \frac{1}{k!}\int \sum_{\sigma\in S_k}(-1)^{\operatorname{sgn}\sigma}\omega^{\sigma(1)}_{i_1 s_1}(x_1;u(x_1);\ldots)\ldots\omega^{\sigma(k)}_{i_k s_k}(x_k;u(x_k);\ldots)$$
$$\alpha^{i_1 s_1;\ldots;i_k s_k}(x_1,\ldots,x_k;u(x_1),\ldots,u(x_k);\ldots)dx_1\ldots dx_k \in \Lambda_0^{\hat{\otimes}k} \qquad (2.3.7)$$

is well defined on $\Lambda_1^{\otimes k}$.

Example 2.3.5 *The value of a translation invariant k-vector α with the components $A^{i_1\ldots i_k}$ on the 1-forms $\delta\bar{f}^1$, ..., $\delta\bar{f}^k$ equals*

$$< \alpha, \delta\bar{f}^1\wedge\ldots\wedge\delta\bar{f}^k >$$
$$= \int \frac{\delta\bar{f}^1}{\delta u^{i_1}(x_1)}\cdots\frac{\delta\bar{f}^k}{\delta u^{i_k}(x_k)}A^{i_1\ldots i_k}(x_1,\ldots,x_k;u(x_1),\ldots,u(x_k);\ldots)dx_1\ldots dx_k$$
$$\in \Lambda_0^{\hat{\otimes}k}. \qquad (2.3.8)$$

The transformation law of components of translation invariant multivectors w.r.t. changes of coordinates on the intersection of two coordinate charts $(U,u^1,\ldots,u^n)$ and $(V,v^1,\ldots,v^n)$ is analogous to the transformation law of components of multivectors on a finite-dimensional manifolds:

$$A_V^{a_1\ldots a_k}(x_1,\ldots x_k;v(u(x_1)),\ldots,v(u(x_k));\frac{\partial v}{\partial u}u_x(x_1),\ldots,\frac{\partial v}{\partial u}u_x(x_k),\ldots)$$
$$= \frac{\partial v^{a_1}}{\partial u^{i_1}}(x_1)\ldots\frac{\partial v^{a_k}}{\partial u^{i_k}}(x_k)A_U^{i_1\ldots i_k}(x_1,\ldots,x_k;u(x_1),\ldots,u(x_k);u_x(x_1),\ldots,u_x(x_k),\ldots).$$
$$(2.3.9)$$

We now proceed to the main definition of *local multivectors*. They are translation invariant multivectors α such that their dependence on $x_1, \ldots, x_k$ is given by a finite order distribution with the support on the diagonal $x_1 = x_2 = \ldots = x_k$

$$A^{i_1\ldots i_k} = \sum_{p_2,p_3,\ldots,p_k\geq 0} B^{i_1\ldots i_k}_{p_2,\ldots,p_k}(u(x_1); u_x(x_1), \ldots)\delta^{(p_2)}(x_1-x_2)\delta^{(p_3)}(x_1-x_3)\ldots\delta^{(p_k)}(x_1-x_k). \tag{2.3.10}$$

The coefficients $B^{i_1\ldots i_k}p_2, \ldots, p_k(u(x_1); u_x(x_1), \ldots)$ are differential polynomials in $\mathcal{A}$ not depending explicitly on x. All the sums at the moment are assumed to be finite. In the next section we will relax this condition. Delta functions and their derivatives and products are defined by the formulae

$$\int f(y)\delta(x-y)dy = f(x), \quad \int f(y)\delta^{(p)}(x-y)dy = f^{(p)}(x) \tag{2.3.11}$$

$$\int f(x_1, \ldots, x_k)\delta^{(p_2)}(x_1-x_2)\delta^{(p_3)}(x_1-x_3)\ldots\delta^{(p_k)}(x_1-x_k)dx_2\ldots dx_k$$
$$= \partial^{p_2}_{x_2}\ldots\partial^{p_k}_{x_k} f(x_1, \ldots, x_k)|_{x_1=x_2=\ldots=x_k}.$$

Lemma 2.3.6 *The value (2.3.7) of a local k-vector α on k 1-forms $\omega^1, \ldots, \omega^k$ is given by*

$$< \alpha, \omega^1 \wedge \ldots \wedge \omega^k >$$
$$= \int B^{i_1\ldots i_k}_{p_2\ldots p_k}(u; u_x, u_{xx}, \ldots)\omega^1_{i_1}(x; u; u_x, \ldots)\partial^{p_2}_x\omega^2_{i_2}(x; u; u_x, \ldots)$$
$$\ldots\partial^{p_k}_x\omega^k_{i_k}(x; u; u_x, \ldots)\, dx \in \Lambda_0. \tag{2.3.12}$$

It gives a well-defined polylinear map

$$\alpha : \Lambda_1^{\otimes k} \to \Lambda_0.$$

In calculations with local multivectors various simple identities for delta-functions will be useful. All of them are simple consequences of the definition (2.3.11). First,

$$f(y)\delta^{(p)}(x-y) = \sum_{q=0}^{p}\binom{p}{q} f^{(q)}(x)\delta^{(p-q)}(x-y). \tag{2.3.13}$$

Next,

$$\delta(x_1-x_2)\ldots\delta(x_1-x_k) = \delta(x_2-x_1)\delta(x_2-x_3)\ldots\delta(x_2-x_k) = \ldots$$
$$= \delta(x_k-x_1)\ldots\delta(x_k-x_{k-1}). \tag{2.3.14}$$

Differentiating (2.3.14) w.r.t. $x_1, \ldots, x_k$ we will obtain relations between products of derivatives of delta-functions.

We leave as a simple exercise for the reader to prove that the space of local multivectors that we denote

$$\Lambda^*_{loc} = \oplus \Lambda^k_{loc}$$

is closed w.r.t. the Schouten - Nijenhuis bracket. Warning: this space is not closed w.r.t. the exterior product! Because of this we were to introduce a wider algebra of multivectors to introduce the definition of the Schouten - Nijenhuis bracket according to the rules one uses in the finite dimensional case.

Example 2.3.7 *The component of a local bivector ϖ has the form*

$$\varpi^{ij} = \sum_{s \geq 0} A_s^{ij}(u(x); u_x(x), \dots)\delta^{(s)}(x-y). \tag{2.3.15}$$

The value of the bivector on two 1-forms $\phi = \phi_i \delta u^i \wedge dx$ and $\psi = \psi_i \delta u^i \wedge dx$ equals

$$\int \phi_i A_s^{ij} \partial_x^s \psi_j dx. \tag{2.3.16}$$

The conditions of antisymmetry of the bivector reads

$$A_s^{ji} = \sum_{t \geq s} (-1)^{t+1} \binom{t}{s} \partial_x^{t-s} A_t^{ij}. \tag{2.3.17}$$

Proof Let us explain how to prove the antisymmetry condition (2.3.17). We must have

$$\sum_s A_s^{ji}(u(y); \dots)\delta^{(s)}(y-x) = -\sum A_s^{ij}(u(x); \dots)\delta^{(s)}(x-y).$$

Using $\delta^{(s)}(y-x) = (-1)^s \delta^{(s)}(x-y)$ and (2.3.13) we obtain (2.3.17). □

Remark 2.3.8 *One can represent the bivector as*

$$A^{ij}(u(x); u_x(x), \dots; \frac{d}{dx})\delta(x-y). \tag{2.3.18}$$

Here the differential operators A^{ij} are

$$A^{ij}(x; u(x); u_x(x), \dots; \frac{d}{dx}) = \sum_s A_s^{ij} \frac{d^s}{dx^s}.$$

For local multivectors of higher rank the language of differential operators was used by Olver [123].

Example 2.3.9 *The Schouten - Nijenhuis bracket of the bivector*

$$\varpi = h^{ij}\delta(x-y),$$

where h^{ij} *is a constant antisymmetric matrix, with* α *of the form (2.3.15) reads*

$$[\varpi,\alpha]^{ijk}_{x,y,z} = \left[\frac{\partial A^{ij}_t}{\partial u^{l,s}}h^{lk} + \sum(-1)^{q+r+s}\begin{pmatrix} q+r+s \\ q \quad r\end{pmatrix}\left(\frac{\partial A^{ki}_{q+r+s}}{\partial u^{l,t-q}}\right)^{(r)} h^{lj}\right.$$
$$\left. + \sum(-1)^{q+r+t}\begin{pmatrix} q+r+t \\ q \quad r\end{pmatrix}\left(\frac{\partial A^{jk}_{s-q}}{\partial u^{l,q+r+t}}\right)^{(r)} h^{li}\right]\delta^{(v)}(x-y)\delta^{(s)}(x-z). \quad (2.3.19)$$

Proof Substituting into (2.3.6) we obtain

$$[\varpi,\alpha]^{ijk}_{x,y,z} = \frac{\partial A^{ij}_t(x)}{\partial u^{l,s}(x)}h^{lk}\delta^{(v)}(x-y)\delta^{(s)}(x-z)$$
$$+\frac{\partial A^{ki}_t(z)}{\partial u^{l,s}(z)}h^{lj}\delta^{(v)}(z-x)\delta^{(s)}(z-y) + \frac{\partial A^{jk}_t(y)}{\partial u^{l,s}(y)}h^{li}\delta^{(v)}(y-z)\delta^{(s)}(y-x). \quad (2.3.20)$$

Here $A^{ij}_t(x)$, $A^{jk}_t(y)$, $A^{ki}_t(z)$ stand for $A^{ij}_t(u(x);\ldots)$, $A^{jk}_t(u(y);\ldots)$, $A^{ki}_t(u(z);\ldots)$ resp. Use the identities (2.3.14)

$$\delta^{(v)}(z-x)\delta^{(s)}(z-y) = (-\partial_x)^t(-\partial_y)^s[\delta(z-x)\delta(z-y)]$$

$$= (-\partial_x)^t(-\partial_y)^s[\delta(x-y)\delta(x-z)] = (-1)^t\sum_{q=0}^{t}\begin{pmatrix} t \\ q\end{pmatrix}\delta^{(s+q)}(x-y)\delta^{(t-q)}(x-z),$$

$$\delta^{(v)}(y-z)\delta^{(s)}(y-x) = (-\partial_x)^s(-\partial_z)^t[\delta(y-z)\delta(y-x)]$$

$$= (-\partial_x)^s(-\partial_z)^t[\delta(x-y)\delta(x-z)] = (-1)^s\sum_{q=0}^{s}\begin{pmatrix} s \\ q\end{pmatrix}\delta^{(s-q)}(x-y)\delta^{(t+q)}(x-z)$$

and also (2.3.13) to arrive at (2.3.19). □

Definition. A *local Poisson structure* on the formal loop space is a local bivector $\varpi \in \Lambda^2_{loc}$ (2.3.15) satisfying $[\varpi,\varpi]=0$.

Adopting the notations common in the physical literature we will represent the Poisson structure in the form

$$\{u^i(x), u^j(y)\} = \sum_s A^{ij}_s(u(x); u_x(x), u_{xx}(x),\ldots)\delta^{(s)}(x-y). \quad (2.3.21)$$

The Poisson bracket of two local functionals $\bar f = \int f(x;u;u_x,\ldots)dx$ and $\bar g = \int g(x;u;u_x,\ldots)dx$ can be written in the following equivalent forms (see above the general theory of multivectors)

$$\{\bar{f},\bar{g}\} =< \varpi, \delta\bar{f}\wedge\delta\bar{g} >= \int\int dxdy \frac{\delta\bar{f}}{\delta u^i(x)}\{u^i(x),u^j(y)\}\frac{\delta\bar{g}}{\delta u^j(y)}$$
$$= \sum_s \int dx \frac{\delta\bar{f}}{\delta u^i(x)} A_s^{ij}(u;u_x,u_{xx},\dots)\left(\frac{\delta\bar{g}}{\delta u^j(x)}\right)^{(s)} \in \Lambda_0. \tag{2.3.22}$$

Therefore it is again a local functional.

The crucial property of local Poisson brackets is that, the Hamiltonian systems

$$u_t^i = -i_{\delta\bar{H}}\varpi = \{u^i(x),\bar{H}\} = A_s^{ij}(u;u_x,u_{xx},\dots)\partial_x^s \frac{\delta\bar{H}}{\delta u^j(x)} \tag{2.3.23}$$

with local translation invariant Hamiltonians

$$\bar{H} = \int H(u;u_x,\dots)dx$$

are translation invariant evolutionary PDEs (2.2.23).

Living in the infinite dimensional loop space we will not impose conditions on the rank of the Poisson bracket. However, in the main examples the corank of the bivector will be finite.

Example 2.3.10 *For a constant antisymmetric matrix h^{ij} the bivector*

$$\{u^i(x),u^j(y)\} = h^{ij}\delta(x-y) \tag{2.3.24}$$

is a local Poisson structure. It is called ultralocal Poisson bracket. This is a symplectic structure on the loop space iff $\det h^{ij} \neq 0$. *The Hamiltonian evolutionary PDEs read*

$$u_t^i = h^{ij}\frac{\delta\bar{H}}{\delta u^j(x)}.$$

Reducing the nodegenerate matrix h^{ij} to the canonical form we arrive at the Hamiltonian formulation of 1+1 dimensional variational problems

$$q_t^i = \frac{\delta\bar{H}}{\delta p_i(x)}, \quad {p^i}_t = -\frac{\delta\bar{H}}{\delta q^i(x)}.$$

Example 2.3.11 *For a constant symmetric matrix η^{ij} the bivector*

$$\{u^i(x),u^j(y)\} = \eta^{ij}\delta'(x-y) \tag{2.3.25}$$

is a local Poisson structure. Under the assumption $\det\eta^{ij}\neq 0$ *this Poisson bracket has n independent Casimirs*

$$\bar{u}^1 = \int u^1\,dx, \dots, \bar{u}^n = \int u^n\,dx. \tag{2.3.26}$$

The annihilator of (2.3.25) is generated by the above Casimirs.

The Hamiltonian evolutionary PDEs read

$$u^i_t = \eta^{ij} \partial_x \frac{\delta \bar{H}}{\delta u^j(x)}. \tag{2.3.27}$$

We finish this section with spelling out the transformation law of coefficients of local bivectors imposed by the general formula (2.3.9). If $A^{ij}_s(u; u_x, \dots)$ and $A^{ab}_t(v; v_x, \dots)$ are the coefficients of a local bivector in two coordinate charts $(U, u^1, \dots, u^n)$ and $(V, v^1, \dots, v^n)$ resp. then, on $U \cap V$ one has

$$A^{ab}_t(v; \frac{\partial v}{\partial u} u_x, \dots) = \sum_{s \geq t} \binom{s}{t} \frac{\partial v^a}{\partial u^i} \left(\frac{\partial v^b}{\partial u^j} \right)^{(s-t)} A^{ij}_s(u; u_x, \dots). \tag{2.3.28}$$

.

Example 2.3.12 *Applying the transformation*

$$u = \frac{1}{4} v^2 \tag{2.3.29}$$

to the bivector

$$\{u(x), u(y)\} = u(x)\delta'(x-y) + \frac{1}{2} u'(x)\delta(x-y) \tag{2.3.30}$$

we obtain a constant Poisson bracket of the form (2.3.25)

$$\{v(x), v(y)\} = \delta'(x-y). \tag{2.3.31}$$

Hence (2.3.30) is itself a Poisson structure. This is the Lie - Poisson bracket on the space dual to the Lie algebra of vector fields on the circle [49].

2.4 Problem of classification of local Poisson brackets

The last example of the previous section is the simplest issue of the problem of reduction of local Poisson brackets to the simplest (possibly, to the constant one) form. In this example the reduction to the constant form was achieved by a change of coordinates in the target space M (M was one-dimensional). We give now another well-known example: to transform the bivector (the *Magri bracket* for the KdV equation)

$$\{u(x), u(y)\} = u(x)\delta'(x-y) + \frac{1}{2} u'(x)\delta(x-y) - \delta'''(x-y) \tag{2.4.1}$$

to the constant form (2.3.31) one is to use the celebrated *Miura transformation*

$$u = \frac{1}{4} v^2 + v'. \tag{2.4.2}$$

Our strategy will be to classify local Poisson brackets on the loop space $\mathcal{L}(M)$ (the target space M will be a ball in this section) with respect to the action of the group of Miura-type transformations.

The problem of reduction of certain classes of Poisson brackets to a canonical form by coordinate transformations was first investigated in [47] for the Poisson brackets of hydrodynamic type and in [48] for the so-called differential geometric Poisson brackets (see also [49] and the references therein). Some results regarding reduction of the local Poisson brackets to the canonical form by using Miura - Bäcklund transformations were obtained in[2], [69], [124], [127] (in the latter non translation invariant Poisson brackets were studied).

We want to classify local Poisson brackets w.r.t. general Miura type transformations of the form

$$u^i \to \tilde{u}^i = F^i(u; u_x, u_{xx}, \ldots). \tag{2.4.3}$$

The problem is that these transformations do not form a group. The main trouble is with inverting such a transformation. E.g., to invert the Miura transformation one is to solve Riccati equation (2.4.2) w.r.t. v. To resolve this problem we will extend the class of Miura-type transformations. Simultaneously we will be to also extend the class of local functionals, vector fields, and Poisson brackets.

2.4.1 Extended formal loop space

Let us introduce gradation on the ring $\mathcal{A}$ of differential polynomials putting

$$\deg u^{i,k} = k, \quad k \geq 1, \quad \deg f(x; u) = 0. \tag{2.4.4}$$

We extend the gradation onto the spaces $\mathcal{A}_{k,l}$ of differential forms by

$$\deg dx = -1, \quad \deg \delta u^{i,s} = s, \;\; s \geq 0.$$

The differentials d and δ preserve the gradation. Introduce a formal indeterminate ϵ of the degree

$$\deg \epsilon = -1.$$

Let us define a subcomplex

$$\hat{\mathcal{A}}_{k,l} \subset \mathcal{A}_{k,l} \otimes \mathbb{C}[[\epsilon], \epsilon^{-1}], \quad \hat{\Lambda}_k \subset \Lambda_k \otimes \mathbb{C}[[\epsilon], \epsilon^{-1}]$$

collecting all the elements of the total degree $k-1$. In particular, the space of local functionals $\hat{\Lambda}_0$ consists of integrals of the form

$$\bar{f} = \int f(u; u_x, u_{xx}, \ldots; \epsilon) dx,$$

$$f(u; u_x, u_{xx}, \ldots) = \sum_{k=0}^{\infty} \epsilon^k f_k(u; u_x, \ldots, u^{(k)}), \quad f_k \in \mathcal{A}, \quad \deg f_k = k. \tag{2.4.5}$$

We will still call such a series differential polynomials when it will not cause confusions. Taking the tensor algebra of $\hat\Lambda_0$ we obtain the ring of functionals on the *extended* formal loop space that we will denote $\hat{\mathcal{L}}(M)$. The vertical differential δ of the bicomplex must be renormalized

$$\delta \mapsto \hat\delta = \frac{1}{\epsilon}\delta.$$

As it follows from Theorem 1.2.1 the bicomplex $(\hat{\mathcal{A}}_{k,l}, d, \hat\delta)$ is exact

The gradation on the vector and multivector fields is defined by

$$\deg\frac{\partial}{\partial x} = 1, \quad \deg\frac{\partial}{\partial u^{i,s}} = -s, \;\; s \geq 0.$$

Observe that ∂_x increases degrees by one:

$$\deg\partial_x f = \deg f + 1.$$

The space $\hat\Lambda^1$ of vector fields on $\hat{\mathcal{L}}(M)$ is obtained by collecting all the elements in $\Lambda^1 \otimes \mathbf{C}[[\epsilon], \epsilon^{-1}]$ of the total degree 1. In particular, the translation invariant evolutionary vector fields are

$$a^i = \sum_{k=0}^{\infty} \epsilon^{k-1} a^i_k(u; u_x, \dots, u^{(k)}), \quad a^i_k \in \mathcal{A}, \quad \deg a^i_k = k. \tag{2.4.6}$$

The corresponding evolutionary system of PDEs reads

$$\begin{aligned} &u^i_t = \epsilon^{-1}\, a^i_0(u) + a^i_1(u; u_x) + \epsilon\, a^i_2(u; u_x, u_{xx}) + O(\epsilon^2) \\ &a^i_1(u; u_x) = v^i_j(u) u^j_x, \\ &a^i_2(u; u_x, u_{xx}) = b^i_j(u) u^j_{xx} + \frac{1}{2} c^i_{jk}(u) u^j_x u^k_x \end{aligned} \tag{2.4.7}$$

etc.

Proceeding in a similar way we introduce the subspace

$$\hat\Lambda^k \subset \Lambda^k \otimes \mathbb{C}[[\epsilon], \epsilon^{-1}]$$

of k-vectors of the total degree k.

Lemma 2.4.1 *The Schouten - Nijenhuis bracket gives a well-defined map*

$$\epsilon\,[\ ,\ ]:\ \hat\Lambda^k \times \hat\Lambda^l \to \hat\Lambda^{k+l-1}.$$

There is an important subtlety with the grading of the local multivectors. Indeed, a local k-vector is a map

$$\Lambda_1^{\otimes k} \to \Lambda_0$$

not

$$\Lambda_1^{\otimes k} \to \Lambda_0^{\hat\otimes k}$$

(cf. the formulae (2.3.7), (2.3.10) and (2.3.12)). Such a map does not respect the grading. So we must assign a nonzero degree to delta-function

$$\deg\delta(x-y) = 1, \quad \deg\delta^{(s)}(x-y) = s+1 \tag{2.4.8}$$

Therefore a general local bivector in $\hat{\Lambda}_{loc}$ will be represented by an infinite sum

$$\{u^i(x), u^j(y)\} = \sum_{k=-1}^{\infty} \epsilon^k \{u^i(x), u^j(y)\}^{[k]} \tag{2.4.9}$$

$$\{u^i(x), u^j(y)\}^{[k]} = \sum_{s=0}^{k+1} A^{ij}_{k,s}(u; u_x, \dots, u^{(s)})\delta^{(k-s+1)}(x-y),$$

$$A^{ij}_{k,s} \in \mathcal{A}, \quad \deg A^{ij}_{k,s} = s, \quad s = 0, 1, \dots, k+1.$$

More explicitly, the first three terms in the expansion (2.4.9) read

$$\{u^i(x), u^j(y)\}^{[-1]} = h^{ij}(u(x))\delta(x-y) \tag{2.4.10}$$

$$\{u^i(x), u^j(y)\}^{[0]} = g^{ij}(u(x))\delta'(x-y) + \Gamma^{ij}_k(u(x))u^k_x\delta(x-y) \tag{2.4.11}$$

$$\{u^i(x), u^j(y)\}^{[1]} = a^{ij}(u(x))\delta''(x-y) + b^{ij}_k(u(x))u^k_x\delta'(x-y)$$

$$+[c^{ij}_k(u(x))u^k_{xx} + \frac{1}{2}d^{ij}_{kl}(u(x))u^k_x u^l_x]\delta(x-y) \tag{2.4.12}$$

where h^{ij}, $g^{ij}(u)$, $\Gamma^{ij}(u)$, $a^{ij}(u)$, $b^{ij}_k(u)$, $c^{ij}_k(u)$, $d^{ij}_{kl}(u)$ are some functions on the manifold M.

Remark 2.4.2 *Our rules of introducing the gradation can be memorized using the following simple trick. Do a rescaling of the independent variable x,*

$$x \mapsto \epsilon x. \tag{2.4.13}$$

The x-derivatives $u^{i,k} = d^k u^i/dx^k$ will change

$$u^{i,k} \mapsto \epsilon^k u^{i,k}. \tag{2.4.14}$$

We also have

$$dx \mapsto \epsilon^{-1} dx. \tag{2.4.15}$$

The delta-function, according to the definition (2.3.11) must be rescaled as

$$\delta(x) \mapsto \epsilon\, \delta(x). \tag{2.4.16}$$

In other words, "delta-function" is not a function but a density. Simultaneously with the rescaling we will also redefine the integrals

$$\int \,.\, dx_1 \dots dx_k \mapsto \epsilon^k \int \,.\, dx_1 \dots dx_k.$$

After such a rescaling we expand all the formulae of the previous two sections in a power series in ϵ to arrive at our grading conventions.

Example 2.4.3 *Rescaling (2.4.13) the KdV equation $u_t = uu_x + u_{xxx}$ one obtains*

$$u_t = \epsilon(uu_x + \epsilon^2 u_{xxx}).$$

One usually introduces slow time variable $t \mapsto \epsilon t$ to recast the last equation into the form

$$u_t = uu_x + \epsilon^2 u_{xxx}. \tag{2.4.17}$$

This is the small dispersion expansion of the KdV equation. The smooth solutions of (2.4.17) describe solutions to KdV slow varying in space and time.

Remark 2.4.4 *Besides the rescaling procedure, one can arrive at the above series (2.4.5), (2.4.6), (2.4.9) considering the continous limits of differential-difference systems. E.g., for the well-known example of Toda lattice*

$$\dot{u}_n = v_n - v_{n-1}, \quad \dot{v}_n = e^{u_{n+1}} - e^{u_n}, \quad n \in \mathbf{Z} \tag{2.4.18}$$

the continuous limit $u_n = u(\epsilon n) = u(x)$, $v_n = v(\epsilon n) = v(x)$ gives an evolutionary system of the form (2.4.7) with an infinite series in the r.h.s.

$$\begin{aligned} u_t &= (v(x) - v(x-\epsilon)) = \epsilon[v' - \frac{1}{2}\epsilon v'' + O(\epsilon^2)], \\ v_t &= e^{u(x+\epsilon)} - e^{u(x)} = \epsilon[(e^u)' + \frac{1}{2}\epsilon (e^u)'' + O(\epsilon^2)]. \end{aligned}$$

Replacing in the Hamiltonian structure

$$\{u_m, u_n\} = \{v_m, v_n\} = 0, \quad \{u_m, v_n\} = \delta_{mn} - \delta_{m,n+1}$$

the Kronecker symbols δ_{mn} by $\epsilon^{-1}\delta(x-y)$, $\delta_{m,n+1}$ by $\epsilon^{-1}\delta(x-y-\epsilon)$, we obtain a Poisson bracket of the (2.4.9) form

$$\begin{gathered} \{u(x), u(y)\} = \{v(x), v(y)\} = 0, \\ \{u(x), v(y)\} = \frac{1}{\epsilon}[\delta(x-y) - \delta(x-y-\epsilon)] = \delta'(x-y) - \frac{\epsilon}{2}\delta''(x-y) + O(\epsilon^2). \end{gathered}$$

2.4.2 Miura group

The last definition will be that to extend the class of Miura-type transformations (2.4.3). Let us consider the transformations

$$\begin{aligned} & u^i \mapsto \tilde{u}^i = \sum_{k=0}^{\infty} \epsilon^k F_k^i(u; u_x, \ldots, u^{(k)}), \quad i = 1, \ldots, n \\ & F_k^i \in \mathcal{A}, \quad \deg F_k^i = k, \\ & \det\left(\frac{\partial F_0^i(u)}{\partial u^j}\right) \neq 0. \end{aligned} \tag{2.4.19}$$

Lemma 2.4.5 *The transformations of the form (2.4.19) form a group. The Lie algebra of the group is isomorphic to the subalgebra $\hat\Lambda^1_{ev}$ of all translation invariant evolutionary vector fileds in $\hat\Lambda^1$ with the Lie bracket operation.*

Example 2.4.6 *Let us invert the clasical Miura transformation*

$$u = \frac{1}{4}v^2 + \epsilon v'$$

using successive approximations. Rewriting the equation in the form

$$v = 2\sqrt{u - \epsilon v'} = 2\sqrt{u} - \epsilon\frac{v'}{\sqrt{u}} + O(\epsilon^2) = 2\sqrt{u} - \epsilon\frac{u'}{u} + O(\epsilon^2)$$

we obtain first two terms of the solution $v = F(u; u', \dots; \epsilon)$.

Remark 2.4.7 *This way of solving the Riccati equation is essentially equivalent to the classical WKB method of solving the related linear second order ODE*

$$\epsilon^2 y'' = \frac{1}{4}u\, y, \quad v = 4\epsilon\frac{y'}{y}.$$

Substituting the above series solution to Riccati we obtain the WKB asymptotic solution to the second order ODE with the small parameter $\epsilon \to 0$

$$y = u^{-1/4}\exp\frac{1}{2\epsilon}\int\sqrt{u}dx\,(1 + O(\epsilon))\,.$$

Definition. The group $\mathcal{G}$ of all the transformations of the form (2.4.19) is called *Miura group.*

The Miura group $\mathcal{G}$ looks to be a natural candidate for the role of the group of "local diffeomorphisms" of the extended formal loop space $\hat{\mathcal{L}}(M)$ (recall that at the moment M is a ball). $\mathcal{G}$ contains the group of diffeomorphisms $Diff(M)$ of the manifold M as a subgroup. It coincides with the semidirect product of $Diff(M)$ and the pro-unipotent subgroup $\mathcal{G}_0$ of Miura-type transformations close to identity,

$$u^i \mapsto u^i + \epsilon\, A^i_j(u)u^j_x + \epsilon^2\left(B^i_j(u)u^j_{xx} + \frac{1}{2}C^i_{jk}(u)u^j_x u^k_x\right) + \dots \tag{2.4.20}$$

The produc in the group $\mathcal{G}_0$ reads

$$\begin{aligned}
&A^i_j = A_1{}^i_j + A_2{}^i_j\\
&B^i_j = B_1{}^i_j + B_2{}^i_j + A_2{}^i_k A_1{}^k_j\\
&C^i_{jk} = C_1{}^i_{jk} + C_2{}^i_{jk} + \frac{1}{2}\left[\partial_s A_2{}^i_j A_1{}^s_k + \partial_s A_2{}^i_k A_1{}^s_j + A_2{}^i_s\partial_k A_1{}^s_j + A_2{}^i_s\partial_j A_1{}^s_k\right]\\
&\dots
\end{aligned} \tag{2.4.21}$$

The following simple statements immediately follow from Lemma 2.4.5.

Lemma 2.4.8 *The class of local functionals (2.4.5), evolutionary PDEs (2.4.7), and local translation invariant multivectors (see the formula (2.4.9) for the bivectors) on the extended formal loop space $\hat{\mathcal{L}}(M)$ is invariant w.r.t. the action of the Miura group.*

Lemma 2.4.9 *An arbitrary vector field a of the form (2.4.6) with $a_0 \neq 0$ can be reduced, by a transformation of the Miura group, to a constan form*

$$a_0 = \text{const}, \quad a_i = 0 \text{ for } i > 0.$$

This is an infinite dimensional analogue of the theorem of "rectifying of a vector field".

Proof By using the theorem of "rectifying of a vector field" on a finite dimensional manifold, we can reduce the vector field $(a_0^1, \dots, a_0^n)$ to a constant one by performing a Miura transformation of the form (2.4.19) with $F_k = 0,\ k \geq 1$. We prove the lemma by induction. Let us assume the vector field a to be of the form

$$a^i = a_0^i + \varepsilon^k\, a_k^i(u, u_x, \dots, u^{(k)}) + \mathcal{O}(\varepsilon^{k+1})$$

with $\deg a_k = k$. Since $a_0 \neq 0$, we can find differential polynomials $F_k^i(u, \dots, u^{(k)})$ of degree k such that

$$a_0^l \frac{\partial F_k^i}{\partial u^l} + a_k^i = 0.$$

Then the Miura transformation

$$\bar{u}^i = u^i + \varepsilon^k\, F_k^i(u, \dots, u^{(k)})$$

reduces the vector field a to the form

$$a^i = a_0^i + \mathcal{O}(\varepsilon^{k+1}).$$

□

2.4.3 (p, q)-brackets on the extended formal loop space

Let us write explicitly down the transformation law of the coefficients of a local Poisson bracket w.r.t. transformations from the Miura group (cf. [123]). Let A^{kl} be the differential operator of the Poisson bracket given in (2.3.18). In the new "coordinates" $\tilde{u}^i$ of the form (2.4.3) the Poisson bracket will be given by the operator

$$\tilde{A}^{ij} = {L^*}_k^i A^{kl} L_l^j \tag{2.4.22}$$

where the matrix-valued operator L_k^i and the adjoint one ${L^*}_k^i$ are given by

$$L_k^i = \sum_s (-\partial_x)^s \circ \frac{\partial \tilde{u}^i}{\partial u^{k,s}}, \quad {L^*}_k^i = \sum_s \frac{\partial \tilde{u}^i}{\partial u^{k,s}} \partial_x^s.$$

Main Problem. To describe the orbits of the action of the Miura group $\mathcal{G}$ on $\hat{\Lambda}^2$.

To our opinion this problem is the natural setup of the problem of classification of Poisson structures of one-dimensional evolutionary PDEs with respect to Miura-type transformations (they are called also Darboux, or Bianchi, or Bäcklund transformations. Our transformations do not involve a change of the independent variable x since we consider only translation invariant PDEs).

Our conjecture is that, for a reasonable class of Poisson brackets to be defined below, the orbits are labelled by certain finite-dimensional geometrical structures on the underlined manifold M. Below we will illustrate this claim describing two orbits being, in a certain sense, generic. The full problem remains open.

We first explain how a local Poisson bracket from $\hat{\Lambda}^2_{loc}$ induces certain finite-dimensional geometrical structures on M.

Lemma 2.4.10 *The subgroup $Diff(M) \subset \mathcal{G}$ acts independently on every term $\{\ ,\ \}^{[k]}$ of the expansion (2.4.9), $k \geq -1$. In particular, the leading term $A^{ij}_{k,0}(u)$ is a $(2,0)$-tensor field on M, symmetric/antisymmetric for even/odd k.*

Proof This follows from the transformation law (2.3.28). The symmetry/antisymmetry of the coefficients follows from the general antisymmetry condition (2.3.17). The lemma is proved. □

Lemma 2.4.11 *The subspaces* span $\left(\{\ ,\}^{[-1]}, \{\ ,\ \}^{[0]}, \dots, \{\ ,\ \}^{[k]}\right)$ *for every k remain invariant w.r.t. to the action of the Miura group $\mathcal{G}$.*

This follows from the explicit formula (2.4.22).

Lemma 2.4.12 *The first non-zero term in the expansion (2.4.9) is itself a local Poisson bracket.*

This is obvious.

Corollary 2.4.13 *The coefficient $h^{ij}(u)$ in (2.4.10) is a Poisson structure on M. This Poisson structure is invariant w.r.t. the action of $\mathcal{G}$ on $\hat{\Lambda}^2_{loc}$.*

We obtain a map

$$\hat{\Lambda}^2_{loc}/\mathcal{G} \to \text{ Poisson structures on } M. \tag{2.4.23}$$

Let us assume that the Poisson structure $h^{ij}(u)$ on M has constant rank $p = 2p_1$. Denote $q := n - p$ the corank of $h^{ij}(u)$.

Definition. (2.4.9) is called (p,q)-*bracket* if the coefficient $g^{ij}(u)$ in (2.4.11) does not degenerate on $Ker\, h^{ij}(u) \subset T^*_u M$ on an open dense subset in $M \ni u$.

Example 2.4.14 *The ultralocal Poisson bracket (2.3.24) with a non-degenerate matrix* h^{ij} *is a* $(n, 0)$*-bracket.*

Example 2.4.15 *The Poisson bracket (2.3.25) with a non-degenerate matrix* η^{ij} *is a* $(0, n)$*-bracket.*

Example 2.4.16 *Let* c_k^{ij} *be the structure constants of a semisimple* n*-dimensional Lie algebra* $\mathfrak{g}$*. The Killing form* η^{ij} *on* $\mathfrak{g}$ *defines a central extension* $\hat{\mathfrak{g}}$ *of* $\mathfrak{g}$ *called Kac - Moody Lie algebra [86]. The Lie - Poisson bracket (2.1.8) on the dual space* $\hat{\mathfrak{g}}^*$

$$\frac{1}{\epsilon}\{u^i(x), u^j(y)\} = \eta^{ij}\delta'(x-y) + \frac{1}{\epsilon}c_k^{ij}u^k\delta(x-y) \tag{2.4.24}$$

can be considered as a Poisson bracket of the form (2.4.9) on the loop space $\hat{\mathcal{L}}(\mathfrak{g}^*)$*. Here* ϵ *is the central charge. This is a* (p, q)*-bracket with*

$$q = \mathrm{rk}\,\mathfrak{g}, \quad p = \dim \mathfrak{g} - \mathrm{rk}\,\mathfrak{g}.$$

Let a (p, q)-Poisson bracket on $\hat{\mathcal{L}}(M)$ of the form (2.4.9) - (2.4.11) be given. We will now construct a *flat metric* on the base of the symplectic foliation of M defined by the finite-dimensional Poisson bracket $h^{ij}(u)$. Let us assume that M is a small ball such that the symplectic foliation defines a fibration

$$M \to N, \quad \dim N = q. \tag{2.4.25}$$

Functions on N are Casimirs of the finite-dimensional Poisson bracket $h^{ij}(u)$. Define first a symmetric bilinear form $(\ ,\ )^*$ on T^*N putting

$$(df_1, df_2)^* := \frac{\partial f_1}{\partial u^i}\frac{\partial f_2}{\partial u^j}g^{ij}(u) \tag{2.4.26}$$

for any two Casimirs of $h^{ij}(u)$. By the assumption this bilinear form does not degenerate. Define the non-degenerate symmetric tensor $(\ ,\ )$ on TN by

$$(\ ,\ ) := [(\ ,\ )^*]^{-1}. \tag{2.4.27}$$

Theorem 2.4.17 *The metric (2.4.27) on* TN *is well-defined and flat.*

Proof The simplest way to prove that the metric (2.4.27) is constant along symplectic leaves and also prove vanishing of the curvature of the metric is the following one. Choose local coordinates $u = (w^a, v^\alpha)$ on M, $a = 1, \ldots, p = 2p_1$, $\alpha = 1, \ldots, q$, $p+q = n$, such that $w^1, \ldots, w^p$ are canonical coordinates on the symplectic leaves and $v^1, \ldots, v^q$

is a system of independent Casimirs of $h^{ij}(u)$. The v's can be considered as coordinates on N. In the local coordinates the Poisson bracket (2.4.9) reads

$$\begin{aligned}\{w^a(x), w^b(y)\} &= \frac{1}{\epsilon}h^{ab}\delta(x-y) + A^{ab}_{00}(u)\delta'(x-y) + A^{ab}_{01}(u,u_x)\delta(x-y) + O(\epsilon)\\ \{v^\alpha(x), w^a(y)\} &= A^{\alpha a}_{00}(u)\delta'(x-y) + A^{\alpha a}_{01}(u,u_x)\delta(x-y) + O(\epsilon)\\ \{v^\alpha(x), v^\beta(y)\} &= g^{\alpha\beta}(u)\delta'(x-y) + \left(\Gamma^{\alpha\beta}_a(u)w^a_x + \Gamma^{\alpha\beta}_\gamma(u)v^\gamma_x\right)\delta(x-y) + O(\epsilon)\end{aligned} \tag{2.4.28}$$

Here h^{ab} is a constant antisymmetric nondegenerate matrix, the $q\times q$ matrix $g^{\alpha\beta}(u)$ coincides with the Gram matrix of the bilinear form (2.4.26) in the basis $dv^1, \ldots, dv^q$. Let us consider the following foliation on the loop space $\hat{\mathcal{L}}(M)$

$$w^a(x) \equiv w^a_0, \quad a = 1, \ldots, p \tag{2.4.29}$$

for arbitrary given numbers $w^1_0, \ldots, w^p_0$. Due to nondegeneracy of h^{ab} this foliation is cosymplectic. The corresponding Dirac bracket $\{\ ,\ \}_D$ can be considered as a Poisson bracket on $\hat{\mathcal{L}}(N)$ since, by definition

$$\{v^\alpha(x), w^a(y)\}_D = \{w^a(x), w^b(y)\}_D = 0.$$

Let us show that the Dirac bracket is a $(0,q)$-bracket on $\hat{\mathcal{L}}(N)$ with the same leading term

$$\{v^\alpha(x), v^\beta(y)\}_D = \{v^\alpha(x), v^\beta(y)\} + O(\epsilon).$$

Introduce the differential operator

$$\Pi_{ab} = h_{ab} - \epsilon\, h_{a\,a'}\left(A^{a'b'}_{00}(u)\frac{d}{dx} + A^{a'b'}_{01}(u,u_x)\right) h_{b'\,b} + O(\epsilon^2)$$

inverse to the operator $\epsilon\, A^{ab}$. Then the Dirac bracket has the form

$$\{v^\alpha(x), v^\beta(y)\}_D = \{v^\alpha(x), v^\beta(y)\} - \epsilon\sum_{a,b=1}^{p} A^{\alpha a}\Pi_{ab}A^{b\beta}\delta(x-y). \tag{2.4.30}$$

We obtain a $(0,q)$ Poisson bracket on $\hat{\mathcal{L}}(N)$ eventually depending on the parameters $w^1_0, \ldots, w^p_0$. The leading term

$$\{v^\alpha(x), v^\beta(y)\}^{[0]}_D = g^{\alpha\beta}(v,w_0)\delta'(x-y) + \Gamma^{\alpha\beta}_\gamma(v,w_0)v^\gamma_x(x-y)$$

is itself a Poisson bracket (the so-called Poisson bracket of hydrodynamic type). According to the theory of such brackets [47] one can choose local coordinates v^α on N in such a way that

$$g^{\alpha\beta} = \text{const}, \quad \Gamma^{\alpha\beta}_\gamma = 0.$$

This proves the theorem. □

An alternative way to prove of the Theorem is to write explicitly down the terms of the order ϵ^{-1} in the Jacobi identity

$$\{\{v^\alpha(x), v^\beta(y)\}, w^a(z)\} + \text{(cyclic)} = 0$$

in order to prove that the leading term in $\{v^\alpha(x), v^\beta(y)\}$ does not depend on w, w_x. Then from the leading term in the Jacobi identity

$$\{\{v^\alpha(x), v^\beta(y)\}, v^\gamma(z)\} + \text{(cyclic)} = 0$$

it follows, as in [47], vanishing of the curvature of the metric $g^{\alpha\beta}(v)$.

We believe that the problem of classification of (p, q)-brackets (and also classification of pencils of (p, q)-brackets) is very important in the Hamiltonian theory of integrable PDEs. In this paper we will mainly consider $(0, n)$-brackets leaving the general case for a subsequent publication.

To illustrate our technique we will begin with a more simple example of $(n, 0)$-brackets.

2.4.4 Classification of $(n, 0)$-brackets

Our first result is

Theorem 2.4.18 *If M is a ball then all $(n, 0)$ Poisson brackets in $\hat{\Lambda}^2_{loc}$ are equivalent w.r.t. the action (2.4.22) of the Miura group $\mathcal{G}$.*

Proof First we choose the Darboux coordinates for the symplectic structure on M. The Poisson bracket in question will read

$$\{u^i(x), u^j(y)\} = \frac{1}{\epsilon} h^{ij}\delta(x-y) + \sum_{k=0}^{\infty} \epsilon^k \{u^i(x), u^j(y)\}^{[k]}. \tag{2.4.31}$$

Next we will try to kill all the terms of the expansion (2.4.31) by transformations of the form (2.4.19) with $F_0^i(u) = u^i$, $i = 1, \ldots, n$. To this end an appropriate version of Poisson cohomology will be useful. We define the Poisson cohomology $H^*(\hat{\mathcal{L}}(M), \varpi)$ for a Poisson structure $\varpi \in \hat{\Lambda}^2_{loc}$ as the cohomology of the complex

$$0 \to \hat{\Lambda}^0_{loc} \xrightarrow{\partial} \hat{\Lambda}^1_{loc} \xrightarrow{\partial} \hat{\Lambda}^2_{loc} \xrightarrow{\partial} \ldots \tag{2.4.32}$$

with the differential $\partial\beta := [\varpi, \beta]$. In the present proof $\varpi = h^{ij}\delta(x-y)$. The cohomology is naturally decomposed into the direct sum

$$H^k = \oplus_{m\geq -1} H^{k,m} \tag{2.4.33}$$

with respect to monomials in ϵ, where $H^{k,m}$ consists of the cocycles proportional to ϵ^m. Denote

$$\tilde{H}^k := \oplus_{m\geq 0} H^{k,m}. \tag{2.4.34}$$

The following obvious statement holds true.

Lemma 2.4.19 *The first non-zero term in the expansion (2.4.31) is a 2-cocycle in the Poisson cohomology $\tilde{H}^2$ of the ultralocal Poisson bracket (2.3.24).*

So, we will be able to kill this first nonzero term of the expansion if we prove that, for any 2-cocycle $\in \hat{\Lambda}^2_{loc}$ of the ultralocal bracket $\varpi := \epsilon\{\ ,\ \}^{[-1]}$, there exists a vector field a of the form (2.4.6) such that the Lie derivative $Lie_a \varpi$ gives the cocycle and $a|_{\epsilon=0} = 0$. This will follow from the following general statement about triviality of cohomology of the ultralocal Poisson bracket.

Lemma 2.4.20 *For M = ball and the ultralocal Poisson bracket ϖ (2.3.24) with det $h^{ij} \neq 0$ the Poisson cohomology $\tilde{H}^1(\hat{\mathcal{L}}(M), \varpi)$, $\tilde{H}^2(\hat{\mathcal{L}}(M), \varpi)$ vanish.*

The first proof of the lemma (and of the lemma 2.4.22 below) was obtained by E. Getzler [75] (also triviality of the higher cohomology has been proved). Independently, L. Degiovanni, F. Magri, V. Sciacca obtained another proof [20]. We have decided to present here our own proofs of triviality of cohomologies that closely follow the finite-dimensional case. Our proof will also be useful in the study of bihamiltonian structures below.

Let us prove first triviality of H^1. Let an evolutionary vector field a with the components $a^1, \ldots, a^n$ be a cocycle. Denote

$$\omega_i = h_{ij} a^j$$

where the constant matrix h_{ij} is inverse to h^{ij}. The condition $\partial a = Lie_a \varpi = 0$ reads

$$\frac{\partial \omega_i}{\partial u^{j,s}} = \sum_{t \geq s} (-1)^t \begin{pmatrix} t \\ s \end{pmatrix} \partial_x^{t-s} \frac{\partial \omega_j}{\partial u^{i,t}}.$$

Using (2.2.27) we conclude that there exists a local functional $\bar{f} = \int f\, dx$ such that

$$\omega_i = \frac{\delta \bar{f}}{\delta u^i(x)}.$$

Therefore the vector field is a Hamiltonian one,

$$a^i = h^{ij} \frac{\delta \bar{f}}{\delta u^j(x)}.$$

Let us now proceed to the proof of triviality of H^2. The idea is very simple: the bivector

$$\varpi + \varepsilon \alpha = h^{ij} \delta(x-y) + \varepsilon \sum_s A_s^{ij} \delta^{(s)}(x-y)$$

satisfies the Jacobi identity

$$[\varpi + \varepsilon\alpha, \varpi + \varepsilon\alpha] = 0 (\mathrm{mod}\, \varepsilon^2)$$

iff the inverse matrix is a closed differential form

$$\frac{1}{2}h_{ij}dx\wedge\delta u^i\wedge\delta u^j+\frac{1}{2}\,\varepsilon\,\omega_{i;js}\,dx\wedge\delta u^i\wedge\delta u^{j,s}\ (\mathrm{mod}\,\varepsilon^2)$$

where

$$\omega_{i;js}:=h_{ip}h_{jq}A_s^{pq}. \tag{2.4.35}$$

Denote

$$\omega=\frac{1}{2}\omega_{i;js}\delta u^i\wedge\delta u^{j,s}.$$

From the condition of closedness $\delta(dx\wedge\omega)=0\in\Lambda_3$ we derive, due to Corollary 2.2.14, existence of a one-form $dx\wedge\phi$, $\phi=\phi_i\delta u^i$ such that $\delta(dx\wedge\phi)=dx\wedge\omega$. The vector field a with the components

$$a^i=h^{ij}\phi_j$$

gives a solution to the equation

$$[\varpi,a]=\alpha.$$

To be on the safe side we will now show, by straightforward calculations, that, indeed, the above geometrical arguments work. First, from the antisymmetry condition (2.3.17) for the bivector α it readily follows the antisymmetry condition (2.2.30) for the 2-form ω with the reduced components (2.4.35). Next, we are to verify that from the cocycle condition $[\varpi,\alpha]=0$ where the Schouten - Nijenhuis bracket $[\varpi,\alpha]$ is written in (2.3.19), it follows closedness (2.2.32) of $dx\wedge\omega\in\Lambda_2$. First we will rewrite the formula for the bracket in a slightly modified form. Differentiating the antisymmetry condition

$$A_s^{ik}=-\sum(-1)^m\begin{pmatrix}m\\ s\end{pmatrix}\partial_x^{m-s}A_m^{ki}$$

w.r.t. $u^{l,t}$ and using the commutators (2.2.5) we obtain

$$\frac{\partial A_s^{ik}}{\partial u^{l,t}}=-\sum(-1)^{q+r+s}\begin{pmatrix}q+r+s\\ q\quad r\end{pmatrix}\left(\frac{\partial A_{q+r+s}^{ki}}{\partial u^{l,t-q}}\right)^r.$$

So the coefficients of the Schouten - Nijenhuis bracket (2.3.19) can be rewritten as follows

$$[\varpi,\alpha]_{x,y,z}^{ijk}=\left[\frac{\partial A_t^{ij}}{\partial u^{l,s}}h^{lk}-\frac{\partial A_s^{ik}}{\partial u^{l,t}}\right.$$
$$\left.+\sum(-1)^{q+r+t}\begin{pmatrix}q+r+t\\ q\quad r\end{pmatrix}\left(\frac{\partial A_{s-q}^{jk}}{\partial u^{l,q+r+t}}\right)^{(r)}h^{li}\right]\delta^{(t)}(x-y)\delta^{(s)}(x-z). \tag{2.4.36}$$

The coeffcient of $\delta^{(t)}(x-y)\delta^{(s)}(x-z)$ must vanish for every t and s. Multiplying this coefficient by $h_{ia}h_{jb}h_{kc}$ we arrive at the condition of closedness (2.2.32) of the 2-form (2.4.35). Using Corollary 2.2.14 we establish existence of differential polynomials ϕ_1, ..., ϕ_n representing the 2-form as in (2.2.31). The translation invariant vector field

$$a^i=h^{ij}\phi_j$$

will satisfy $\partial a=\alpha$. This proves the lemma, and also the theorem. □

2.4.5 Classification of $(0,n)$-brackets

Let us now proceed to considering the Poisson structures in $\hat\Lambda^2$ with identically vanishing leading term $\{\ ,\ \}^{[-1]}$. As above, the first nonzero term (2.4.11) is itself a Poisson bracket. The leading coefficient $g^{ij}(u)$ of it determines a symmetric tensor field on M invariant w.r.t. the action of the Miura group. This gives a map

$$\hat\Lambda^2/\mathcal{G} \to \text{symmetric tensors on } M.$$

Theorem 2.4.21 *Let M be a ball. Then the only invariant of a $(0,n)$ Poisson bracket in $\hat\Lambda^2$ with respect to the action of the Miura group is the signature of the quadratic form $g^{ij}(u)$.*

Proof. The symmetric nondegenerate tensor $g^{ij}(u)$ defines a *pseudoriemannian metric*

$$g_{ij}(u)du^i du^j, \quad (g_{ij}) = \left(g^{ij}\right)^{-1}$$

on the manifold M. From the general theory of [47] of the Poisson brackets of the form (2.4.11) it follows that the Riemann curvature of the metric vanishes, and that the coefficient $\Gamma^{ij}_k(u)$ in (2.4.11) is related to the Christoffel coefficients $\Gamma^k_{ij}(u)$ of the Levi-Civita connection for the metric by

$$\Gamma^{ij}_k = -g^{is}\Gamma^j_{sk}.$$

Using standard arguments of differential geometry we deduce that, locally coordinates $v^1(u), \ldots, v^n(u)$ exists such that, in the new coordinates the metric becomes constant

$$\frac{\partial v^k}{\partial u^i}\frac{\partial v^l}{\partial u^j}g^{ij}(u) = \eta^{kl} = \text{const}.$$

The Christoffel coefficients in these coordinates vanish. Of course, all constant symmetric matrices η^{kl} of a given signature are equivalent w.r.t. linear changes of coordinates.

We have reduced the proof of the theorem to reducing to the normal form (2.3.25) the Poisson bracket

$$\{u^i(x), u^j(y)\} = \eta^{ij}\delta'(x-y) + \sum_{k=1}^{\infty}\epsilon^k\{u^i(x), u^j(y)\}^{[k]} \tag{2.4.37}$$

by the transformations of the form (2.4.19) with $F^i_0 =$id. As in the proof of Theorem 2.4.18 the latter problem is reduced to proving triviality of the second Poisson cohomology of the Poisson bracket α of the form (2.3.25). Triviality of the cohomology is somewhat surprising from the point of view of finite-dimensional Poisson geometry. Indeed, as we have seen above this Poisson bracket degenerates. So we will be to also prove that all the cocycles are tangent to the leaves of the symplectic foliation (see the end of section 2.3).

Lemma 2.4.22 *For M = ball all the cocycles in $H^1(\hat{\mathcal{L}}(M))$ and $H^2(\hat{\mathcal{L}}(M))$ vanishing at $\epsilon = 0$ are trivial.*

Denote, like in (2.4.34),

$$\tilde{H}^k = \oplus_{m>0} H^{k,m}. \tag{2.4.38}$$

We are to prove that $\tilde{H}^1 = \tilde{H}^2 = 0$.

Let us begin with proving triviality of $\tilde{H}^1$. Using (2.3.5) we obtain

$$\partial_a \varpi = Lie_a \varpi^{ij} = -\partial_x \sum_{t\geq 0} (-1)^t \eta^{ip} \left(\frac{\partial a^j}{\partial u^{p,t}} \right)^{(v)} \delta(x-y)$$

$$- \sum_{r\geq 0} \left[\frac{\partial a^i}{\partial u^{p,r}} \eta^{pj} + \sum_{t\geq r} (-1)^t \binom{t+1}{r+1} \eta^{ip} \left(\frac{\partial a^j}{\partial u^{p,t}} \right)^{(t-r)} \right] \delta^{(r+1)}(x-y). \tag{2.4.39}$$

Since

$$\frac{\delta \bar{a}^j}{\delta u^p(x)} = \sum_{t\geq 0} (-1)^t \left(\frac{\partial a^j}{\partial u^{p,t}} \right)^{(v)}$$

is a differential polynomial in $\Lambda_0 \otimes \mathbb{C}[[\epsilon]]$ of the degree 0 vanishing at $\epsilon = 0$, from vanishing of the coefficient in front of $\delta(x-y)$ we derive that

$$\frac{\delta \bar{a}^j}{\delta u^p(x)} = 0, \quad j,\, p = 1, \dots, n.$$

Using Example 2.2.5 we derive existence of differential polynomials b^j s.t.

$$a^j = \partial_x b^j, \quad j = 1, \dots, n.$$

This is the crucial point in the proof: we have shown that the vector field a is tangent to the level surface of the Casimirs (2.3.26). The remaining part of the proof is rather straightforward. Using (2.2.5) and also the Pascal triangle identity

$$\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$$

we rewrite the coefficient of $\delta^{(r+1)}(x-y)$ in the form

$$\partial_x \left[\frac{\partial \omega_k}{\partial u^{l,r}} - \sum_{t\geq r} (-1)^t \binom{t+1}{r} \left(\frac{\partial \omega_l}{\partial u^{k,t}} \right)^{(t-r)} \right]$$

$$+ \frac{\partial \omega_k}{\partial u^{l,r-1}} + (-1)^r \frac{\partial \omega_l}{\partial u^{k,r-1}} = 0.$$

Here

$$\omega_k = \eta_{li} b^i, \quad (\eta_{ij}) = (\eta^{ij})^{-1},$$

the last two terms are not present for $r=0$. As above, for $r=0$ we derive that

$$\frac{\partial \omega_k}{\partial u^l} = \sum_{t\geq 0}(-1)^t \left(\frac{\partial \omega_l}{\partial u^{k,t}}\right)^{(v)}.$$

Proceeding by induction in r we prove that the 1-form $\int dx \wedge \omega_i \, \delta u^i$ is closed. Using the Volterra criterion we derive existence of a differential polynomial f s.t. $\omega = \delta \int f\, dx$. Hence

$$a^i = \eta^{ij}\partial_x \frac{\delta \bar{f}}{\delta u^j(x)}.$$

We proved triviality of $\tilde{H}^1$.

Let us proceed to prove the triviality of $\tilde{H}^2$. The condition $\partial\alpha = 0$ for α of the form (2.3.15) can be computed similarly to Example 2.3.9. We obtain a system of equations

$$\begin{aligned} &\frac{\partial A_t^{ij}}{\partial u^{l,s-1}}\eta^{lk} + \sum (-1)^{q+r+s}\begin{pmatrix} q+r+s \\ q \;\; r \end{pmatrix}\left(\frac{\partial A^{ki}_{q+r+s}}{\partial u^{l,t-q-1}}\right)^{(r)}\eta^{lj} \\ &+\sum(-1)^{q+r+t}\begin{pmatrix} q+r+t \\ q \;\; r\end{pmatrix}\left(\frac{\partial A^{jk}_{s-q}}{\partial u^{l,q+r+t-1}}\right)^{(r)}\eta^{li} = 0 \\ &\text{for any } i,j,k,s,t \end{aligned} \qquad (2.4.40)$$

(it is understood that the terms with $s-1$, $t-q-1$ or $t+q+r-1$ negative do not appear in the sum). Recall that the crucial point in the proof of triviality of the 2-cocycle is to establish validity of (2.1.18) for the Casimirs (2.3.26) of ϖ. Explicitly, we need to show that

$$\alpha(\delta\bar{u}^i, \delta\bar{u}^j) = \int A_0^{ij}\, dx = 0 \;\; \text{for any } i,j. \qquad (2.4.41)$$

We first use (2.4.40) for $s=t=0$ to prove that

$$\partial_x \sum_r (-1)^r \left(\frac{\partial A_0^{jk}}{\partial u^{l,r}}\right)^{(r)}\eta^{li} = 0.$$

Hence

$$A_0^{jk} = \partial_x B^{jk}$$

for some differential polynomial B^{jk}. This implies (2.4.41). The rest of the proof is identical to the proof of Lemma 2.1.3. We first construct the vector field z (see the proof of the lemma 2.1.3). To this end we use the equation (2.4.40) for $s=0$, $t>0$:

$$\sum_{q,r}(-1)^{q+r}\begin{pmatrix} q+r \\ r\end{pmatrix}\left(\frac{\partial A^{ki}_{q+r}}{\partial u^{l,t-q-1}}\right)^{(r)}\eta^{lj} + \sum_r (-1)^{t+r}\begin{pmatrix} t+r \\ r\end{pmatrix}\left(\frac{\partial A_0^{jk}}{\partial u^{l,t+r-1}}\right)^{(r)}\eta^{li} = 0.$$

Differentiating the antisymmetry condition

$$A_0^{ik} = \sum (-1)^{r+1} \left(A_r^{ki}\right)^{(r)}$$

w.r.t. $u^{l,t-1}$ we identify the first term of the previous equation with

$$-\frac{\partial A_0^{ik}}{\partial u^{l,t-1}} \eta^{lj}.$$

The resulting equation coincides with the condition $\partial a^k = 0$ of closedness of the 1-cocycle

$$(a^k)^i = A_0^{ik}$$

for every $k = 1, \ldots, n$ (see (2.4.39) for the explicit form of this condition). Using the first part of Lemma we arrive at existence of n differential polynomials $q^1, \ldots, q^n$ s.t.

$$A_0^{ik} = \eta^{is} \partial_x \frac{\delta \bar{q}^k}{\delta u^s(x)}. \tag{2.4.42}$$

The last step, as in the proof of Lemma 2.1.3, is to change the cocycle α to a cohomological one to obtain a closed 2-cocycle

$$\alpha \mapsto \alpha + \partial z =: \alpha'$$

for

$$z = q^i \frac{\partial}{\partial u^i}.$$

The new 2-cocycle α' will have the same form as above with $A_0^{ij} = 0$. Denote

$$g_{i;js} := \eta_{ip} \eta_{jq} A_s^{ij}, \quad s \geq 1.$$

We will now show existence of differential polynomials $\omega_{i;j0}$, $\omega_{i;j1}, \ldots$ s.t.

$$\begin{aligned} g_{i;j1} &= \partial_x \omega_{i;j0}, \\ g_{i;js} &= \partial_x \omega_{i;j,s-1} + \omega_{i;j,s-2} \quad \text{for } s \geq 2. \end{aligned} \tag{2.4.43}$$

From (2.4.40) for $s = 1$, $t = 0$ we obtain

$$\partial_x \sum_r (-1)^r \left(\frac{\partial A_1^{jk}}{\partial u^{l,r}} \right)^{(r)} = 0.$$

As we already did many times, from the last equation it follows that

$$\sum_r (-1)^r \left(\frac{\partial A_1^{jk}}{\partial u^{l,r}} \right)^{(r)} = 0.$$

This shows existence of $\omega_{i;j0}$. Using (2.4.40) for $s=1$ and $t>0$ we inductively prove existence of the differential polynomials $\omega_{i;j,t-1}$. Actually, we can obtain

$$\omega_{i;jl} = \sum_{s\geq l+2} \partial_x^{s-l-2} g_{i;js}. \tag{2.4.44}$$

From this it readily follows that the coefficients $\omega_{i;js}$ satisfy the antisymmetry conditions (2.2.30). Thus they determine a 2-form ω.

Let us prove that the 2-form ω is closed. Denote

$$J_{ijk;st} := \left(\sum_{m=s}^{t+s} \sum_{r=0}^{m-s} + \sum_{m\geq t+s+1} \sum_{r=0}^{t} \right) (-1)^m \left(\begin{array}{c} m \\ r\ s \end{array} \right) \partial_x^{m-r-s} \frac{\partial \omega_{j;k,t-r}}{\partial u^{i,m}}$$

$$+\frac{\partial \omega_{i;j,s}}{\partial u^{k,t}} - \frac{\partial \omega_{i;k,t}}{\partial u^{j,s}}$$

the l.h.s. of the equation (2.2.32) of closedness of a 2-form. Let us show that the coefficient of $\delta^{(v)}(x-y)\delta^{(s)}(x-z)$ in (2.4.40) is equal to

$$\partial_x J_{ijk;t-1,s-1} + J_{ijk;t-1,s-2} + J_{ijk;t-2,l-1}. \tag{2.4.45}$$

To this end we replace the second sum in (2.4.40) by

$$-\frac{\partial A_s^{ik}}{\partial u^{l,t-1}} \eta^{lj}.$$

Lowering the indices by means of η_{ij} and using (2.4.43) we obtain (2.4.45). From vanishing of (2.4.45) we inductively deduce that $J_{ijk;st}=0$ for all $i,j,k=1,\ldots,n$ and all $s,t\geq 0$ (observe that the coefficients $J_{ijk;t0}=J_{ijk;0s}=0$ due to our assumption $A_0^{ij}=0$. This proves that the 2-form ω is closed. So $\omega = \delta \int dx \wedge \phi$ for some 1-form $\phi = \phi_i \delta u^i$. Introducing the vector field

$$a^i = \eta^{ik}\phi_k$$

we finally obtain, for the original cocycle α,

$$\alpha = \partial(a - z).$$

Theorem is proved.

3 Bihamiltonian geometry of loop spaces

3.1 Bihamiltonian structures and hierarchies of commuting flows

3.1.1 Poisson pencils and bihamiltonian recursion procedure: summary of the finite-dimensional case

Definition. A *bihamiltonan structure* on the manifold P is a 2-dimensional linear subspace in the space of Poisson structures on P.

Choosing two points $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ of the subspace we obtain that the linear combination

$$a_1\{\ ,\ \}_1 + a_2\{\ ,\ \}_2 \tag{3.1.1}$$

with arbitrary constant coefficients a_1, a_2 is again a Poisson bracket. This reformulation is usually referred to as *the compatibility condition* of the two Poisson brackets. It is spelled out as vanishing of the Schouten - Nijenhuis bracket

$$[\{\ ,\ \}_1, \{\ ,\ \}_2] = 0. \tag{3.1.2}$$

An importance of bihamiltonian structures for recursive constructions of integrable systems was discovered by F.Magri [104] in the analysis of the so-called Lenard scheme of constructing the KdV integrals. The basic idea of these constructions is given by the following simple

Lemma 3.1.1 *Let H_0, H_1, ..., be a sequence of functions on P satisfying the recursion relation*

$$\{\ .\ , H_{p+1}\}_1 = \{\ .\ , H_p\}_2, \quad p = 0, 1, \ldots \tag{3.1.3}$$

Then

$$\{H_p, H_q\}_1 = \{H_p, H_q\}_2 = 0, \quad p,\ q = 0, 1, \ldots$$

For convenience of the reader we reproduce the proof of the lemma. Let $p < q$ and $q - p = 2m$ for some $m > 0$. Using the recursion and antisymmetry of the brackets we obtain

$$\{H_p, H_q\}_1 = \{H_p, H_{q-1}\}_2 = -\{H_{q-1}, H_p\}_2 = -\{H_{q-1}, H_{p+1}\}_1 = \{H_{p+1}, H_{q-1}\}_1.$$

Iterating we arrive at

$$\{H_p, H_q\}_1 = \ldots = \{H_{p+m}, H_{q-m}\}_1 = 0$$

since $p + m = q - m$. Doing similarly in the case $q - p = 2m + 1$ we obtain

$$\{H_p, H_q\}_1 = \ldots = \{H_n, H_{n+1}\}_1 = \{H_n, H_n\}_2 = 0$$

where $n = p + m = q - m - 1$. The commutativity $\{H_p, H_q\}_2 = 0$ easily follows from the recursion. The Lemma is proved.

We have not used yet the compatibility condition of the two brackets. It turns out to be crucial in contructing the Hamiltonians satisfying the recursion relation (3.1.3). There are two essentially different realizations of the recursive procedure.

The first one applies to the case when the bihamiltonian structure is *symplectic* , i.e. $N = 2n$ and the Poisson structures of the affine line (3.1.1) do not degenerate for generic a_1, a_2. Without loss of generality one may assume nondegeneracy of $\{\ ,\ \}_1$. The *recursion operator*

$$\mathcal{R} : TP \to TP$$

is defined by

$$\mathcal{R} := \{\ ,\ \}_2 \cdot \{\ ,\ \}_1^{-1}. \tag{3.1.4}$$

The main recursion relation (3.1.3) can be rewritten in the form

$$dH_{p+1} = \mathcal{R}^* dH_p, \quad p = 0,\, 1, \ldots \tag{3.1.5}$$

where

$$\mathcal{R}^* : T^*P \to T^*P$$

is the adjoint operator.

Theorem 3.1.2 *[104, 112] The Hamiltonians*

$$H_p := \frac{1}{p+1} \operatorname{tr} \mathcal{R}^{p+1}, \quad p \geq 0$$

satisfy the recursion (3.1.5).

Clearly there are at most n independent of these commuting functions. We say that the bihamiltonian symplectic structure is *generic* if exactly n of these functions are independent. Let us denote $\lambda_i = \lambda_i(x)$ the eigenvalues of the recursion operator. Since the characteristic polynomial of $\mathcal{R}$ is a perfect square

$$\det(\mathcal{R} - \lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)^2.$$

only n of these eigenvalues can be distinct, say, $\lambda_1 = \lambda_1(x)$, $\ldots$, $\lambda_n = \lambda_n(x)$. For generic bihamiltonian symplectic structure these are independent functions on $P \ni x$.

Theorem 3.1.3 *[104, 105, 112] Let $\{\ ,\ \}_{1,2}$ be a generic symplectic bihamiltonian structure. Then*

1) All the commuting Hamiltonians

$$H_p = \frac{1}{p+1} \operatorname{tr} \mathcal{R}^{p+1} = \frac{1}{p+1} \sum_{i=1}^{n} \lambda_i^{p+1}(x), \quad p = 0, 1, \ldots, n-1$$

generate completely integrable systems on P.

2) The eigenvalues $\lambda_i(x)$ can be included in a coordinate system $\lambda_1, \mu_1, \dots, \lambda_n, \mu_n$ in order to reduce the two Poisson structures to a block diagonal form where the i-th block in $\{\ ,\ \}_1$ and in $\{\ ,\ \}_2$ reads, respectively

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}, \quad i = 1, \dots, n.$$

The last formula gives the normal form of a generic symplectic bihamiltonian structure. Therefore all such structures are equivalent w.r.t. the group of local diffeomorphisms.

Let us now consider the degenerate situation. We assume that the Poisson structure (3.1.1) has constant rank for generic a_1 and a_2. Without loss of generality we may assume that

$$k = \text{corank}\{\ ,\}_1 = \text{corank}(\{\ ,\}_1 + \epsilon\{\ ,\}_2) \qquad (3.1.6)$$

for an arbitrary sufficiently small ϵ.

Let us first prove the following useful property of bihamiltonian structures of the constant rank.

Lemma 3.1.4 *Let the bihamiltonian structure satisfy (3.1.6). Then the Casimirs of $\{\ ,\}_1$ commute w.r.t. $\{\ ,\}_2$.*

Proof Let $2m$ be the rank of $\{\ ,\}_1$. We first reduce the matrix of this bracket to the canonical constant block diagonal form. Denote (h^{ab}) the matrix of the second Poisson bracket in these coordinates. Let us now choose two integers i, j such that $2m < i < j \le N = 2m + k$ and form a $(2m+1) \times (2m+1)$ minor of the matrix $\{\ ,\}_1 + \epsilon\{\ ,\}_2$ by adding i-th column and j-th row to the principal $2m \times 2m$ minor standing in the first $2m$ columns and first $2m$ rows. The condition (3.1.6) is equivalent to vanishing of the determinants of all these minors. It is easy to see that the determinant in question is equal to $-\epsilon\, h^{ij} + O(\epsilon^2)$. Therefore $h^{ij} = 0$ for all pairs (i, j) greater than $2m$. The lemma is proved. □

Corollary 3.1.5 *For a compatible pair of Poisson brackets of the constant* rank $(\{\ ,\ \}_2 - \lambda\{\ ,\ \}_1) = \text{rank}\{\ ,\ \}_1$, $\lambda \to \infty$,

$$\{\ ,\ \}_2 \in H^2(P, \{\ ,\ \}_1)$$

is a trivial cocycle.

Proof What $\{\ ,\ \}_2$ is a cocycle w.r.t. the Poisson cohomology of $(P, \{\ ,\ \}_1)$ follows from (3.1.2). To prove triviality use commutativity of the Casimirs of the first Poisson bracket and also Lemma 2.1.3. □

A bihamiltonian structure with a marked line $\lambda\{\ ,\ \}_1$ is called *Poisson pencil.* Choosing another Poisson bracket $\{\ ,\ \}_2$ of the pencil one can represent the brackets of the bihamiltonian structure in the form

$$\{\ ,\ \}_\lambda := \{\ ,\ \}_2 - \lambda\{\ ,\ \}_1. \tag{3.1.7}$$

The representation (3.1.7) is well-defined up to an affine change of the parameter λ

$$\lambda \mapsto a\,\lambda + b.$$

In the case of Poisson pencils of constant rank the corank of $\{\ ,\ \}_\lambda$ equals k for $\lambda \to \infty$. The recursive construction of the commuting flows in this case is given by the following simple statement (cf. [104, 70, 126]).

Theorem 3.1.6 *Under the assumption (3.1.6) the coefficients of the Taylor expansion*

$$c^\alpha(x,\lambda) = c^\alpha_{-1}(x) + \frac{c^\alpha_0(x)}{\lambda} + \frac{c^\alpha_1(x)}{\lambda^2} + \dots, \quad \lambda \to \infty \tag{3.1.8}$$

of the Casimirs $c^\alpha(x,\lambda)$, $\alpha = 1,\dots,k$ of the Poisson bracket $\{\ ,\ \}_\lambda$ commute with respect to both the Poisson brackets

$$\{c^\alpha_p, c^\beta_q\}_{1,2} = 0, \quad \alpha,\beta = 1,\dots,k, \quad p,q \geq -1.$$

Proof Spelling out the definition of the Casimirs

$$\{\ .\ , c^\alpha\}_\lambda = 0$$

for the coefficients of the expansion (3.1.8) we must have first that

$$\{\ .\ , c^\alpha_{-1}\}_1 = 0. \tag{3.1.9}$$

That is, the leading coefficients of the Taylor expansions are Casimirs of $\{\ ,\ \}_1$. For the subsequent coefficients we get the recursive relations

$$\{\ .\ , c^\alpha_{p+1}\}_1 = \{\ .\ , c^\alpha_p\}_2, \quad p = -1,\, 0,\, 1, \dots \tag{3.1.10}$$

From (3.1.10) and Theorem 1 it follows that

$$\{c^\alpha_p, c^\alpha_q\}_{1,2} = 0, \quad p,q \geq -1.$$

The commutativity $\{c^\alpha_p, c^\beta_q\}_{1,2} = 0$ for $\alpha \neq \beta$ easily follows from the same recursion trick and from commutativity of the Casimirs

$$\{c^\alpha_{-1}, c^\beta_{-1}\}_2 = 0 \tag{3.1.11}$$

proved in Lemma 3.1.4. The theorem is proved. □

Example 3.1.7 *According to Corollary 3.1.5 there exists a vector field Z such that*

$$Lie_Z\{\ ,\ \}_1 = \{\ ,\ \}_2.$$

We say, following [12] that the bihamiltonian structure is exact *if the vector field Z can be chosen in such a way that*

$$(Lie_Z)^2\{\ ,\ \}_1 = 0. \tag{3.1.12}$$

For an exact bihamiltonian structure the generating functions (3.1.8) of the commuting Hamiltonians $c_p^\alpha(x)$ have the form

$$c^\alpha(x;\lambda) = \exp(-Z/\lambda)\, c_{-1}^\alpha(x) = c_{-1}^\alpha(x) - \frac{1}{\lambda}\partial_Z c_{-1}^\alpha(x) + \frac{1}{\lambda^2}\partial_Z^2 c_{-1}^\alpha(x)\dots \tag{3.1.13}$$

for every $\alpha = 1, \dots, k$.

This formula can be easily proved by choosing a system of local coordinates $x^1, \dots, x^N$ on the phase space P such that the vector field Z corresponds to the shift along x^1. In these coordinates the tensor of the first Poisson bracket depends linearly on x^1 and the second Poisson bracket is x^1-independent. The Poisson pencil $\{\ ,\ \}_\lambda$ is obtained from $\{\ ,\ \}_1$ by the shift $x^1 \mapsto x^1 - 1/\lambda$ and by multiplication by $-\lambda$.

Conversely, if, in a given coordinate system, $\{\ ,\ \}_1$ depends linearly on one of the coordinates and $\{\ ,\ \}_2$ does not depend on this coordinate then the bihamiltonian structure is exact. In particular this trick can be applied to the standard linear Lie - Poisson structures on the dual spaces to Lie algebras (cf [108]). In this case it was called in [116] the method of argument translation.

All our bihamiltonian structures on the loop spaces to be studied below will be exact. However, at the moment we do not see their Lie algebraic origin.

The construction of Theorem 3.1.6 for a bihamiltonian structure of the constant corank k produces k chains of pairwise commuting bihamiltonian flows

$$\frac{dx}{dt^{\alpha,p}} = \{x, c_p^\alpha\}_1 = \{x, c_{p-1}^\alpha\}_2, \quad \alpha = 1, \dots, k, \quad p = 0, 1, 2, \dots \tag{3.1.14}$$

The chains are labeled by the Casimirs c_{-1}^α of the first Poisson bracket. The level p in each chain corresponds to the number of iterations of the recursive procedure (we will keep using this expression although the recursion operator is not defined in the degenerate case). All the family of commuting flows organized by the above recursion procedure is called *the hierarchy* determined by the bihamiltonian structure.

The hierarchy structure of the constructed family of commuting flows depends non-trivially on the choice of $\{\ ,\ \}_1$ in the Poisson pencil (3.1.1). On the contrary, a different choice of the second Poisson bracket in the pencil produces a triangular linear transformation of the commuting Hamiltonians, i.e., to the Hamiltonians of the level p it will be added a linear combination of the Hamiltonians of the lower levels.

In the finite dimensional case we are discussing now all the chains of the hierarchy will be finite. In other words, the generating functions (3.1.8) of the commuting flows will become polynomials after multiplication by a suitable power of λ (the degrees of an appropriate system of these polynomials correspond to the *type* of the bihamiltonian structure [126]). A simple necessary and sufficient condition of complete integrability of the flows of the hierarchy was found by A.Brailov and A.Bolsinov (see in [8]). The problem of normal forms of degenerate bihamiltonian structures has been studied by I.M.Gelfand and I.Zakharevich [70], [71] for the case of the corank 1 and by I.Zakharevich [149] and A.Panasyuk [126] for higher coranks.

3.1.2 Construction of bihamiltonian hierarchies on the extended loop spaces

In the remaining part of the paper we will study bihamiltonian structures on the extended loop spaces $\hat{\mathcal{L}}(M)$ assuming M to be a n-dimensional ball. Moreover we restrict ourselves at considering $(0,n)$ bihamiltonian structures, i.e., of the form

$$\{u^i(x),u^j(y)\}_{1,2}=\sum_{k=0}^{\infty}\epsilon^k\{u^i(x),u^j(y)\}^{[k]}_{1,2},\quad i,j=1,\dots,n \tag{3.1.15}$$

with

$$\{u^i(x),u^j(y)\}^{[k]}_{1,2}=\sum_{s=0}^{k+1}A^{ij}_{k,s_{1,2}}(u;u_x,\dots,u^{(s)})\delta^{(k-s+1)}(x-y) \tag{3.1.16}$$

$$A^{ij}_{k,s_{1,2}}(u;u_x,\dots,u^{(s)})\in\mathcal{A},\quad \deg A^{ij}_{k,s_{1,2}}=s,$$

both satisfying the condition

$$\det A^{ij}_{0,0_1}(u)\neq 0,\quad \det A^{ij}_{0,0_2}(u)\neq 0. \tag{3.1.17}$$

Example 3.1.8 *For $n=1$ take*

$$\{u(x),u(y)\}_2=u(x)\delta'(x-y)+\frac{1}{2}u'\delta(x-y)-\frac{1}{4}\epsilon^2\delta'''(x-y) \tag{3.1.18}$$

This is the Lie - Poisson bracket on the dual space to the Virasoro algebra (see Example 2.1.8 above). It depends linearly on $u(x)$. Taking the Lie derivative of this bracket along the vector field $\partial/\partial u$ we obtain another Poisson bracket

$$\{u(x),u(y)\}_1=\delta'(x-y). \tag{3.1.19}$$

This is an exact bihamiltonian structure in the sense of Example 3.1.7 (the roles of $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ have been interchanged, $Z=\partial/\partial u$). The Casimir $c([u];\lambda)$ of the Poisson pencil $\{\ ,\ \}_2-\lambda\{\ ,\ \}_1$ is determined from the following third order equation

$$-\frac{\epsilon^2}{4}y'''+u\,y'+\frac{1}{2}u'y=\lambda\,y' \tag{3.1.20}$$

where

$$y := \frac{\delta c([u];\lambda)}{\delta u(x)}.$$

Starting from the Casimir

$$I_{-1}[u] = \int u\, dx$$

of (3.1.19) and applying the recursive procedure of the previous section we produce an infinite sequence of commuting Hamiltonians of the KdV hierarchy

$$\frac{\partial u}{\partial t^k} = \partial_x \frac{\delta I_k[u]}{\delta u(x)} = \left[-\frac{1}{4}\epsilon^2\partial_x^3 + u\partial_x + \frac{1}{2}u_x\right]\frac{\delta I_{k-1}}{\delta u(x)}, \quad k = 0, 1, \dots \tag{3.1.21}$$

where the first few equations of the hierarchy and their Hamiltonians have the form

$$\begin{aligned} I_0[u] &= \int \frac{u^2}{2}dx, \quad I_1 = \frac{1}{4}\int\left(\epsilon^2\frac{u'^2}{2} + u^3\right)\, dx, \\ I_2[u] &= \frac{1}{16}\int\left(\epsilon^4\frac{u''^2}{2} + 5\epsilon^2 u\, u'^2 + \frac{5}{2}u^4\right)\, dx\,, \dots, \end{aligned} \tag{3.1.22}$$

$$\begin{aligned} \frac{\partial u}{\partial t^0} &= u_x, \quad \frac{\partial u}{\partial t^1} = \frac{1}{4}(6u\, u' - \epsilon^2 u'''), \\ \frac{\partial u}{\partial t^2} &= \frac{1}{16}[30u^2u' - 10\epsilon^2(2u'u'' + u\, u''') + \epsilon^4 u^V]\,, \dots \end{aligned} \tag{3.1.23}$$

The above algorithm of constructing the KdV hierarchy is due to Lenard (see in [67]). The bihamiltonian interpretation of it is due to Magri [104]. The generating function of the Hamiltonian densities

$$\frac{1}{2\,\chi_R} = \frac{1}{2\sqrt{\lambda}} + \sum_{j=0}^{\infty}\frac{R_j}{\lambda^{\frac{2j+1}{2}}} \tag{3.1.24}$$

$$H_k = -4\int R_{k-2}dx \tag{3.1.25}$$

that is, the density of the Casimir of the Poisson pencil $\{\ ,\ \}_2 - \lambda\{\ ,\ \}_1$ coincides, according to I.M.Gelfand and L.A.Dickey [68] with the diagonal of the resolvent of the Lax operator

$$L = -\epsilon^2\partial_x^2 + u(x).$$

The series in (3.1.24) is to be understood in the sense of the asymptotic expansion at $\lambda \to \infty$. The coefficients of the asymptotic expansion do not depend on the choice of the boundary conditions.

Let us return back to the general case (3.1.15)–(3.1.17). We will apply now the results of the first part of the paper to prove existence of the commuting hierarchy for an arbitrary bihamiltonian structure of the above form. Indeed, according to these results for every λ the Poisson bracket

$$\{u^i(x), u^j(y)\}_\lambda := \{u^i(x), u^j(y)\}_2 - \lambda\{u^i(x), u^j(y)\}_1 \qquad (3.1.26)$$

can be reduced to the constant form (2.3.25). We will now prove that the coefficients of the Taylor expansion of the reducing transformation are densities of the commuting Hamiltonians.

Theorem 3.1.9 *There exists a transformation of the form*

$$u^i \mapsto \tilde{u}^i = F^i(u; u_x, u_{xx}, \dots; \epsilon; \lambda) = \sum_{p=-1}^{\infty} \frac{f_p^i(u; u_x, u_{xx}, \dots; \epsilon)}{\lambda^{p+1}} \qquad (3.1.27)$$

reducing the Poisson bracket (3.1.26) to the normal form

$$\{\tilde{u}^i(x), \tilde{u}^j(y)\}_\lambda = -\lambda\, \eta^{ij}\delta'(x-y). \qquad (3.1.28)$$

The coefficients of the expansion

$$f_p^i(u; u_x, u_{xx}, \dots; \epsilon) \in \mathcal{A} \otimes \mathbb{C}[[\epsilon]]$$

are densities of pairwise commuting integrals

$$\bar{f}_p^i := \int f_p^i(u; u_x, u_{xx}, \dots; \epsilon)\, dx \qquad (3.1.29)$$

$$\{\bar{f}_p^i, \bar{f}_q^j\}_{1,2} = 0, \quad i, j = 1, \dots, n, \quad p, q = -1, 0, 1, 2, \dots$$

Proof Let us prove first that the reducing transformation for the Poisson pencil (3.1.26) can be chosen in the form of the series (3.1.27). Let

$$u^i \mapsto \hat{u}^i = f_{-1}^i(u; u_x, \dots; \epsilon)$$

be the Miura-type transformation reducing the first Poisson bracket to the normal form $\eta^{ij}\delta'(x-y)$ with a constant nodegenerate symmetric matrix η^{ij}. The compatibility condition of the Poisson pencil says that the second Poisson bracket is a 2-cocycle of the first one. Due to triviality of the 2-cohomology there exists an infinitesimal transformation

$$\hat{u}^i \mapsto \hat{u}^i + \frac{\hat{f}_0^i(u; u_x, \dots; \epsilon)}{\lambda}$$

reducing the pencil to the form

$$-\lambda\left[\eta^{ij}\delta'(x-y) + O\left(\frac{1}{\lambda^2}\right)\right].$$

Iterating this procedure we will kill all the terms in the $1/\lambda$ expansion of the pencil (3.1.26). The superposition of the Miura-type transformations gives the series (3.1.27).

To prove commutativity of the functionals (3.1.29) it suffices to observe that

$$\int \tilde{u}^i dx = \sum_{p=-1}^{\infty} \frac{\bar{f}_p^i}{\lambda^{p+1}}, \quad i = 1, \ldots, n$$

are the Casimirs of the Poisson pencil (3.1.26). This immediately follows from the reduced form of it (3.1.28). The theorem is proved. □

Corollary 3.1.10 *Every bihamiltonian structure of the form (3.1.15) - (3.1.17) generates a commuting bihamiltonian hierarchy of evolutionary PDEs*

$$\frac{\partial u^i}{\partial t^{j,p}} = \{u^i(x), \bar{f}_p^j\}_1 = \{u^i(x), \bar{f}_{p-1}^j\}_2, \quad j = 1, \ldots, n, \quad p = 0, 1, 2, \ldots \tag{3.1.30}$$

Example 3.1.11 *Let us apply the above procedure to the bihamiltonian structure (3.1.19), (3.1.18) of the KdV hierarchy. We already know that (3.1.18) is reduced to the constant form by means of the Miura transformation $u \mapsto \chi$,*

$$i\epsilon\chi' - \chi^2 = u.$$

The whole Poisson pencil (3.1.26) built of (3.1.19) and (3.1.18) is obtained from (3.1.18) by a shift $u \mapsto u - \lambda$ (i.e., this is an exact bihamiltonian structure). Therefore the reducing transformation for the pencil has the form

$$u \mapsto \tilde{u} = -2\sqrt{\lambda}[\chi - \sqrt{\lambda}],$$
$$\{\tilde{u}(x), \tilde{u}(y)\}_\lambda = -\lambda\, \delta'(x - y). \tag{3.1.31}$$

Here χ is the unique solution to the Riccati equation

$$i\epsilon\chi' - \chi^2 = u - \lambda$$

of the form

$$\chi = k + \sum_{m=1}^{\infty} \frac{\chi_m}{k^m}, \quad k = \sqrt{\lambda}.$$

The coefficients χ_m are polynomials in u, $\epsilon u'$, $\epsilon^2 u''$ etc.,

$$\chi_1 = -u/2, \quad \chi_2 = -i\epsilon u'/4, \quad \chi_3 = \frac{1}{8}(\epsilon^2 u'' - u^2), \ldots$$

The even coefficients all are total derivatives; the odd ones give the Hamiltonians of the KdV hierarchy

$$\frac{\partial u}{\partial t^k} = \partial_x \frac{\delta I_k}{\delta u(x)}$$

where

$$I_k = -4 \int \chi_{2k+3} dx, \quad k = 0, 1, \ldots$$

(we redenote $f_k^1 = -4\chi_{2k+3}$, $t^k = t^{1,k}$).

This algorithm of constructing the conservation laws of the KdV equation was found by Gardner, Kruskal and Miura in [118]. The equivalence of the family of KdV integrals produced by the Lenard scheme to those produced by the Gardner, Kruskal and Miura algorithm was established by G.Wilson [144]. To our best knowledge the Hamiltonian nature of this algorithm was not discussed in the literature.

Remark 3.1.12 *It can be shown that the even part*

$$\sum \frac{\chi_{2l}}{\lambda^l}$$

of the reducing transformation can be gauged out by a Miura-type transformation preserving the canonical form (3.1.31).

3.2 The leading order of $(0, n)$ bihamiltonian structures

The main problem we address in this paper is the classification of Poisson pencils of the form (3.1.15)-(3.1.17) on the extended loop space w.r.t. the action of Miura group. Actually we will add below further restrictions on the class of bihamiltonian structures to be classified.

As we already did above in solving the problem of normal forms of a single Poisson bracket on the extended loop space, we can try to classify Poisson pencils by successive approximations, i.e., first reducing to a normal form the leading term of the series (3.1.15), then studying 2-cocycles on this leading term etc.

It is clear that the leading term

$$\{u^i(x), u^j(y)\}^{[0]}_\lambda = \{u^i(x), u^j(y)\}^{[0]}_2 - \lambda\{u^i(x), u^j(y)\}^{[0]}_1 \tag{3.2.1}$$

is itself a Poisson pencil. Let us redenote the coefficients of the pencil as follows

$$\begin{aligned}\{u^i(x), u^j(y)\}^{[0]}_1 &= g^{ij}_1(u(x))\delta'(x-y) + \Gamma^{ij}_{1\,k}(u)u^k_x\delta(x-y),\\ \{u^i(x), u^j(y)\}^{[0]}_2 &= g^{ij}_2(u(x))\delta'(x-y) + \Gamma^{ij}_{2\,k}(u)u^k_x\delta(x-y).\end{aligned} \tag{3.2.2}$$

According to our assumption (3.1.17) the leading coefficients $g^{ij}_1(u)$ and $g^{ij}_2(u)$ define two flat geometries on M. Under what conditions these two flat geometries correspond to a bihamiltonian structure (3.2.2) on M? It is clear that the two flat geometries must be members of a *flat pencil*, i.e., the linear combination $\Gamma^{ij}_{2\,k} - \lambda\Gamma^{ij}_{1\,k}$ of the Christoffel coefficients of the two metrics gives the Christoffel coefficients of the linear combination of the metrics $g^{ij}_2 - \lambda g^{ij}_1$ and the latter is a flat (contravariant) metric for an arbitrary λ. The notion of the flat pencil was introduced by one of the authors in [40] (see also [41], [43]). It was shown in [40] that flat pencils are parametrized by solutions to a certain system of nonlinear PDEs that resembles the equations of associativity. Let us briefly recall this parametrization.

Definition. A *quasi-Frobenius structure* on a manifold M is a pair $(f, <\ ,\ >)$ where $f = f_\alpha(v)dv^\alpha$ is a one-form and $<\ ,\ >$ is a symmetric bilinear nondegenerate form on the tangent planes. This form will be called metric on M. The following conditions must hold true.

1) the metric $< \, , \, >$ is flat

2) in the flat coordinates $v^1, \ldots, v^n$ for the flat metric,

$$< \partial_\alpha, \partial_\beta >= \eta_{\alpha\beta} = \text{const}, \quad \partial_\alpha := \frac{\partial}{\partial v^\alpha}$$

the multiplication table

$$\partial_\alpha \cdot \partial_\beta = \partial_\alpha \partial_\gamma f_\beta(v) \eta^{\gamma\mu} \partial_\mu \tag{3.2.3}$$

(here $(\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}$) defines on the tangent spaces T_vM a structure of algebra satisfying the following conditions

1)

$$(a \cdot b) \cdot c = (a \cdot c) \cdot b. \tag{3.2.4}$$

2) The bilinear form $< \, , \, >$ is *invariant* on the algebra T_vM, i.e.

$$< a \cdot b, c >=< a, c \cdot b > . \tag{3.2.5}$$

Here a, b, c are arbitrary three vectors in T_vM.

3) The bilinear form on the cotangent bundle defined by

$$(dv^\alpha, dv^\beta) = \eta^{\alpha\lambda}\eta^{\beta\mu}(\partial_\lambda f_\mu + \partial_\mu f_\lambda) \tag{3.2.6}$$

is an invariant form for the dual algebra

$$dv^\alpha \cdot dv^\beta = \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\gamma f_\mu(v)\, dv^\gamma$$

on T_v^*M.

Warning: the name "quasi-Frobenius manifold" was also used in different senses in [38, 109]!

Theorem 3.2.1 *For an arbitrary quasi-Frobenius manifold the two metrics* $< \, , \, >$ *and* $(\, , \,)$ *form a flat pencil. Conversely, any flat pencil can locally be obtained in such a way.*

Remark 3.2.2 *The metric* $(\, , \,)$ *could be degenerate. However, the linear combination* $(\, , \,) - \lambda < \, , \, >$ *does not degenerate for generic* λ*.*

Example 3.2.3 *In the particular case of quadratic functions in* v

$$f_\beta(v) = \frac{1}{2} c_{\beta\lambda\mu} v^\lambda v^\mu$$

the structure constants

$$c^\gamma_{\alpha\beta} = \eta^{\gamma\nu} c_{\nu\alpha\beta}$$

define on the linear space $M = T_v M$ *a structure of* Novikov algebra*, i.e., an algebra satisfying (3.2.4) and also* left-symmetric*, i.e.,*

$$a \cdot (b \cdot c) - (a \cdot b) \cdot c = b \cdot (a \cdot c) - (b \cdot a) \cdot c \tag{3.2.7}$$

(the algebras satisfying the last identity are called Vinberg algebras or also pre-Lie algebras). The inner product $< \, , \, >$ *satisfying (3.2.5) is called an invariant bilinear form on the Novikov algebra. The above second Poisson bracket in this case is a linear Poisson bracket of hydrodynamic type. The theory of such Poisson brackets was first studied by A.Balinsky and S.P.Novikov [3].*

Proof of the theorem see in [40], [41].

The problem of local classification of quasi-Frobenius manifolds can be reduced to the theory of the following system of nonlinear PDEs

$$\partial_\alpha \partial_\lambda f_\beta \eta^{\lambda\mu} \partial_\mu \partial_\delta f_\gamma = \partial_\alpha \partial_\lambda f_\gamma \eta^{\lambda\mu} \partial_\mu \partial_\delta f_\beta \tag{3.2.8}$$

$$(\partial_\alpha f_\lambda + \partial_\lambda f_\alpha) \eta^{\lambda\mu} \partial_\mu \partial_\beta f_\gamma = (\partial_\beta f_\lambda + \partial_\lambda f_\beta) \eta^{\lambda\mu} \partial_\mu \partial_\alpha f_\gamma. \tag{3.2.9}$$

Very recently O. Mokhov [119] and E.Ferapontov [62] proved integrability of this system.

As it was essentially shown in [40] (see the precise formulation in [43]) that, under certain quasihomogeneity assumption the quasi-Frobenius structure on M reduces to a Frobenius one. In the next section we impose a somewhat different additional constraint onto the bihamiltonian structure (3.2.2) that will also give a correspondence between the normal forms of the bihamiltonian structures (3.1.15) on the extended loop spaces $\mathcal{L}(M)$ and Frobenius structures on M.

3.3 Tau-structures, tau-covers and normal coordinates of bihamiltonian hierarchies of PDEs

The densities of the commuting hamiltonians of the hierarchy (3.1.30) constructed in the Section 3.1.2 admit certain freedom in their definition. Indeed, instead of the densities f^i_p one can take

$$h_{i,p} = \sum_{q=-1}^{p} \sum_{j=1}^{n} a_{ij,pq} f^j_q + \text{total derivative} \tag{3.3.1}$$

with arbitrary constant coefficients $a_{ij,pq}$ satisfying the following nondegeneracy condition

$$\det\left(a_{ij,pp}\right) \neq 0, \quad p = -1, 0, 1, \ldots. \tag{3.3.2}$$

The flows of the new hierarchy

$$\frac{\partial}{\partial \tilde{t}^{i,p}} = \{\ .\ , \bar{h}_{i,p}\}_1, \quad \bar{h}_{i,p} := \int h_{i,p} dx, \quad i = 1, \ldots, n, \quad p = 0, 1, 2, \ldots \tag{3.3.3}$$

still commute.

Definition. 1). A *tau-structure* for a Poisson pencil (3.1.26) is a choice of the densities of the commuting hamiltonians in the class of equivalence w.r.t. the transformations (3.3.1) s.t. the 1-form

$$\omega = \sum_{p=0}^{\infty} \sum_{i=1}^{n} h_{i,p-1} d\tilde{t}^{i,p} \tag{3.3.4}$$

is closed, i.e.,

$$\frac{\partial h_{i,p-1}(u; u_x, \ldots; \epsilon)}{\partial t^{j,q}} = \frac{\partial h_{j,q-1}(u; u_x, \ldots; \epsilon)}{\partial t^{i,p}}. \tag{3.3.5}$$

A Poisson pencil admitting a tau-structure is called *tau-symmetric*.

2). We say that the tau-symmetric Poisson pencil is *compatible with spatial translations* if

$$\{\ .\ , \bar{h}_{1,0}\}_1 = a_{11,00}\{\ .\ , \bar{f}_0^1\}_1 = \frac{\partial}{\partial x}. \tag{3.3.6}$$

Clearly tau-symmetry is a geometric property of a Poisson pencil. Indeed, if a Poisson pencil $\{\ ,\ \tilde{\}}_\lambda$ is obtained from $\{\ ,\ \}_\lambda$ by means of a Miura-type transformation

$$u^i \mapsto \tilde{u}^i = F^i(u; u_x, \ldots; \epsilon)$$

then the densities $h_{i,p}(u; u_x, \ldots; \epsilon)$ satisfying (3.3.1), (3.3.5) considered as functions of the new coordinates $\tilde{u}^i$ will give a tau-structure for the pencil $\{\ ,\ \tilde{\}}_\lambda$ with the same coefficients $a_{ij,pq}$.

The functions

$$h_{i,-1} = h^{[0]}_{i,-1}(u) + \epsilon\, h^{[1]}_{i,-1}(u; u_x) + \ldots, \quad i = 1, \ldots, n \tag{3.3.7}$$

will be of particular importance for working with tau-symmetric bihamiltonan hierarchies.

Lemma 3.3.1 *The functions (3.3.7) define a Miura-type transformation*

$$u^i \mapsto \tilde{u}_i := h_{i,-1}, \quad i = 1, \ldots, n. \tag{3.3.8}$$

The functionals $\int \tilde{u}_1 dx, \ \ldots, \ \int \tilde{u}_n dx$ *are Casimirs of* $\{\ ,\ \}_1$.

Proof From the definition we have

$$h_{i,-1}(u;u_x,\dots;\epsilon) = \sum_j a_{ij,-1\,-1} f^j_{-1}(u;u_x,\dots;\epsilon) + \text{total derivative}$$

where the functions $f^j_p(u;u_x,u_{xx},\dots;\epsilon) \in \mathcal{A}_{0,0}(\epsilon)$ were constructed in the proof of Theorem 3.1.9. In particular

$$h^{[0]}_{i,-1}(u) = \sum_j a_{ij,-1\,-1} {f^j_{-1}}^{[0]}(u)$$

where

$$f^j_{-1}(u;u_x,\dots) = {f^j_{-1}}^{[0]}(u) + O(\epsilon).$$

Since $u^j \mapsto f^j_{-1}(u;u_x,\dots)$ is the transformation reducing $\{\ ,\ \}_1$ to the normal form, the nondegeneracy

$$\det\left(\frac{\partial {f^j_{-1}}^{[0]}(u)}{\partial u^k}\right) \neq 0$$

holds true. This implies the nondegeneracy

$$\det\left(\frac{\partial {h_{i,-1}}^{[0]}(u)}{\partial u^k}\right) \neq 0$$

also for the transformation (3.3.8). It remains to observe that the functionals

$$\int f^j_{-1}(u;u_x,\dots;\epsilon)\,dx$$

are Casimirs of the first Posson bracket. Taking an invertible linear combination of them and adding a total derivative will still give a system of Casimirs. □

Definition. The dependent variables $\tilde{u}_1\ \dots,\ \tilde{u}_n$ are called *normal coordinates* on $\mathcal{L}(M)$ w.r.t. the given tau-structure.

The first Poisson bracket in the normal coordinates has the form

$$\{\tilde{u}_i(x),\tilde{u}_j(y)\}_1 = \eta_{ij}\delta'(x-y) + O(\epsilon) \tag{3.3.9}$$

with a constant invertible matrix (η_{ij}). The variables

$$\tilde{u}^i := \eta^{ij}\tilde{u}_j, \quad \left(\eta^{ij}\right) := \left(\eta_{ij}\right)^{-1}$$

will also be called the normal coordinates. The tilde over the normal coordinates will often be omitted. The equations of the hierarchy in the normal coordinates are written in the following form

$$\frac{\partial u_i}{\partial t^{j,q}} = \partial_x \Omega_{i,0;j,q}(u;u_x,\dots;\epsilon) \tag{3.3.10}$$

where the matrix $\Omega_{i,p;j,q}(u; u_x, \dots; \epsilon)$ is defined in the formula (3.3.13) below.

We will settle below the problem of ambiguity in the choice of tau-structure of a given Poisson pencil under certain additional assumption about the pencil. In particular we will describe the freedom in the choice of normal coordinates.

Let us denote

$$\mathcal{K} = \mathcal{K}(M; \{\ ,\ \}_1, \{\ ,\ \}_2) := \cap_\lambda \mathrm{Ker}\ (\{\ ,\ \}_2 - \lambda \{\ ,\ \}_1) \in \Lambda_0(M)$$

the subspace spanned by the commuting Hamiltonians $\bar{h}_{i,p}$ of the hierarchy. This can be lifted to a subspace

$$\hat{\mathcal{K}} = \in \mathcal{A}_{0,1}(M)$$

w.r.t. to the factorization map $\mathcal{A}_{0,1}(M) \xrightarrow{\pi} \Lambda_0(M)$ (all the notations as in Section 2). Namely, the Hamiltonian $\bar{h}_{i,p}$ lifts to the density $h_{i,p}$ satisfying (3.3.3). We will now define an important symmetric *product map*

$$* : \hat{\mathcal{K}} \times \hat{\mathcal{K}} \to \mathcal{A}_{0,0}(M) \tag{3.3.11}$$

as follows. Due to commutativity of the flows there exists an infinite matrix of the densities of the fluxes of the conserved quantities

$$\Omega_{i,p;j,q}(u; u_x, u_{xx}, \dots; \epsilon) = \sum_{k=0}^{\infty} \epsilon^k \Omega^{[k]}_{i,p;j,q}(u; u_x, \dots, u^{(k)}) \in \mathcal{A}_{0,0}, \tag{3.3.12}$$
$$1 \le i, j \le n, \quad p, q = 0, 1, 2, \dots$$

such that

$$\frac{\partial h_{i,p-1}(u; u_x, \dots; \epsilon)}{\partial t^{j,q}} = \partial_x \Omega_{i,p;j,q}(u; u_x, u_{xx}, \dots; \epsilon) \tag{3.3.13}$$

(we will omit tilde over the time variables always assuming the hierarchy to be written in the normal coordinates). Observe the shift $p \mapsto p-1$ in the level of the conserved quantity $\bar{h}_{i,p-1}$. The matrix $\Omega_{i,p;j,q}$ is determined up to adding of constants.

Lemma 3.3.2 *The collection of the densities $h_{i,p}(u; u_x, \dots; \epsilon)$ of the commuting Hamiltonians $\bar{h}_{i,p}$ of the form (3.3.1), (3.3.2) is a tau-structure* iff *the matrix $\Omega_{i,p;j,q} = \Omega_{i,p;j,q}(u; u_x, \dots; \epsilon)$ is symmetric*

$$\Omega_{i,p;j,q} = \Omega_{j,q;i,p}. \tag{3.3.14}$$

Proof The condition of closedness of the 1-form (3.3.4) written in (3.3.5) is equivalent to the symmetry (3.3.14). □

We define the product map (3.3.11) on the basis of the space $\mathcal{K}$ putting

$$(h_{i,p}, h_{j,q}) \mapsto h_{i,p} * h_{j,q} := \Omega_{i,p+1;j,q+1}. \tag{3.3.15}$$

All the equations of the constructed hierarchy have the form (1.7). We define solutions to the hierarchy as vector functions

$$u^i = u^i(\mathbf{t};\epsilon) = \sum_{k=0}^{\infty} \epsilon^k u^i_{[k]}(\mathbf{t}), \quad i = 1,\dots,n, \quad \mathbf{t} = \left(t^{i,p}\right)_{1\le i\le n,\ p=0,1,\dots}$$

satisfying all the equations of the hierarchy as formal series in ϵ. Compatibility of the equations of the hierachy gives a possibility to solve them simultaneously. Actually there is a problem of the definition of an appropriate class of functions of infinite number of variables. This will be done below where we will also construct the solution to the Cauchy problem for the hierarchy with the initial data in a suitable class of functions of x.

Corollary 3.3.3 *For every solution*

$$u^i = u^i(\mathbf{t};\epsilon), \quad i = 1,\dots,n$$

of a tau-symmetric hierarchy there exists a function $\tau = \tau(\mathbf{t};\epsilon)$ *such that*

$$\Omega_{i,p;j,q}(u(\mathbf{t};\epsilon); u_x(\mathbf{t};\epsilon),\dots;\epsilon) = \epsilon^2 \frac{\partial^2 \log\tau}{\partial t^{i,p}\partial t^{j,q}}.$$

The function $\tau(\mathbf{t};\epsilon)$ *is determined uniquely up to a transformation of the form*

$$\tau(\mathbf{t};\epsilon) \mapsto e^{\frac{1}{2\epsilon^2}\sum Q_{i,p;j,q}(\epsilon)t^{i,p}t^{j,q}}\tau(\mathbf{t};\epsilon)$$

with a constant symmetric matrix $Q_{i,p;j,q}(\epsilon)$.

The function τ will be called *tau-function* of a given tau-structure of the bihamiltonian hierarchy.

For a tau-structure compatible with spatial translations we have

$$\frac{\partial}{\partial t^{1,0}} = \frac{\partial}{\partial x}.$$

Therefore the solution to the hierarchy, when written in the normal coordinates reads

$$u_i(\mathbf{t};\epsilon) = \epsilon^2 \frac{\partial^2 \log\tau}{\partial x \partial t^{i,0}}, \quad i = 1,\dots,n. \tag{3.3.16}$$

In the study of symmetries of integrable PDEs it will be useful the following extension of the PDEs to systems with an infinite family of dependent variables f, $f_{i,p}$, $i = 1,\dots,n$, $p = 0,\ 1,\ \dots$, $u_1,\ \dots,\ u_n$:

$$\begin{aligned}
\epsilon \frac{\partial f}{\partial t^{j,q}} &= f_{j,q}\\
\epsilon \frac{\partial f_{i,p}}{\partial t^{j,q}} &= \Omega_{i,p;j,q}(u; u_x,\dots;\epsilon)\\
\frac{\partial u_i}{\partial t^{j,q}} &= \partial_x \Omega_{i,0;j,q}(u; u_x,\dots;\epsilon).
\end{aligned} \tag{3.3.17}$$

Definition. The system (3.3.17) of PDEs is called *tau-cover* of the hierarchy (3.3.10).

It is clear that the hierarchy (3.3.17) is still commutative.

The notion of tau-function was discovered by Date, Jimbo, Kashiwara and Miwa [17] in their study of KP hierarchy. In the next section we will construct the tau-structure for the KdV hierarchy.

3.4 Example: tau-structure of KdV and heat kernel expansion

Let us define the differential polynomials

$$h_k = h_k(u, \epsilon\, u_x, \epsilon^2 u_{xx}, \dots, \epsilon^{2k+2} u^{(2k+2)})$$

as the Seeley coefficients [132] of the Lax operator $L = -\epsilon^2 \partial_x^2 + u$, i.e., as the coefficients of the asymptotic expansion at $z \to 0$ of the diagonal of the heat kernel

$$< x \,|\, e^{-z\,(-\epsilon^2 \partial_x^2 + u(x))} \,|\, x > \sim \frac{1}{\sqrt{4\,\pi\, z}} \sum_{k=0}^{\infty} (-z)^k h_{k-2}, \quad h_{-2} = 1. \tag{3.4.1}$$

Note that the coefficients of the asymptotic expansion (3.4.1) do not depend on the choice of the boundary conditions for the Lax operator [132]. They are related to the gradients of the old Hamiltonians by

$$h_k := \prod_{i=1}^{k+1} \left(i + \frac{1}{2}\right)^{-1} \frac{\delta I_{k+1}}{\delta u(x)}, \quad k = -1, 0, 1, \dots \tag{3.4.2}$$

We want to prove that these differential polynomials define a tau-structure of the KdV hierarchy, i.e. the following symmetry holds true

$$\{h_{k-1}, \bar{h}_l\}_1 = \{h_{l-1}, \bar{h}_k\}_1. \tag{3.4.3}$$

Observe that, redefining the flows of the hierarchy as follows

$$\frac{\partial u}{\partial \tilde{t}^k} := \{u(x), \bar{h}_k\}_1 = \partial_x \frac{\delta \bar{h}_k}{\delta u(x)}, \quad \bar{h}_k = \int h_k dx, \quad k = 0, 1, \dots \tag{3.4.4}$$

we obtain the same hierarchy up to a change of normalization of the flows

$$\frac{\partial u}{\partial \tilde{t}^k} = \prod_{i=1}^{k} \left(i + \frac{1}{2}\right)^{-1} \frac{\partial u}{\partial t^k}. \tag{3.4.5}$$

The symmetry (3.4.3) means that the new densities are coefficients of a closed 1-form

$$\omega = h_{-1} d\tilde{t}^0 + h_0 d\tilde{t}^1 + h_1 d\tilde{t}^2 + \dots. \tag{3.4.6}$$

Explicitly the first few Hamiltonians and the corresponding flows of the hierarchy are

$$h_{-1} = u, \quad h_0 = \frac{u^2}{2} - \epsilon^2 \frac{u''}{6}, \quad h_1 = \frac{u^3}{6} - \frac{\epsilon^2}{12}(u'^2 + 2u\, u'') + \epsilon^4 \frac{u^{IV}}{60},$$

$$h_2 = \frac{u^4}{24} - \frac{\epsilon^2}{12}(u\, u'^2 + u^2 u'') + \frac{\epsilon^4}{120}(3u''^2 + 4u'u''' + 2u\, u^{IV}) - \frac{\epsilon^6}{840} u^{VI}$$

$$u_{\tilde t^0} = u', \quad u_{\tilde t^1} = u\, u' - \frac{\epsilon^2}{6} u''',$$

$$u_{\tilde t^2} = \frac{u^2 u'}{2} - \frac{\epsilon^2}{6}(2u'u'' + u\, u''') + \frac{\epsilon^4}{60} u^V.$$

To prove that the Seeley coefficients define a tau-structure of the KdV hierarchy let us recall some identities for the generating functional

$$p = p(\lambda) = k + \sum_{m=1}^{\infty} \frac{\bar\chi_m}{k^m}, \quad k = \sqrt{\lambda}$$

(see, e.g., [33]). Let us introduce the "real part"

$$\chi_R := k + \sum_{j=0}^{\infty} \frac{\chi_{2j+1}}{k^{2j+1}}.$$

Then we have

$$\frac{\delta p}{\delta u(x)} = -\frac{1}{2\chi_R}$$

and

$$\frac{dp}{d\lambda} = \int \frac{1}{2\chi_R} dx.$$

From the last two formulae it immediately follows that

$$\int \frac{\delta \bar\chi_{2j+3}}{\delta u(x)} dx = \left(j + \frac{1}{2}\right) \bar\chi_{2j+1}.$$

This proves that the new flows (3.4.4) are proportional to the equations of the KdV hierarchy with the coefficient of proportionality given in (3.4.5). From the same formula it follows that the new densities of the Hamiltonians satisfy the recursion similar to (3.1.21) but with a different normalization

$$\left(-\frac{1}{4}\epsilon^2 \partial_x^3 + u\partial_x + \frac{1}{2} u_x\right) \frac{\delta \bar h_{j-1}}{\delta u(x)} = \left(j + \frac{1}{2}\right) \partial_x \frac{\delta \bar h_j}{\delta u(x)}.$$

All this means that the new Hamiltonians can be obtained from the old ones by multiplying a nonzero constant and by adding of a total derivative.

The generating function of the densities h_k coincides with

$$w(x,\lambda) := \frac{\sqrt{\lambda}}{\chi_R(x,\lambda)} = 1 + \frac{h_{-1}}{2\lambda} + \sum_{k=2}^{\infty} \frac{(2k-1)!!h_{k-2}}{(2\lambda)^k}.$$

We are now to prove the symmetry

$$\frac{\partial h_{i-1}}{\partial \tilde{t}^j} = \frac{\partial h_{j-1}}{\partial \tilde{t}^i} \tag{3.4.7}$$

for each pair (i, j). We will use the following formula [33]

$$\{\chi_R(x,\lambda), p(\mu)\}_1 = \frac{1}{8\,(\mu-\lambda)} \left(\frac{\chi_R(x,\lambda)}{\chi_R(x,\mu)}\right)' \tag{3.4.8}$$

(here $' = d/dx$, μ is an indeterminate). The both sides of the formula are understood as formal series in inverse powers of $\sqrt{\mu}$. From this it easily follows that

$$-\frac{1}{4} \sum_{i,j=0}^{\infty} \frac{(2i+1)!!\,(2j+1)!!}{(2\,\mu)^{i+1}\,(2\,\lambda)^{j+1}} \frac{\partial h_{j-1}}{\partial \tilde{t}^i} = \frac{1}{\mu-\lambda}[w(x,\mu)w'(x,\lambda) - w'(x,\mu)w(x,\lambda)].$$

The symmetry of the r.h.s. proves (3.4.7).

Using the formula (28) from [68] we can represent the r.h.s. as the total derivative of the function

$$\begin{aligned} &\frac{2}{\mu-\lambda}[w(x,\mu)w'(x,\lambda) - w'(x,\mu)w(x,\lambda)] \\ &= \partial_x \frac{1}{(\mu-\lambda)^2}\left[w''(\mu)w(\lambda) + w(\mu)w''(\lambda) - w'(\mu)w'(\lambda)\right. \\ &\quad \left. +2(\lambda+\mu-2\,u)\,w(\mu)w(\lambda) - 2(\lambda+\mu)\right]. \end{aligned} \tag{3.4.9}$$

It would be interesting to derive the symmetry (3.4.7) directly from the properties of the heat kernel (cf. the recent paper [111] where an analogous symmetry of the derivatives of the Green function of the Dirichlet boundary value problem on the plane has been derived from the Hadamard variational formula for the Green function).

Remark 3.4.1 *In [13] it was suggested an approach to the theory of tau-functions of the KP hierarchy and of its reductions. This approach is based on the theory of exact Poisson pencils (see Example 3.1.7 above). Let*

$$\{\ ,\ \}_\lambda = \{\ ,\ \}_2 - \lambda\{\ ,\ \}_1$$

be an exact Poisson pencil

$$Lie_Z\{\ ,\ \}_2 = \{\ ,\ \}_1, \quad Lie_Z\{\ ,\ \}_1 = 0$$

for some vector field Z. Let us assume that the vector field Z satisfies the conditions of the lemma 2.4.9. Doing if necessary a Miura-type transformation we may assume that

$$Z = \frac{\partial}{\partial u^1}.$$

Let

$$u^i \mapsto \tilde{u}^i = F^i(u^1, \dots, u^n; u^1_x, \dots, u^n_x, \dots; \epsilon)$$

be the reducing transformation for the Poisson bracket $\{\ ,\ \}_2$. Then the λ-dependent reducing transformation for the pencil described in Theorem 3.1.9 must have the form

$$u^i \mapsto \tilde{u}^i = F^i(u^1 - \lambda, \dots, u^n; u^1_x, \dots, u^n_x, \dots; \epsilon) \tag{3.4.10}$$

assuming that the r.h.s. is analytic at $\lambda = \infty$. If this is the case then we can derive, following [13] that

$$\frac{\delta \bar{f}^i_p}{\delta u^1(x)} = p\, f^i_{p-1} + \text{total derivative.} \tag{3.4.11}$$

Indeed, from

$$\frac{\partial}{\partial \lambda} \int F^i(u^1 - \lambda, \dots, u^n; u^1_x, \dots, u^n_x, \dots; \epsilon)\, dx = -\int \frac{\delta \bar{F}^i}{\delta u^1(x)} dx$$

it follows that

$$\frac{\partial}{\partial \lambda} F^i(u^1 - \lambda, \dots, u^n; u^1_x, \dots, u^n_x, \dots; \epsilon) = -\frac{\delta}{\delta u^1(x)} \int F^i\, dx + \text{total derivative.}$$

From (3.4.11) it follows that the coefficients

$$m^i_{p-1} := \frac{\delta \bar{f}^i_p}{\delta u^1(x)} \tag{3.4.12}$$

of the expansion of $\delta F^i / \delta u^1(x)$ are densities of conserved quantities of the bihamiltonian hierarchy. They were used in [13] in order to define the tau-function.

In the particular case of KdV one is to use, following the above prescription, the differential polynomials

$$m_{k-1} := \frac{\delta I_k}{\delta u(x)},$$

$$\int m_{k-1} dx = \frac{2k+1}{2} I_{k-1}$$

in order to define the tau-function of the KdV hierarchy

$$m_{k-1} = \epsilon^2 \frac{\partial^2 \log \tau}{\partial x\, \partial t_k}.$$

Instead of the symmetry (3.4.3) one obtains

$$\frac{2k+3}{2}\{m_{k-1}, \bar{m}_l\}_1 = \frac{2l+3}{2}\{m_{l-1}, \bar{m}_k\}_1. \tag{3.4.13}$$

The main problem with extending this approach to more general class of exact Poisson pencils is that to describe the analytic properties for large λ of the reducing transformation defined for small λ by the formula (3.4.10). Our axioms imply that this transformation always has a regular singularity at $\lambda = \infty$ described by the formulae (3.6.66) -(3.6.68) below.

3.5 From tau-structures to Frobenius manifolds

Here we will construct the main invariant of an arbitrary tau-symmetric Poisson pencil of $(0, n)$ brackets on $\mathcal{L}(\hat{\mathcal{M}})$ with respect to the action of Miura group. We will prove that every such a bihamiltonian structure satisfying certain genericity assumption determines a Frobenius structure on M.

Actually our invariant will depend only on the leading order (3.2.2) of the bihamiltonian structure. We will prove the following general result: classes of equivalence of tau-symmetric bihamiltonian structures of the form (3.2.2) on $\mathcal{L}(M)$ satisfying certain genericity assumption are in one-to-one correspondence with Frobenius structures on M.

All the calculations will be done in the system of flat coordinates $v^1, \ldots, v^n$ for the metric g_1^{ij}. Denote $\eta^{\alpha\beta}$ the (constant) components of the metric g_1^{ij} and $g^{\alpha\beta}(v)$ and $\Gamma_\gamma^{\alpha\beta}(v)$ the components of the metric g_2^{ij} and the Levi-Civita connection for this metric. Recall (see Section 3.2 above) that, in these coordinates $g^{\alpha\beta}(v)$ and $\Gamma_\gamma^{\alpha\beta}(v)$ are expressed in terms of certain functions $f_1(v), \ldots, f_n(v)$ as follows

$$\begin{aligned} \Gamma_\gamma^{\alpha\beta}(v) &= \partial^\alpha \partial_\gamma f^\beta(v), \\ g^{\alpha\beta}(v) &= \partial^\alpha f^\beta(v) + \partial^\beta f^\alpha(v). \end{aligned} \tag{3.5.1}$$

Here all raising and lowering of the Greek indices is to be done with the help of the matrix $(\eta^{\alpha\beta})$ and the inverse one $(\eta_{\alpha\beta}) = (\eta^{\alpha\beta})^{-1}$. E.g.,

$$\partial^\alpha := \eta^{\alpha\lambda}\partial_\lambda.$$

We first observe that the densities of the commuting flows of the hierarchy determined by the bihamiltonian structure (3.2.2) are just functions on M. They are determined by the following procedure.

Lemma 3.5.1 *The bihamiltonian hierarchy determined by (3.2.2) in the flat coordinates $(v^1, \ldots, v^n)$ for $\{\ ,\ \}_1^{[0]}$ has the form*

$$\frac{\partial v^\beta}{\partial t^{\alpha,p}} = \partial_x \left(\partial^\beta \phi_{\alpha,p}(v)\right)$$

where the functions $\phi_{\alpha,p}(v)$ on M are the coefficients of the series solution

$$\phi_\alpha(v;\lambda) = \sum_{p=-1}^{\infty} \frac{\phi_{\alpha,p}(v)}{\lambda^{p+1}}$$

to the linear system

$$\left(g^{\beta\epsilon}(v) - \lambda\,\eta^{\beta\epsilon}\right)\partial_\epsilon\partial_\gamma\phi + \Gamma^{\beta\epsilon}_\gamma(v)\partial_\epsilon\phi = 0, \quad \beta,\,\gamma = 1,\dots,n \tag{3.5.2}$$

such that

$$\phi_{\alpha,-1}(v) = v_\alpha := \eta_{\alpha\epsilon}v^\epsilon.$$

Proof The system (3.5.2) is just the spelling of the definition

$$\{\ .\ , \bar\phi(v;\lambda)\}^{[0]}_\lambda = 0$$

of the Casimir of the pencil. □

Observe that the system (3.5.2) coincides with the equations for the flat coordinates for the flat pencil $g_2^{ij} - \lambda\,g_1^{ij}$ written in the flat coordinates for g_1^{ij}.

Corollary 3.5.2 *The flows of the lower level of the hierarchy have the form*

$$\frac{\partial v^\beta}{\partial t^{\alpha,0}} = \partial_x(\partial^\beta f_\alpha(v)).$$

Proof For the coefficient $\phi_{\alpha,0}(v)$ of λ^0 we obtain

$$\partial^\beta\partial_\gamma\phi_{\alpha,0} = \Gamma^{\beta\epsilon}_\gamma\eta_{\epsilon\alpha} = \partial^\beta\partial_\gamma f_\alpha(v)$$

where the functions $f_\alpha(v)$ were defined in (3.5.1). So

$$\phi_{\alpha,0}(v) = f_\alpha(v) + \text{linear}.$$

Adding of a function linear in v does not affect the flat pencil neither the Hamiltonian equations. □

By the assumption the densities $h^{[0]}_{\alpha,p}(v)$ are related to the densities $\phi_{\alpha,p}(v)$ constructed in Lemma 3.5.1 by an invertible triangular transformation with constant coefficients. In particular we may assume that

$$h^{[0]}_{\alpha,-1}(v) = v_\alpha$$

and

$$h^{[0]}_{\alpha,0}(v) = a^\beta_\alpha f_\beta(v) + \text{linear function}. \tag{3.5.3}$$

Actually the linear function in the last formula can be absorbed by a redefinition of $f_\beta(v)$.

Theorem 3.5.3 *Let (3.2.2) be a tau-symmetric bihamiltonian structure. Introduce the matrices*

$$a = (a^\beta_\alpha), \quad b := a^{-1} = (b^\beta_\alpha), \quad b^{\alpha\beta} := \eta^{\alpha\lambda} b^\beta_\lambda.$$

Then there exists a function $F(v)$ s.t.

$$\Gamma^{\alpha\beta}_\gamma(v) = b^{\beta\epsilon}\partial^\alpha\partial_\gamma\partial_\epsilon F(v), \tag{3.5.4}$$

$$g^{\alpha\beta}(v) = b^{\alpha\epsilon}\partial_\epsilon\partial^\beta F(v) + b^{\beta\epsilon}\partial_\epsilon\partial^\alpha F(v) + g_0^{\alpha\beta}. \tag{3.5.5}$$

The function $F(v)$ satisfies associativity equations

$$\partial_\alpha\partial_\beta\partial_\lambda F(v)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\delta F(v) = \partial_\delta\partial_\beta\partial_\lambda F(v)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\alpha F(v), \tag{3.5.6}$$

$$\alpha, \beta, \gamma, \delta = 1, \dots, n. \tag{3.5.7}$$

The function $F(v)$ is determined uniquely up to adding of at most quadratic polynomial.

Proof The first of the symmetry conditions (3.3.5) reads

$$\frac{\partial v_\alpha}{\partial t^{\beta,0}} = \frac{\partial v_\beta}{\partial t^{\alpha,0}}.$$

Due to (3.5.3) it implies

$$a^\epsilon_\alpha \partial_\beta f_\epsilon(v) = a^\epsilon_\beta \partial_\alpha f_\epsilon(v).$$

Therefore there exists a function $F(v)$ s.t.

$$a^\epsilon_\alpha f_\epsilon(v) = \partial_\alpha F(v).$$

Hence

$$f_\alpha(v) = b^\lambda_\alpha \partial_\lambda F(v).$$

The freedom in adding of linear functions to $f_\alpha(v)$ corresponds to the freedom in adding of a quadratic polynomial to $F(v)$.

It remains to prove that the algebra with the structure constants

$$c^\gamma_{\alpha\beta}(v) := \partial_\alpha\partial_\beta\partial^\gamma F(v)$$

is associative for any $t \in M$. Indeed, from the quasi-Frobenius property

$$\Gamma^{\alpha\beta}_\epsilon \Gamma^{\epsilon\gamma}_\delta = \Gamma^{\alpha\gamma}_\epsilon \Gamma^{\epsilon\beta}_\delta$$

for the Christoffel coefficients

$$\Gamma^{\alpha\beta}_\gamma(v) = c^\alpha_{\epsilon\gamma}(v) b^{\beta\epsilon}$$

we obtain

$$b^{\beta\lambda} b^{\gamma\mu}\left[c^\alpha_{\epsilon\lambda}(v)c^\epsilon_{\mu\delta}(v) - c^\alpha_{\epsilon\mu}(v)c^\epsilon_{\lambda\delta}(v)\right] = 0.$$

This proves associativity. □

Due to the Theorem the formula

$$\partial_\alpha \cdot \partial_\beta := c_{\alpha\beta}^\gamma(v)\partial_\gamma \tag{3.5.8}$$

where

$$c_{\alpha\beta}^\gamma(v) := \eta^{\gamma\epsilon}\partial_\epsilon\partial_\alpha\partial_\beta F(v) \tag{3.5.9}$$

defines on the tangent planes T_vM a structure of commutative associative algebra.

We will now put into the game the condition of compatibility with spatial translations.

Lemma 3.5.4 *If the tau-structure is compatible with spatial translations then the function $F(v)$ constructed in Theorem 3.5.3 satisfies*

$$\partial_1\partial_\alpha\partial_\beta F(v) = \eta_{\alpha\beta}.$$

Proof By definition we must have

$$a_1^\epsilon \partial_\beta f_\epsilon(v) = v_\beta + \text{const},$$

i.e.,

$$\partial_1\partial_\beta F(v) = v_\beta + v_\beta^0 \tag{3.5.10}$$

for a constant shift v_β^0. Adding if necesary a quadratic polynomial to $F(v)$ we end the proof. □

According to the lemma the vector field

$$e := \partial/\partial v^1 \tag{3.5.11}$$

is the unity of the Frobenius algebra on T_vM at every point $v \in M$. Actually we have a somewhat stronger condition

$$\frac{\partial}{\partial \tilde{t}^{1,0}} = c\frac{\partial}{\partial t^{1,0}} = \frac{\partial}{\partial x}, \quad a_1^\alpha = c\,\delta_1^\alpha \tag{3.5.12}$$

for some nonzero constant c.

We are now to construct the Euler vector field on M. Let us introduce the linear function $\varphi(v) = v_1 := \eta_{1\epsilon}v^\epsilon$. The gradient of this function w.r.t. to the second metric we will denote E.

Lemma 3.5.5 *The components $E^\alpha(v)$ of the vector field $E = E^\alpha(v)\partial_\alpha$ are the following linear functions on M*

$$E^\alpha(v) = c\,v^\alpha + \bar{b}_\beta^\alpha(v^\beta + v_0^\beta) + \eta_{1\epsilon}g_0^{\epsilon\alpha} \tag{3.5.13}$$

where $\bar{b}_\beta^\alpha$ is the matrix of the operator adjoint to b_β^α,

$$\bar{b}_\beta^\alpha := \eta^{\alpha\lambda}\eta_{\beta\mu}b_\lambda^\mu.$$

Proof By definition

$$E^\alpha = g^{\alpha\epsilon}(v)\partial_\epsilon\varphi(v) = \eta_{1\epsilon}g^{\alpha\epsilon}(v) = \partial^\alpha f_1(v) + \partial_1 f^\alpha(v) + \eta_{1\epsilon}g_0^{\epsilon\alpha}.$$

Using $b_1^\alpha = c\,\delta_1^\alpha$ we obtain

$$\partial^\alpha f_1 = c\,v^\alpha.$$

Representing

$$f^\alpha(v) = \bar{b}^\alpha_\beta \partial^\beta F(v)$$

and using (3.5.10) we end the proof. □

Lemma 3.5.6 *Denote (,) the bilinear form on T_v^*M corresponding to the second metric. Then*

$$(\omega_1, \omega_2) = i_E\omega_1 \cdot \omega_2. \tag{3.5.14}$$

Here ω_1, $\omega_2 \in T_v^*M$ are two 1-forms on M. There product is the 1-form defined via the product of tangent vectors on M with the help of the isomorphism

$$\eta : T_vM \to T_v^*M.$$

Proof From the condition of symmetry of the Levi-Civita connection

$$g^{\alpha\epsilon}\Gamma_\epsilon^{\beta\gamma} = g^{\beta\epsilon}\Gamma_\epsilon^{\alpha\gamma}$$

we derive that

$$g^{\alpha\epsilon}c_\epsilon^{\beta\gamma} = g^{\beta\epsilon}c_\epsilon^{\alpha\gamma}.$$

Multiplying the last equation by $\partial_\beta\varphi(v)$ and taking the sum w.r.t. β we obtain

$$E^\epsilon c_\epsilon^{\alpha\gamma} = g^{\alpha\gamma}.$$

□

Definition. We say that a solution $F(v)$ to the associativity equations (3.5.6) is *rigid* if every constant invariant symmetric bilinear form on the algebras T_vM is proportional to $< \, , >$. A bihamiltonian structure (3.1.15) - (3.1.17) possesing of a tau-structure is called rigid if the corresponding solution to the associativity equations is.

Every cubic solution to the associativity equations is not rigid: taking an arbitrary constant linear form l we define a constant invariant symmetric bilinear form by

$$(a, b)_l := l(a \cdot b), \quad a,\ b \in T_vM.$$

Observe that the structure constants $c_{\alpha\beta}^\gamma(v)$ of such a Frobenius manifold do not depend on v. Conversely, nonrigid solutions to the associativity equations can be characterized by the following statement.

Lemma 3.5.7 *A solution $F(v)$ to the associativity equations is not rigid if and only if there exists a constant (in the flat coordinates v^α) vector field w non proportional to e such that*

$$\partial_w F(v) = \text{at most quadratic polynomial in } v. \quad (3.5.15)$$

Proof . Let (,) be a constant invariant symmetric bilinear form on T_vM non proportional to $<\ ,\ >$. We introduce a constant linear form

$$l(a) := (a, e)$$

and a constant vector $w = (w^\alpha)$ dual to l, i.e. $l(a) =< w, a >$,

$$w^\alpha = \eta^{\alpha\beta} l_\beta, \quad l_\beta := l(\partial_\beta).$$

Since

$$(\partial_\alpha, \partial_\beta) = c^\gamma_{\alpha\beta}(v) l_\gamma,$$

the sum

$$w^\epsilon c^\gamma_{\alpha\epsilon}(v)$$

does not depend on v. Differentiating the last equation along v^β we obtain

$$\partial_w c^\gamma_{\alpha\beta}(v) = 0.$$

That means validity of (3.5.15). Inverting the arguments, we also prove the converse statement. □

There is a simple way to generate new solutions to the equations of associativity starting from a non rigid solution.

Lemma 3.5.8 *Let $F(v)$ be a solution to the equations of associativity and w be the constant vector field satisfying (3.5.15). There locally exists a function $F^w(v)$ and a constant antisymmetric linear operator $\rho = (\rho^\alpha_\beta)$ such that*

$$\nabla F^w(v) = w \cdot \nabla F(v) + \frac{1}{2}\rho(v) \quad (3.5.16)$$

where

$$(\rho(v))^\alpha = \rho^\alpha_\beta v^\beta.$$

The function $F^w(v)$ satisfies associativity equations (3.5.6) with the same $\eta^{\alpha\beta}$.

Proof Let us introduce the following 1-form

$$\omega_\gamma := q_{\gamma\nu}\nabla^\nu F(v) = w^\epsilon c^\nu_{\gamma\epsilon}(v)\partial_\nu F(v).$$

Here $q_{\gamma\nu} := w^\epsilon c_{\gamma\epsilon\nu}(v)$ by assumption is a constant symmetric matrix. Differentiating the above equation twice along v^α and v^β and using the associativity we obtain

$$\partial_\alpha\partial_\beta\omega_\gamma = w^\nu c^\epsilon_{\nu\alpha}(v) c_{\epsilon\beta\gamma}(v).$$

Due to the symmetry of the r.h.s. in β and γ, there exists a constant skew-symmetric matrix $\rho_{\beta\gamma}$ such that

$$\partial_\beta \omega_\gamma - \partial_\gamma \omega_\beta = \rho_{\beta\gamma}.$$

The one-form

$$\omega_\gamma - \frac{1}{2}\rho_{\gamma\nu} v^\nu$$

is closed. Therefore it is locally equal to the differential of a function we denote $F^w(v)$.

To prove the last statement of Lemma we observe that

$$\partial_\alpha \partial_\beta \partial_\lambda F^w(v) \eta^{\lambda\mu} \partial_\mu \partial_\gamma \partial_\delta F^w(v) = \partial_\alpha \partial_\beta \partial_\lambda F(v) \eta_w^{\lambda\mu} \partial_\mu \partial_\gamma \partial_\delta F(v)$$

where

$$\eta_w^{\lambda\mu} = q^\lambda_{\lambda'} q^\mu_{\mu'} \eta^{\lambda'\mu'} = (w \cdot w)^\nu c_\nu^{\lambda\mu}(v)$$

is a constant invariant symmetric bilinear form on $T_v M$. This proves validity of the associativity equations for the new function $F^w(v)$. □

Theorem 3.5.9 *For a solution $F(v)$ to the associativity equations corresponding to a tau-symmetric bihamiltonian structure compatible with spatial translations the constant vector $w = (w^\alpha)$*

$$w^\alpha = 2c\,\delta_1^\alpha + \bar b_1^\alpha \tag{3.5.17}$$

satisfies (3.5.15). The function $F(v)$ satisfies

$$\partial_E F(v) = 2F^w(v) + \text{quadratic} \tag{3.5.18}$$

The Lie bracket of the vector fields E, e equals

$$[e, E] = w - c\, e. \tag{3.5.19}$$

Proof From the equation (3.5.1) we obtain

$$E^\epsilon \partial_\epsilon \partial_\alpha \partial_\beta F = b_\alpha^\epsilon \partial_\epsilon \partial_\beta F + b_\beta^\epsilon \partial_\epsilon \partial_\alpha F + g_{0\alpha\beta}.$$

We can rewrite it as

$$\partial_\alpha \partial_\beta [E^\epsilon \partial_\epsilon F] = Q_\alpha^\epsilon \partial_\epsilon \partial_\beta F + Q_\beta^\epsilon \partial_\epsilon \partial_\alpha F + g_{0\alpha\beta} \tag{3.5.20}$$

where

$$Q_\alpha^\beta = c\,\delta_\alpha^\beta + b_\alpha^\beta + \bar b_\alpha^\beta,$$

the constant c was defined in (3.5.12). The compatibility conditions

$$\partial_\gamma [Q_\alpha^\epsilon \partial_\epsilon \partial_\beta F + Q_\beta^\epsilon \partial_\epsilon \partial_\alpha F] = \partial_\beta [Q_\alpha^\epsilon \partial_\epsilon \partial_\gamma F + Q_\gamma^\epsilon \partial_\epsilon \partial_\alpha F]$$

implies

$$Q_\beta^\epsilon c_{\epsilon\alpha\gamma}(v) = Q_\gamma^\epsilon c_{\epsilon\alpha\beta}(v).$$

This means invariance of the constant symmetric bilinear form

$$< Q\, x, y >= c < x, y > + < b\, x, y > + < x, b\, y > .$$

Therefore the matrix Q^α_β can be represented in the form

$$Q^\alpha_\beta = w^\epsilon c^\alpha_{\epsilon\beta}(v), \quad w^\alpha = Q^\alpha_1.$$

This gives the formula (3.5.17) since $b^\alpha_1 = c\,\delta^\alpha_1$. Using Lemma 3.5.7 we arrive at the proof of (3.5.15).

To prove (3.5.18) we differentiate (3.5.20) along v^γ and use associativity to obtain

$$\partial_\alpha\partial_\beta\partial_\gamma(\partial_E F(v)) = 2Q^\epsilon_\gamma c_{\epsilon\alpha\beta}(v) = \partial_\alpha\partial_\beta[2(w\cdot\nabla F(v))_\gamma] = 2\partial_\alpha\partial_\beta\partial_\gamma F^w(v).$$

The Theorem is proved. □

Recall [41] that a solution $F(v)$ to the associativity equations (3.5.6) possesing a constant unity vector field e defines on $M \ni v$ a structure of Frobenius manifold if a linear vector field E exists on M s.t.

$$\partial_E F(v) = (3-d)F(v) + \text{quadratic}, \quad [e, E] = e. \tag{3.5.21}$$

Here d is a constant called *the charge* of the Frobenius manifold. The linear vector field is called *Euler vector field* of the Frobenius manifold. If a linear vector field E satisfies

$$\partial_E F(v) = k\, F(v) + \text{quadratic}, \quad [e, E] = 0 \tag{3.5.22}$$

for some constant k then we will call M *degenerate Frobenius manifold*. The theory of degenerate Frobenius manifolds is simpler than the theory of Frobenius manifolds due to presence of a commutative group of algebra automorphisms generated by e, E. In the particular semisimple case a degenerate Frobenius manifold can be described in terms of Prym theta-functions of plane algebraic curves of the degree $n = \dim M$ with an involution (see Appendix below).

Lemma 3.5.10 *If the constant vector field (3.5.17) is proportional to e,*

$$w = (c+\kappa)e$$

for some constant κ then, for $\kappa \neq 0$ the function $\kappa\, F(v)$, defines on M a Frobenius structure with the Euler vector field $\kappa^{-1}E$, the unity e and the charge

$$d = 1 - \frac{2\,c}{\kappa}. \tag{3.5.23}$$

If $\kappa = 0$ then $(F(v), E)$ define on M a structure of degenerate Frobenius manifold.

Proof For $w = (c+\kappa)e$

$$F^w(v) = (c+\kappa)F(v) + \text{quadratic}.$$

So (3.5.18) gives

$$\partial_E F(v) = 2(c+\kappa)F(v) + \text{quadratic}, \quad [e, E] = \kappa\, e.$$

For $\kappa = 0$ the last equation coincides with (3.5.22) with $k = 2c$. For $\kappa \neq 0$ the renormalization $E \mapsto \kappa^{-1}E$, $F \mapsto \kappa\, F$ gives (3.5.21). Observe that, due to the renormalization of $F(v)$ the formula (3.5.14) for the second metric remains unchanged.
□

Corollary 3.5.11 *Every rigid tau-symmetric bihamiltonian structure on $\mathcal{L}(M)$ of the form (3.2.2) compatible with spatial translations in some coordinates can be reduced to one of the following two normal forms.*

1)

$$\begin{aligned}\{v^\alpha(x), v^\beta(y)\}_1 &= \eta^{\alpha\beta}\delta'(x-y),\\ \{v^\alpha(x), v^\beta(y)\}_2 &= g^{\alpha\beta}(v(x))\delta'(x-y) + \Gamma^{\alpha\beta}_\gamma(v(x))v^\gamma_x\delta(x-y)\end{aligned} \tag{3.5.24}$$

where

$$g^{\alpha\beta}(v) = E^\epsilon(v)c^{\alpha\beta}_\epsilon(v)$$

is the intersection form of a Frobenius structure on M

$$\Gamma^{\alpha\beta}_\gamma(v) = c^{\alpha\epsilon}_\gamma(v)\left(\frac{1}{2} - \mu\right)^\beta_\epsilon$$

$$\mu := \frac{2-d}{2} - \nabla E.$$

2)

$$\begin{aligned}\{v^\alpha(x), v^\beta(y)\}_1 &= \eta^{\alpha\beta}\delta'(x-y),\\ \{v^\alpha(x), v^\beta(y)\}_2 &= g^{\alpha\beta}(v(x))\delta'(x-y) + \Gamma^{\alpha\beta}_\gamma(v(x))v^\gamma_x\delta(x-y)\end{aligned} \tag{3.5.25}$$

where

$$g^{\alpha\beta}(v) = E^\epsilon(v)c^{\alpha\beta}_\epsilon(v)$$

is the intersection form of a degenerate Frobenius structure on M

$$\Gamma^{\alpha\beta}_\gamma(v) = -c^{\alpha\epsilon}_\gamma(v)\mu^\beta_\epsilon$$

$$\mu := \frac{k}{2} - \nabla E.$$

We will now show that, under the assumption of semisimplicity the bihamiltonian structure can be reduced to the direct sum of the structures (3.5.24), (3.5.25) even without assumption of rigidity.

Definition. The bihamiltonian structure (3.1.15) - (3.1.17) is called *semisimple* if the characteristic equation

$$\det(g_2^{ij}(u) - \lambda\, g_1^{ij}(u)) = 0$$

has pairwise distinct roots for any $u \in M$.

We will see below that semisimplicity guarantees integrability of the bihamiltonian hierarchy (3.1.30).

Lemma 3.5.12 *For a semisimple tau-symmetric bihamiltonian structure compatible with spatial translations the Frobenius algebra constructed in Theorem 3.5.3 is semisimple.*

Proof According to Lemma 3.5.6 the linear operator

$$(g_2^{ij}) \cdot (g_1^{ij})^{-1}$$

coincides with the operator of multiplication by the vector field E. Therefore the Frobenius algebra on T_vM contains at least one element with simple spectrum. Hence all the elements of the algebra are semisimple. □

Lemma 3.5.13 *Every semisimple solution to the associativity equations possesing of a constant unity can be decomposed into a direct sum of rigid solutions.*

Proof Let us consider the subspace W of all constant vector fields w satisfying (3.5.15). It contains the unity vector field e. Observe that the operator of multiplication of vectors of T_vM by any vector $w \in W$ has constant matrix in the basis of flat coordinates on M. Therefore W is a finite dimensional subalgebra in the Frobenius algebra of vector fields on M.

Due to semisimplicity the commuting operators of multiplication by the vectors w from W can be simultaneously reduced to the diagonal form. Let $\lambda_1(w), \ldots, \lambda_m(w)$ be the pairwise distinct eigenvalues of these operators. We consider these eigenvalues as elements of the dual space W^*. We define

$$TM = TM_1 \oplus \ldots \oplus TM_m$$

as the decomposition of the tangent bundle into the direct sum of the corresponding eigensubspaces. It is easy to see that this is an orthogonal decomposition w.r.t. the bilinear form $< \, , \, >$ and that the products of vectors from TM_i and TM_j are all zeroes for $i \neq j$. In this way we obtain the submanifolds $M_1, \ldots, M_m$. The solution $F(v)$ decomposes into the sum

$$F(v) = F_1(v_1) + \ldots + F_m(v_m), \quad v_s \in M_s$$

up to adding of at most quadratic polynomial.

Denote e_s, E_s, w_s the projections of the vectors e, E, w resp. onto the s-th factor. The vector e_s is the unity of the Frobenius algebra on M_s. The eigenvalues of the operator of multiplication by $w_s - \lambda_s(w) e_s$ restricted onto TM_s are zeroes. Due to semisimplicity this implies that

$$w_s = \lambda_s(w) e_s.$$

Therefore M_s is rigid. □

Corollary 3.5.14 *Let $F(v)$, $v \in M$ be a semisimple solution to the associativity equations possesing a constant unity vector field e and a linear vector field E satisfying (3.5.18) where the constant vector field w satisfies (3.5.15). Then M is isomorphic to the direct product*

$$M = M_1 \times M_2 \times \ldots \times M_m \tag{3.5.26}$$

of Frobenius manifolds or degenerate Frobenius manifolds. The vector field w must have the form

$$w = c\, e + \sum_{s=1}^{m} \kappa_s e_s,$$

where $e = e_1 + \ldots + e_m$ is the decomposition of the unity vector fields w.r.t. the product structure (3.5.26), and $\kappa_1, \ldots, \kappa_m$ are some constants. The factors with $\kappa_s = 0$ correspond to degenerate Frobenius manifolds and the factors with $\kappa_s \neq 0$ correspond to the Frobenius manifold M_s with

$$d_s = 1 - \frac{2\, c}{\kappa_s}$$

and with the Euler vector field

$$E_s := \kappa_s^{-1} \mathrm{pr}_s E.$$

Proof Decomposing the constant vector field w given in (3.5.17) w.r.t. the decomposition (3.5.26) we obtain, due to rigidity of the factors

$$w = c + \sum_{i=1}^{m} \kappa_s e_s.$$

where $\kappa_1, \ldots, \kappa_m$ are some constants. Decomposing also (3.5.18), (3.5.19) we obtain

$$\partial_{E_s} F_s = 2(c + \kappa_s) F_s + \text{quadratic}, \quad [e_s, E_s] = \kappa_s e_s.$$

The factors with $\kappa_s = 0$ give degenerate Frobenius manifolds; those with $\kappa_s \neq 0$ after a renormalization like in Lemma 3.5.10 are Frobenius manifolds. $\square$

It remains to only investigate the freedom in the choice of the tau-structure. We will do it for the rigid Poisson pencil (3.2.2).

Lemma 3.5.15 *For a rigid tau-symmetric Poisson pencil of the form (3.2.2) the tau-structure is determined uniquely up to simultaneous linear transformations*

$$h_{\alpha,p}(v) \mapsto A_\alpha^\beta h_{\beta,p}, \quad p = -1, 0, 1, \ldots \tag{3.5.27}$$

and transformations of the form

$$h_{\alpha,p}(v) \mapsto \rho^{p+1} h_{\alpha,p}(v) + \sum_{q=0}^{p} \rho^{p-q} B_{\alpha,q}^\gamma h_{\gamma,p-q-1}, \quad p = 0, 1, \ldots \tag{3.5.28}$$

with some nonzero ρ and a collection of $n \times n$ constant matrices $B_{\alpha,p}^\beta$, $p = 0, 1, \ldots$.

Proof Indeed, let $\tilde h_{\alpha,p}(v) := \tilde h_{\alpha,p}^{[0]}$ be another tau-structure. Let us first assume that

$$\tilde h_{\alpha,-1} = h_{\alpha,-1} = v_\alpha.$$

Let

$$\tilde h_{\alpha,0} = A_\alpha^\beta h_{\beta,0} + B_\alpha^\beta v_\beta.$$

From

$$\frac{\partial v_\alpha}{\partial t^{\beta,0}} = c_{\alpha\beta\gamma}(v) v_x^\gamma$$

using the symmetry

$$\frac{\partial v_\alpha}{\partial \tilde t^{\beta,0}} = \frac{\partial v_\beta}{\partial \tilde t^{\alpha,0}}$$

we derive

$$A_\alpha^\lambda c_{\lambda\beta\gamma}(v) = A_\beta^\lambda c_{\lambda\alpha\gamma}(v).$$

For $\gamma = 1$ the last equation implies that the constant bilinear form $< A\,x, y >$ on T_vM is symmetric. From the whole equation it follows that this form is invariant. Due to rigidity it must be proportional to $< \, , \, >$. Hence $A_\alpha^\beta = \rho\, \delta_\alpha^\beta$ for some nonzero constant ρ.

Let us redenote $B_{\alpha,0}^\beta := B_\alpha^\beta$. From the next symmetry condition

$$\frac{\partial v_\alpha}{\partial \tilde t^{\beta,1}} = \frac{\partial \tilde h_{\beta,0}}{\partial \tilde t^{\alpha,0}}$$

we derive that

$$\frac{\partial}{\partial \tilde t^{\alpha,1}} = \rho^2 \frac{\partial}{\partial t^{\alpha,1}} + \rho\, B_{\alpha,0}^\gamma \frac{\partial}{\partial t^{\gamma,0}}.$$

Hence

$$\tilde h_{\alpha,1}(v) = \rho^2 h_{\alpha,1}(v) + \rho\, B_{\alpha,0}^{\gamma} h_{\gamma,0}(v) + B_{\alpha,1}^{\gamma} v_\gamma$$

where $B_{\alpha,1}^{\gamma}$ is a new constant matrix. The proof of the lemma can be finished using induction. □

Observe that, conversely, an arbitrary transformation of the form (3.5.27), (3.5.28) map a tau-structure to another one.

We postpone for a subsequent publication the study of integrable hierarchies corresponding to degenerate Frobenius manifolds. In the remaining part of this paper we will assume that the $(0, n)$ Poisson pencil (3.2.2) defines on M a structure of a semisimple Frobenius manifold. Actually, we will see that not an arbitrary Frobenius manifold can be obtained starting from a tau-symmetric Poisson pencil (3.2.2). The restriction is that, the spectrum of the Frobenius manifold must contain no half-integers. However, an arbitrary semisimple Frobenius manifold generates an integrable hierarchy according to the construction of the next section. If the spectrum contains no half-integers then the hierarchy admits a tau-symmetric bihamiltonian structure. Integrable hierarchies corresponding to the Frobenius manifolds with half-integers in the spectrum can be considered as the closure of our construction. For example, these hierarchies are conjecturally in the theory of Gromov - Witten invariants of smooth projective varieties of odd complex dimension (see below).

3.6 From Frobenius manifold to the Principal Hierarchy

Let M be a n-dimensional Frobenius manifold. In this section we construct the integrable hierarchy on $\mathcal{L}(M)$ of the first order quasilinear systems

$$\frac{\partial v^i}{\partial t} = A^i_j(v)\frac{\partial v^j}{\partial x}, \quad i = 1, \dots, n. \tag{3.6.1}$$

Under certain assumptions about the eigenvalues of the gradient ∇E of the Euler vector field E we will show that the hierarchies are generated by the bihamiltonian structure of the form (3.5.24). We will also construct the conservation laws for the hierarchy and, for an appropriate class of its solutions we compute the tau-function. Finally, for the case of semisimple Frobenius manifold we will prove completeness of our system of conservation laws.

3.6.1 Commuting bihamiltonian flows on the loop space of a Frobenius manifold

Let us concentrate first at the hamiltonian systems w.r.t. the Poisson bracket $\{\ ,\ \}_1$. Recall that, in the flat coordinates $v^1, \dots, v^n$ for the metric $<\ ,\ >$ on M the Poisson bracket has the form

$$\{v^\alpha(x), v^\beta(y)\}_1 = \eta^{\alpha\beta}\delta'(x-y).$$

The flow with the hamiltonian $\bar{f} = \int f(v)\, dx$ reads

$$v_t = \{v(x), \bar{f}\}_1 = \partial_x \nabla f.$$

This is a first order quasilinear PDE with the matrix of coefficients

$$A^\alpha_\beta(v) = \nabla^\alpha \nabla_\beta f(v).$$

Denote $A(M)$ the space of smooth functions $f(v)$ on M satisfying

$$\nabla_a \nabla_b f = \nabla_{a\cdot b} Lie_e f \tag{3.6.2}$$

for arbitrary two vector fields a, b on M. Here e is the unity vector field on M.

Theorem 3.6.1 *1) For arbitrary two functions f, $g \in A(M)$ the hamiltonian flows*

$$v_t = \{v(x), \bar{f}\}_1, \quad v_s = \{v(x), \bar{g}\}_1 \tag{3.6.3}$$

commute. 2) For any $f \in A(M)$ there exists $g \in A(M)$ such that

$$\{\ .\ , \bar{f}\}_1 = \{\ .\ , \bar{g}\}_2.$$

Here the second Poisson bracket has the form (3.5.24).

Proof The commutator $(v_t)_s - (v_s)_t$ of the vector fields (3.6.3) reads

$$(v^\alpha_t)_s - (v^\alpha_s)_t = \partial_x \left[\partial_\beta \nabla^\alpha f \partial_x \nabla^\beta g - \partial_\beta \nabla^\alpha g \partial_x \nabla^\beta f \right].$$

Using (3.6.2) we rewrite the expression in the brackets as follows

$$\begin{aligned} &\partial_\beta \nabla^\alpha f \partial_x \nabla^\beta g - \partial_\beta \nabla^\alpha g \partial_x \nabla^\beta f \\ &= \left[c^\alpha_{\beta\lambda} c^\beta_{\gamma\mu} (Lie_e \nabla^\lambda f Lie_e \nabla^\mu g - Lie_e \nabla^\lambda g Lie_e \nabla^\mu f \right] v^\gamma_x = 0 \end{aligned}$$

due to associativity.

To prove the bihamiltonian property of the flows with the densities of hamiltonians in $A(M)$ it suffices, due to Lemma 2.4.22, to prove that these flows are symmetries of the second Poisson bracket. We leave this calculation as an exercise for the reader.
□

Remark 3.6.2 *To find the hamiltonian $\bar{g}$ for the hamiltonian flow*

$$v_t = \{v(x), \bar{f}\}_1 = \{v(x), \bar{g}\}_2,$$

$f(v) \in A(M)$, w.r.t. the second Poisson bracket (3.5.24) one is to find a solution $g(v)$ to (3.6.2) satisfying also

$$Lie_E \nabla g + \frac{3-d}{2} \nabla g = \nabla Lie_e f. \tag{3.6.4}$$

Our nearest goal is to construct a basis in a suitable subspace of $A(M)$. Such a basis is just the hierarchy we promised to construct.

3.6.2 Spectrum of Frobenius manifold, Levelt basis of deformed flat coordinates, and Hamiltonians of the Principal Hierarchy

Denote $\tilde\nabla$ the deformed flat connection on $M \times \mathbb{C}^*$

$$\begin{aligned}\tilde\nabla_u v &= \nabla_u v + z\, u\cdot v, \quad u, v \in TM, \quad z \in \mathbf{C}^* \\ \tilde\nabla_{\frac{d}{dz}} v &= \partial_z v + E\cdot v - \frac{1}{z}\mathcal{V} v\end{aligned}$$

where

$$\mathcal{V} := \frac{2-d}{2} - \nabla E \tag{3.6.5}$$

is an antisymmetric operator w.r.t. $<\ ,\ >$. The unity vector field e is an eigenvector of this operator with the eigenvalue

$$\mathcal{V} e = -\frac{d}{2} e.$$

Lemma 3.6.3 *Let $f = f(v, z)$ be a* horizontal function *for the connection (3.6.5), i.e.,*

$$\tilde\nabla\, df = 0. \tag{3.6.6}$$

Then

$$f(v, z) \in A(M) \ \text{ for any } z. \tag{3.6.7}$$

Proof Spelling the first half of the horizontality equation $\tilde\nabla_\alpha df = 0$ one obtains

$$\partial_\alpha \partial_\beta f = z\, c_{\alpha\beta}^\gamma(v) \partial_\gamma f.$$

In particular,

$$\partial_1 \partial_\gamma f = z \partial_\gamma f.$$

These two equations imply (3.6.2). □

To construct a basis in $A(M)$ we will use the coefficients of expansion at $z = 0$ of the *deformed flat coordinates*. By definition these are n independent horizontal functions $\tilde v_1(v; z), \ldots, \tilde v_n(v; z)$, that is,

$$\det\left(\frac{\partial \tilde v_\alpha(v; z)}{\partial v^\beta}\right) \neq 0$$

and the functions satisfy

$$\tilde\nabla d\tilde v_\alpha = 0, \quad \alpha = 1, \ldots, n. \tag{3.6.8}$$

We now describe a particular system of solutions to (3.6.8).

First, using the Levi-Civita connection for the flat metric $< \, , \, >$ we obtain a natural trivialization of the tangent bundle

$$TM \simeq M \times V$$

where V is a n-dimensional complex space with a symmetric nondegenerate bilinear form that we denote by the same symbol $< \, , \, >$. For an arbitrary linear operator $A : V \to V$ denote $A^* : V \to V$ the adjoint operator

$$< A^* x, y >=< x, Ay > \quad \text{for any} \quad x, y \in V.$$

The horizontal $(1,1)$-tensor $\mathcal{V}$ becomes a linear operator on V satisfying

$$\mathcal{V}^* = -\mathcal{V}.$$

The horizontal unity vector field e gives a distinguished eigenvector of $\mathcal{V}$ in V that we also denote e. After such a trivialization the deformed flat connection becomes a flat connection on the trivial bundle $M \times \mathbb{C}^* \times V$. The equation (3.6.8) for the gradients

$$Y(v; z) := \nabla \tilde{v}(v; z)$$

of deformed flat coordinates takes the form of a system

$$\partial_\alpha Y = z C_\alpha(v) Y, \quad \alpha = 1, \ldots, n \tag{3.6.9}$$

$$\partial_z Y = \left(\mathcal{U}(v) + \frac{\mathcal{V}}{z} \right) Y. \tag{3.6.10}$$

Here $C_\alpha(v)$ and $\mathcal{U}(v)$ are the linear operators in V of multiplication by $\partial/\partial v^\alpha$ and E resp. These operators are symmetric,

$$C_\alpha^* = C_\alpha, \quad \mathcal{U}^* = \mathcal{U}.$$

To fix a system of the deformed flat coordinates we are to choose a basis in the space of solutions to the system (3.6.9), (3.6.10). Such a basis corresponds to a choice of a representative in the equivalence class of *normal forms* of the system (3.6.10) near $z = 0$ (see details in [42]). The parameters of such a normal form are called *monodromy at the origin* of the Frobenius manifold. Let us first recall the description of the parameters.

Definition. The *spectrum* of a Frobenius manifold is a quadruple $(V, < \, , \, >, \hat{\mu}, R)$ where V is a n-dimensional linear space over $\mathbb{C}$ equipped with a symmetric non-degenerate bilinear form $< \, , \, >$, semisimple antisymmetric linear operator $\hat{\mu} : V \to V$, $< \hat{\mu}\, a, b > + < a, \hat{\mu}\, b >= 0$ and a nilpotent linear operator $R : V \to V$ satisfying the following properties. First,

$$R^* = -e^{\pi i \hat{\mu}} R e^{-\pi i \hat{\mu}} \tag{3.6.11}$$

Observe the following consequence of (3.6.11)

$$R e^{2\pi i \hat{\mu}} = e^{2\pi i \hat{\mu}} R. \tag{3.6.12}$$

The operator R must also be $\hat\mu$-*nilpotent*, i.e., it must preserve a natural flag in V associated to the operator $\hat\mu$. The flag is constructed as follows.

For a given $\rho \in Spec\, e^{2\pi i \hat\mu}$ denote $\mu_{\max} = \mu_{\max}(\rho) \in Spec\, \hat\mu$ the eigenvalue with the maximal real part satisfying $e^{2\pi i \mu_{\max}} = \rho$. For every nonnegative integer m define the subspace

$$V^m := \oplus_{0\leq k\leq m} \oplus_{\rho\in Spec\, e^{2\pi i \hat\mu}} \mathrm{Ker}\, [\hat\mu - (\mu_{\max}(\rho) - k)\cdot 1] \subset V. \tag{3.6.13}$$

Clearly

$$0 = V^{-1} \subset V^0 \subset V^1 \subset \dots V \tag{3.6.14}$$

For sufficiently large m one has $V^m = V$. Let $0 = k_1 \leq k_2 < k_3 < \dots < k_l$ be all the integers such that

$$V^{k_i - 1} \neq V^{k_i}.$$

Denoting

$$F_i := V^{k_i}, i = 1, \dots, l \tag{3.6.15}$$

we obtain a flag

$$0 = F_0 \subset F_1 \subset \dots \subset F_l = V. \tag{3.6.16}$$

Definition. The flag (3.6.16) is called *Levelt flag* associated with $(V, <\ ,\ >, \hat\mu)$. The operator $R : V \to V$ is called $\hat\mu$-*nilpotent* if the Levelt flag is invariant

$$R(F_j) \subset F_j, \quad j = 0, 1, \dots, l. \tag{3.6.17}$$

By the construction the operator R satisfies

$$z^{\hat\mu} R z^{-\hat\mu} = R_0 + R_1 z + R_2 z^2 + \dots \tag{3.6.18}$$

where the coefficients of the matrix valued polynomial are nilpotent operators R_0, R_1, ... such that

$$R = R_0 + R_1 + \dots \tag{3.6.19}$$

and

$$R_k(V^j) \subset V^{j-k}. \tag{3.6.20}$$

Observe the following useful identity

$$z^{\hat\mu} R_k z^{-\hat\mu} = z^k R_k, \quad k = 0, 1, \dots \tag{3.6.21}$$

$$[\hat\mu, R_k] = k\, R_k, \quad k = 0, 1, \dots. \tag{3.6.22}$$

The spelling of the equation (3.6.11) for the coefficients R_k reads

$$R_k^* = (-1)^{k+1} R_k, \quad k = 0, 1, \dots. \tag{3.6.23}$$

Any polynomial of the matrices R_k can be uniquely decomposed as follows

$$P(R_0, R_1, \dots) = [P(R_0, R_1, \dots)]_0 + [P(R_0, R_1, \dots)]_1 + \dots \tag{3.6.24}$$

$$z^{\hat\mu}[P(R_0,R_1,\ldots)]_k z^{-\hat\mu} = z^k[P(R_0,R_1,\ldots)]_k. \tag{3.6.25}$$

The last ingredient of the monodromy at the origin is an eigenvector $e \in V$ of $\hat\mu$ satisfying $R_0\, e = 0$. It will be needed later on.

We will now explain how to associate a 5-tuple $(V, <\ ,\ >, \hat\mu, R, e)$ to a Frobenius manifold. The linear space V with a symmetric nondegenerate bilinear form $<\ ,\ >$ and a vector $e \in V$ have already been constructed above. Denote $\hat\mu : V \to V$ the semisimple part of the operator $\mathcal{V}$, i.e.,

$$\hat\mu := \oplus_{\mu \in Spec\, \mathcal{V}} \mu\, P_\mu \tag{3.6.26}$$

where $P_\mu : V \to V_\mu$ is the projector of V onto the root subspace of $\mathcal{V}$

$$V = \oplus_{\mu \in Spec\, \mathcal{V}} V_\mu, \quad V_\mu := \mathrm{Ker}\,(\mathcal{V} - \mu \cdot 1)^n,$$

$P_\mu(V_{\mu'}) = 0$ for $\mu \neq \mu'$, $P_\mu|_{V_\mu} = \mathrm{id}_{V_\mu}$. Clearly the operator $\hat\mu$ is antisymmetric, $\hat\mu^* = -\hat\mu$. Denote R_0 the nilpotent part of $\mathcal{V}$

$$\mathcal{V} = \hat\mu + R_0.$$

Other operators R_1, R_2, ... are not determined by $\mathcal{V}$ only. They appear only in presence of resonances, i.e., pairs of eigenvalues μ, μ' of $\mathcal{V}$ such that $\mu - \mu' \in \mathbf{Z}_{>0}$.

Let us choose a basis $e_1, \ldots, e_n$ in V such that $e_1 = e$. The matrices of the linear operators $\hat\mu$ and R we will denote by the same symbols.

Theorem 3.6.4 *For a sufficiently small ball $B \in M$ there exists a fundamental matrix of solutions to the system (3.6.9), (3.6.10) of the form*

$$Y(v;z) = \Theta(v;z) z^{\hat\mu} z^R \tag{3.6.27}$$

such that the matrix valued function $\Theta(v;z) : V \to V$ is analytic on $B \times \mathbb{C}$ satisfying

$$\Theta(v;0) \equiv 1 \tag{3.6.28}$$

$$\Theta^*(v;-z)\Theta(v;z) \equiv 1. \tag{3.6.29}$$

This was proved in [42] for the case of diagonalizable $\mathcal{V}$. The general case can be settled in a similar way. Note that a branch of logarithm $\log z$ is to be fixed in order to define the matrices $z^{\hat\mu} := e^{\hat\mu \log z}$ and $z^R := e^{R \log z}$. The latter matrix is polynomial in $\log z$. The fundamental matrix (3.6.27) is analytic on the universal covering $\widetilde{B \times \mathbb{C}^*}$.

Remark 3.6.5 *Forgetting about the first part (3.6.9) of the linear system and also about the orthogonality (3.6.29) we arrive, for a fixed v, at a distinguished fundamental matrix for the system of linear differential equations (3.6.10) with rational coefficients. It essentially coincides with the fundamental matrix constructed by F.R.Gantmakher [66] and A.H.M.Levelt [102]. The decomposition (3.6.19) corresponds to the Levelt's* *flag in the space of solutions* $\mathbf{V}$ *to (3.6.10) determined by the following non-archimedian valuation function*

$$\nu(Y) := \text{maximal integer } m \text{ such that } \lim_{z\to 0} z^{-r} Y(z) = 0 \text{ for any real } r < m \quad (3.6.30)$$

for a non-zero solution $Y = Y(z)$ *(it is understood that, during the limit in (3.6.30)* z *goes to zero within an arbitrary fixed sector of the universal covering of* $\mathbb{C}^*$*), and* $\nu(0) = \infty$ *(see details in [102]). If*

$$\infty = \nu_0 > \nu_1 > \ldots > \nu_l$$

are all the values of the valuation function then the Levelt's flag

$$0 = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \ldots \subset \mathbf{F}_l = \mathbf{V}$$

in the space of solutions is defined by

$$\mathbf{F}_k := \{y \in \mathbf{V} \,|\, \nu(y) \geq \nu_k\}\,. \quad (3.6.31)$$

The flag is invariant w.r.t. the monodromy around $z = 0$ *transformation given in the basis (3.6.27) by the matrix*

$$M_0 = e^{2\pi i \hat{\mu}} e^{2\pi i R}. \quad (3.6.32)$$

The fundamental matrix (3.6.27) maps the flag (3.6.16) to the flag (3.6.31).

Let us now describe, following [42], the ambiguity in the choice of the fundamental matrix (3.6.27). Denote $P(V, <\,,\,>, \hat{\mu}, e) \subset Aut\, V$ the group of linear transformations $\Delta : V \to V$ satisfying

$$z^{\hat{\mu}} \Delta z^{-\hat{\mu}} = \text{polynomial in } z \quad (3.6.33)$$

$$\Delta^* e^{\pi i \hat{\mu}} \Delta = e^{\pi i \hat{\mu}} \quad (3.6.34)$$

$$\Delta\, e = e. \quad (3.6.35)$$

The group acts on the fundamental matrices of the form (3.6.27) by the formulae

$$R \mapsto \Delta^{-1} R \Delta \quad (3.6.36)$$

$$\Theta(v; z) \mapsto \Theta(v; z)(\Delta_0 + z\Delta_1 + \ldots) \quad (3.6.37)$$

where

$$z^{\hat{\mu}} \Delta z^{-\hat{\mu}} = \Delta_0 + z\Delta_1 + \ldots \quad (3.6.38)$$

Theorem 3.6.6 *Two fundamental matrices of the form (3.6.27) correspond to the same Frobenius manifold* iff *they are related by the transformation (3.6.36) - (3.6.37).*

Definition. Two 5-tuples $(V_i, <\ ,\ >_i, \hat{\mu}_i, e_i, R_i)$, $i = 1, 2$, are called *equivalent* if there exists an isomorphism $\phi : V_1 \to V_2$ and $\Delta \in P(V_1, <\ ,\ >_1, \hat{\mu}_1)$ such that

$$\phi^* <\ ,\ >_2 = <\ ,\ >_1$$

$$\hat{\mu}_2 \phi = \phi \hat{\mu}_1$$

$$\phi(e_1) = e_2$$

$$\phi^{-1} R_2 \phi = \Delta^{-1} R_1 \Delta.$$

Definition. Class of equivalence of the parameters in (3.6.27) is called *monodromy at $z = 0$* of the Frobenius manifold.

Columns of the fundamental matrix (3.6.27) are gradients of a system of deformed flat coordinates. Due to constancy of $\hat{\mu}$, R also columns of $\Theta(v; z) = \left(\Theta^\alpha_\beta(v; z)\right)$ are gradients of some analytic functions on $B \times \mathbb{C}$ that we denote $\theta_1(v; z), \ldots, \theta(v; z)$

$$\Theta^\alpha_\beta(v; z) = \nabla^\alpha \theta_\beta(v; z).$$

We obtain a system of deformed flat coordinates of the form

$$(\tilde{v}_1(v; z), \ldots, \tilde{v}_n(v; z)) = (\theta_1(v; z), \ldots, \theta_n(v; z)) z^{\hat{\mu}} z^R. \tag{3.6.39}$$

Definition. We will call (3.6.39) *Levelt system of deformed flat coordinates* on M at $z = 0$.

Denote $\theta_{\alpha,p}$ the coefficients of the Taylor expansions of the analytic part of a Levelt system of deformed flat cordinates

$$\theta_\alpha(v; z) = \sum_{p=0}^\infty \theta_{\alpha,p}(v) z^p, \quad \alpha = 1, \ldots, n. \tag{3.6.40}$$

The coefficients $\theta_{\alpha,p}$ are determined from the recursion procedure

$$\partial_\lambda \partial_\mu \theta_{\alpha,p}(v) = c^\nu_{\lambda\mu}(v) \partial_\nu \theta_{\alpha,p-1}(v), \quad p > 0, \tag{3.6.41}$$

$$\theta_{\alpha,0} = v_\alpha \equiv \eta_{\alpha\epsilon} v^\epsilon \tag{3.6.42}$$

with the additional constraints for the matrices $\Theta_p(v) := (\nabla^\alpha \theta_{\beta,p}(v))$ given by

$$(p+1)\Theta_{p+1}(v) + [\Theta_{p+1}(v), \mathcal{V}] = \mathcal{U}(v)\Theta_p(v) - \sum_{k \geq 1} \Theta_{p-k+1}(v) R_k, \quad p = 0, 1, \ldots. \tag{3.6.43}$$

In particular,

$$\theta_{\alpha,1}(v) = \frac{\partial F(v)}{\partial v^\alpha} \tag{3.6.44}$$

$$\theta_{1,2}(v) = \frac{\partial F(v)}{\partial v^\epsilon} v^\epsilon - 2F(v). \tag{3.6.45}$$

In these formulae $F(v)$ is the potential of the Frobenius manifold.

Denote $A_0(M) \subset A(M)$ the subspace of all solutions to (3.6.2) polynomial in the first coordinate v^1. It is a dense subspace when restricting to functions on a compact in M.

Lemma 3.6.7 *The coefficients $\theta_{\alpha,p}(v)$ form a basis in $A_0(M)$, i.e., every solution $f(v)$ to (3.6.2) polynomial in v^1 can be uniquely represented as a finite linear combination*

$$f(v) = \sum_{\alpha,p} c^{\alpha,p}\theta_{\alpha,p}(v)$$

with constant coefficients $c^{\alpha,p}$.

Proof Let us first prove that $\theta_{\alpha,p}(v)$ are polynomials in v^1. Indeed, from (3.6.41) it follows that

$$\partial_1\theta_{\alpha,p}(v) = \theta_{\alpha,p-1}(v), \quad p > 0,$$
$$\partial_1\theta_{\alpha,0}(v) = \eta_{\alpha 1}.$$

Polynomiality follows from these equations. Moreover, we can compute the leading terms of the polynomials:

$$\theta_{\alpha,k+1}(v) = \eta_{\alpha 1}\frac{(v^1)^{k+1}}{(k+1)!} + \sum_{\gamma\neq 1}\eta_{\alpha\gamma}v^\gamma\,\frac{(v^1)^k}{k!} + \text{terms of lower degrees in } v^1. \quad (3.6.46)$$

From the above equations and from (3.6.41) it also follows that $\theta_{\alpha,p}(v)$ satisfies (3.6.2). So, $\theta_{\alpha,p}(v) \in A_0(M)$ for $\alpha = 1,\ldots,n$, $p = 0, 1, \ldots$.

Let us now prove that these functions form a basis in $A_0(M)$. We use induction w.r.t. the degree of $f(v) \in A_0(M)$ as a polynomial in v^1. For the polynomials of the degree 0 the equation (3.6.2) gives

$$\partial_\alpha\partial_\beta f(v) = 0.$$

So $f(v)$ is a linear function of the flat coordinates

$$f(v) = \sum c^{\alpha,0}\theta_{\alpha,0}(v) + \text{const}.$$

Assuming the Lemma already proved for the polynomials of the degree $\deg_{v^1} f(v) \leq k-1$ consider

$$f(v) = f_0(\bar v) + f_1(\bar v)v^1 + \ldots + f_k(\bar v)\frac{(v^1)^k}{k!}$$

where

$$\bar v = (v^2, \ldots v^n).$$

Then from (3.6.2) we deduce that $f_k(\bar v)$ is a linear function

$$f_k(\bar v) = a^\epsilon v_\epsilon + b$$

where $a^1, \ldots, a^n$, b are some constant coefficients. Due to independence of f_k on v^1 the coefficients a^ϵ satisfy

$$a^\epsilon\eta_{\epsilon 1} = 0.$$

Using (3.6.46) we show that the polynomial

$$f'(v) := f(v) - a^\epsilon \theta_{\epsilon,k+1}(v) - b\,\eta^{1\,\alpha}\theta_{\alpha,k}(v) \in A_0(M)$$

has degree in v^1 less than k.

It remains to prove linear independence of the functions $\theta_{\alpha,p}(v)$. Assume that a nontrivial linear combination

$$\sum_{p=0}^{m}\sum_{\alpha=1}^{n} c^{\alpha,p}\theta_{\alpha,p}(v) = 0$$

and not all the coefficients $c^{\alpha,m}$ vanish. Applying the operator ∂_1^m we obtain

$$\sum_{\alpha=1}^{n} c^{\alpha,m} v_\alpha = \text{const}.$$

This contradicts independency of the flat coordinates. □

We arrive at the main construction of this section, i.e., at an infinite family of commuting flows

$$\frac{\partial v}{\partial t^{\alpha,p}} = \{v(x), H_{\alpha,p}\}_1 = \partial_x \nabla\theta_{\alpha,p+1}(v) = \nabla\theta_{\alpha,p}(v)\cdot v_x, \quad H_{\alpha,p} := \bar\theta_{\alpha,p+1}, \quad p = 0, 1, \ldots. \tag{3.6.47}$$

In particular from (3.6.44) it follows that

$$\frac{\partial v}{\partial t^{1,0}} = \frac{\partial v}{\partial x} \tag{3.6.48}$$

$$\frac{\partial v^\alpha}{\partial t^{\beta,0}} = c^\alpha_{\beta\gamma}(v) v^\gamma_x. \tag{3.6.49}$$

From (3.6.45) we also obtain that

$$\frac{\partial v}{\partial t^{1,1}} = v \cdot v_x \tag{3.6.50}$$

In the last formula we identify the vector $v \in M$ of the flat coordinates with the tangent vector $v \in TM$ having the same components.

Definition. The hierarchy (3.6.47) of the first order quasilinear evolutionary PDEs on $\mathcal{L}(M)$ is called *the Principal Hierarchy* corresponding to the Frobenius manifold M.

The product map (3.3.11), (3.3.15) is given by the following multiplication table

$$\theta_{\alpha,p} * \theta_{\beta,q} = \Omega_{\alpha,p;\beta,q}(v) \tag{3.6.51}$$

where the generating function of the coefficients $\Omega_{\alpha,p;\beta,q}(v)$ is given by

$$\begin{aligned}\sum \Omega_{\alpha,p;\beta,q}(v) z^p w^q &= \frac{< \nabla\theta_\alpha(v;z), \nabla\theta_\beta(v;w) > - \eta_{\alpha\beta}}{z+w} \\ &= \sum_{k=1}^{\infty} (-1)^k \frac{(w+z)^{k-1}}{k!} < \nabla\theta_\alpha(v;z), \partial_z^k \nabla\theta_\beta(v;-z) > .\end{aligned} \tag{3.6.52}$$

Example 3.6.8 *For one-dimensional Frobenius manifold* $F(v) = \frac{1}{6}v^3$. *Here*

$$\Theta(v,z) = e^{z\,v}, \quad \theta_{1,p} = \frac{v^{p+1}}{(p+1)!}, \quad p = 0, 1, \dots. \tag{3.6.53}$$

Redenoting the times $t^{1,p} =: t^p$ *we obtain the hierarchy*

$$\frac{\partial v}{\partial t^p} = \frac{v^p}{p!} v_x \tag{3.6.54}$$

In particular

$$\frac{\partial v}{\partial t^1} = v\, v_x.$$

This equation (after the change of the sign of the time $t^1 \mapsto -t^1$*) is sometimes called dispersionless KdV or nonviscous Burgers equation. It also coincides with the Riemann simple wave equation. The hierarchy (3.6.54) is the simplest example of the hierarchies in our considerations. We suggest to call it Riemann hierarchy.*

The product map of two polynomials (or power series) in v *is given by the following formula*

$$f(v) * g(v) = \int dv\, f'(v) g'(v).$$

Example 3.6.9 *For the two-dimensional Frobenius manifolds the only parameter is the charge* d. *It is convenient to introduce parameter* κ *s.t.*

$$d = 1 - \frac{2}{\kappa}.$$

For generic $\kappa \neq -1, 0, 1$

$$F = \frac{1}{2}(v^1)^2 v^2 + \frac{(v^2)^{\kappa+1}}{\kappa^2 - 1}. \tag{3.6.55}$$

The deformed flat coordinates can be expressed via modified Bessel functions. The normalized system of deformed flat coordinates (3.6.39) reads

$$\tilde v_1 = z^{-\frac{1}{2}} \sqrt{v^2}\, e^{z\,v^1}\, \Gamma(1+\kappa^{-1})\, (\kappa - \kappa^{-1})^{-\frac{1}{2\kappa}} I_{\frac{1}{\kappa}}[\frac{2\,z\,\sqrt{\kappa^2-1}}{\sqrt{\kappa}} (v^2)^{\frac{\kappa}{2}}]$$
$$= e^{z\,v^1} \left[\sum_{m\ge 0} \frac{\Gamma(1+\kappa^{-1})}{\Gamma(m+1+\kappa^{-1})} (\kappa-\kappa^{-1})^m (v^2)^{m\,\kappa+1} \frac{z^{2\,m}}{m!} \right] z^{-\frac{1}{2}+\frac{1}{\kappa}},$$
$$\tilde v_2 = z^{-\frac{1}{2}} \sqrt{v^2}\, e^{z\,v^1} \Gamma(1-\kappa^{-1}) (\kappa-\kappa^{-1})^{\frac{1}{2\kappa}} I_{-\frac{1}{\kappa}}[\frac{2\,z\,\sqrt{\kappa^2-1}}{\sqrt{\kappa}} (v^2)^{\frac{\kappa}{2}}] - z^{-\frac{1}{\kappa}-\frac{1}{2}}$$
$$= z^{-1} \left[e^{z\,v^1} \sum_{m\ge 0} \frac{\Gamma(1-\kappa^{-1})}{\Gamma(m+1-\kappa^{-1})} (\kappa-\kappa^{-1})^m (v^2)^{\kappa\,m} \frac{z^{2\,m}}{m!} - 1 \right] z^{\frac{1}{2}-\frac{1}{\kappa}}.$$

The matrices $\hat\mu$ and R in (3.6.39) are given by

$$\hat\mu = \begin{pmatrix} -\frac{1}{2}+\frac{1}{\kappa} & 0 \\ 0 & \frac{1}{2}-\frac{1}{\kappa} \end{pmatrix}, \quad R = 0.$$

The hamiltonian flow (3.6.50) after changing of the sign of the time variable $t = -t^{1,1}$ and redenoting $v^1 = v$, $v^2 = \rho$ coincides with the equations of motion of one-dimensional polytropic gas with the equation of state $p = \frac{\kappa}{\kappa+1}\rho^{\kappa+1}$:

$$\begin{aligned} & v_t + \left(\frac{v^2}{2} + \rho^\kappa\right)_x = 0 \\ & \rho_t + (\rho\, v)_x = 0. \end{aligned}$$

The bihamiltonian structure (3.5.24) reads

$$\begin{aligned} \{v(x), v(y)\}_\lambda^{[0]} &= 2\rho^{\kappa-1}(x)\,\delta'(x-y) + \left(\rho^{\kappa-1}\right)_x \delta(x-y), \\ \{v(x), \rho(y)\}_\lambda^{[0]} &= (v(x)-\lambda)\,\delta'(x-y) + \frac{1}{\kappa}\, v'(x)\,\delta(x-y), \\ \{\rho(x), \rho(y)\}_\lambda^{[0]} &= \frac{1}{\kappa}\left(2\,\rho(x)\,\delta'(x-y) + \rho'(x)\,\delta(x-y)\right). \end{aligned} \tag{3.6.56}$$

This bihamiltonian structure for the polytropic gas equations has been found by P.Olver [125]. The above formulae remain valid also for the exceptional values $\kappa = \pm 1$ where the expression for the potential of the Frobenius manifold is to be modified. For the particular value $\kappa = 3$ the Frobenius manifold (3.6.55) corresponds to the A_2 topological minimal model [25].

Example 3.6.10 *The exceptional 2-dimensional Frobenius manifold with the charge $d = 1$*

$$F = \frac{1}{2}\,(v^1)^2\, v^2 + e^{v^2} \tag{3.6.57}$$

corresponds to the quantum cohomology of $\mathbf{CP}^1$ [38] (it will also be called $\mathbf{CP}^1$ sigma-model). The deformed flat coordinates can also be expressed via modified Bessel functions. We have

$$\hat\mu = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \tag{3.6.58}$$

So the normalized system (3.6.39) reads

$$(\tilde v_1(v;z), \tilde v_2(v;z)) = (\theta_1(v;z), \theta_2(v;z)) \begin{pmatrix} z^{-\frac{1}{2}} & 0 \\ 0 & z^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2\log z & 1 \end{pmatrix} \tag{3.6.59}$$

where

$$\theta_1(v;z) = -2\, e^{zv^1} \left(K_0(2ze^{\frac{1}{2}v^2}) + (\log z + \gamma) I_0(2ze^{\frac{1}{2}v^2})\right)$$

$$= -2e^{z\,v^1}\sum_{m\geq 0}(\gamma - \frac{1}{2}v^2 + \psi(m+1))e^{m\,v^2}\frac{z^{2\,m}}{(m!)^2}, \tag{3.6.60}$$

$$\theta_2(v;z) = z^{-1}\,e^{zv^1}\,I_0(2ze^{\frac{1}{2}\,v^2}) - z^{-1}$$

$$= z^{-1}\left(\sum_{m\geq 0} e^{m\,v^2+z\,v^1}\frac{z^{2\,m}}{(m!)^2} - 1\right). \tag{3.6.61}$$

Here γ denotes Euler's constant, $\psi(z)$ stands for the digamma function.

The $-t^{1,1}$-flow has still the meaning of one-dimensional isentropic fluid (polytropic gas) with the equation of state $p = (\rho^2 - 2\rho + 2)e^\rho$. It is more instructive to look at the $t = t^{2,0}$-flow. Using (3.6.49) we obtain the so-called long wave limit of the Toda lattice equation

$$\rho_{tt} - (e^\rho)_{xx} = 0. \tag{3.6.62}$$

The bihamiltonian structure reads

$$\{v(x), v(y)\}^{[0]}_\lambda = 2\,e^{\rho(x)}\,\delta'(x-y) + \left(e^{\rho(x)}\right)_x\,\delta(x-y),$$

$$\{v(x), \rho(y)\}^{[0]}_\lambda = (v(x) - \lambda)\,\delta'(x-y),$$

$$\{\rho(x), \rho(y)\}^{[0]}_\lambda = 2\,\delta'(x-y).$$

Remark 3.6.11 *In the setting of two-dimensional topological field theory [27, 23, 24, 146] the Frobenius manifold is called small phase space. The basis of the Hamiltonian densities $\theta_{\alpha,p}(v)$ coincides with particular two point tree level correlators*

$$\theta_{\alpha,p}(v) = <\tau_p(\phi_\alpha)\tau_0(\phi_1)>_0$$

as functions on the small phase space. Here ϕ_1 corresponds to the identity operator. Other two point tree level correlators are

$$<\tau_p(\phi_\alpha)\tau_q(\phi_\beta)>_0 = \theta_{\alpha,p} * \theta_{\beta,q}.$$

To extend these formulae on the big phase space one is to evaluate these functions on the topological solution $v = v(\mathbf{t})$ (see (3.6.89), (3.6.90) below) of the Principal Hierarchy.

We are to clarify an important point about the bihamiltonian nature of the hierarchy (3.6.47). Although all these equations, as it has been proved in Theorem 3.6.1, are bihamiltonian flows w.r.t. the Poisson pencil (3.5.24), their hamiltonian densities in $A(M)$ not always belong to $A_0(M)$. For example, the spatial translation flow on $\mathcal{L}(M)$, M being the Frobenius manifold of the Example 3.6.10 above

$$v_{1t} = v_{1x}, \quad v_{2t} = v_{2x}$$

w.r.t. the first Poisson bracket has the hamiltonian $\bar{v_1 v_2}$ with the density $v_1 v_2 \in A_0(M)$. But the hamiltonian density of this flow w.r.t. the second Poisson bracket

$$g = \frac{1}{4}v_2^2 - \left[\frac{1}{2}v_2 - \log v_1 + \sqrt{v_1^2 - 4\,e^{v_2}}\right]^2$$

does not belong to $A_0(M)$. We will show below that the hamiltonians of all but a finite number of the flows of the hierarchy (3.6.47) w.r.t. the second Poisson bracket belong to $A_0(M)$.

3.6.3 Periods of the Frobenius manifold and the bihamiltonian recursion for the Principal Hierarchy

The first step will be to show that the hamiltonians of the hierarchy (3.6.47) are obtained by a triangular transformation from those given by the bihamiltonian recursion procedure presented in Section 3.1.2 above. To do this we are to find the reducing transformation for the Poisson pencil $\{\ ,\ \}_\lambda^{[0]} = \{\ ,\ \}_2^{[0]} - \lambda\{\ ,\ \}_1^{[0]}$ and to express the coefficients of the expansion of this reducing transformation in terms of the hamiltonian densities $\theta_{\alpha,p}(v)$.

Definition. The functions $p = p(v; \lambda)$ satisfying

$$(\nabla^* - \lambda\,\nabla)dp = 0 \tag{3.6.63}$$

are called *periods* of the Frobenius manifold. The system (3.6.63) is called *the Gauss-Manin system* on the Frobenius manifold [41].

Here ∇^* is the Levi-Civita connection for the metric $(\ ,\ )$. The connection is well-defined outside the discriminant $\Sigma \subset M$ (see details in [41]).

Choosing a system of n independent periods we obtain a system of flat coordinates $p^1(v;\lambda), \dots, p^n(v;\lambda)$ for the flat pencils of metrics $(\ ,\ )_\lambda := (\ ,\ ) - \lambda < , >$

$$\left(dp^i(v;\lambda), dp^j(v;\lambda)\right)_\lambda = G^{ij} \tag{3.6.64}$$

for some constant nondegenerate matrix G^{ij}. According to the results of Section 2.4.5, choosing $p^i(v;\lambda)$, $i = 1, \dots, n$ as a new system of depending variables we obtain a reduction of the Poisson pencil to the canonical form

$$\{p^i(v(x);\lambda), p^j(v(y);\lambda)\}_\lambda = G^{ij}\delta'(x-y).$$

Using results of [45] we will produce a particular system of independent periods specified according to their behaviour for large λ. For technical reasons it will be convenient to label these periods by *lower* indices; the roles of upper and lower indices therefore will be interchanged in the subsequent formulae.

Theorem 3.6.12 *Let M be a semisimple Frobenius manifold such that the spectrum of $\mathcal{V}$ does not contain half-integers. Then the transformation $p_\alpha = p_\alpha(v;\lambda)$ reducing the Poisson pencil to the constant form*

$$\{p_\alpha(v(x);\lambda), p_\beta(v(y);\lambda)\}_\lambda = G_{\alpha\beta}\delta'(x-y)$$

with a constant matrix $G = (G_{\alpha\beta})$

$$G = -2\pi\,\eta\left[e^{\pi\, i\, R}e^{\pi\, i\,\hat\mu} + e^{-\pi\, i\, R}e^{-\pi\, i\,\hat\mu}\right]^{-1} \tag{3.6.65}$$

is given by the formula

$$\mathbf{p}(v;\lambda) = (p_1(v;\lambda),\dots,p_n(v;\lambda)) = (\pi_1(v;\lambda),\dots,\pi_n(v;\lambda))\lambda^{-\frac{1}{2}-\hat\mu}\lambda^{-R} \tag{3.6.66}$$

where the vector function $\pi(v;\lambda) = (\pi_1(v;\lambda),\dots,\pi_n(v;\lambda))$ *is analytic for sufficiently large* $|\lambda|$*. It has the following Taylor expansion at* $\lambda=\infty$

$$\pi(v;\lambda) = \pi^{(0)}(\lambda) + \sum_{m=0}^{\infty}\lambda^{-m}\sum_{p+q=m}\theta_p(v)\Gamma_q(R,\hat\mu+m+\frac{1}{2}), \tag{3.6.67}$$

$$\pi^{(0)} = \lambda\sum_{q\geq 0}\omega_1\,\Gamma_q(R,\hat\mu+q-\frac{1}{2}), \quad \omega_1 := (\eta_{11},\eta_{12},\dots,\eta_{1n}). \tag{3.6.68}$$

In this formula $\theta_p(v) = (\theta_{1,p}(v),\dots,\theta_{n,p}(v))$*. The decomposition of the matrix polynomial* $e^R = [e^R]_0 + [e^R]_1 + \dots$ *was defined in (3.6.24), (3.6.25). The matrices* $\Gamma_q(R,\hat\mu+a)$ *for an arbitrary complex number* a *such that the spectrum of the matrix* $\hat\mu + a\,\mathrm{id}$ *does not contain negative integers are defined by*

$$\Gamma_q(R,\hat\mu+a) = \left[e^{R\,\partial_\nu}\right]_q\Gamma(\hat\mu+a+\nu)_{\nu=0}. \tag{3.6.69}$$

Proof As we already know, the reducing transformation for the Poisson pencil $\{\ ,\ \}_2 - \lambda\{\ ,\ \}_1$ is given by a system of the flat coordinates for the flat pencil of metrics $(\ ,\ ) - \lambda <\ ,\ >$. The latter can be obtained [41] by applying the Laplace-type integrals to the deformed flat coordinates

$$p_\alpha(v;\lambda) = \oint e^{-\lambda z}\tilde v_\alpha(v;z)\,\frac{dz}{\sqrt{z}}, \quad \alpha = 1,\dots,n.$$

Here the symbol of loop integral means just the possibility of integration by parts dropping the boundary terms. On a semisimple Frobenius manifold the above loop integral can be regularized as follows. Consider the integral

$$p_\alpha^{(\nu)}(v;\lambda) = \int_0^{\infty e^{i\varphi}} e^{-\lambda z}\tilde v_\alpha(v;z)\,\frac{dz}{z^{\frac{1}{2}-\nu}} \tag{3.6.70}$$

along the ray $\arg z = \varphi$ on the complex z-plane. Here ν is a complex parameter. The integral converges at $z=0$ for $\mathrm{Re}\,\nu >> 0$. It also converges at $z = e^{i\varphi}\infty$ for sufficiently large $|\lambda| > r$ for some $r = r(v)$ due to the exponential behaviour of the deformed flat coordinates at $z=\infty$ (here the semisimplicity of the Frobenius manifold plays the crucial role!). Rotating the argument φ we obtain an analytic continuation of the integral onto the universal covering of the disc $r < |\lambda| < \infty$. It remains to analytically continue the integral into the point of interest $\nu = 0$ to obtain the needed functions

$$p_\alpha(v;\lambda) := p_\alpha^{(0)}(v;\lambda).$$

We will do the needed analytic continuation just integrating the terms of the expansion (3.6.39), (3.6.40) and then setting x to zero. This can be done using the following calculations

$$\int_0^{\infty e^{i\varphi}} e^{-\lambda z} z^{\hat\mu+p+\nu-\frac{1}{2}} z^R dz = \int_0^{\infty} e^{-t} t^{\hat\mu+p+\nu-\frac{1}{2}} t^{R_0+\frac{R_1}{\lambda}+\frac{R_2}{\lambda^2}+\dots} dt\,\lambda^{-(\hat\mu+p+\nu+\frac{1}{2})}\lambda^{-R}$$

$$= \sum_{q\geq 0}\sum_{k\geq 0}\int_0^\infty e^{-t} t^{\hat\mu+p+\nu-\frac12}\lambda^{-q}\frac{[R^k]_q \log^k t}{k!} dt\, \lambda^{-(\hat\mu+p+\nu+\frac12)}\lambda^{-R}$$

$$= \sum_{q\geq 0}\sum_{k\geq 0}\frac{1}{k!}\partial_\nu^k\Gamma(\hat\mu+p+\nu+\frac12)\frac{[R^k]_q}{\lambda^q}\lambda^{-(\hat\mu+p+\nu+\frac12)}\lambda^{-R}$$

$$= \sum_q [e^{R\partial_\nu}]_q\Gamma(\hat\mu+p+q+\nu+\frac12)\, \lambda^{-(\hat\mu+p+q+\nu+\frac12)}\lambda^{-R}$$

where we used the following obvious integral

$$\int_0^\infty e^{-t} t^{s-1}\log^k t\, dt = \partial_s^k\Gamma(s)$$

and also the commutation relation

$$f(\hat\mu)[P(R_0, R_1, \ldots)]_q = [P(R_0, R_1, \ldots)]_q f(\hat\mu+q)$$

valid for an arbitrary polynomial of the matrices R_i and for an arbitrary analytic function f.

We see that the coefficients of the above series are meromorphic functions on the complex ν-plane with the poles at

$$\nu \in \cup_{\mu\in Spec\,\hat\mu}\cup_{k>0}(-\mu-k-\frac12). \tag{3.6.71}$$

For every pole in (3.6.71) only finite number of the coefficients of the series become infinite. Therefore the sum of the series is a meromorphic function in x with poles at (3.6.71). Due to the assumption about the spectrum of $\mathcal{V}$ = spectrum of $\hat\mu$ the value $\nu = 0$ is not a pole of this series. Setting $\nu = 0$ we obtain the proof of (3.6.66), (3.6.67). Note that the term $\pi_\alpha^{(0)}(\lambda)$ does not depend on v. It drops from the Gauss - Manin system (3.6.63). We choose this term in such a way to have the identity

$$\frac{\partial p(v;\lambda)}{\partial\lambda} = -\frac{\partial p(v;\lambda)}{\partial v^1} \tag{3.6.72}$$

valid. □

Remark 3.6.13 *The Poisson brackets $\{\pi_\alpha(v(x);\lambda), \pi_\beta(v(y);\lambda)\}_\lambda$ are also constant but they depend on λ:*

$$\{\pi_\alpha(v(x);\lambda), \pi_\beta(v(y);\lambda)\}_\lambda = \tilde g_{\alpha\beta}(\lambda)\delta'(x-y)$$

$$\left(\tilde G_{\alpha\beta}\right) = -2\pi\,\lambda\,\eta\left[e^{\pi\, i\,(R_0+\frac{R_1}{\lambda}+\ldots)}e^{\pi\, i\,\hat\mu} + e^{-\pi\, i\,(R_0+\frac{R_1}{\lambda}+\ldots)}e^{-\pi\, i\,\hat\mu}\right]^{-1}.$$

We are now able to write explicitly down the bihamiltonian recursion relation for the Principal Hierarchy.

Theorem 3.6.14 *Under the assumptions of the theorem 3.6.12 the following recursion relation holds true*

$$\mathcal{R}\frac{\partial}{\partial t^{p-1}} = \frac{\partial}{\partial t^p}\left(p + \hat{\mu} + \frac{1}{2}\right) + \sum_{k=0}^{p} \frac{\partial}{\partial t^{p-k}} R_k. \tag{3.6.73}$$

Here

$$\frac{\partial}{\partial t^p} := \left(\frac{\partial}{\partial t^{1,p}}, \dots, \frac{\partial}{\partial t^{n,p}}\right).$$

Proof Applying the recursion operator to the hamiltonian flow

$$v_t = \{v(x), \bar{f}\}_1 = \partial_x \nabla f(v), \quad \bar{f} := \int f(v)\, dx$$

with an arbitrary Hamiltonian density $f(v)$ one obtains

$$\mathcal{R}^\alpha_\beta\, v^\beta_t = \nabla^{*\alpha} \partial_\gamma f(v)\, v^\gamma_x.$$

For the generating function of the flows of Principal Hierarchy we take

$$f = \tilde{v}(v; z)$$

and use the identity [41]

$$\nabla^* d\tilde{v} = (\partial_z - \frac{1}{2\, z}) \nabla d\tilde{v} \tag{3.6.74}$$

to arrive at the needed recursion relation. □

If the spectrum of $\hat{\mu}$ contains half-integers then the Gauss - Manin system has solutions polynomial in λ. The recursion operator (3.6.73) becomes degenerate. In other words, although the flows of the Principal Hierarchy remain bihamiltonian, their Hamiltonians are not described by the bihamiltonian recursion procedure.

More specifically, if $\frac{1}{2} \in \operatorname{Spec} \hat{\mu}$ then the two Poisson brackets have common Casimirs. E.g., in the particular example of the $\mathbf{CP}^1$ model the variable v_2 is the density of a Casimir for both Poisson brackets.

3.6.4 Solutions to the Principal Hierarchy and their tau-functions

We will now describe a natural class of solutions to the hierarchy (3.6.47) and compute explicitly their tau-functions.

We will consider *analytic solutions* of the hierarchy, i.e., power series in the variables $\mathbf{t} := (t^{\alpha,p})$ with the coefficients in $\mathbb{C}[[\epsilon]]$

$$v = v(x, \mathbf{t}, \epsilon) = a_0(\epsilon) + \sum_{k>0} a_{\alpha_1,p_1;\dots;\alpha_k,p_k}(\epsilon) t^{\alpha_1,p_1} \dots t^{\alpha_k,p_k}|_{t^{1,0} \mapsto t^{1,0}+x}. \tag{3.6.75}$$

Analyticity of the function in infinite number of variables is to be understood as follows. Setting $t^{\alpha,p} = 0$ for all $\alpha = 1, \dots, n$, $p \geq N$ for an arbitrary positive integer N we must obtain a power series in the finite number of variables x and $t^{\beta,q}$, $\beta = 1, \dots, n$, $0 \leq q < N$ with the coefficients in $\mathbb{C}[[\epsilon]]$. Every term of the formal power series in ϵ must be a *convergent* series in x and $t^{\beta,q}$, $\beta = 1, \dots, n$, $0 \leq q < N$ in a ball near the origin of the space $\mathbb{C}^{n\,N+1}$ (the size of the ball may depend on N).

To make it possible the substitution of the solution to the equations of the hierarchy the vector $a_0(0)$ must be a point in the Frobenius manifold M. The vector v_x for $\epsilon = 0$, $\mathbf{t} = 0$, $x = 0$ can be considered as an element of the tangent space at this point:

$$v_x(x = 0, \mathbf{t} = 0, \epsilon = 0) \in T_{v=a_0(0)}M. \tag{3.6.76}$$

Definition. The solution (3.6.75) is called *monotone* at the origin if the vector (3.6.76) is an invertible element of the algebra $T_{v=a_0(0)}M$.

We will now construct a dense subset in the space of analytic monotone solutions.

Let us fix a point $v_0 \in M$ and a collection of formal power series $c^{\alpha,p}(\epsilon) \in \mathbb{C}[[\epsilon]]$ with constant coefficients, $\alpha = 1, \dots, n$, $p \geq 1$ with only finite number of them being nonzero such that the multiplication operator

$$m_0 := \left(\cdot \nabla \sum_{p\geq 1} c^{\alpha,p}(0)\theta_{\alpha,p-1}(v) \right)_{v=v_0} : T_{v_0}M \to T_{v_0}M \tag{3.6.77}$$

is invertible. We are to also fix n series $c^{\alpha,0}(\epsilon) \in \mathbb{C}[[\epsilon]]$ such that

$$c^{\alpha,0}(0) := -\nabla^\alpha \sum_{p\geq 1} c^{\beta,p}(0)\theta_{\beta,p}(v_0).$$

The solution to the hierarchy will be defined by the following system of equations

$$x\, e + \sum_{p\geq 0} \tilde{t}^{\alpha,p} \nabla \theta_{\alpha,p}(v) = 0 \tag{3.6.78}$$

where

$$\tilde{t}^{\alpha,p} := t^{\alpha,p} - c^{\alpha,p}(\epsilon). \tag{3.6.79}$$

Theorem 3.6.15 *1) There exists a unique solution to the system (3.6.78) in the form (3.6.75) with $a_0(0) = v_0$. It satisfies the equations of the hierarchy (3.6.47). 2) The solutions of the form (3.6.75) are dense in the space of analytic monotone solutions to the hierarchy.*

The first part of the theorem is an analogue of the Tsarev's generalized hodograph transform in the theory of integrable systems of hydrodynamic type [140] adapted for the case of hierarchies of these systems. A construction of a dense set of solutions for certain particular classes of systems of hydrodynamic type has been obtained in [35], [141].

Proof Differentiating (3.6.78) w.r.t. $v^1, \dots, v^n$ and setting $x = 0$, $\mathbf{t} = 0$, $\epsilon = 0$ we obtain the Jacobi matrix of the system coinciding with the nondegenerate matrix $-m_0$ of the operator (3.6.77). Therefore existence and uniqueness of an analytic solution of (3.6.78) with $t^{\alpha,p} = 0$ for $p \geq N$

$$v = v_0 + \sum_{k>0} v_0^{[k]} \epsilon^k + \sum_{k>0} \sum_{0 \leq p_1, \dots, p_k < N} a_{\alpha_1,p_1;\dots;\alpha_k,p_k}(\epsilon) t^{\alpha_1,p_1} \dots t^{\alpha_k,p_k} |_{t^{1,0} \mapsto t^{1,0}+x}$$

for every positive N is an immediate consequence of the implicit function theorem. Differentiating (3.6.78) w.r.t. x and $t^{\alpha,p}$ we obtain

$$w \cdot v_x = -e$$

$$w \cdot \partial_{t^{\alpha,p}} v = -\nabla\theta_{\alpha,p}(v)$$

where the operator of multiplication by the vector

$$w := \sum \tilde{t}^{\beta,q} \nabla\theta_{\beta,q-1}(v)$$

is invertible for small $t^{\beta,q}$ and for v close to v_0 due to our choice of v_0 and of the constants $c^{\beta,q}(\epsilon)$. The equations of the hierarchy

$$\partial_{t^{\alpha,p}} v = \nabla\theta_{\alpha,p}(v) \cdot \partial_x v$$

readily follow by dividing over w.

Let us prove density of the constructed solutions. Let $v = v(x, \mathbf{t}, \epsilon)$ be a monotone analytic solution to the hierarchy s.t. $v(0,0,0) = v_0 \in M$. From the monotonicity condition and from the level zero equations of the hierarchy

$$\partial_{t^{\alpha,0}} v^\beta = c^\beta_{\alpha\gamma}(v) \partial_x v^\gamma \tag{3.6.80}$$

it follows that

$$\det\left(\frac{\partial v^\alpha(0,0,0)}{\partial t^{\beta,0}}\right) \neq 0.$$

Restricting this solution onto the subspace $t^{\alpha,p} = 0$ for $p > 0$ and using the nondegeneracy of the Jacobian we can rewrite this restriction in the following implicit form

$$\begin{aligned} x + \ t^{1,0} &= f^1(v,\epsilon)\ , \\ t^{2,0} &= f^2(v,\epsilon)\ , \\ &\dots\ , \\ t^{n,0} &= f^n(v,\epsilon)\ . \end{aligned} \tag{3.6.81}$$

Here $f^1(v,\epsilon), \dots, f^n(v,\epsilon)$ are some formal power series in ϵ with the coefficients analytic in a ball near $v = v_0$.

Lemma 3.6.16 *The functions $f^\alpha(v,\epsilon)$ have the form*

$$f^\alpha(v,\epsilon) = \nabla^\alpha f(v,\epsilon) \tag{3.6.82}$$

for some function $f(v,\epsilon) \in A(M)$ Conversely, every function $f(v,\epsilon)$ satisfying (3.6.2) defines a solution to the Principal Hierarchy (3.6.47) in the implicit form

$$x\, e + \sum t^{\alpha,p}\, \nabla\theta_{\alpha,p}(v) = \nabla f(v,\epsilon). \tag{3.6.83}$$

Proof Differentiating (3.6.81) w.r.t. x and $t^{\alpha,0}$ and using again the level zero part (3.6.80) of the equations of the hierarchy we obtain

$$\delta^\alpha_\beta = \frac{\partial f^\alpha}{\partial v^\epsilon} c^\epsilon_{\beta\gamma} \frac{\partial v^\gamma}{\partial x}.$$

The last equation can be recast into the form

$$e_\alpha = \nabla f_\alpha \cdot v_x$$

where we denote

$$f_\alpha(v,\epsilon) := \eta_{\alpha\beta} f^\beta(v,\epsilon).$$

In particular,

$$v_x = (\nabla f_1)^{-1}.$$

Hence

$$\nabla f_\alpha = e_\alpha \cdot \nabla f_1,$$

i.e.,

$$\partial_\beta f_\alpha = c^\gamma_{\alpha\beta} \partial_\gamma f_1.$$

The symmetry in α and β proves closedness of the one-form

$$f_\alpha(v,\epsilon) dv^\alpha = d\, f(v,\epsilon)$$

and also implies the equation (3.6.2). □

To finish the proof of the Theorem we just approximate near $v = v_0$ the coefficients of the ϵ-expansion of the solution $f(v,\epsilon)$ to the system (3.6.2) by the coefficients of the expansion of another solution $\tilde f(v,\epsilon)$ to (3.6.2) polynomial in v^1. Applying Lemma 3.6.7 to the function $\tilde f(v,\epsilon)$ (with possibly the adding to it a linear in v^α term) we obtain a finite linear combination

$$\tilde f(v,\epsilon) = \sum_{p\geq 1} c^{\alpha,p}(\epsilon)\theta_{\alpha,p}$$

for some constants $c^{\alpha,p}(\epsilon)$. We now use these constants together with $\nabla^\alpha \tilde f(v_0, 0) = -c^{\alpha,0}(\epsilon)$ to produce a solution (3.6.78) to the hierarchy. Let us denote this solution $\tilde v(x, \mathbf{t}, \epsilon)$. It will approximate the given number of the coefficients of the expansion of the given solution to $v(x, \mathbf{t}, \epsilon)$ to (3.6.83) in a power series in ϵ when restricted onto a finite-dimensional subspace $t^{\alpha,p} = 0$ for $p \geq N$ for sufficiently small $|x|$ and $|t^{\alpha,p}|$, $0 \leq p < N$. The Theorem is proved. □

The equation (3.6.78) can be rewritten as the following stationary point equation

$$\nabla \Phi_{x,\mathbf{t},c(\epsilon)}(v) = 0 \tag{3.6.84}$$

where the function $\Phi_{x,\mathbf{t},c(\epsilon)}(v)$ on M depending on the parameters x, $\mathbf{t} = (t^{\alpha,p})$ and $c(\epsilon) = (c^{\alpha,p}(\epsilon))$ has the form

$$\Phi_{x,\mathbf{t},c(\epsilon)}(v) = \sum \tilde{t}^{\alpha,p} \theta_{\alpha,p}(v)|_{\tilde{t}^{1,0} \mapsto \tilde{t}^{1,0}+x}. \tag{3.6.85}$$

(The solution depends also on the choice of the point $v_0 \in M$ such that

$$\nabla^\alpha \Phi_{0,\mathbf{0},c(0)}(v_0) = 0, \quad \alpha = 1, \dots, n.$$

However, locally v_0 is uniquely determined by $c(0)$ due to invertibility of the vector $\nabla \sum c^{\alpha,p}(0)\theta_{\alpha,p-1}(v_0) \in T_{v_0}M$.) As we have just proved the dependence of the stationary point that we denote $v(x, \mathbf{t}, c(\epsilon))$ on the parameters x and $\mathbf{t}$ satisfies the equations of the hierarchy (3.6.47). The representation (3.6.84) will be useful in all calculations with the solutions of the hierarchy and with their tau-functions. Observe that $\Phi_{x,\mathbf{t},c(\epsilon)}(v)$ can be considered as a vector of the space $\mathcal{K} \otimes \mathbb{C}[[\epsilon]]$ of the densities of the conservation laws of the hierarchy depending explicitly on x, $\mathbf{t}$ and ϵ.

Theorem 3.6.17 *The tau-function of the solution $v(x, \mathbf{t}, c(\epsilon))$ defined by (3.6.84) has the form*

$$\begin{aligned} \mathcal{F}_0(x, \mathbf{t}, c(\epsilon)) = \epsilon^2 \log \tau &= \frac{1}{2} \Phi_{x,\mathbf{t},c(\epsilon)}(v) * \Phi_{x,\mathbf{t},c(\epsilon)}(v)|_{v=v(x,\mathbf{t},c(\epsilon))} \\ &= \frac{1}{2} \sum \tilde{t}^{\alpha,p} \tilde{t}^{\beta,q} \Omega_{\alpha,p;\beta,q}(v(x, \mathbf{t}, c(\epsilon))). \end{aligned} \tag{3.6.86}$$

The first derivatives of the tau-function w.r.t. the times of the hierarchy are given by the formula

$$\epsilon^2 \partial_{t^{\alpha,p}} \log \tau = \theta_{\alpha,p}(v) * \Phi_{x,\mathbf{t},c(\epsilon)}(v)|_{v=v(x,\mathbf{t},c(\epsilon))} = \sum \tilde{t}^{\beta,q} \Omega_{\alpha,p;\beta,q}(v(x, \mathbf{t}, c(\epsilon))). \tag{3.6.87}$$

Recall that the product map $*$ was defined in (3.3.11), (3.3.15), (3.6.51).

Proof Differentiating (3.6.86) w.r.t. $t^{\gamma,r}$ and using

$$\nabla(\theta_{\alpha,p} * \theta_{\beta,q}) = \nabla\theta_{\alpha,p} \cdot \nabla\theta_{\beta,q}$$

we obtain

$$\epsilon^2 \partial_{t^{\gamma,r}} \log \tau = \sum \tilde{t}^{\beta,q} \theta_{\gamma,r} * \theta_{\beta,q}|_{\tilde{t}^{1,0} \mapsto \tilde{t}^{1,0}+x} + \frac{1}{2} < \nabla\Phi_{x,\mathbf{t},c(\epsilon)}(v) \cdot \nabla\Phi_{x,\mathbf{t},c(\epsilon)}(v) \cdot \nabla\theta_{\gamma,r}(v), v_x > .$$

The second part of the formula vanishes for $v = v(x, \mathbf{t}, c)$ due to (3.6.84). This proves (3.6.87). Repeating the trick we obtain

$$\epsilon^2 \frac{\partial^2 \log \tau}{\partial t^{\alpha,p} \partial t^{\beta,q}} = \Omega_{\alpha,p;\beta,q}.$$

The theorem is proved. □

Example 3.6.18 *The particular solution to the Principal Hierarchy specified by the constants*

$$c^{\alpha,p} = \delta_1^\alpha\, \delta_1^p \tag{3.6.88}$$

will be called topological solution. *It is specified by the following fixed point equation*

$$v = \nabla \Phi_{x,\mathbf{t}}(v), \quad \Phi_{x,\mathbf{t}}(v) = \sum \bar{t}^{\alpha,p} \theta_{\alpha,p}(v). \tag{3.6.89}$$

The expansion of the topological solution has the form

$$v^\alpha(t) = t^{\alpha,0} + \sum_{k\geq 1,\, p_i \geq 1} A^\alpha_{\beta_1,q_1;\dots;\beta_k,q_k}(t^{1,0}, \dots, t^{n,0})\, t^{\beta_1,q_1} \dots t^{\beta_k,q_k}, \tag{3.6.90}$$

the coefficients are determined recursively by (3.6.89). For example, we have

$$A^\alpha_{\beta,q} = \left. \frac{\partial \theta_{\beta,q}}{\partial v_\alpha} \right|_{v_\gamma = t^{\gamma,0}}, \quad A^\alpha_{\beta_1,q_1;\beta_2,q_2} = \left. \frac{1}{2} \frac{\partial^2 \theta_{\beta_1,q_1}}{\partial v_\alpha \partial v_\gamma} \frac{\partial \theta_{\beta_2,q_2}}{\partial v_\gamma} \right|_{v_\xi = t^{\xi,0}}.$$

As it was shown in [37], the logarithmic derivatives of the tau-function of the topological solution satisfy the genus zero topological recursion relations [27]. In order to formulate these recursion relations we introduce the symbols ("the genus zero correlation functions")

$$\langle\langle \tau_{p_1}(\phi_{\alpha_1}) \tau_{p_2}(\phi_{\alpha_2}) \dots \tau_{p_k}(\phi_{\alpha_k}) \rangle\rangle_0 := \epsilon^k \frac{\partial^k \log \tau}{\partial t^{\alpha_1,p_1} \partial t^{\alpha_2,p_2} \dots \partial t^{\alpha_k,p_k}}. \tag{3.6.91}$$

They are functions of all the times $t^{\alpha,p}$. The following identities hold true for these functions

$$\begin{aligned} &\langle\langle \tau_p(\phi_\alpha) \tau_q(\phi_\beta) \tau_r(\phi_\gamma) \rangle\rangle_0 \\ &= \langle\langle \tau_{p-1}(\phi_\alpha) \tau_0(\phi_\nu) \rangle\rangle_0 \eta^{\nu\mu} \langle\langle \tau_0(\phi_\mu) \tau_q(\phi_\beta) \tau_r(\phi_\gamma) \rangle\rangle_0. \end{aligned} \tag{3.6.92}$$

In topological sigma-models expanding the function $\log \tau$ *at the point of classical limit one obtains from (3.6.92) the corresponding identities for the intersection numbers of the Gromov - Witten Mumford - Morita - Miller classes on $\bar{\mathcal{M}}_{0,k}$ [146, 128].*

On the small phase space $t^{\alpha,p} = 0$ *for* $p > 0$ *the logarithm of the tau-function coincides [37] with the potential of the Frobenius manifold*

$$\log \tau|_{t^{\alpha,0} = v^\alpha,\ t^{\alpha,p} = 0,\ p > 0} = \frac{1}{\epsilon^2} F(v). \tag{3.6.93}$$

The formula

$$F(v) = \frac{1}{2}\Omega_{1,1;1,1}(v) - v^\alpha \Omega_{\alpha,0;1,1} + \frac{1}{2} v^\alpha v^\beta \Omega_{\alpha,0;\beta,0}(v) \tag{3.6.94}$$

was used in the derivation of (3.6.93). This formula, together with (3.6.52)gives an expression of $F(v)$ via gradients of the functions $\theta_{\alpha,p}(v)$, $0 \le p \le 3$.

Note that the quasihomogeneity axiom was not used in the proofs of these statements. So, the above relations remain valid also for degenerate Frobenius manifolds.

Remark 3.6.19 *The Principal Hierarchy appears also in the so-called symplectic field theory of Ya. Eliashberg, A. Givental and H. Hofer [60] at the genus zero approximation. For example, the long wave limit of the Toda lattice essentially appears in their calculation of the genus zero Gromov - Witten invariants of the projective plane. We are going to consider the new problems of the theory of integrable systems inspired by [60] in a subsequent publication.*

3.6.5 Complete integrability of the Principal Hierarchy corresponding to a semisimple Frobenius manifold

We are now to prove *completeness* of the system

$$H_{\alpha,p} = \int \theta_{\alpha,p+1} dx$$

of conservation laws of the hierarchy (3.6.47).

Let us assume that

$$I = \int h(v; v_x, \ldots, v^{(m)}) dx$$

is a conservation law of the hierarchy,

$$\{I, \bar\theta_{\alpha,p}\}_1 = 0, \quad \alpha = 1, \ldots, n, \quad p = 0, 1, 2, \ldots.$$

Lemma 3.6.20 *I is a conservation law of the hierarchy (3.6.47)* iff

$$\frac{\partial}{\partial v^{\alpha,k}} \frac{\delta I}{\delta v(x)} = 0, \quad \alpha = 1, \ldots, n, \quad k > 0. \tag{3.6.95}$$

Proof Denote

$$W^\alpha = \eta^{\alpha\beta} \frac{\delta I}{\delta v^\beta(x)}.$$

Then I gives a conservation law for the dispersionless hierarchy iff

$$\frac{\delta}{\delta v^\alpha(x)} \left(W^\gamma \, \partial_x \frac{\partial \theta_\beta(v; z)}{\partial v^\gamma} \right) = 0, \quad \alpha, \beta = 1, \ldots, n$$

identically in z. Using

$$\frac{\partial^2\theta_\beta(v;z)}{\partial v^\lambda \partial v^\mu} = z\, c^\nu_{\lambda\mu}(v)\frac{\partial\theta_\beta(v;z)}{\partial v^\nu}$$

we obtain, after division by z,

$$\frac{\partial\left(W^\gamma\, c^\rho_{\gamma\sigma}\right)}{\partial v^\alpha}\,\frac{\partial\theta_\beta}{\partial v^\rho}\,v^\sigma_x + z\,W^\gamma\,c^\rho_{\gamma\sigma}\,c^\nu_{\alpha\rho}\frac{\partial\theta_\beta}{\partial v^\nu}v^\sigma_x - \partial_x\left(\frac{\partial W^\gamma}{\partial v^\alpha_x}\,c^\rho_{\gamma\sigma}\,\frac{\partial\theta_\beta}{\partial v^\rho}\,v^\sigma_x + W^\gamma\,c^\rho_{\gamma\alpha}\,\frac{\partial\theta_\beta}{\partial v^\rho}\right)$$
$$+\sum_{k=2}^{2m}(-1)^k\partial_x^k\left(\frac{\partial W^\gamma}{\partial v^{\alpha,k}}\;c^\rho_{\gamma\sigma}\,\frac{\partial\theta_\beta}{\partial v^\rho}\,v^\sigma_x\right)$$
$$=\frac{\partial\left(W^\gamma\, c^\rho_{\gamma\sigma}\right)}{\partial v^\alpha}\,\frac{\partial\theta_\beta}{\partial v^\rho}\,v^\sigma_x - \partial_x\left(W^\gamma\,c^\rho_{\gamma\alpha}\right)\,\frac{\partial\theta_\beta}{\partial v^\rho} + \sum_{k=1}^{2m}(-1)^k\partial_x^k\left(\frac{\partial W^\gamma}{\partial v^{\alpha,k}}\;c^\rho_{\gamma\sigma}\,\frac{\partial\theta_\beta}{\partial v^\rho}\,v^\sigma_x\right)$$

Multiplying by the inverse matrix of $\left(\frac{\partial\theta_\beta(v;z)}{\partial v^\xi}\right)$ we arrive at a polynomial of degree $2m$ in z with the coefficient of z^{2m} given by

$$\frac{\partial W^\gamma}{\partial v^{\alpha,2m}}c^{\rho_1}_{\gamma\sigma}\,c^{\rho_2}_{\sigma_1\rho_1}\,c^{\rho_3}_{\sigma_2\rho_2}\dots c^{\rho_{2m}}_{\sigma_{2m-1}\rho_{2m-1}}\,c^{\xi}_{\sigma_{2m}\rho_{2m}}\,v^\sigma_x\,v^{\sigma_1}_x\dots v^{\sigma_{2m}}_x$$
$$=\frac{\partial W^\gamma}{\partial v^{\alpha,2m}}\sum_i \psi_{i\gamma}\,\psi_i^\xi\,u_{i,x}^{2m+1}$$

where u_i are the canonical coodinates. The vanishing of the above expression yields

$$\frac{\partial W^\gamma}{\partial v^{\alpha,2m}} = 0.$$

In a similar way, we prove inductively that

$$\frac{\partial W^\gamma}{\partial v^{\alpha,k}} = 0, \quad k = 1,\dots 2m-1.$$

Therefore W^γ does not depend on the x-derivatives of v^α. □

Theorem 3.6.21 *Let $I = \int h(v; v_x, \dots, v^{(m)})dx$ be a conservation law of the hierarchy (3.6.47) polynomial in v^1. Then*

$$h(v; v_x, \dots, v^{(m)}) = \sum c^{\alpha,p}\theta_{\alpha,p}(v) + \text{total derivative} \tag{3.6.96}$$

where only finite number of the constant coefficients $c^{\alpha,p}$ is not equal to zero.

Proof This follows from the above lemma and from the lemma 3.6.7. □

3.7 Quasitrivial bihamiltonian structures

After having settled the problem of normal forms of the leading term of the expansion of the Poisson pencil (3.1.15)-(3.1.16) we now address the problem of construction and classification of the higher order terms. Recall that we want to classify the Poisson pencils up to the action of the Miura-type transformations independent of the parameter λ of the pencil. More precisely,

Definition. Two Poisson pencils

$$\{u^i(x), u^j(y)\}_\lambda = \sum_{k=0}^{\infty} \epsilon^k \left[\{u^i(x), u^j(y)\}_2^{[k]} - \lambda \{u^i(x), u^j(y)\}_1^{[k]} \right] \tag{3.7.1}$$

and

$$\{v^i(x), v^j(y)\}_\lambda = \sum_{k=0}^{\infty} \epsilon^k \left[\{v^i(x), v^j(y)\}_2^{[k]} - \lambda \{v^i(x), v^j(y)\}_1^{[k]} \right] \tag{3.7.2}$$

are called *equivalent* if there exists a Miura-type transformation

$$v^i = \sum_{k=0}^{\infty} \epsilon^k F_k^i(u; u_x, \ldots, u^{(k)}), \quad i = 1, \ldots, n$$

transforming (3.7.1) to (3.7.2) for every λ. The Poisson pencil (3.7.1) is called *trivial* if it is equivalent to (3.7.2) with $\{\ ,\ \}_{1,2}^{[k]} = 0$ for $k > 0$.

As in Section 2.4 above, the infinitesimal description of the space of classes of equivalence of Poisson pencils with a given leading term can be done in terms of certain cohomology. More precisely, the two Poisson brackets $\{\ ,\ \}_{1,2}^{[0]}$ induce two anticommuting differentials ∂_1 and ∂_2 on multivectors,

$$\partial_i = \left[\ .\ , \{\ ,\ \}_i^{[0]} \right], \quad i = 1, 2,$$

$$\partial_1^2 = \partial_2^2 = \partial_1 \partial_2 + \partial_2 \partial_1 = 0.$$

As we already know both the differentials have trivial cohomology.

Lemma 3.7.1 *Let us denote*

$$\begin{aligned} H^k(\mathcal{L}(M); \partial_1, \partial_2) &:= \operatorname{Ker} \partial_1 \partial_2|_{\Lambda^{k-1}} / \left(\operatorname{Im}\partial_1 + \operatorname{Im}\partial_2\right), \quad k > 1, \\ H^1(\mathcal{L}(M); \partial_1, \partial_2) &:= \operatorname{Ker} \partial_1 \partial_2|_{\Lambda^0} \\ H^0(\mathcal{L}(M); \partial_1, \partial_2) &:= \operatorname{Ker} \partial_1|_{\Lambda^0} \cap \operatorname{Ker} \partial_2|_{\Lambda^0}. \end{aligned} \tag{3.7.3}$$

The zero cohomology coincides with the algebra of common Casimirs for the two Poisson brackets. The first cohomology coincides with the space of bihamiltonian vector fields for the Poisson pencil (3.2.2). The second cohomology coincides with classes of equivalence of the infinitesimal deformations of the Poisson pencil modulo infinitesimal Miura-type transformations.

Proof The interpretation of the zero cohomology is straightforward by the definition. For a local functional $\bar{h}$, $h = h(u; u_x, \ldots; \epsilon) \in \mathcal{A}$ the condition $\partial_1\partial_2\bar{h} = 0$ means that $\partial_2\bar{h} \in \operatorname{Ker}\partial_1$. Due to the triviality of the first cohomology of ∂_1 the last condition implies existence of a local functional $\bar{f}$ such that

$$\partial_2\bar{h} = \partial_1\bar{f}.$$

That is, the vector field $\partial_2\bar{h}$ is a bihamiltonian one. Obviously, the converse statement is also true.

Let us now look at the infinitesimal deformations of the Poisson pencil. Without loss of generality we may assume that the perturbation of $\{\ ,\ \}_1^{[0]}$ is trivial, due to triviality of the second cohomology of ∂_1. The infinitesimal deformation of $\{\ ,\ \}_2^{[0]}$ must be annihilated by ∂_2 and also by ∂_1, due to the compatibility condition of the Poisson brackets. So the deformation of the Poisson pencil must be of the form

$$\{\ ,\ \}_1^{[0]} \mapsto \{\ ,\ \}_1^{[0]} + \mathcal{O}(\epsilon^2),\quad \{\ ,\ \}_2^{[0]} \mapsto \{\ ,\ \}_2^{[0]} + \epsilon\,\partial_1 X + \mathcal{O}(\epsilon^2),\quad \partial_2\partial_1 X = 0. \quad (3.7.4)$$

This transformation is trivial if it can be generated by another vector field Y. This means that

$$\partial_1 Y = 0,\quad \partial_2 Y = \partial_1 X.$$

The first of the two equations implies $Y = -\partial_1\bar{a}$ for some local functional $\bar{a}$. The second one proves existence of another local functional $\bar{b}$ such that $X = \partial_2\bar{a} + \partial_1\bar{b}$. □

Similar arguments prove the following simple statement.

Theorem 3.7.2 *The classes of equivalence of bihamiltonian structures on the loop space with the given* $\{\ ,\ \}_{1,2}^{[0]}$ *are in one-to-one correspondence with classes of equivalence of vector fields*

$$X,\quad X|_{\epsilon=0} = 0$$

satisfying

$$\partial_1\left(-\partial_2 X + \frac{1}{2}[X, \partial_1 X]\right) = 0 \quad (3.7.5)$$

modulo shifts along the $\{\ ,\ \}_1^{[0]}$*-hamiltonian vector fields*

$$X \mapsto \exp[\operatorname{ad}_{\partial_1 h}]X$$

with ϵ *dependent Hamiltonian* h.

We leave the proof of this statement to the reader.

We will call the groups (3.7.3) *the bihamiltonian cohomology* of the pencil (3.2.2). The calculation of the bihamiltonian cohomology seem to be a nontrivial problem.

Another problem to be fixed is the one of obstructions to extension of a given infinitesimal deformation (3.7.4) to a global one. It is easy to see from (3.7.5) that the first obstruction is the class of equivalence of the cocycle

$$[X, \partial_1 X] \in H^3(\mathcal{L}(M); \partial_1, \partial_2). \tag{3.7.6}$$

The analysis of this and higher obstructions seems to be an interesting problem of infinite dimensional Poisson geometry.

Example 3.7.3 *For $n = 1$ all deformations upto the fourth order of the Poisson pencil*

$$\{u(x), u(y)\}_\lambda = (u - \lambda)\delta'(x - y) + \frac{1}{2}u_x\delta(x - y) \tag{3.7.7}$$

have been classified by P.Lorenzoni [103]. They are parametrized by one arbitrary function $f = f(u)$ of one variable as follows

$$\begin{aligned}
&\{u(x), u(y)\}_\lambda = (u - \lambda)\delta'(x - y) + \frac{1}{2}u_x\delta(x - y)\\
&+\epsilon^2\left[-2f\delta'''(x - y) - 3\partial_x f\delta''(x - y) - \partial_x^2 f\delta'(x - y)\right]\\
&+\epsilon^4\left[4g\delta^V(x - y) + 10\partial_x g\delta^{IV}(x - y) + (20\,\partial_x^2 g - 8\,hu_{xx})\delta'''(x - y)\right.\\
&\left.+(20\partial_x g - 12\partial_x(hu_{xx}))\delta''(x - y) + (6\partial_x^4 g - 4\,\partial_x^2(hu_{xx}))\delta'(x - y)\right]\\
&+\mathcal{O}(\epsilon^5).
\end{aligned} \tag{3.7.8}$$

In the r.h.s. of this fomula

$$u = u(x),\ u_x = u_x(x),\ u_{xx} = u_{xx}(x), \quad g = ff', \quad h = ff'' + f'^2, \quad f = f(u(x)).$$

In particular, the obstruction (3.7.6) is trivial for an arbitrary infinitesimal deformation of the order ϵ^2. All the above Poisson pencils are inequivalent for different $f(u)$. In particular, for $f(u) = c$ one obtains the KdV Poisson pencil.

$$\{u(x), u(y)\}_\lambda = [u(x) - \lambda]\,\delta'(x - y) + \frac{1}{2}u'\delta(x - y) + c\,\epsilon^2\delta'''(x - y).$$

We will now impose the main restriction onto the class of Poisson pencils that will allow us to get rid of the above unpleasant cohomological problems. Let us extend the class of Miura-type transformations.

Definition. The transformations of the form

$$u^i \mapsto v^i = u^i + \sum_{k=1}^{\infty}\epsilon^k F_k^i(u; u_x, \dots, u^{(n_k)}), \quad i = 1, \dots, n \tag{3.7.9}$$

where the coefficients F_k^i are quasihomogeneous of the degree k *rational* functions in the derivatives $u_x, \dots, u^{(n_k)}$ will be called *quasi-Miura transformations*. The Poisson pencil (3.1.15)– (3.1.17) is called *quasitrivial* if there exists a quasi-Miura transformation reducing the pencil to its leading term (3.2.2).

We emphasize that the coefficients of the Poisson pencil are still to be polynomials in the derivatives. All the denominators must disappear after the transformation.

Remark 3.7.4 *As it was proved in [103], the deformation (3.7.8) is quasitrivial up to the order 4. This suggests that* **all** *Poisson pencils of the form (3.1.16) corresponding to a semisimple Frobenius structure in* $\{\ ,\ \}_\lambda^{[0]}$ *could be quasitrivial. To our opinion, the problem of quasitriviality deserves further investigation.*

In the next section we will show quasitriviality of the KdV hierarchy. Even in this simplest example the quasitriviality is something unobserved before.

3.8 Quasitriviality of KdV and genus expansion in topological gravity

To construct a transformation that maps any solution v of the Riemann hierarchy (3.6.54) to a solution u of the KdV hierarchy we will proceed following [46]. Every solution v of the Riemann hierarchy can be represented in the standard implicit form (3.6.78), i.e.,

$$x + \tilde{t}_0 + \tilde{t}_1 v + \tilde{t}_2 \frac{v^2}{2} + \tilde{t}_3 \frac{v^3}{6} + \ldots = 0. \tag{3.8.1}$$

Here

$$\tilde{t}_p = t_p - c_p$$

where the constants c_p correspond to the choice of the solution (in this section we will systematically suppress the explicit dependence of the coefficients c_p on ϵ). Representing the equation (3.8.1) in the variational form (3.6.84) we obtain

$$\Phi'_{x,\mathbf{t},c}(v) = 0, \quad \Phi'_{x,\mathbf{t},c}(v) = (x + \tilde{t}_0)v + \tilde{t}_1 \frac{v^2}{2!} + \ldots. \tag{3.8.2}$$

Let $\bar{h}_p = \int h_p(u; u_x, \ldots)dx$ be the Hamiltonians of the KdV hierarchy normalized as in (3.4.2), i.e.

$$h_p = \frac{u^{p+2}}{(p+2)!} + \epsilon^2(\text{terms with derivatives}), \quad p \geq -1.$$

Let us construct a functional depending on the same parameters x, $\mathbf{t}$, c

$$I_{x,\mathbf{t},c}[u] = \int \left((x + \tilde{t}_0)u + \sum_{p>0} \tilde{t}_p h_{p-1}(u; u_x, \ldots) \right). \tag{3.8.3}$$

At the moment we do not care about the precise meaning of the integral. We will be interested only in the Euler - Lagrange equation

$$\frac{\delta}{\delta u(x)} I_{x,\mathbf{t},c}[u] = \sum_k (-1)^k \partial_x^k \frac{\partial}{\partial u^{(k)}} \left((x + \tilde{t}_0)u + \sum_{p>0} \tilde{t}_p h_{p-1}(u; u_x, \ldots) \right)$$

$$= x + \tilde{t}_0 + \sum_{p>0} \tilde{t}_p \frac{\delta \bar{h}_{p-1}}{\delta u(x)} = 0. \tag{3.8.4}$$

The first few terms of the Euler - Lagrange equation (3.8.4) read

$$x + \tilde t_0 + \tilde t_1 u + \tilde t_2 \left(\frac{u^2}{2} - \epsilon^2 \frac{u''}{6} \right) + \tilde t_3 \left(\frac{u^3}{6} - \frac{\epsilon^2}{12} (u'^2 + 2u\, u'') + \epsilon^4 \frac{u^{IV}}{60} \right) + \ldots = 0.$$

Truncating $t_p = 0$ for $p \geq N$ (and assuming that only finite number of the constants c_p is distinct from zero) we obtain an ODE for the function $u = u(x)$ depending on the times $t_0, \ldots, t_{N-1}$ and on the constants $c = (c_0, c_1, \ldots)$ as on the parameters.

Lemma 3.8.1 *(cf. [120]) The space of solutions to the Euler - Lagrange equation (3.8.4) is invariant w.r.t. the flows of the KdV hierarchy.*

Proof Let $u_0(x)$ be a solution to the differential equation (3.8.4) with $t_p = t_p^0$, $p = 0, 1, \ldots$. We are to prove that the solution to the Cauchy problem for the KdV hierarchy with the initial data

$$u(x, \mathbf{t})|_{t_p = t_p^0,\ p=0,1,\ldots} = u_0(x)$$

will satisfy the same ODE (3.8.4). □

Let $v = v(x, \mathbf{t}, c)$ be the solution to the Riemann hierarchy determined by (3.8.2) such that $v'(0, \mathbf{0}, c) \neq 0$. (The solution depends on the choice of a simple root v_0 of the polynomial $\sum c_p \frac{v_0^p}{p!} = 0$. It will be understood that such a choice has already been done.)

Lemma 3.8.2 *There exists a unique solution to (3.8.4) in the form of power series in* ϵ^2

$$u = v + \epsilon^2 u^{[1]} + \epsilon^4 u^{[2]} + \ldots. \qquad (3.8.5)$$

Proof We plug (3.8.5) into the equation (3.8.4)) and compute recursively the terms of the expansion. For example, for the first correction we obtain

$$u^{[1]} = -\frac{1}{24} \frac{2\, v'' \left(\tilde t_2 + \tilde t_3 v + \tilde t_4 \frac{v^2}{2} + \ldots \right) + v'^2 \left(\tilde t_3 + \tilde t_4 v + \ldots \right)}{\tilde t_1 + \tilde t_2 v + \tilde t_3 \frac{v^2}{2} + \ldots}.$$

□

Corollary 3.8.3 *The solution (3.8.5) to (3.8.4) satisfies equations of the KdV hierarchy.*

Thus we obtain a map

$$\text{the stationary point (3.8.2)} \mapsto \text{the stationary function (3.8.5) of (3.8.4)} \qquad (3.8.6)$$

transforming solutions of the Riemann hierarchy to the solutions to the KdV hierarchy. We will now show that this is a quasitriviality transformation.

First we will prove

Lemma 3.8.4 *There exist universal (i.e., independent on the choice of the solution v to the Riemann hierarchy) polynomials $P^{[2k]}(v', v'', \dots v^{(3k)})$ quasihomogeneous of the degree $6\,k-2$ such that the transformation (3.8.6) is given by*

$$v \mapsto u = v + \sum_{k\geq 1} \epsilon^{2\,k} (v')^{2-4\,k} P^{[2k]}(v', v'', \dots, v^{(3\,k)}). \tag{3.8.7}$$

Proof It is technically convenient to return to the original normalization of Example 3.1.11 for the KdV hierarchy

$$\frac{\partial u}{\partial t_k} = \partial_x \frac{\delta I_k}{\delta u(x)}$$

where the generating function of the densities of the KdV integrals is to be determined from the Riccati equation

$$i\epsilon\,\chi' - \chi^2 = u - \lambda,$$

$$\chi = k + \sum_{m=1}^{\infty} \frac{\chi_m}{k^m}, \quad k = \sqrt{\lambda},$$

$$I_k = -4 \int \chi_{2k+3} dx.$$

We can rewrite the Euler - Lagrange equation (3.8.4) in the following form. Introduce the series

$$\mathbf{t}(\lambda) := \tilde{t}_0 + \sum_{k=1}^{\infty} \frac{2^k}{(2k-1)!!} \tilde{t}_k \lambda^k.$$

Let us also introduce the linear operator *Res* acting on the series in inverse powers of $k = \sqrt{\lambda}$ by

$$Res\, f := res_{k=\infty} \mathbf{t}(\lambda) f\, dk.$$

Then the Euler - Lagrange equation reads

$$Res \frac{\delta \int \chi\, dx}{\delta u(x)} = 0. \tag{3.8.8}$$

In a similar way, the variational equation (3.8.2) can be written as

$$Res \frac{1}{\sqrt{\lambda - v}} = 0.$$

We will now expand the variational derivatives in powers of ϵ^2. Using the formula

$$\frac{\delta \int \chi\, dx}{\delta u(x)} = -\frac{1}{2\chi_R},$$

where χ_R is the real part of χ, $\chi = \chi_R + i\chi_I$, $\chi_I = \frac{1}{2}\frac{\chi_R'}{\chi_R}$, we rewrite (3.8.8) as

$$Res\, \frac{1}{\chi_R} = 0. \tag{3.8.9}$$

Using differential equation for $f := 1/\chi_R$,

$$f^2 + \frac{\epsilon^2}{\lambda - u}\left[\frac{1}{2}f''f - \frac{1}{4}f'^2\right] = \frac{1}{\lambda - u}$$

(a consequence of Riccati) we can expand $1/\chi_R$ in the series of the form

$$\frac{1}{\chi_R} = \sigma + \frac{1}{8}\epsilon^2\left(\sigma\,\sigma'^2 - 2\,\sigma^2\,\sigma''\right)$$

$$+\frac{1}{128}\epsilon^4\left(3\,\sigma\,\sigma'^4 - 12\,\sigma^2\,\sigma'^2\,\sigma'' + 12\,\sigma^3\,\sigma''^2 + 16\,\sigma^3\,\sigma'\,\sigma''' + 8\,\sigma^4\,\sigma^{IV}\right) + O(\epsilon^6)$$

where

$$\sigma = \frac{1}{\sqrt{\lambda - u}}.$$

Let us now compute the first correction $u^{[1]}$ in the expansion (3.8.5). Within the order ϵ^2 the equation (3.8.9) reads

$$Res\left[\sigma + \frac{1}{8}\epsilon^2\left(\sigma\,\sigma'^2 - 2\,\sigma^2\,\sigma''\right)\right] = O(\epsilon^4). \tag{3.8.10}$$

Denote

$$\sigma_0 := \frac{1}{\sqrt{\lambda - v}}.$$

We must expand the above equation and retain the linear in ϵ^2 terms. Substituting

$$u = v + \epsilon^2 u^{[1]} + \dots$$

in σ we obtain

$$\sigma = \sigma_0 + \frac{\epsilon^2}{2}u^{[1]}\sigma_0^3 + \dots$$

In the second term in (3.8.10) we may replace $\sigma \to \sigma_0$. Next, we are to observe the following simple rules for differentiating σ_0:

$$\sigma_0' = \frac{1}{2}\sigma_0^3, \quad \sigma_0'' = \frac{1}{2}v''\sigma_0^3 + \frac{3}{4}v'^2\sigma_0^5, \dots.$$

So, the equation (3.8.10) can be rewritten as

$$Res\left[\frac{1}{2}u^{[1]}\sigma_0^3 - \frac{1}{8}v''\sigma_0^5 + \frac{5}{32}v'^2\sigma_0^7\right] = 0.$$

It remains to calculate the residues of the form

$$Res\,\sigma_0^{2k+1} = \frac{2^k}{(2k-1)!!}Q_k$$

where the rational functions Q_k in the derivatives are defined recursively

$$Q_{k+1} = \frac{1}{v'} Q_k', \quad Q_1 = -\frac{1}{v'}.$$

To prove the last formula it suffices to observe that

$$Res \frac{d^k \sigma_0}{dv^k} = \tilde{t}_k + \tilde{t}_{k+1} v + \ldots = Q_k,$$

and to compute

$$\frac{d^k \sigma_0}{dv^k} = \frac{(2k-1)!!}{2^k} \sigma_0^{2k+1}.$$

Finally we obtain the needed formula in the form

$$u^{[1]} = \frac{1}{Q_1} \left[\frac{1}{6} v'' Q_2 - \frac{1}{12} v'^2 Q_3 \right] = -\frac{1}{12} (\log v')''.$$

It is clear how to proceed with higher corrections. We want to emphasize that the expressions

$$u^{[k]} = \frac{P^{[2k]}(v', v'', \ldots, v^{(3k-2)})}{v'^{4k-2}}, \quad k \geq 1$$

do not depend on v explicitly. □

Corollary 3.8.5 *The correspondence (3.8.6)*

$$\left\{ \begin{array}{c} \text{solutions to (3.8.2)} \\ v(x, \mathbf{t}, \epsilon) = v_0(x, \mathbf{t}) + \epsilon v_1(x, \mathbf{t}) + \ldots \end{array} \right\} \mapsto \left\{ \begin{array}{c} \text{solution to (3.8.4)} \\ u(x, \mathbf{t}, \epsilon) = u_0(x, \mathbf{t}) + \epsilon u_1(x, \mathbf{t}) + \ldots \end{array} \right\}$$

is a quasi-Miura transformation

$$\begin{aligned} u &= F(v; v_x, v_{xx}, \ldots; \epsilon) \\ &= v - \frac{\epsilon^2}{12} (\log v')'' + \epsilon^4 \left[\frac{v^{IV}}{288 v'^2} - \frac{7 v'' v'''}{480 v'^3} + \frac{v''^3}{90 v'^4} \right]'' + O(\epsilon^6). \end{aligned} \qquad (3.8.11)$$

We are now to prove that the quasi-Miura transformation is one-to-one. Loosely speaking we want to prove that an arbitrary monotone function $u(x, \epsilon)$ satisfies an ODE (3.8.4) possibly of infinite order with $\mathbf{t} = 0$ and appropriate coefficients c_p that may depend on ϵ. More precisely,

Lemma 3.8.6 *Let $u = u(x, \epsilon) \in \mathbb{C}[[x, \epsilon]]$ be an arbitrary power series satisfying $u_x(0, 0) \neq 0$. Then there exist unique power series*

$$c_p(\epsilon) = c_p^{(0)} + \epsilon\, c_p^{(1)} + \epsilon^2 c_p^{(2)} + \ldots, \quad p = 0, 1, 2, \ldots$$

such that the following identity in the ring $\mathbb{C}[[x, \epsilon]]$ holds true

$$x = c_0(\epsilon) + \sum_{p>0} c_p(\epsilon) h_{p-2}(u; u_x, \ldots, \epsilon^{2p-2} u^{(2p-2)}). \qquad (3.8.12)$$

Proof The leading coefficients $c_p^{(0)}$ must satisfy

$$\sum_p c_p^{(0)} \frac{u_0^p(x)}{p!} = x$$

where $u_0(x) = u(x,0)$. Therefore they are equal to the derivatives of the inverse function

$$c_p^{(0)} = \left.\frac{d^p x}{du_0^p}\right|_{u_0=0}, \quad p = 0,1,2,\ldots.$$

Proceeding by induction we obtain

$$\sum_p c_p^{(k)} \frac{u_0^p(x)}{p!} = f_k(x)$$

where $f_k(x)$ is a polynomial in

$$c_q^{(i)}, \quad i = 0,\ldots,k-1$$

and in

$$u_j(x) = \left.\frac{d^j u(x,\epsilon)}{d\epsilon^j}\right|_{\epsilon=0}, \quad j = 0,\ldots,k$$

and their derivatives in x. Therefore

$$c_p^{(k)} = \left.\frac{d^p f_k(x)}{du_0^p}\right|_{u_0=0}, \quad p = 0,1,2,\ldots.$$

□

Corollary 3.8.7 *The transformation (3.8.6) establishes a one-to-one correspondence between solutions $v(x,\mathbf{t},\epsilon)$ to the Riemann hierarchy satisfying $v_x(0,0,0) \neq 0$ and solutions $u(x,\mathbf{t},\epsilon)$ to the KdV hierarchy satisfying $u_x(0,0,0) \neq 0$.*

Proof Let $c_p(\epsilon)$ be the coefficients determined according to Lemma 3.8.6 by $u(x,0,\epsilon)$. According to Lemma 3.8.1 the solution $u(x,\mathbf{t},\epsilon)$ to the KdV hierarchy satisfies

$$x + t_0 - c_0(\epsilon) + \sum_{p>0} (t_p - c_p(\epsilon)) \frac{\delta \bar{h}_{p-1}}{\delta u(x)} = 0$$

identically in $\mathbf{t}$. Let $v = v(x,\mathbf{t},c(\epsilon))$ be the solution (3.8.2) to the Riemann hierarchy determined by these coefficients. By the construction the quasi-Miura transformation maps v to $u(x,\mathbf{t},\epsilon)$. □

We will now prove

Lemma 3.8.8 *The quasi-Miura (3.8.7) $v \mapsto u = F(v; v_x, \dots; \epsilon)$ transforms the vector fields of the Riemann hierarchy to those of the KdV hierarchy.*

Proof Let

$$\hat{K}_j(u; u_x, \dots; \epsilon) = \sum_m \epsilon^{2m} u_x^{-p_m} \hat{K}_j^{[2m]}(u; u_x, \dots, u^{(q_m)}) = \left(\sum_s \frac{\partial F}{\partial v^{(s)}} \partial_x^{s+1} \right) \frac{v^{j+1}}{(j+1)!}$$

be the result of application of the quasi-Miura transform to the flows of the Riemann hierarchy. Here $K_j^{[2m]}(u; u_x, \dots, u^{(n_m)})$ are some polynomials in the derivatives. The precise values of the positive numbers p_m and q_m (that also depend on j) is not important. Denote

$$K_j(u; u_x, \dots, u^{2j+1}; \epsilon) = \sum_{m=0}^{2j} \epsilon^{2m} K_j^{[2m]}(u; u_x, \dots, u^{(2m+1)})$$

the r.h.s. of the j-th equation of the KdV hierarchy. According to the Lemma 3.8.6 the identity

$$\sum_{m=0}^{2j} \epsilon^{2m} K_j^{[2m]}(u; u_x, \dots, u^{(2m+1)}) = \sum_m \epsilon^{2m} u_x^{-p_m} \hat{K}_j^{[2m]}(u; u_x, \dots, u^{(q_m)})$$

holds true for an arbitrary monotone solution $u(x, \mathbf{t}, \epsilon)$ to the KdV hierarchy. From this it easily follows that

$$u_x^{-p_m} \hat{K}_j^{[2m]}(u; u_x, \dots, u^{(q_m)}) = K_j^{[2m]}(u; u_x, \dots, u^{(2m+1)}).$$

□

Denote

$$\hat{h}_k(v; v_x, \dots; \epsilon) = h_k(u; \epsilon\, u_x, \dots, \epsilon^{2k+2} u^{(2k+2)}), \quad k = -1, 0, 1, \dots,$$

the functions in the derivatives obtained from the Hamiltonian densities of KdV by the inverse to the quasi-Miura transformation (3.8.7).

Lemma 3.8.9

$$\hat{h}_k = \frac{v^{k+2}}{(k+2)!} + \text{total derivative}.$$

Proof Applying the inverse to the quasi-Miura transformation to the infinitesimal form of the conservation law

$$\frac{\partial h_k(u, \epsilon\, u_x, \dots)}{\partial t_j} = \frac{\partial \Omega_{k+1,j}(u, \epsilon\, u_x, \dots)}{\partial x}$$

(here $\Omega_{k+1,j}(u,\ldots)$ is the density of the flux of the conserved quantity along the j-th time defined in (3.3.13)) we obtain, according to Lemma 3.8.4

$$\frac{\partial \hat{h}_k(v, v_x, \ldots; \epsilon)}{\partial t_j} = \frac{\partial \hat{\Omega}_{k+1,j}(v, v_x, \ldots; \epsilon)}{\partial x}.$$

In the last equation the time derivative is taken w.r.t. the Riemann hierarchy; the functions $\hat{\Omega}_{k+1,j}(v, v_x, \ldots; \epsilon)$ are obtained from $\Omega_{k+1,j}(u, \epsilon\, u_x, \ldots)$ by the same inverse quasi-Miura. Therefore $\hat{h}_k(v, v_x, \ldots; \epsilon)$ is the density of a conservation law for the Riemann hierarchy. Due to the completeness theorem 3.6.21 it must coincide with the standard density $v^{k+2}/(k+2)!$ up to a total derivative in x. □

We are now ready to prove the main result of this section.

Theorem 3.8.10 *The (inverse to) the quasi-Miura transformation (3.8.7) transforms the Magri Poisson pencil (3.1.18), (3.1.18) to the Poisson pencil (3.7.7) for the Riemann hierarchy.*

Proof Applying the inverse quasi-Miura to the first Poisson bracket of the KdV we obtain a Poisson bracket

$$\{v(x), v(y)\}\hat{}_1 = \sum \epsilon^{2m} A_{2m,s}(v; v_x, \ldots,)\delta^{(2m-s+1)}(x-y).$$

From Lemma 3.8.9 it follows that

$$\frac{\delta \int \hat{h}_k}{\delta v(x)} = \frac{v^{k+1}}{(k+1)!}, \quad k = 0, 1, \ldots.$$

From Lemma 3.8.8 it follows that

$$\sum \epsilon^{2m} A_{2m,s}(v; v_x, \ldots,)\partial_x^{2m-s+1} \frac{v^{k+1}}{(k+1)!} = \partial_x \frac{v^{k+1}}{(k+1)!}$$

for all non-negative k. Multiplying the last equation by z^{k+1}, where z is an indeterminate, and summing in k we obtain

$$\sum \epsilon^{2m} A_{2m,s}(v; v_x, \ldots,)\partial_x^{2m-s+1} e^{z\, v(x,\epsilon)} = \partial_x e^{z\, v(x,\epsilon)}$$

for all z and for an arbitrary function $v(x, \epsilon)$. From this it easily follows that $\{\ ,\ \}\hat{}_1 = \{\ ,\ \}_1$.

Applying the inverse quasi-Miura to the second Poisson bracket for KdV we obtain a Poisson bracket

$$\{v(x), v(y)\}\hat{}_2 = \sum \epsilon^{2m} B_{2m,s}(v; v_x, \ldots,)\delta^{(2m-s+1)}(x-y).$$

that must satisfy the recursion relation (3.6.73) for the Riemann hierarchy

$$\sum \epsilon^{2m} B_{2m,s}(v; v_x, \dots,)\partial_x^{2m-s+1}\frac{v^k}{k!} = (k+\frac{1}{2})\partial_x \frac{v^{k+1}}{(k+1)!}.$$

Multiplying by z^{k+1} and summing in k we obtain the identity

$$\sum \epsilon^{2m} B_{2m,s}(v; v_x, \dots,)\partial_x^{2m-s+1} e^{z\, v} = \left(z\, v + \frac{1}{2}\right) v_x e^{z\, v}$$

valid for any z and for an arbitrary function $v = v(x, \epsilon)$. This proves that $\{ \, , \, \}\hat{}_2 = \{ \, , \, \}_2$. The Theorem is proved. □

Our approach can easily be extended to prove quasitriviality of the Gelfand - Dickey hierarchy (also called nKdV). We will study in a separate publication the problem of quasitriviality of other hierarchies of integrable 1+1 PDEs, in particular of the Drinfeld - Sokolov hierarchies of D and E type and of Toda lattice.

We will now prove that, in addition to Lemma 3.8.4, the following statement.

Lemma 3.8.11 *There exists a function*

$$\Delta f = \Delta f(v', v'', \dots; \epsilon^2) = \sum_{k=1}^{\infty} \epsilon^{2k-2} \Delta f^{[k]}(v', \dots, v^{(3k-2)}) \qquad (3.8.13)$$

where

$$\Delta f^{[1]}(v') = -\frac{1}{12}\log v'$$

and $\Delta f^{[k]}(v', \dots, v^{(3k-2)})$ is a quasi-homogeneous function in the derivatives of the degree $2k-2$ such that the correspondence (3.8.11) is represented as

$$v \mapsto u = v + \epsilon^2 \partial_x^2 \Delta f(v', v'', \dots, ; \epsilon^2). \qquad (3.8.14)$$

Proof (cf. the proof of Theorem 3.9.1 below). We already know from Lemma 3.8.9 that

$$h_{p-1}(u; u', \dots, u^{(p-1)}; \epsilon) = \frac{v^{p+1}}{(p+1)!} + \epsilon \partial_x g_{p-1}(v, v', \dots; \epsilon)$$

for some functions $g_k(v, v', \dots; \epsilon)$. Using the tau-symmetry

$$\frac{\partial h_{p-1}}{\partial t_q} = \frac{\partial h_{q-1}}{\partial t_p}$$

$$\frac{\partial}{\partial t_q}\frac{v^{p+1}}{(p+1)!} = \frac{\partial}{\partial t_p}\frac{v^{q+1}}{(q+1)!}$$

of the KdV hierarchy and of the Riemann hierarchy we obtain

$$\frac{\partial}{\partial x}\left[\frac{\partial g_{p-1}}{\partial t_q} - \frac{\partial g_{q-1}}{\partial t_p}\right] = 0.$$

This implies existence of a function

$$\Delta f = \sum_{k=1}^{\infty} \epsilon^{2k-2} \Delta f^{[k]}(v, v', \dots, v^{(3k-2)})$$

such that

$$g_{p-1}(v, v', \dots; \epsilon) = \epsilon \frac{\partial \Delta f}{\partial t_p} = \epsilon \sum \frac{\partial \Delta f}{\partial v^{(i)}} \left(\frac{v^{p+1}}{(p+1)!} \right)^{(i+1)}.$$

In particular, the quasi-Miura transformation itself reads

$$u = v + \epsilon \partial_x g_0 = v + \epsilon^2 \partial_x^2 \Delta f$$

where we may assume, due to quasihomogeneity of the terms $u^{[k]}$ in the derivatives that

$$\Delta f = \Delta f(v', v'', \dots; \epsilon^2)$$

does not depend explicitly on v. Lemma is proved. □

Example 3.8.12 *The topological solution to the Riemann hierarchy is determined by the equation*

$$v = t_0 + t_1 v + t_2 \frac{v^2}{2} + t_3 \frac{v^3}{6} + \dots \tag{3.8.15}$$

(we omit x identifying x with t_0). The tau-function of this solution

$$\log \tau_0 = \frac{1}{\epsilon^2} \left(\frac{t_0^3}{6} + \frac{t_0^3\, t_1}{6} + \frac{t_0^3\, t_1^2}{6} + \frac{t_0^3\, t_1^3}{6} + \frac{t_0^3\, t_1^4}{6} + \frac{t_0^4\, t_2}{24} + \frac{t_0^4\, t_1\, t_2}{8} \right.$$
$$\left. + \frac{t_0^4\, t_1^2\, t_2}{4} + \frac{t_0^5\, t_2^2}{40} + \frac{t_0^5\, t_3}{120} + \frac{t_0^5\, t_1\, t_3}{30} + \frac{t_0^6\, t_4}{720} + \dots \right). \tag{3.8.16}$$

Applying the quasi-Miura transformation (3.8.11) we obtain, after changing the normalization

$$\epsilon^2 \mapsto -\frac{\epsilon^2}{2}$$

a solution to the KdV hierarchy with the tau-function (1.7). We will show below in Section 3.10.4 this series coincides with the Witten - Kontsevich generating function of the Mumford - Morita - Miller intersection numbers on the moduli spaces $\bar{\mathcal{M}}_{g,n}$ of all genera

$$\mathcal{F} = \sum_{g=0}^{\infty} \epsilon^{2g-2} \mathcal{F}_g \tag{3.8.17}$$

where

$$\mathcal{F}_g = \sum_{n=1}^{\infty} \frac{1}{n!} t_{p_1} \dots t_{p_n} < \phi_{p_1} \dots \phi_{p_n} >_g \tag{3.8.18}$$

$$< \phi_{p_1} \dots \phi_{p_2} >_g = \int_{\bar{\mathcal{M}}_{g,n}} c_1^{p_1}(\mathcal{L}_1) \wedge \dots \wedge c_1^{p_n}(\mathcal{L}_n) \tag{3.8.19}$$

where $\mathcal{L}_i$ is the tautological line bundle over the moduli space $\bar{\mathcal{M}}_{g,n}$ of stable algebraic curves C of genus g with n punctures $x_1, \dots, x_n$, i.e., the fiber of $\mathcal{L}_i$ coincides with the cotangent line $T^*_{x_i}C$.

The idea to express the terms $\mathcal{F}_1$, $\mathcal{F}_2$, ..., of the genus expansion (3.8.17) as functions of v', v'', ... where $v = v(t)$ is the solution (3.8.15) is due to Dijkgraaf and Witten [27]. This idea proved to be fruitful also in other models of 2D topological field theory [56, 58, 59, 152, 110, 78].

Example 3.8.13 *Applying (3.8.11) to the monotone at $x = 0$ function $v = v(x)$*

$$x = \sqrt{v} J_1(2\sqrt{v}) = \sum_{m=0}^{\infty} (-1)^m \frac{v^{m+1}}{m!(m+1)!} \tag{3.8.20}$$

one obtains

$$u(x, \epsilon) = \pi^6 \sum_{g=0}^{\infty} \left(\frac{\epsilon}{\pi^3}\right)^{2g} \sum_n Vol(\mathcal{M}_{g,n}) \left(\frac{x}{\pi^2}\right)^n \tag{3.8.21}$$

where $Vol(\mathcal{M}_{g,n})$ is the Weil - Petersson volume of the moduli space of punctured algebraic curves. This is a reformulation of the result of P. Zograf [152] (see also [110]).

3.9 Properties of quasitrivial Poisson pencils

Let

$$\{u^\alpha(x), u^\beta(y)\}_\lambda = \sum_{k \geq 0} \epsilon^k \{u^\alpha(x), u^\beta(y)\}_\lambda^{[k]}$$

be a quasitrivial Poisson pencil written in the normal coordinates with the leading term

$$\{u^\alpha(x), u^\beta(y)\}_\lambda^{[0]} = \left(g^{\alpha\beta}(u(x)) - \lambda\, \eta^{\alpha\beta}\right) \delta'(x-y) + \Gamma_\gamma^{\alpha\beta}(u) u_x^\gamma \delta(x-y)$$

determined by a n-dimensional rigid semisimple Frobenius manifold M (see the formula (3.5.24) above). Let

$$u_\alpha = v_\alpha + \sum_{k>0} \epsilon^k F_\alpha^{[k]}(v; v_x, \dots, v^{(n_k+2)}) \tag{3.9.1}$$

be the quasitriviality transformation for the pencil:

$$\sum_{k \geq 0} \epsilon^k \{u^\alpha(x), u^\beta(y)\}_\lambda^{[k]} = \left(g^{\alpha\beta}(v(x)) - \lambda\, \eta^{\alpha\beta}\right) \delta'(x-y) + \Gamma_\gamma^{\alpha\beta}(v)\, v_x^\gamma\, \delta(x-y).$$

Here we lower the indices as usual by means of the constant matrix $\eta_{\alpha\beta}$, $F_\alpha^{[k]}(v; v_x, \dots, v^{(n_k)})$ are some functions rational in the derivatives.

Our first result is

Theorem 3.9.1 *Let the quasi-Miura transformation (3.9.1) depend polynomially on v^1. Then there exists a function*

$$\mathcal{F}(v; v_x, \ldots, ; \epsilon) = \sum_{k>0} \epsilon^k \mathcal{F}^{[k]}(v; v_x, \ldots, v^{(n_k)}) \tag{3.9.2}$$

such that the quasitriviality has the form

$$u_\alpha = v_\alpha + \partial_x \partial_{t^{\alpha,0}} \mathcal{F}(v; v_x, \ldots, ; \epsilon). \tag{3.9.3}$$

Moreover, the tau-structure for the pencil $\{\ ,\ \}_\lambda$ written in the coordinates v has the form, up to an equivalence (3.5.27), (3.5.28),

$$h_{\alpha,p}(v; v_x, \ldots; \epsilon) = \theta_{\alpha,p}(v) + \partial_x \partial_{t^{\alpha,p}} \mathcal{F}(v; v_x, \ldots, ; \epsilon). \tag{3.9.4}$$

We recall (see Section 3.6.2 above) that polynomiality in v^1 means that every coefficient $F_\alpha^{[k]}(v; v_x, \ldots, v^{(n_k)})$ is a polynomial in v^1 of the degree that may depend on k.

Proof By the definition of the normal coordinates $\bar{u}_\alpha$ is a Casimir of $\{\ ,\ \}_1$. Since $\{\ ,\ \}_1$ is obtained from $\{\ ,\ \}_1^{[0]}$ by the change of coordinates (3.9.1), and $\bar{v}_\alpha$ is a Casimir of $\{\ ,\ \}_1^{[0]}$, it follows that $\bar{v}_\alpha$ is also a Casimir of $\{\ ,\ \}_1$. Hence the difference $u_\alpha - v_\alpha = O(\epsilon)$ is a conserved quantity for the Principal Hierarchy. Due to Lemma 3.6.20 $u_\alpha - v_\alpha$ must be a total derivative (polynomiality in the derivatives assumption can be eliminated by considering arbitrary functions in the derivatives). Hence the quasitriviality transformation must have the form

$$u_\alpha = v_\alpha + \partial_x f_{\alpha,0}(v; v_x, \ldots; \epsilon)$$

for some function $f_{\alpha,0}(v; v_x, \ldots; \epsilon)$.

Let $h_{\alpha,p}$ be the densities of the commuting Hamiltonians corresponding to a choice of a tau-structure for the Poisson pencil $\{\ ,\ \}_\lambda$ satisfying a recursion relation

$$\{\ .\ , \bar{h}_{\alpha,p}\}_2 = \sum_{q=0}^{p} A_{\alpha,p}^{\beta,q} \{\ .\ , \bar{h}_{\beta,q+1}\}_1$$

with some constant coefficients $A_{\alpha,p}^{\beta,q}$. We have $h_{\alpha,0} = u_\alpha$ since u_α are the normal coordinates for the chosen tau-structure. Rewriting the densities in the v-coordinates

$$h_{\alpha,p} = h_{\alpha,p}(v; v_x, \ldots; \epsilon) = \sum_{k\geq 0} \epsilon^k h_{\alpha,p}^{[k]}(v; v_x, \ldots)$$

we obtain the same recursion relation

$$\{\ .\ , \int h_{\alpha,p}(v; v_x, \ldots; \epsilon)\, dx\}_2^{[0]} = \sum_{q=0}^{p} A_{\alpha,p}^{\beta,q} \{\ .\ , \int h_{\beta,q+1}(v; v_x, \ldots; \epsilon)\, dx\}_1^{[0]}$$

with the initial data

$$\int h_{\alpha,0}(v; v_x, \dots; \epsilon)\, dx = \int v_\alpha\, dx.$$

Therefore the Hamiltonians $\int h_{\alpha,p}(v; v_x, \dots; \epsilon)\, dx$ are linear combinations of the standard Hamiltonians $\int \theta_{\beta,0}(v)\, dx, \ \dots, \ \int \theta_{\beta,p}(v)\, dx$ of the hierarchy (3.6.47). It follows that the leading terms $h_{\alpha,p}(v)^{[0]}$ give a tau-structure of the standard hierarchy. It must be related to the standard tau-structure by a transformation of the form (3.5.27), (3.5.28). Modifying the Hamiltonians $h_{\alpha,p}$ by the inverse transformation we obtain an equivalent tau-structure for the Poisson pencil $\{\ ,\ \}_\lambda$ satisfying

$$h_{\alpha,p}(v; v_x, \dots; \epsilon) = \theta_{\alpha,p}(v) + \epsilon\, \partial_x f_{\alpha,p}(v; v_x, \dots; \epsilon)$$

where

$$f_{\alpha,p}(v; v_x, \dots; \epsilon) = \sum_{k>0} \epsilon^{k-1} f_{\alpha,p}^{[k]}(v; v_x, \dots)$$

for some functions $f_{\alpha,p}^{[k]}(v; v_x, \dots)$. By the definition of the tau-structure we have

$$\frac{\partial f_{\alpha,p}(v; v_x, \dots; \epsilon)}{\partial t^{\beta,q}} = \frac{\partial f_{\beta,q}(v; v_x, \dots; \epsilon)}{\partial t^{\alpha,p}}.$$

In particular,

$$\frac{\partial f_{1,0}(v; v_x, \dots; \epsilon)}{\partial t^{\beta,q}} = \frac{\partial f_{\beta,q}(v; v_x, \dots; \epsilon)}{\partial x}. \tag{3.9.5}$$

So $f_{1,0}(v; v_x, \dots; \epsilon)$ is an integral of the hierarchy (3.6.47). Due to the polynomiality assumption it must be a total derivative of some function that we denote $\mathcal{F}$

$$f_{1,0}(v; v_x, \dots; \epsilon) = \partial_x \mathcal{F}(v; v_x, \dots; \epsilon).$$

From (3.9.5) we get

$$\partial_x \partial_{t^{\beta,q}} \mathcal{F} = \partial_x f_{\beta,q}.$$

This proves the theorem. □

We will next obtain upper estimates for the order of the highest derivative in $\mathcal{F}^{[k]}$, and we will also describe the explicit form of the first three terms of this expansion.

Let us first consider the infinitesimal deformation of the Poisson pencil $\{v^\alpha(x), v^\beta(y)\}_\lambda^{[0]}$ caused by a quasi-Miura transformation

$$v_\alpha \mapsto w_\alpha = v_\alpha + \epsilon\, \frac{\partial^2 \mathcal{F}(v; v_x, \dots, v^{(l)})}{\partial x \partial t^{\alpha,0}} + O(\epsilon^2). \tag{3.9.6}$$

We have changed the notations for the dependent functions of the hierarchy since the variables $u^i = u^i(v)$, $i = 1, \dots, n$, will be reserved for denoting the canonical coordinates on the Frobenius manifold. As in the Section 2, we denote $v^{\alpha,s}$ and $w^{\alpha,s}$ the jet coordinates,

$$v^{\alpha,s} = \frac{\partial v^\alpha}{\partial x^s}, \quad w^{\alpha,s} = \frac{\partial w^\alpha}{\partial x^s},$$

$$v^{\alpha,0} = v^\alpha, \quad w^{\alpha,0} = w^\alpha, \quad v^{\alpha,1} = v_x^\alpha, \quad w^{\alpha,1} = w_x^\alpha, \quad \dots.$$

Lemma 3.9.2 *The deformed Poisson pencil has the form*

$$\begin{aligned}
&\{w^\alpha(x), w^\beta(y)\}_1 = \eta^{\alpha\beta}\,\delta'(x-y)\\
&\quad +\epsilon\left(W^{\alpha\beta}(w, w_x, \ldots)\,\delta^{(K_l)}(x-y) + R^{\alpha\beta}(w, w_x, \ldots)\,\delta^{(K_l-1)}(x-y) + \ldots\right)\\
&\quad +\mathcal{O}(\epsilon^2),
\end{aligned}\tag{3.9.7}$$

$$\begin{aligned}
&\{w^\alpha(x), w^\beta(y)\}_2 = g^{\alpha\beta}(w(x))\,\delta'(x-y) + \Gamma^{\alpha\beta}_\gamma(w(x))\,w^\gamma_x\,\delta(x-y)\\
&\quad +\epsilon\left(S^{\alpha\beta}(w, w_x, \ldots)\,\delta^{(K_l)}(x-y) + Q^{\alpha\beta}(w, w_x, \ldots)\,\delta^{(K_l-1)}(x-y) + \ldots\right)\\
&\quad +\mathcal{O}(\epsilon^2),
\end{aligned}\tag{3.9.8}$$

where the integer K_l is equal to $l+3$ when $l = 2\,m$ and it is equal to $l+2$ when $l = 2m-1$.

The proof can be obtained by a simple straightforward computation.

Lemma 3.9.3 *Let $l = 2\,m$, then in the deformed Poisson bracket (3.9.8) the term $S^{\alpha\beta}(w, w_x, \ldots)\,\delta^{(l+3)}(x-y)$ does not appear* iff *$\mathcal{F}$ does not depend on $v^{\alpha,2\,m}$, $\alpha = 1, \ldots, n$.*

Proof From the form of the quasi-Miura transformation we see that the functions $S^{\alpha\beta}(v, v_x, \ldots)$ are given by the formulae

$$\left(g^{\alpha\nu}\,c^{\gamma\beta}_\nu + g^{\beta\nu}\,c^{\gamma\alpha}_\nu\right)\frac{\partial\mathcal{F}}{\partial v^{\gamma,2m}} = 2\,g^{\alpha\nu}\,c^{\gamma\beta}_\nu\frac{\partial\mathcal{F}}{\partial v^{\gamma,2m}}$$

So outside the discriminant $\det(g^{\alpha\beta}) = 0$ of the Frobenius manifold the above expression vanishes *iff* $\frac{\partial\mathcal{F}}{\partial v^{\gamma,2m}} = 0$. The lemma is proved. □

Lemma 3.9.4 *Let l be an odd positive integer, $l = 2\,m-1$. Denote $h_\alpha = \frac{\partial\mathcal{F}}{\partial v^{\alpha,2m-1}}$, then*

$$\begin{aligned}
W^{\alpha\beta} &= -2m\,\eta^{\alpha\gamma}\,c^{\beta\xi}_\gamma\,\partial_x h_\xi + \eta^{\alpha\gamma}\,\frac{\partial h_\gamma}{\partial t_{\beta,0}} + \eta^{\beta\gamma}\,\frac{\partial h_\gamma}{\partial t_{\alpha,0}} + 2\,\eta^{\alpha\gamma}\,c^{\beta\xi}_\gamma\,\frac{\partial\mathcal{F}}{\partial v^{\xi,2m-2}}\\
&\quad +2m\,\partial_x(\eta^{\alpha\gamma}\,c^{\beta\xi}_\gamma)h_\xi.
\end{aligned}\tag{3.9.9}$$

$$\begin{aligned}
S^{\alpha\beta} &= -2m\,g^{\alpha\gamma}\,c^{\beta\xi}_\gamma\,\partial_x h_\xi + g^{\alpha\gamma}\,\frac{\partial h_\gamma}{\partial t_{\beta,0}} + g^{\beta\gamma}\,\frac{\partial h_\gamma}{\partial t_{\alpha,0}} + 2\,g^{\alpha\gamma}\,c^{\beta\xi}_\gamma\,\frac{\partial\mathcal{F}}{\partial v^{\xi,2m-2}}\\
&\quad +\left(2m\,\partial_x(g^{\alpha\gamma}\,c^{\beta\xi}_\gamma) + \partial_\gamma g^{\alpha\beta}\,c^{\gamma\xi}_\nu\,v^\nu_x\right)h_\xi,
\end{aligned}\tag{3.9.10}$$

and

$$c^\lambda_{\alpha\beta}\left(S^{\alpha\beta} - \mathcal{U}^\alpha_\gamma\,W^{\gamma\beta}\right) = (2m+1)\,c^\lambda_{\alpha\beta}\,c^{\alpha\gamma}_\nu\,c^{\beta\xi}_\gamma\,v^\nu_x\,\frac{\partial\mathcal{F}}{\partial v^{\xi,2m-1}},\quad 1 \le \lambda \le n,\tag{3.9.11}$$

where $\mathcal{U}^\alpha_\beta = E^\gamma\,c^\alpha_{\gamma\beta}$ is the matrix of the operator of multiplication by the Euler vector field. In particular, the left hand sides of the last equalities vanish iff *$\mathcal{F}$ does not depend on $v^{\alpha,2\,m-1}$, $\alpha = 1, \ldots, n$.*

Here we use notations

$$\frac{\partial}{\partial t_{\alpha,p}} := \eta^{\alpha\beta}\frac{\partial}{\partial t^{\beta,p}}$$

for the linear combinations of the flows of the hierarchy (3.6.47).

Proof The identities (3.9.9) and (3.9.10) are deduced from their definitions. The equalities in (3.9.11) follow directly from (3.9.9) and (3.9.10). To prove the last statement of the lemma, let us rewrite the right hand side of the identity (3.9.11) in the canonical coordinates $u^i = u^i(v^1, \dots, u^n)$, $1 \le i \le n$ (see Section 3.10.5 for the definition of the canonical coordinates and of the functions $\psi_{i,\alpha}$ to be used below). It becomes equal to

$$(2m+1)\eta^{\lambda\nu}\,\psi_{i\nu}\,\frac{u^i_x}{\psi^3_{i1}}\,\frac{\partial\mathcal{F}}{\partial u^{i,2m-1}}.$$

Since $\det(\eta^{\lambda\nu}\psi_{i\nu}) \neq 0$, we deduce that the left hand sides of the above equalities vanish *iff* $\mathcal{F}$ does not depend on $u^{i,2m-1}$, $1 \le i \le n$ (equivalently, $\mathcal{F}$ does not depend on $v^{\alpha,2m-1}$, $1 \le \alpha \le n$). Lemma is proved. □

Theorem 3.9.5 *Let*

$$\{w^\alpha(x), w^\beta(y)\}_1 = \eta^{\alpha\beta}\delta'(x-y) + \sum_{i\ge 1}\epsilon^i\,\{w^\alpha(x), w^\beta(y)\}_1^{[i]} \tag{3.9.12}$$

$$\{w^\alpha(x), w^\beta(y)\}_2 = g^{\alpha\beta}(w(x))\,\delta'(x-y) + \Gamma^{\alpha\beta}_\gamma(w(x))\,w^\gamma_x\,\delta(x-y) + \sum_{i\ge 1}\epsilon^i\,\{w^\alpha(x), w^\beta(y)\}_2^{[i]} \tag{3.9.13}$$

be a quasitrivial Poisson pencil. Here $\{w^\alpha(x), w^\beta(y)\}_1^{[i]}$, $\{w^\alpha(x), w^\beta(y)\}_2^{[i]}$ *have the form*

$$\{w^\alpha(x), w^\beta(y)\}_1^{[i]} = \sum_{l=0}^{i+1} H^{\alpha\beta}_{i,l}(w; w_x, w_{xx}, \dots, w^{(l)})\,\delta^{(i+1-l)}(x-y),$$
$$\{w^\alpha(x), w^\beta(y)\}_2^{[i]} = \sum_{l=0}^{i+1} K^{\alpha\beta}_{i,l}(w; w_x, w_{xx}, \dots, w^{(l)})\,\delta^{(i+1-l)}(x-y), \tag{3.9.14}$$

and $H^{\alpha\beta}_{i,l}, K^{\alpha\beta}_{i,l}$ *are quasihomogeneous polynomials in the derivatives of the degree* l. *Then the quasi-triviality transformation*

$$v_\alpha \mapsto w_\alpha = v_\alpha + \sum_{k\ge 1}\epsilon^k\,\frac{\partial^2\mathcal{F}^{[k]}(v^1, \dots, v^n; v^{1,1}, \dots, v^{n,1}; \dots, v^{1,m_k}, \dots, v^{n,m_k})}{\partial x\partial t^{\alpha,0}}, \tag{3.9.15}$$

must have the property

$$m_{2g}, \quad m_{2g+1} \le 3g-2. \tag{3.9.16}$$

Moreover, modulo additive constants, we

$$\mathcal{F}^{[1]} = 0, \tag{3.9.17}$$

$$\mathcal{F}^{[2]} = \sum_{i=1}^{n} a_i \, \log(u_x^i) + f(v), \tag{3.9.18}$$

$$\mathcal{F}^{[3]} = \sum_{i=1}^{n} f_i(v) \, u_x^i. \tag{3.9.19}$$

where a_i are constants, $f(v)$ and $f_i(v)$ are some functions, and $u^1(v)$, $\dots$, $u^n(v)$ are the canonical coordinates on M.

Proof From Lemma 3.9.3 and Lemma 3.9.4 we see that $\mathcal{F}^{[1]}$ must be a constant, and $\mathcal{F}^{[2]}$, $\mathcal{F}^{[3]}$ only depend on $v^1, \dots, v^n; v_x^1, \dots, v_x^n$. We are to prove that $\mathcal{F}^{[2g]}, \mathcal{F}^{[2g+1]}$ depend at most on $v^1, \dots, v^n; v^{1,1}, \dots, v^{n,1}; \dots; v^{1,3g-2}, \dots, v^{n,3g-2}$ for $g \geq 2$. This can be done by induction. Assume that the statement is true for $g \leq N-1$, we need to prove the validity of the above property for $g = N$. Express the left and right hand sides of (3.9.12) and (3.9.13) in the v-coordinates by using the quasi-Miura transformation (3.9.15) and compare the ϵ^{2N} terms of both sides. We denote, as in Lemma 3.9.2, by $W^{\alpha\beta}$ and $S^{\alpha\beta}$ the coefficients of the highest order derivatives of the delta function in the ϵ^{2N} term that are contributed by $\mathcal{F}^{[2]}, \dots, \mathcal{F}^{[2N-1]}$ in the left hand sides of (3.9.12) and (3.9.13) respectively. When N is even this is the coefficients of $\delta^{(3N+1)}(x-y)$ and when N is odd this is the coefficient of $\delta^{(3N)}(x-y)$. While on the right hand side of (3.9.12) and (3.9.13) the highest order derivatives of the delta function in the ϵ^{2N} term is $\delta^{(2N+1)}(x-y)$ due to the form of $\{\ ,\ \}_2^{[2N]}$. So, in order that the equality (3.9.13) holds true, the $\mathcal{F}^{[2N]}$ term in the quasi-Miura transformation (3.9.15) must be responsible for the killing of the $S^{\alpha\beta}\,\delta^{(3N+1)}(x-y)$ term when N is even and the $S^{\alpha\beta}\,\delta^{(3N)}(x-y)$ term when N is odd in the left hand sides of (3.9.12) and (3.9.13). We first consider the case when N is odd. In this case, by using Lemma 3.9.3 and Lemma 3.9.4 we immediately deduce that $\mathcal{F}^{[2g]}$ depends at most on $v^1, \dots, v^n; v^{1,1}, \dots, v^{n,1}, \dots, v^{1,3N-2}, \dots, v^{n,3N-2}$. Now let us consider the case when N is even. From Lemma 3.9.3 we see that $\mathcal{F}^{[2N]}$ depends at most on $v^\alpha, v^{\alpha,1}, \dots, v^{\alpha,3N-1}$, $\alpha = 1, \dots, n$. We are to prove that actually it does not depend on $v^{1,3N-1}, \dots, v^{n,3N-1}$. To prove this let us compute $W^{\alpha\beta}$ and $S^{\alpha\beta}$ to obtain

$$S^{\alpha\beta} = \sum_{k=1}^{N-1} \frac{\partial \mathcal{F}^{[2k]}}{\partial v^{\gamma,3k-2}} \frac{\partial \mathcal{F}^{[2N-2k]}}{\partial v^{\nu,3(N-k)-2}} \, c_\sigma^{\alpha\gamma} \, c_\rho^{\beta\nu} \, g^{\sigma\rho},$$

$$W^{\alpha\beta} = \sum_{k=1}^{N-1} \frac{\partial \mathcal{F}^{[2k]}}{\partial v^{\gamma,3k-2}} \frac{\partial \mathcal{F}^{[2N-2k]}}{\partial v^{\nu,3(N-k)-2}} \, c_\sigma^{\alpha\gamma} \, c_\rho^{\beta\nu} \, \eta^{\sigma\rho},$$

and

$$c_{\alpha\beta}^\lambda \left(S^{\alpha\beta} - \mathcal{U}_\gamma^\alpha \, W^{\gamma\beta} \right) = 0.$$

So from Lemma 3.9.4 we see that $\mathcal{F}^{[2N]}$ indeed does not depend on $v^{1,3N-1}, \dots, v^{n,3N-1}$. In a similar way, we can prove that $\mathcal{F}^{[2N+1]}$ depends at most on $v^1, \dots, v^n$; $v^{1,1}, \dots, v^{n,1}$;

$\ldots; v^{1,3M-2}, \ldots, v^{n,3M-2}$. We have thus finished the procedure of induction and proved (3.9.16).

We next prove that $\mathcal{F}^{[2]}$ must have the form (3.9.18). Our assumption on $\{\ ,\ \}_1^{[2]}$, $\{\ ,\ \}_2^{[2]}$ implies that $H_{2,0}^{\alpha\beta}$ and $K_{2,0}^{\alpha\beta}$ are functions of $u^1, \ldots, u^n$. Rewrite both sides of (3.9.12) and (3.9.13) in the v-coordinates by using the quasi-Miura transformation (3.9.15) and compare the coefficients of $\epsilon^2\delta'''(x-y)$, we obtain by using (3.9.11) the following identities:

$$3\, c_{\alpha\beta}^{\lambda}\, c_{\nu}^{\alpha\gamma}\, c_{\gamma}^{\beta\xi}\, v_x^{\nu}\, \frac{\partial \mathcal{F}^{[2]}}{\partial v^{\xi,1}} = c_{\alpha\beta}^{\lambda} \left(K_{2,0}^{\alpha\beta} - \mathcal{U}_{\gamma}^{\alpha}\, H_{2,0}^{\gamma\beta} \right), \quad 1 \le \lambda \le n. \tag{3.9.20}$$

In the canonical coordinates the left hand sides of the above identities have the expressions

$$\eta^{\lambda\nu} \sum_{i=1}^{n} \frac{\psi_{i\nu}}{\psi_{i1}^3}\, u_x^i\, \frac{\partial \mathcal{F}^{[2]}}{\partial u_x^i}.$$

So from (3.9.20) we see that there exist functions $a_1(u), \ldots, a_n(u)$ of $u^1, \ldots, u^n$ such that

$$u_x^i\, \frac{\partial \mathcal{F}^{[2]}}{\partial u_x^i} = a_i(u), \quad 1 \le i \le n, \tag{3.9.21}$$

which yields

$$\mathcal{F}^{[2]} = \sum_{i=1}^{n} a_i(u)\, \log(u_x^i) + f(u) \tag{3.9.22}$$

for certain function f of $u^1, \ldots, u^n$. Now let's prove that $a_i(u)$ are constants. Indeed, the coefficients of $\epsilon^2\,\delta'''(x-y)$ in the left hand side of (3.9.13) written in the v-coordinates have the expressions

$$\begin{aligned} &-2\, g^{\alpha\gamma}\, c_{\gamma}^{\beta\xi}\, \partial_x \left(\frac{\partial \mathcal{F}^{[2]}}{\partial v_x^{\xi}} \right) + g^{\alpha\gamma}\, \frac{\partial}{\partial t_{\beta,0}} \left(\frac{\partial \mathcal{F}^{[2]}}{\partial t^{\gamma,1}} \right) + g^{\beta\gamma}\, \frac{\partial}{\partial t_{\alpha,0}} \left(\frac{\partial \mathcal{F}^{[2]}}{\partial t^{\gamma,1}} \right) \\ &\quad + \left(2m\, \partial_x (g^{\alpha\gamma}\, c_{\gamma}^{\beta\xi}) + \partial_{\gamma} g^{\alpha\beta}\, c_{\nu}^{\gamma\xi}\, v_x^{\nu} \right) \frac{\partial \mathcal{F}^{[2]}}{\partial v_x^{\xi}} + 2\, g^{\alpha\gamma}\, c_{\gamma}^{\beta\xi}\, \frac{\partial \mathcal{F}^{[2]}}{\partial v^{\xi}} \end{aligned} \tag{3.9.23}$$

Substituting the formula (3.9.22) into the above expressions, we see that the first four summands are rational polynomials in the x-derivatives of $u^1, \ldots, u^n$, and the last summand is a linear combination of $\log(u_x^1), \ldots, \log(u_x^n)$. Since (3.9.23) should be functions of $u^1, \ldots, u^n$ only, we deduce that

$$g^{\alpha\gamma}\, c_{\gamma}^{\beta\xi}\, \frac{\partial a_i}{\partial v^{\xi}} = 0, \quad 1 \le i \le n.$$

Putting $\beta = n$ in the above equations we obtain

$$\frac{\partial a_i}{\partial v^{\xi}} = 0, \quad 1 \le i, \xi \le n$$

for generic point v when $\det(g^{\alpha\beta}(v)) \neq 0$. So $a_1(u), \ldots, a_n(u)$ are constants.

We will now prove that $\mathcal{F}^{[3]}$ has the form (3.9.19). Since $\mathcal{F}^{[3]}$ only depends on $v^1, \dots, v^n; v_x^1, \dots, v_x^n$, the highest order of the derivatives of the delta-function in $\{\ ,\ \}_2^{[3]}$ is 3, and the coefficients of $\delta'''(x-y)$ depend linearly on $v_x^1, \dots, v_x^n$ since $\deg K_{3,1}^{\alpha\beta} = 1$. An approach similar to the one given in the derivation of (3.9.21) yields

$$u_x^i \frac{\partial \mathcal{F}^{[3]}}{\partial u_x^i} = \sum_{k=1}^n b_k^i(u)\, u_x^k, \quad 1 \le i \le n. \tag{3.9.24}$$

By using the compatibility condition $\frac{\partial}{\partial u_x^i}\left(\frac{\partial}{\partial u_x^j}\mathcal{F}^{[3]}\right) = \frac{\partial}{\partial u_x^j}\left(\frac{\partial}{\partial u_x^i}\mathcal{F}^{[3]}\right)$ we have

$$b_k^i(u) = 0, \quad i \ne k$$

which yields

$$\mathcal{F}^{[3]} = \sum_{i=1}^n b_i^i(u)\, u_x^i + h(u).$$

Here $h(u)$ is certain function of $u^1, \dots, u^n$. To prove that $h(u)$ is a constant we use the explicit expression for $K_{3,1}^{\alpha\beta}$ which is given by the right hand side of (3.9.10) with $m = 1$ and $\mathcal{F} = \mathcal{F}^{[3]}$. From this expression we get

$$g^{\alpha\gamma} c_\gamma^{\beta\xi} \frac{\partial h(u)}{\partial v^\xi} = 0, \quad 1 \le \alpha, \beta \le n$$

which implies that $h(u)$ is a constant. The theorem is proved. □

Since $\mathcal{F}^{[1]}$ is a constant and $\mathcal{F}^{[3]}$ is a polynomial in the x-derivatives of v^α, the quasitrivial bihamiltonian structure (3.9.12) and (3.9.13) is equivalent to a quasitrivial bihamiltonian structure whose quasitriviality transformation does not contain the ϵ and ϵ^3 terms. The equivalence is established by the Miura transformation

$$u_\alpha \mapsto u_\alpha - \epsilon^3 \frac{\partial^2 \mathcal{F}^{[3]}(v, v_x)}{\partial x \partial t^{\alpha,0}}.$$

We were not be able to prove that, for an arbitrary quasitriviality transformation all the terms with odd powers of ϵ can be gauged out by a Miura transformation. In the next section we will prove that this is the case under an additional assumption about the structure of the symmetry algebra of the hierarchy.

3.10 Virasoro symmetries

In this section we will show that the Principal Hierarchy (3.6.47) on $\mathcal{L}(M^n)$ of a n-dimensional Frobenius manifold M^n always admits a rich algebra of symmetries isomorphic to the half of the Virasoro algebra with the central charge n. The Virasoro algebra is constructed in terms of the spectrum of the Frobenius manifold. The operators of the Virasoro algebra act by nonlinear first order differential operators on the tau-cover of the hierarchy. We will characterize general solution of the Principal Hierarchy in terms of the action of the Virasoro algebra. Due to quasitriviality the action of the Virasoro algebra can be extended to the full hierarchy (1.7). Our last condition requires linearity of this action of Virasoro onto the tau-function of the full hierarchy. We prove that, for a semisimple Frobenius manifold M^n this condition uniquely determines the quasitriviality transformation (3.7.9). For the semisimple Frobenius manifolds coinciding with quantum cohomology of a smooth projective variety X we identify our condition of linearization of the Virasoro symmetries with the Virasoro constraints conjectured by T.Eguchi *et al.* [56] in the theory of the higher genus Gromov - Witten invaraints of X.

3.10.1 From Galilean invariance to Virasoro symmetries of the Principal Hierarchy

This subsection is based on the paper [52]. We also find a nice generating formula for the Virasoro symmetries that will be useful in subsequent calculations.

Let us begin with the following

Definition. The PDE

$$\frac{\partial v^\alpha}{\partial s} = B^\alpha(x, \mathbf{t}, v; v_x, \dots, ; \epsilon) \tag{3.10.1}$$

is called (infinitesimal) *symmetry* of the hierarchy (3.6.47) if it commutes with all the flows of the hierarchy

$$\frac{\partial}{\partial s}\frac{\partial v}{\partial t^{\alpha,p}} = \frac{\partial}{\partial t^{\alpha,p}}\frac{\partial v}{\partial s}.$$

According to our definition the flows of the hierarchy themselves are symmetries. In this case the r.h.s. does not depend on x, $\mathbf{t}$. Less trivial example is given by

Lemma 3.10.1 *The flow*

$$\frac{\partial v}{\partial s} = e + \sum_{p=1}^{\infty} t^{\alpha,p}\frac{\partial v}{\partial t^{\alpha,p-1}} \tag{3.10.2}$$

is a symmetry of the Principal Hierarchy (3.6.47).

Here e is the unity vector field on the Frobenius manifold.

Proof (cf. [52]) Let us consider first the flow

$$\frac{\partial v}{\partial s} = e.$$

Using

$$\partial_1\theta_{\alpha,p+1} = \theta_{\alpha,p}, \quad p \geq 0, \quad \partial_1\theta_{\alpha,0} = \eta_{\alpha 1}$$

we obtain

$$\frac{\partial}{\partial s}\frac{\partial v}{\partial t^{\alpha,p}} - \frac{\partial}{\partial t^{\alpha,p}}\frac{\partial v}{\partial s} = \frac{\partial v}{\partial t^{\alpha,p-1}},\ p>0, \quad \frac{\partial}{\partial s}\frac{\partial v}{\partial t^{\alpha,0}} = \frac{\partial}{\partial t^{\alpha,0}}\frac{\partial v}{\partial s}.$$

The term $\sum_{p=1}^{\infty} t^{\alpha,p}\frac{\partial v}{\partial t^{\alpha,p-1}}$ in (3.10.2) compensates the noncommutativity of the above flow with the equations of the hierarchy (cf. [64]). □

Example 3.10.2 *For the Riemann equation*

$$v_t = v\, v_x$$

the above symmetry coincides with the infinitesimal form

$$v_s = 1 + t\, v_x$$

of the Galilean transformation

$$x \mapsto x + c\,t, \quad t \mapsto t, \quad v \mapsto v + c.$$

Here c is an arbitrary parameter.

Also in the general case we will call (3.10.2) the Galilean symmetry of the hierarchy (3.6.47). It is natural to produce an infinite chain of other symmetries by applying the recursion operator

$$\mathcal{R}\frac{\partial}{\partial s_{m-1}} = \frac{\partial}{\partial s_m}, \quad m \geq 0 \tag{3.10.3}$$

where we redenote $\partial/\partial s \mapsto \partial/\partial s_{-1}$ the Galilean symmetry (3.10.2). Such symmetries were discovered in [16] for the case of KdV (also the idea appeared already in [82]). It was shown in [154] that, for the symplectic bihamiltonian structures the above chain satisfies the Virasoro commutation relations

$$\left[\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j}\right] = (j-i)\frac{\partial}{\partial s_{i+j}}, \quad i, j \geq -1 \tag{3.10.4}$$

if

$$Lie_{\partial/\partial s_0}\mathcal{R} = \mathcal{R}.$$

We cannot apply directly this result to the Principal Hierarchy. Indeed, the Poisson pencil (3.5.24) is not symplectic. From practical point of view the recursion procedure (3.10.3) produces nonlocal flows, starting from $m = 1$. For the simplest example of Riemann hierarchy

$$\frac{\partial v}{\partial s_0} = v + \frac{x}{2}v_x + \sum_{k\geq 0}\left(k+\frac{1}{2}\right) t_k \frac{\partial v}{\partial t_k}$$

$$\frac{\partial v}{\partial s_1} = v^2 + \frac{3}{4}x^2\frac{\partial v}{\partial t_1} + \frac{1}{4}\partial_x^{-1} v + \sum_{k\geq 0}\left(k+\frac{1}{2}\right)\left(k+\frac{3}{2}\right) t_k \frac{\partial v}{\partial t_{k+1}}$$

etc.

The general problem of dealing with the nonlocalities in the bihamiltonian recursion procedure has been analyzed, in the style of formal variational calculus, in [89, 133]. For the case of Principal Hierarchy it was shown in [52] that the bihamiltonian recursion procedure can be used to produce Virasoro symmetries of the tau-cover (3.3.17) of the Principal Hierarchy. We present here this result in a slightly modified form, using generating functions.

Let us consider first the nonresonant case, i.e., assuming that the spectrum of $\hat\mu$ contains no half-integers. Let $p_\alpha(v;\lambda)$, $\alpha = 1, \dots, n$ be a basis of independent periods (e.g., the one described in Theorem 3.6.12). Introduce the constant Gram matrix

$$G^{\alpha\beta} := \left(\frac{\partial}{\partial p_\alpha}, \frac{\partial}{\partial p_\beta}\right)_\lambda = \frac{\partial v^\sigma}{\partial p_\alpha} g_{\sigma\epsilon}(v;\lambda)\frac{\partial v^\epsilon}{\partial p_\beta}, \quad (g_{\sigma\epsilon}(v;\lambda)) = \left(g^{\sigma\epsilon}(v) - \lambda\eta^{\sigma\epsilon}\right)^{-1}.$$

Recall that, if the basis of independent periods is chosen as in Theorem 3.6.12, then

$$\left(G^{\alpha\beta}\right) = -\frac{1}{2\pi}(e^{\pi i R}e^{\pi i\hat\mu} + e^{-\pi i R}e^{-\pi i\hat\mu})\eta^{-1}. \tag{3.10.5}$$

Introduce *action functions*

$$s_\alpha = s_\alpha(x, f, \partial f/\partial\mathbf{t};\lambda)$$

$$\begin{aligned} s_\alpha = \int^x p_\alpha(v;\lambda)dx &= x\,\omega_1 \sum_{q\geq 0}\frac{\Gamma_q(R, \hat\mu + q - \frac12)}{\lambda^{q-1}}\lambda^{-\frac12-\hat\mu}\lambda^{-R} \\ &+ \sum_{m\geq 0}\sum_{p+q=m}\frac{\partial f}{\partial t^p}\frac{\Gamma_q(R,\hat\mu+m+\frac12)}{\lambda^m}\lambda^{-\frac12-\hat\mu}\lambda^{-R}. \end{aligned} \tag{3.10.6}$$

Put

$$\begin{aligned} S_\alpha &= S_\alpha(x, \mathbf{t}, f, \partial f/\partial\mathbf{t};\lambda) \\ &= s_\alpha + \sum_m \lambda^{m+1}\sum_{p-q=m}(-1)^p t_p \Gamma_q(R, \hat\mu - m - \frac12)\lambda^{-\frac12-\hat\mu}\lambda^{-R} \\ &= \int_0^\infty \frac{dz}{\sqrt z}e^{-\lambda z}\left[\sum_{p\geq 0}\frac{\partial f}{\partial t^p}z^p + \sum_{q\geq 0}(-1)^q\bar t_q z^{-q-1}\right] z^{\hat\mu}z^R. \end{aligned} \tag{3.10.7}$$

Here we use short notations for the following row vectors

$$\frac{\partial}{\partial t^p} := \left(\frac{\partial}{\partial t^{1,p}}, \dots, \frac{\partial}{\partial t^{n,p}}\right),$$

$$t_q := (t_{1,q}, \dots, t_{n,q}), \quad t_{\alpha,q} := \eta_{\alpha\beta}t^{\beta,q},$$

and we denote

$$\bar t_q = t_q,\ q > 0, \quad \bar t_0 = t_0 + x\,\omega_1.$$

The Laplace integrals in (3.10.7) are defined as in Theorem 3.6.12.

It will be always assumed that the periods $p = p(v; \lambda)$ are chosen in such a way that

$$\partial_\lambda p = -\partial_1 p. \tag{3.10.8}$$

The functions S_α, $\alpha = 1, \ldots, n$ satisfy the following simple identities

$$\partial_x \frac{\partial}{\partial \lambda} S_\alpha = \frac{\partial p_\alpha}{\partial \lambda} = -\partial_1 p_\alpha \tag{3.10.9}$$

$$\frac{\partial}{\partial t^{\gamma,0}} \frac{\partial}{\partial \lambda} S_\alpha = -\partial_\gamma p_\alpha. \tag{3.10.10}$$

In the proof of the second formula one is to use the equation

$$\partial_\alpha \partial_\beta p = c_{\alpha\beta}^\gamma(v) \partial_\gamma \partial_1 p.$$

Theorem 3.10.3 *The flows $\partial/\partial s_m$, $m \geq -1$, are defined on the tau-covering of the Principal Hierarchy by the following generating formula*

$$\frac{\partial}{\partial s} = \sum_{m \geq -1} \frac{1}{\lambda^{m+2}} \frac{\partial}{\partial s_m} \tag{3.10.11}$$

$$\frac{\partial f}{\partial s} = \left[-\frac{1}{2} \frac{\partial S_\alpha}{\partial \lambda} G^{\alpha\beta} \frac{\partial S_\beta}{\partial \lambda} \right]_+ \tag{3.10.12}$$

$$\frac{\partial f_{\gamma,p}}{\partial s} = \frac{\partial}{\partial t^{\gamma,p}} \frac{\partial f}{\partial s} \tag{3.10.13}$$

$$\frac{\partial v_\gamma}{\partial s} = \left[\partial_x \partial_\gamma p_\alpha \, G^{\alpha\beta} \frac{\partial S_\beta}{\partial \lambda} \right]_+ - \eta_{\gamma\epsilon} \left((E - \lambda\, e)^{-1} \right)^\epsilon \tag{3.10.14}$$

In these formulae $[\ \ ]_+$ means the regular part of the expansion of the function vanishing at $\lambda = \infty$.

Proof We first derive the part of the symmetries not containing explicitly the times.

Lemma 3.10.4 *Let $B_{-1} = (B_{-1,\alpha})$, $B_0 = (B_{0,\alpha})$, $B_1 = (B_{1,\alpha})$, $B_2 = (B_{2,\alpha})$, $\ldots$, be the r.h.s. of the flows*

$$\frac{\partial v_\alpha}{\partial s_m} = B_{m,\alpha}$$

defined recursively by

$$B_m = \mathcal{R} B_{m-1}, \ m \geq 0, \quad B_{-1,\alpha} = \eta_{\alpha,1}. \tag{3.10.15}$$

Here

$$\mathcal{R}_\beta^\alpha = \mathcal{U}_\beta^\alpha + \eta_{\beta\gamma} \Gamma_\epsilon^{\alpha\gamma} v_x^\epsilon \partial_x^{-1} \tag{3.10.16}$$

is the recursion operator,

$$\mathcal{R} = \{\ ,\ \}_2 \{\ ,\ \}_1^{-1}.$$

Then the generating function

$$B_\lambda := \frac{B_{-1}}{\lambda} + \frac{B_0}{\lambda^2} + \frac{B_1}{\lambda^3} + \dots$$

reads

$$(B_\lambda)_\alpha = -\frac{1}{2}\partial_{t^{\alpha,0}}\partial_x \left[\partial_\lambda \int^x p^a dx\, G_{ab} \partial_\lambda \int^x p^b\right]. \tag{3.10.17}$$

Here $p^a = p^a(v;\lambda)$, $a = 1, \dots, n$ is a system of independent periods of the Frobenius manifold, the constant Gram matrix G_{ab} is defined by

$$G_{ab} := \frac{\partial v^\alpha}{\partial p^a} \frac{\partial v^\beta}{\partial p^b} \left(g^{\alpha\beta}(v) - \lambda\, \eta^{\alpha\beta}\right)^{-1}.$$

Proof Denote ϖ_1 and ϖ_2 the tensors of the first and the second Poisson brackets respectively. Then

$$B_\lambda = \frac{1}{\lambda}\left(1 - \frac{\mathcal{R}}{\lambda}\right)^{-1} B_{-1} = -\varpi_1(\varpi_2 - \lambda\, \varpi_1)^{-1} B_{-1}.$$

Let us first compute

$$b := (\varpi_2 - \lambda\, \varpi_1)^{-1} B_{-1}.$$

To this end we are to solve the following linear inhomogeneous equation

$$(g^{\alpha\beta} - \lambda\, \eta^{\alpha\beta})\partial_x b_\beta + \Gamma^{\alpha\beta}_\epsilon v^\epsilon_x b_\beta = \delta^\alpha_1. \tag{3.10.18}$$

Denote

$$\phi^a_\alpha := \partial_\alpha p^a, \quad a = 1, \dots, n$$

the basis of solutions of the corresponding linear homogeneous equation. Applying variation of constants we obtain

$$b_\alpha = \phi^a_\alpha G_{ab} \int^x \partial_1 p^b dx.$$

Therefore

$$(B_\lambda)_\alpha = -\partial_x \left(\partial_\alpha p^a G_{ab} \int^x \partial_1 p^b dx\right). \tag{3.10.19}$$

Using

$$\partial_{t^{\alpha,0}} \int^x p\, dx = \partial_\alpha p$$

valid for an arbitrary period $p = p(v;\lambda)$ (cf. (3.10.10) above) we complete the proof of the lemma. □

To complete the proof of the theorem we are to apply the operator

$$\frac{1}{\lambda}\left(1-\frac{\mathcal{R}}{\lambda}\right)^{-1} = -\varpi_1(\varpi_2 - \lambda\,\varpi_1)^{-1}$$

at the sum $\sum t^{\epsilon,r}\frac{\partial v}{\partial t^{\epsilon,r-1}}$. We leave this part of calculations as an exercise to the reader.

The final step in the proof of the theorem is to check that, indeed, the flows (3.10.12)–(3.10.14) are symmetries of the tau-covering of the Principal Hierarchy satisfying the Virasoro commutation relations. This can be done by identifying the coefficients of the expansion of the flows (3.10.14) with the symmetries obtained in [52]). The theorem is proved. □

In the resonant case we can regularize the formula (3.10.12)–(3.10.14) as follows: Introduce

$$S_\alpha^{(\nu)} = \int_0^\infty \frac{dz}{z^{\frac{1}{2}-\nu}}\, e^{-\lambda\, z}\left[\sum_{p\geq 0}\frac{\partial f}{\partial t^p} z^p + \sum_{q\geq 0}(-1)^q \bar{t}_q z^{-q-1}\right] z^{\hat\mu} z^R. \tag{3.10.20}$$

Define the deformed Gram matrix by

$$G^{\alpha\beta}(\nu) := -\frac{1}{2\,\pi}\left[\left(e^{\pi\, i\, R}e^{\pi\, i\,(\hat\mu+\nu)} + e^{-\pi\, i\, R}e^{-\pi\, i\,(\hat\mu+\nu)}\right)\,\eta^{-1}\right]^{\alpha\beta}. \tag{3.10.21}$$

Then we define the regularized symmetry by

$$\frac{\partial f}{\partial s} = \lim_{\nu\to 0}\left[-\frac{1}{2}\frac{\partial S_\alpha^{(\nu)}}{\partial\lambda}G^{\alpha\beta}(\nu)\frac{\partial S_\beta^{(-\nu)}}{\partial\lambda}\right]_+. \tag{3.10.22}$$

Existence of the limit and the Virasoro commutation relations can be proved representing the r.h.s. in the form similar to (3.10.31).

3.10.2 Free field realization of the Virasoro algebra

Let $(\mathcal{L}, <\ ,\ >, \hat\mu, R)$ be the spectrum of a n-dimensional Frobenius manifold. We will construct here a representation of the Virasoro algebra in the ring of functions of infinite number of variables $\mathbf{t} = (t^{\alpha,p})$. The construction is isomorphic to that of [52] but the formulae are much simpler.

Let us choose a basis $e_1, \ldots, e_n$ in $\mathcal{L}$, denote $\eta_{\alpha\beta} =< e_\alpha, e_\beta >$. We introduce the Heisenberg algebra with the generators

$$a_{\alpha,p}, \quad \alpha = 1,\ldots,n, \quad p \in \mathbf{Z} + \frac{1}{2}$$

and the commutation relations

$$[a_{\alpha,p}, a_{\beta,q}] = (-1)^{p-\frac{1}{2}}\eta_{\alpha\beta}\delta_{p+q,0}. \tag{3.10.23}$$

Introduce the row vectors

$$\mathbf{a}_p = (a_{1,p}, \ldots, a_{n,p})$$

and put

$$\phi_\alpha(\lambda) = \left(\int_0^\infty \frac{dz}{z} e^{-\lambda z} \sum_{p\in \mathbf{Z}+\frac{1}{2}} \mathbf{a}_p z^{p+\hat{\mu}} z^R \right)_\alpha , \quad \alpha = 1, \ldots, n. \tag{3.10.24}$$

We consider first the nonresonant case where the spectrum of $\hat{\mu}$ contains no half-integers. In this case the Virasoro operators are given by the following generating function

$$T(\lambda) = \sum_{m\in\mathbf{Z}} \frac{L_m}{\lambda^{m+2}} = -\frac{1}{2} :\partial_\lambda \phi_\alpha \, G^{\alpha\beta} \partial_\lambda \phi_\beta \; : + \frac{1}{4\,\lambda^2} \mathrm{tr} \left(\frac{1}{4} - \hat{\mu}^2 \right). \tag{3.10.25}$$

Here

$$G^{\alpha\beta} = -\frac{1}{2\,\pi} \left[\left(e^{\pi\, i\, R} e^{\pi\, i\, \hat{\mu}} + e^{-\pi\, i\, R} e^{-\pi\, i\, \hat{\mu}} \right) \eta^{-1} \right]^{\alpha\beta}, \tag{3.10.26}$$

the normal ordering is defined by

$$\begin{aligned} &: a_{\alpha,p} a_{\beta,q} := a_{\beta,q} a_{\alpha,p} \quad \text{if } q < 0,\ p > 0, \\ &: a_{\alpha,p} a_{\beta,q} := a_{\alpha,p} a_{\beta,q} \quad \text{otherwise.} \end{aligned}$$

Lemma 3.10.5 *The operators L_m satisfy Virasoro commutation relations*

$$[L_i, L_j] = (i-j) L_{i+j} + n\, \frac{i\,(i^2-1)}{12} \delta_{i+j,0}. \tag{3.10.27}$$

In more general resonant case we regularize the formula (3.10.25) as follows. Introduce the operator-valued functions

$$\phi_\alpha^{(\nu)}(\lambda) = \left(\int_0^\infty \frac{dz}{z^{1-\nu}}\, e^{-\lambda z} \sum_{p\in \mathbf{Z}+\frac{1}{2}} \mathbf{a}_p z^{p+\hat{\mu}} z^R \right)_\alpha , \quad \alpha = 1, \ldots, n. \tag{3.10.28}$$

and the Gram matrix $G^{\alpha\beta}(\nu)$ as in (3.10.21). Here ν is an arbitrary complex parameter. Put

$$T^{(\nu)}(\lambda) = \sum_{m\in\mathbf{Z}} \frac{L_m^{(\nu)}}{\lambda^{m+2}} = -\frac{1}{2} :\partial_\lambda \phi_\alpha^{(\nu)} \, G^{\alpha\beta}(\nu) \partial_\lambda \phi_\beta^{(-\nu)} \; : + \frac{1}{4\,\lambda^2} \mathrm{tr} \left(\frac{1}{4} - \hat{\mu}^2 \right). \tag{3.10.29}$$

Lemma 3.10.6 *Let k be the minimal positive integer such that $R^k = 0$. Then there exist the limits*

$$L_m := \lim_{\nu\to 0} L_m^{(\nu)}, \quad m \geq -1$$
$$L_m := \lim_{\nu\to 0} \nu^k L_m^{(\nu)}, \quad m < -1. \tag{3.10.30}$$

These operators satisfy the following commutation relations

$$[L_i, L_j] = 0, \quad i, j < -1, \text{ or } i + j \geq -1, \text{ but } (i+1)(j+1) < 0,$$
$$[L_i, L_j] = (i-j)L_{i+j}, \quad i + j < -1, \ (i+1)\,(j+1) < 0, \quad \text{or } i, j \geq -1.$$

The proof easily follows form the explicit formula

$$T^{(\nu)}(\lambda) = \frac{1}{2\pi} \times$$
$$\times \sum_{p,q\in\mathbf{Z}+\frac{1}{2}} \sum_{r\geq 0} :\, \mathbf{a}_p \left[e^{R\,\partial_\nu}\right]_r \frac{\Gamma(\hat\mu+\nu+p+r+1)\cos\pi(\hat\mu+\nu)\Gamma(-\hat\mu-\nu+q+1)}{\lambda^{p+q+r+2}}\mathbf{a}^q \ :$$
$$+\frac{1}{4\,\lambda^2}\mathrm{tr}\left(\frac{1}{4}-\hat\mu^2\right). \tag{3.10.31}$$

In this formula

$$\mathbf{a}^q = \left(a^{1,q}, \ldots, a^{n,q}\right)^T$$

is a column vector with the entries

$$a^{\alpha,q} := \eta^{\alpha\beta} a_{\beta,q}.$$

□

To prove the commutation relations of the Virasoro operators it suffices to compute the commutator

$$[T(\lambda_1), T(\lambda_2)] = \frac{1}{2}\sum_{p,q}\sum_{s}(-1)^{s-\frac{1}{2}} :\mathbf{a}_p \left(M^p_{-s}(\lambda_2)M^s_q(\lambda_1) - M^p_{-s}(\lambda_1)M^s_q(\lambda_2)\right)\,\mathbf{a}^q :$$
$$+\frac{1}{2}\sum_{p,q>0}(-1)^{p+q+1}(M^p_q(\lambda_1)M^{-q}_{-p}(\lambda_2) - M^p_q(\lambda_2)M^{-q}_{-p}(\lambda_1).$$

Here

$$M^p_q(\lambda) = \sum_{r\geq 0} N^p_q(r)\,\lambda^{-p-q-r-2} \tag{3.10.32}$$

with

$$N^p_q(r) =$$
$$\frac{1}{\pi}\,\left[e^{R\,\partial_\nu}\right]_r\left(\Gamma(\hat\mu+\nu+p+r+1)\cos\pi(\hat\mu+\nu)\Gamma(-\hat\mu-\nu+q+1)\right)\Big|_{\nu=0}. \tag{3.10.33}$$

This can be easily done using the identity

$$\sum_{r_1+r_2=m+n-p-q} (-1)^{p+r_2-n-\frac{1}{2}} N^p_{n-p-r_2}(r_2)\, N_q^{-n+p+r_2}(r_1)$$
$$+ \sum_{r_1+r_2=m+n-p-q} (-1)^{m-p-r_2-\frac{1}{2}} N^p_{-m+p+r_2}(r_2)\, N_q^{m-p-r_2}(r_1)$$
$$= (m-n)\, N_q^p(m+n-p-q).$$

A natural representation of the Heisenberg algebra (3.10.23) is obtained as follows

$$\begin{aligned} a_{\alpha,p} &= \epsilon \frac{\partial}{\partial t^{\alpha,p-\frac{1}{2}}}, \quad p > 0, \\ a_{\alpha,p} &= \epsilon^{-1}(-1)^{p+\frac{1}{2}}\eta_{\alpha\beta} t^{\beta,-p-\frac{1}{2}}, \quad p < 0. \end{aligned} \tag{3.10.34}$$

In this representation $T(\lambda)$ and L_m become linear second order differential operators

$$T(\lambda) = T(\epsilon^{-1}\mathbf{t}, \epsilon\partial/\partial\mathbf{t}; \lambda) \tag{3.10.35}$$

$$\begin{aligned} L_m &= L_m(\epsilon^{-1}\mathbf{t}, \epsilon\partial/\partial\mathbf{t}) \\ &= \epsilon^2 \sum a_m^{\alpha,p;\beta,q} \frac{\partial^2}{\partial t^{\alpha,p}\partial t^{\beta,q}} + \sum b_{m\,\beta,q}^{\;\alpha,p}\, t^{\beta,q} \frac{\partial}{\partial t^{\alpha,p}} + \epsilon^{-2} c^m_{\alpha,p;\beta,q}\, t^{\alpha,p}\, t^{\beta,q} \end{aligned} \tag{3.10.36}$$

for some constant coefficients $a_m^{\alpha,p;\beta,q}$, $b_{m\,\beta,q}^{\;\alpha,p}$, $c^m_{\alpha,p;\beta,q}$ depending on m and on the spectrum of the Frobenius manifold.

Example 3.10.7 *For $n = 1$ (the KdV theory) the Virasoro operators (3.10.36) coincide, up to normalization of the independent variables, with the well known in the KdV theory [26, 65, 88] realization of the Virasoro algebra*

$$\begin{aligned} L_m &= \frac{\epsilon^2}{2} \sum_{k+l=m-1} \frac{(2k+1)!!\,(2l+1)!!}{2^{m+1}} \frac{\partial^2}{\partial t_k \partial t_l} \\ &\quad + \sum_{k\geq 0} \frac{(2k+2m+1)!!}{2^{m+1}(2k-1)!!}\, t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{16}\delta_{m,0}, \quad m \geq 0, \\ L_{-1} &= \sum_{k\geq 1} t^k \frac{\partial}{\partial t_{k-1}} + \frac{1}{2\epsilon^2} t_0^2, \\ L_{-m} &= \frac{1}{2\,\epsilon^2} \sum_{k+l=m-1} \frac{2^{m-1}}{(2k-1)!!(2l-1)!!} t_k\, t_l \\ &\quad + \sum_{k\geq 0} \frac{2^{m-1}(2k+1)!!}{(2m+2k-1)!!}\, t_{k+m} \frac{\partial}{\partial t_k}, \quad m > 1. \end{aligned} \tag{3.10.37}$$

Example 3.10.8 *For the Frobenius manifold (3.6.57) (the* $\mathbf{CP}^1$*-model) the regularized operators (3.10.29) read*

$$\begin{aligned}
L_m &= \frac{\epsilon^2}{2}\sum_{k=1}^{m-1} k!\,(m-k)!\frac{\partial^2}{\partial t^{2,k-1}\partial t^{2,k-m-1}} \\
&+\sum_{k\geq 1}\frac{(m+k)!}{(k-1)!}\left(t^{1,k}\frac{\partial}{\partial t^{1,m+k}}+t^{2,k-1}\frac{\partial}{\partial t^{2,k-1}}\right)+2\sum_{k\geq 0}\alpha_m(k)t^{1,k}\frac{\partial}{\partial t^{2,m+k-1}}, \quad m>0 \\
L_0 &= \sum_{k\geq 1} k\left(t^{1,k}\frac{\partial}{\partial t^{1,k}}+t^{2,k-1}\frac{\partial}{\partial t^{2,k-1}}\right) \\
L_{-1} &= \sum_{k\geq 1} t^{\alpha,k}\frac{\partial}{\partial t^{\alpha,k-1}}+\frac{1}{\epsilon^2}t^{1,0}t^{2,0} \\
L_{-m} &= \frac{1}{\epsilon^2}\sum_{k=1}^{m-1}\frac{t^{1,k}\,t^{1,m-k}}{(k-1)!(m-k-1)!}, \quad m>1.
\end{aligned} \tag{3.10.38}$$

Here the integer coefficients $\alpha_m(k)$ *are defined by*

$$\alpha_m(0)=m!, \quad \alpha_m(k)=\frac{(m+k)!}{(k-1)!}\sum_{j=k}^{m+k}\frac{1}{j}, \ k>0.$$

Lemma 3.10.9 *The generating symmetry flow (3.10.12) can be represented as follows*

$$\frac{1}{\epsilon^2}\frac{\partial f}{\partial s}=\left[T(\epsilon^{-1}\bar{\mathbf{t}},\epsilon\partial/\partial\mathbf{t};\lambda)\exp(\frac{f}{\epsilon^2})\right]_+ +O(1), \tag{3.10.39}$$

$$\bar{\mathbf{t}}=(\bar{t}^{\alpha,p}).$$

The proof is straightforward.

Remark 3.10.10 *After this paper was finished a very interesting work of A. Givental appeared [79]. In particular, an elegant realization of our Virasoro operators* L_m *with* $m\geq -1$ *was obtained in [79].*

3.10.3 Virasoro symmetries and solutions of the Principal Hierarchy

Let $v=v(x,\mathbf{t},c(\epsilon))$ be a solution to the Principal Hierarchy specified by the series $c(\epsilon)=(c^{\alpha,p}(\epsilon))$ in ϵ with constant coefficients as in (3.6.75). Put

$$f(x,\mathbf{t},c(\epsilon))=\mathcal{F}_0(x,\mathbf{t})=\log\tau, \quad f_{\alpha,p}(x,\mathbf{t},c(\epsilon))=\partial_{t^{\alpha,p}}\log\tau. \tag{3.10.40}$$

Here the tau-function and its first derivatives for the solution $v=v(x,\mathbf{t},c(\epsilon))$ are defined by the formulae (3.6.86), (3.6.87).

Theorem 3.10.11 *The functions $v = v(x, \mathbf{t}, c(\epsilon))$, $f = f(x, \mathbf{t}, c(\epsilon))$, $f_{\alpha,p} = f_{\alpha,p}(x, \mathbf{t}, c(\epsilon))$ are the stationary points of the following symmetries of the Principal Hierarchy*

$$\frac{\partial f}{\partial \tilde{s}} = \left[-\frac{1}{2}\frac{\partial S_\alpha(\lambda, \tilde{\mathbf{t}})}{\partial \lambda} G^{\alpha\beta} \frac{\partial S_\beta(\lambda, \tilde{\mathbf{t}})}{\partial \lambda}\right]_+ = 0 \tag{3.10.41}$$

$$\frac{\partial f_{\gamma,p}}{\partial \tilde{s}} = \frac{\partial}{\partial t^{\gamma,p}}\left[-\frac{1}{2}\frac{\partial S_\alpha(\lambda, \tilde{\mathbf{t}})}{\partial \lambda} G^{\alpha\beta} \frac{\partial S_\beta(\lambda, \tilde{\mathbf{t}})}{\partial \lambda}\right]_+ = 0 \tag{3.10.42}$$

$$\frac{\partial v_\gamma}{\partial \tilde{s}} = \left[\partial_x \partial_\gamma p_\alpha\, G^{\alpha\beta} \frac{\partial S_\beta(\lambda, \tilde{\mathbf{t}})}{\partial \lambda}\right]_+ - \eta_{\gamma\epsilon}\left((E - \lambda\, e)^{-1}\right)^\epsilon = 0 \tag{3.10.43}$$

where $\tilde{\mathbf{t}} = (\tilde{t}^{\alpha,p})$,

$$\tilde{t}^{\alpha,p} = t^{\alpha,p} - c^{\alpha,p}(\epsilon). \tag{3.10.44}$$

Observe that the difference between the "shifted symmetries" (3.10.43) and the original ones (3.10.14) is a linear combination of the flows of the Principal Hierarchy. In other words, the solution $v = v(x, \mathbf{t}, c(\epsilon))$ satisfies the following infinite family of constraints

$$\frac{\partial v}{\partial s} = b_{m\,\beta,q}^{\;\;\alpha,p}\, c^{\beta,q}\, \frac{\partial v}{\partial t^{\alpha,p}}, \quad m \geq -1. \tag{3.10.45}$$

The theorem was proved in [52] for the particular case $c^{\alpha,p} = \delta^\alpha_1\, \delta^p_1$. The proof can be repeated also in the general case without major changes. The crucial point in the proof is the identity

$$\partial_{E^{m+1}} \Omega_{\alpha,p;\beta,q} = 2 \sum a_m^{\lambda,k;\epsilon,l}\, \Omega_{\alpha,p;\lambda,k}\, \Omega_{\epsilon,l;\beta,q} + b_{m;\alpha,p}^{\lambda,k}\, \Omega_{\lambda,k;\beta,q} + b_{m;\beta,q}^{\lambda,k}\, \Omega_{\lambda,k;\alpha,p} + 2\, c^m_{\alpha,p;\beta,q}. \tag{3.10.46}$$

valid for any $m \geq -1$.

The expanded form of the stationary equations (3.10.41) reads

$$\sum a_m^{\alpha,p;\beta,q} \frac{\partial f}{\partial t^{\alpha,p}} \frac{\partial f}{\partial t^{\beta,q}} + \sum b_{m\,\beta,q}^{\;\;\alpha,p} \tilde{t}^{\beta,q} \frac{\partial f}{\partial t^{\alpha,p}} + \sum c^m_{\alpha,p;\beta,q} \tilde{t}^{\alpha,p} \tilde{t}^{\beta,q} = 0, \quad m \geq -1. \tag{3.10.47}$$

The constant coefficients are the same as in (3.10.36). The stationary equations for the first derivatives of the tau-function and for v_α are obtained from the above by differentiation.

3.10.4 Linearization of Virasoro symmetries

Let us consider a quasitrivial bihamiltonian tau-symmetric hierarchy

$$\frac{\partial w_\alpha}{\partial t^{\alpha,p}} = \{w_\alpha(x), H_{\alpha,p}\}_1 \tag{3.10.48}$$

obtained from the Principal Hierarchy by a quasi-Miura transformation

$$v_\alpha \mapsto w_\alpha = v_\alpha + \partial_x \partial_{t^{\alpha,0}} \sum_{k\geq 1} \epsilon^k \mathcal{F}^{[k]}(v; v_x, \dots, v^{(n_k)}). \tag{3.10.49}$$

(For technical reasons we have changed the notations of the dependent variables of the hierarchy from $u^1, \dots, u^n$ to $w^1, \dots, w^n$.)

Theorem 3.10.12 *The flows* $\frac{\partial}{\partial \hat{s}_m}$ *defined by the generating function*

$$\frac{\partial}{\partial \hat{s}} = \sum_{m\geq -1} \frac{1}{\lambda^{m+2}} \frac{\partial}{\partial \hat{s}_m} \tag{3.10.50}$$

$$\frac{\partial \log\tau}{\partial \hat{s}} = \frac{\partial f}{\partial s} + \sum_{k\geq 1} \epsilon^k \sum_{r=0}^{n_k} \frac{\partial \mathcal{F}^{[k]}}{\partial v^{\alpha,r}} \partial_x^r \frac{\partial v^\alpha}{\partial s} \tag{3.10.51}$$

$$\frac{\partial}{\partial \hat{s}} \frac{\partial \log\tau}{\partial t^{\alpha,p}} = \frac{\partial}{\partial t^{\alpha,p}} \frac{\partial \log\tau}{\partial \hat{s}} \tag{3.10.52}$$

$$\frac{\partial w_\alpha}{\partial \hat{s}} = \epsilon^2 \partial_x \partial_{t^{\alpha,0}} \frac{\partial \log\tau}{\partial \hat{s}} \tag{3.10.53}$$

$$\tag{3.10.54}$$

are symmetries of the tau-cover of the hierarchy (3.10.48). They satisfy the Virasoro commutation relations (3.10.4). Every solution of the hierarchy, its tau-function and the first derivatives of it is a stationary point of the symmetries with the shifted times as in (3.10.41)–(3.10.43).

Proof This is obtained by applying the "change of coordinates" (3.10.49) to Theorems 3.10.3 and 3.10.11. □

Definition. We say that the quasitriviality (3.10.49) *linearizes the Virasoro symmetries* if there exists linear differential operators

$$\hat{L}_m = \hat{L}_m(\epsilon^{-1}\mathbf{t}, \epsilon\frac{\partial}{\partial \mathbf{t}}), \quad m \geq -1 \tag{3.10.55}$$

with coefficients polynomial in $\{\epsilon^{-1} t^{\alpha,p}\}$ such that

$$\frac{\partial \tau}{\partial \hat{s}_m} = \hat{L}_m \tau, \quad m \geq -1. \tag{3.10.56}$$

Example 3.10.13 *The quasi-triviality (3.8.11)*

$$\begin{aligned} v \mapsto u &= v + \epsilon^2 \partial_x^2 \Delta f(v', v'', \dots; \epsilon^2) \\ &= v + \frac{\epsilon^2}{24} (\log v')'' + \epsilon^4 \left(\frac{v^{IV}}{1152\, v'^2} - \frac{7\, v'' v'''}{1920\, v'^3} + \frac{v''^3}{360\, v'^4} \right)'' + O(\epsilon^6). \end{aligned} \tag{3.10.57}$$

for the KdV hierarchy satisfies the condition of linearization of Virasoro constraints.

Observe that we have changed the normalization

$$\epsilon^2 \mapsto -\frac{\epsilon^2}{2}.$$

Proof From independence of Δf of v and from Lemma 3.8.8 it follows that (3.10.57) transforms the Galilean symmetry

$$\frac{\partial v}{\partial s_{-1}} = 1 + \sum_{p\geq 1} t_p \frac{\partial v}{\partial t_{p-1}} \tag{3.10.58}$$

of the Riemann hierarchy to the Galilean symmetry

$$\frac{\partial u}{\partial s_{-1}} = 1 + \sum_{p\geq 1} t_p \frac{\partial u}{\partial t_{p-1}} \tag{3.10.59}$$

of the KdV hierarchy. For higher symmetries we use, due to Theorem 3.8.10, that the quasi-Miura (3.10.57) transforms the recursion

$$\frac{\partial v}{\partial s_m} = \left(v + \frac{1}{2} v' \partial_x^{-1}\right) \frac{\partial v}{\partial s_{m-1}}, \quad m \geq 0$$

for the symmetries of the Riemann hierarchy to the recursion

$$\frac{\partial u}{\partial s_m} = \left(\frac{1}{8}\epsilon^2 \partial_x^2 + u + \frac{1}{2} u' \partial_x^{-1}\right) \frac{\partial u}{\partial s_{m-1}}, \quad m \geq 0 \tag{3.10.60}$$

for the symmetries of the KdV hierarchy. As it was shown in [26], the symmetries generated by the recursion procedure (3.10.60), (3.10.59) have the form

$$\frac{\partial u}{\partial s_m} = \epsilon^2 \partial_x^2 \frac{L_m \tau}{\tau}, \quad m \geq -1 \tag{3.10.61}$$

where the Virasoro operators L_m are defined in (3.10.37). Observe that our normalization of the flows of the KdV hierarchy differs from that of [26]. Our flows satisfy the recursion relation

$$\left(p + \frac{1}{2}\right) \frac{\partial u}{\partial t_p} = \left[\frac{\epsilon^2}{8} \partial_x^2 + u + \frac{1}{2} u' \partial_x^{-1}\right] \frac{\partial u}{\partial t_{p-1}}, \quad p \geq 1.$$

□

Let us assume that the quasitriviality transformation (3.10.49) is such that $F_1 = 0$. This can always be done according to Theorem 3.9.5.

Lemma 3.10.14 *The operators $\hat{L}_m$ in (3.10.55) must have the form*

$$\hat{L}_m = L_m + \kappa_0 \, \delta_{m,0}, \quad m \geq -1 \tag{3.10.62}$$

where the Virasoro operators L_m were defined in Section 3.10.2 and κ_0 is a constant.

Proof By definition the principal symbol of the operator $\hat{L}_m$ coincides with

$$\sum a_m^{\alpha,p;\beta,q} P_{\alpha,p} P_{\beta,q} + \sum b_{m\,\beta,q}^{\ \alpha,p} \tilde{t}^{\beta,q} P_{\alpha,p} + \sum c^m_{\alpha,p;\beta,q} \tilde{t}^{\alpha,p} \tilde{t}^{\beta,q}, \ m \geq -1.$$

Here the variables $P_{\alpha,p}$ correspond to $\frac{\partial}{\partial t^{\alpha,p}}$. Due to vanishing of the terms of the order ϵ^{-1} in the expansion of the tau-function we conclude that

$$\hat{L}_m = \epsilon^2 \sum a_m^{\alpha,p;\beta,q} \frac{\partial^2}{\partial t^{\alpha,p} \partial t^{\beta,q}} + \sum b_{m\,\beta,q}^{\ \alpha,p} \bar{t}^{\beta,q} \frac{\partial}{\partial t^{\alpha,p}} + \epsilon^{-2} c^m_{\alpha,p;\beta,q} \bar{t}^{\alpha,p} \bar{t}^{\beta,q} + \kappa_m = L_m + \kappa_m$$

for some constants κ_m. Using the Virasoro commutation relations

$$\left[\frac{\partial}{\hat{\partial} s_i}, \frac{\partial}{\partial \hat{s}_j} \right] = (j-i) \frac{\partial}{\partial \hat{s}_{i+j}}, \quad i, j \geq -1$$

and

$$[L_i, L_j] = (j-i) L_{i+j}, \quad i, j \geq -1$$

we derive that $\kappa_m = 0$ for $m \neq 0$. □

Let $v = v(x, \mathbf{t}, c(\epsilon))$ be a solution to the Principal Hierarchy described in Section 3.6.4. We call it *admissible* w.r.t. the quasitriviality transformation if all the denominators of the rational functions $\partial_x \partial_{t^{\alpha,0}} \mathcal{F}^{[k]}(v; v_x, \ldots, v^{(n_k)})$ do not vanish at the point $x = 0$, $\mathbf{t} = 0$. We will see below that any monotone solution to the Principal Hierarchy is admissible for our class of quasitriviality transformations.

Denote $w = w(x, \mathbf{t}, c(\epsilon))$ the solution to the full hierarchy (3.10.48) obtained from an admissible solution $v = v(x, \mathbf{t}, c(\epsilon))$ by the quasitriviality transformation (3.10.49). All these will be called *admissible solutions to the full hierarchy.* Applying to an admissible solution $w(x, \mathbf{t}, c(\epsilon))$ the quasi-Miura inverse to (3.10.49) one obtains a solution $v(x, \mathbf{t}, c(\epsilon))$ of the form (3.6.75). The tau-function of the solution $w(x, \mathbf{t}, c(\epsilon))$ has the form

$$\tau(x, \mathbf{t}, c(\epsilon)) = \exp\,[\epsilon^{-2} \mathcal{F}_0(x, \mathbf{t}, c(\epsilon)) + \sum_{k \geq 1} \epsilon^{k-2} \mathcal{F}^{[k]}(v; v_x, \ldots, v^{(n_k)})|_{v = v(x, \mathbf{t}, c(\epsilon))}] \quad (3.10.63)$$

where the function $\mathcal{F}_0(x, \mathbf{t}, c(\epsilon))$ has been defined in (3.6.86).

Lemma 3.10.15 *The tau-function (3.10.63) of any admissible solution $w = w(x, \mathbf{t}, c(\epsilon))$ to the full hierarchy satisfies the following system of* Virasoro constraints

$$\hat{L}_m(\epsilon^{-1} \tilde{\mathbf{t}}, \epsilon \frac{\partial}{\partial \mathbf{t}})\, \tau(x, \mathbf{t}, c(\epsilon)) = 0, \quad m \geq -1 \qquad (3.10.64)$$

where

$$\tilde{t}^{\alpha,p} = t^{\alpha,p} - c^{\alpha,p}(\epsilon) + x\, \delta_1^\alpha \delta_0^p.$$

Proof Substituting the expansion (3.10.63) into the definition of the linearization

$$\hat{L}_m \tau = \frac{\partial \tau}{\partial \hat{s}_m}$$

and collecting the coefficients of ϵ^{k-2} we obtain, for every $k \geq 0$

$$\sum_{l=0}^{k} a_m^{\alpha,p;\beta,q} \frac{\partial \mathcal{F}^{[l]}}{\partial t^{\alpha,p}} \frac{\partial \mathcal{F}^{[k-l]}}{\partial t^{\beta,q}} + a_m^{\alpha,p;\beta,q} \frac{\partial^2 \mathcal{F}^{[k-2]}}{\partial t^{\alpha,p} \partial t^{\beta,q}} + b_{m}{}_{\beta,q}^{\alpha,p} t^{\beta,q} \frac{\partial \mathcal{F}^{[k]}}{\partial t^{\alpha,p}}$$

$$+\delta_{k,0} \sum c^m_{\alpha,p;\beta,q} \bar{t}^{\alpha,p} \bar{t}^{\beta,q} + \kappa_0 \delta_{m,0} \delta_{k,2} = \frac{\partial \mathcal{F}^{[k]}}{\partial s_m}. \tag{3.10.65}$$

In this equation we identify $\mathcal{F}_0 =: \mathcal{F}^{[0]}$. The equality with $k = 0$ holds true due to the definition of the symmetry (3.10.51)–(3.10.53). For $k > 0$ we can use (3.10.45) in order to rewrite the r.h.s. in the form

$$\frac{\partial \mathcal{F}^{[k]}}{\partial s_m} = \sum \frac{\partial \mathcal{F}^{[k]}}{\partial v^{\gamma,r}} \partial_x^r \frac{\partial v^\gamma}{\partial s_m} = \sum \frac{\partial \mathcal{F}^{[k]}}{\partial v^{\gamma,r}} \partial_x^r b_{m}{}_{\beta,q}^{\alpha,p} c^{\beta,q} \frac{\partial v^\gamma}{\partial t^{\alpha,p}} = b_{m}{}_{\beta,q}^{\alpha,p} c^{\beta,q} \frac{\partial \mathcal{F}^{[k]}}{\partial t^{\alpha,p}}.$$

This proves the lemma. □

From the above statements it follows

Theorem 3.10.16 *The logarithm of the tau-function*

$$\mathcal{F} := \log \tau$$

of any admissible solution of the full hierarchy satisfies the following system of equations

$$\sum_{p,q>0} \left[\frac{\epsilon^2}{2} \frac{\partial}{\partial t^{p-1/2}} M_q^p(\lambda) \frac{\partial}{\partial t_{q-1/2}} \mathcal{F} + \frac{\partial \mathcal{F}}{\partial t^{p-1/2}} M_q^p(\lambda) \frac{\partial \mathcal{F}}{\partial t_{q-1/2}} \right.$$

$$\left. (-1)^{p-\frac{1}{2}} \tilde{t}^{p-\frac{1}{2}} M_{-p}^q(\lambda) \frac{\partial \mathcal{F}}{\partial t_{q-1/2}} + \frac{1}{2\epsilon^2} (-1)^{p+q+1} \tilde{t}^{p-\frac{1}{2}} M_{-p}^{-q}(\lambda) \tilde{t}_{q-\frac{1}{2}} + \frac{\kappa_0}{\lambda^2} \right]_+ = 0 \tag{3.10.66}$$

for any λ.

Example 3.10.17 *For $n = 1$ the equation (3.10.66) coincides with the loop equation in topological gravity [26, 65] (for the particular solution with $c_j = \delta_{j,1}$) or in the double scaling limit [11, 29, 81] of the Hermitean matrix model at the k-th multicritical point (for the particular solution with $c_j = \delta_{j,k+1}$, $k \geq 1$) [18] (see also [1, 107]).*

Motivating by this example, we introduce

Definition. The equation (3.10.66) is called *generalized loop equation* in the theory of integrable hierarchies.

In Section 3.10.6 we will develop a technique to obtain a universal "perturbative solution" to the loop equation. To this end we need first to recall some technical tricks of using canonical coordinates on semisimple Frobenius manifolds.

3.10.5 Working with Frobenius manifolds in the canonical coordinates

In this Section we summarize, following [37, 41] a very efficient technique of working with semisimple Frobenius manifolds based on introduction of *canonical coordinates*.

Let M be a semisimple Frobenius manifold. Denote $M_{s\,s} \subset M$ the open dense subset in M consisting of all points $v \in M$ s.t. the operator of multiplication by the Euler vector field

$$E(v)\cdot\ : T_vM \to T_vM$$

has simple spectrum. Denote $u_1(v)$, $\dots$, $u_n(v)$ the eigenvalues of this operator, $v \in M_{s\,s}$. The mapping

$$M_{s\,s} \to \left(\mathbb{C}^n \setminus \cup_{i<j}(u_i = u_j)\right)/S_n, \quad v \mapsto (u_1(v), \dots, u_n(v))$$

is an unramified covering. Therefore one can use the eigenvalues as local coordinates on $M_{s\,s}$. The vectors $\partial/\partial u_i$, $i = 1, \dots, n$ are basic idempotents of the algebra T_vM for any $v \in M_{s\,s}$

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}.$$

Orthogonality

$$\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle = 0,\ i \neq j$$

readily follows from the above multiplication table. We call u_1, $\dots$, u_n canonical coordinates on $M_{s\,s}$. We will never use summation over repeated indices when working in the canonical coordinates.

Choosing locally branches of the square roots

$$\psi_{i1}(u) := \sqrt{< \partial/\partial u_i, \partial/\partial u_i >},\ i = 1, \dots, n \tag{3.10.67}$$

we obtain a transition matrix $\Psi = (\psi_{i\alpha}(u))$ from the basis $\partial/\partial v^\alpha$ to the orthonormal basis

$$\psi_{11}^{-1}(u)\frac{\partial}{\partial u_1}, \psi_{21}^{-1}(u)\frac{\partial}{\partial u_2}, \dots, \psi_{n1}^{-1}(u)\frac{\partial}{\partial u_n} \tag{3.10.68}$$

of the normalized idempotents

$$\frac{\partial}{\partial v^\alpha} = \sum_{i=1}^n \frac{\psi_{i\alpha}(u)}{\psi_{i1}(u)} \frac{\partial}{\partial u_i}. \tag{3.10.69}$$

Equivalently, the Jacobi matrix has the form

$$\frac{\partial u_i}{\partial v^\alpha} = \frac{\psi_{i\alpha}}{\psi_{i1}}. \tag{3.10.70}$$

The matrix $\Psi(u)$ satisfies orthogonality condition

$$\Psi^T(u)\Psi(u) \equiv \eta, \quad \eta = (\eta_{\alpha\beta}), \quad \eta_{\alpha\beta} := \left\langle \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta} \right\rangle.$$

The lengths (3.10.67) coincide with the first column of this matrix. So, the metric $< \, , \, >$ in the canonical coordinates reads

$$< \, , \, >= \sum_{i=1}^{n} \psi_{i1}^2(u) du_i^2. \tag{3.10.71}$$

Denote $V(u) = (V_{ij}(u))$ the matrix of the antisymmetric operator $\mathcal{V}$ (3.6.5) w.r.t. the orthonormal frame

$$V(u) := \Psi(u) \, \mathcal{V} \, \Psi^{-1}(u). \tag{3.10.72}$$

It is a solution to the following system of commuting time-dependent Hamiltonian flows on the Lie algebra $so(n)$ equipped with the standard Lie - Poisson bracket (2.1.8)

$$\frac{\partial V}{\partial u_i} = \{V, H_i(V;u)\}, \quad i = 1, \ldots, n \tag{3.10.73}$$

with quadratic Hamiltonians

$$H_i(V;u) = \frac{1}{2} \sum_{j \neq i} \frac{V_{ij}^2}{u_i - u_j}. \tag{3.10.74}$$

The matrix $\Psi(u)$ satisfies

$$\frac{\partial \Psi}{\partial u_i} = V_i(u)\Psi, \quad V_i(u) := \mathrm{ad}_{E_i} \mathrm{ad}_U^{-1}(V(u)). \tag{3.10.75}$$

Here $U = \mathrm{diag}\,(u_1, \ldots, u_n)$, the matrix unity E_i has the entries

$$(E_i)_{ab} = \delta_{ai}\delta_{ib}.$$

The *isomonodromic tau-function* of the semisimple Frobenius manifold is defined by

$$d \log \tau_I(u) = \sum_{i=1}^{n} H_i(V(u);u) du_i. \tag{3.10.76}$$

It is an analytic function on a suitable unramified covering of M_{ss}.

The system (3.10.73) coincides with the equations of isomonodromy deformations of the following linear differential operator with rational coefficients

$$\frac{dY}{dz} = \left(U + \frac{V}{z} \right) Y. \tag{3.10.77}$$

The latter is nothing but the last component of the deformed flat connection (3.6.5) written in the orthonormal frame (3.10.68). The integration of (3.10.73), (3.10.75) and, more generally, the reconstruction of the Frobenius structure can be reduced to a solution of certain Riemann - Hilbert problem [42].

In the canonical coordinates the intersection form $(\, , \,)_\lambda$, having in the flat coordinates the Gram matrix

$$\left(g^{\alpha\beta}(v) - \lambda\,\eta^{\alpha\beta}\right)^{-1},$$

becomes equal to

$$(\, , \,)_\lambda = \sum_{i=1}^{n} \frac{\psi_{i1}^2(u)}{u_i - \lambda} du_i^2. \tag{3.10.78}$$

The differential equations for the periods $p = p(v;\lambda)$ (i.e., the Gauss - Manin system in the terminology of [41]) can be recast into the form

$$\begin{aligned} \frac{\partial \phi_i}{\partial u_j} &= -\frac{V_{ij}}{u_i - u_j}\phi_j, \quad j \neq i \\ \frac{\partial \phi_i}{\partial u_i} &= \frac{1}{\lambda - u_i}\left[\frac{1}{2}\phi_i + \sum_s V_{is}\phi_s\right] + \sum_s \frac{V_{is}}{u_i - u_s}\phi_s \end{aligned} \tag{3.10.79}$$

where the functions $\phi_i = \phi_i(v;\lambda)$ are defined by

$$\frac{\partial p(v(u);\lambda)}{\partial u_i} = \psi_{i1}(u)\phi_i(v(u);\lambda). \tag{3.10.80}$$

These functions also satisfy the following equations which will be used later:

$$\frac{\partial \phi_i}{\partial \lambda} = \frac{\phi_i}{2(u_i - \lambda)} + \sum_k \frac{V_{ik}\,\phi_k}{u_i - \lambda}. \tag{3.10.81}$$

For the basis of periods $p_\alpha = p_\alpha(v;\lambda)$ such that

$$G^{\alpha\beta} = \left(\frac{\partial}{\partial p_\alpha}, \frac{\partial}{\partial p_\beta}\right)_\lambda$$

we put

$$\frac{\partial p_\alpha(v(u);\lambda)}{\partial u_i} = \psi_{i1}(u)\phi_{i\alpha}(u;\lambda). \tag{3.10.82}$$

From (3.10.78) and from the tensor law for the metric $(\, , \,)_\lambda$ the orthogonality condition follows

$$\phi_{i\alpha}(v(u);\lambda)G^{\alpha\beta}\phi_{j\beta}(v(u);\lambda) = \frac{\delta_{ij}}{u_i - \lambda}. \tag{3.10.83}$$

3.10.6 Loop equation on the jet space of a semisimple Frobenius manifold

In this section we develop the main tool for computing the "perturbative solution" of the loop equation. In particular we will prove that the loop equation (3.10.66) uniqely determine the quasitriviality transformation (3.10.49).

The coefficients $\mathcal{F}^{[k]}$ for $k>0$ are to be determined from the recursion relations

$$2\, a_m^{\alpha,p;\beta,q} \frac{\partial \mathcal{F}_0}{\partial t^{\alpha,p}} \frac{\partial \mathcal{F}^{[k]}}{\partial t^{\beta,q}} + b_{m\,\beta,q}^{\alpha,p} \tilde{t}^{\beta,q} \frac{\partial \mathcal{F}^{[k]}}{\partial t^{\alpha,p}}$$

$$= -\sum_{l=1}^{k-1} a_m^{\alpha,p;\beta,q} \frac{\partial \mathcal{F}^{[l]}}{\partial t^{\alpha,p}} \frac{\partial \mathcal{F}^{[k-l]}}{\partial t^{\beta,q}} - a_m^{\alpha,p;\beta,q} \frac{\partial^2 \mathcal{F}^{[k-2]}}{\partial t^{\alpha,p} \partial t^{\beta,q}} - \kappa_0 \delta_{m,0} \delta_{k,2}, \quad m \geq -1 \quad (3.10.84)$$

that must hold true for an arbitrary solution $v = v(x, \mathbf{t}, c(\epsilon))$. The l.h.s. of this equation is a linear differential operator acting on $\mathcal{F}^{[k]}$. We will call it *linearized Virasoro constraint*. The coefficients of the linear differential operator depend on the choice of the solution $v = v(x, \mathbf{t}, c(\epsilon))$. Let us compute the generating function (3.10.66) of the linearized Virasoro constraints.

Lemma 3.10.18 *Let $\mathcal{F}(v; v_x, \ldots, v^{(N)})$ be an arbitrary function on the jet space, the functions $\mathcal{F}_0(x, \mathbf{t}, c(\epsilon))$ be as in (3.6.86). Then*

$$\sum_{m=-1}^{\infty} \frac{1}{\lambda^{m+2}} \left[2\, a_m^{\alpha,p;\beta,q} \frac{\partial \mathcal{F}_0}{\partial t^{\alpha,p}} \frac{\partial \mathcal{F}}{\partial t^{\beta,q}} + b_{m\,\beta,q}^{\alpha,p} \tilde{t}^{\beta,q} \frac{\partial \mathcal{F}}{\partial t^{\alpha,p}} \right]$$

$$= \sum_{r=0}^{N} \frac{\partial \mathcal{F}}{\partial v^{\gamma,r}} \partial_x^r \left(\frac{1}{E-\lambda} \right)^{\gamma} + \sum_{r=1}^{N} \frac{\partial \mathcal{F}}{\partial v^{\gamma,r}} \sum_{k=1}^{r} \binom{r}{k} \partial_x^{k-1} \partial_1 p_\alpha G^{\alpha\beta} \partial_x^{r-k+1} \partial^\gamma p_\beta. \quad (3.10.85)$$

Proof According to (3.10.41)–(3.10.43), the l.h.s. of the last equation can be written in the form

$$\left[\frac{\partial S_\alpha(\tilde{\mathbf{t}}, \lambda)}{\partial \lambda} G^{\alpha\beta} \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \left(\frac{\partial \mathcal{F}}{\partial t^p} z^{p+1} z^{\hat{\mu}} z^R \right)_\beta \right]_+$$

$$= \left[\frac{\partial S_\alpha(\tilde{\mathbf{t}}, \lambda)}{\partial \lambda} G^{\alpha\beta} \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \left(\sum_{r=0}^{N} \frac{\partial \mathcal{F}}{\partial v^{\gamma,r}} \partial_x^r \frac{\partial v^\gamma}{\partial t^p} z^{p+1} z^{\hat{\mu}} z^R \right)_\beta \right]_+$$

$$= \left[\frac{\partial S_\alpha(\tilde{\mathbf{t}}, \lambda)}{\partial \lambda} G^{\alpha\beta} \sum_{r=0}^{N} \frac{\partial \mathcal{F}}{\partial v^{\gamma,r}} \partial_x^r \partial_x \partial^\gamma p_\beta \right]_+$$

$$= \left[-\sum_{r=0}^{N} \frac{\partial \mathcal{F}}{\partial v^{\gamma,r}} \partial_x^r \left(\frac{\partial S_\alpha(\tilde{\mathbf{t}}, \lambda)}{\partial \lambda} G^{\alpha\beta} \partial_x \partial_{t_{\gamma,0}} \frac{\partial S_\beta(\tilde{\mathbf{t}}, \lambda)}{\partial \lambda} \right) \right.$$

$$\left. - \sum_{r=1}^{N} \frac{\partial \mathcal{F}}{\partial v^{\gamma,r}} \sum_{k=1}^{r} \binom{r}{k} \partial_x^k \frac{\partial S_\alpha(\tilde{\mathbf{t}}, \lambda)}{\partial \lambda} G^{\alpha\beta} \partial_x^{r-k+1} \partial^\gamma p_\beta \right]_+$$

$$= \sum_{r=0}^{N} \frac{\partial \mathcal{F}}{\partial v^{\gamma,r}} \partial_x^r \left(\frac{1}{E-\lambda} \right)^{\gamma} + \sum_{r=1}^{N} \frac{\partial \mathcal{F}}{\partial v^{\gamma,r}} \sum_{k=1}^{r} \binom{r}{k} \partial_x^{k-1} \partial_1 p_\alpha G^{\alpha\beta} \partial_x^{r-k+1} \partial^\gamma p_\beta. \quad (3.10.86)$$

The lemma is proved. □

We will now rewrite the linearized Virasoro constraint operator in the canonical coordinates $u_1, \ldots, u_n$ on the Frobenius manifold.

For any integer $r > 0$ denote

$$K_r^\gamma(u; u_x, \ldots, u^{(r)}; \lambda) = \partial_x^r \left(\frac{1}{E-\lambda}\right)^\gamma + \sum_{k=1}^r \binom{r}{k} \partial_x^{k-1}\partial_1 p_\alpha \, G^{\alpha\beta}\, \partial_x^{r-k+1}\partial^\gamma p_\beta \quad (3.10.87)$$

where $p_\alpha = p_\alpha(v(u); \lambda)$, $p_\beta = p(v(u); \beta)$.

Lemma 3.10.19 *The differential polynomial $K_r(u; u_x, \ldots, u^{(r)}; \lambda)$ is a rational function in λ with poles of the order $r+1$ at $\lambda = u_1, \ldots, \lambda = u_n$. It is regular at $\lambda = \infty$. The coefficient of the highest order pole $(\lambda - u_i)^{-r-1}$ is equal to*

$$-\psi_{i1}(u)\psi_i{}^\gamma(u)\,(u_i')^r \left[r! + 2^{-r}\sum_{k=1}^r \binom{r}{k}(2k-3)!!(2r-2k+1)!!\right]. \quad (3.10.88)$$

Proof For $v \in M_{ss}$ the vector functions $\phi_i(v;\lambda) = (\phi_1(v;\lambda), \ldots, \phi_n(v;\lambda))^T$ are solutions of a Fuchsian system in λ with regular singularities at $\lambda = u_1, \ldots, \lambda = u_n, \lambda = \infty$. The monodromy group of this Fuchsian system preserves the invariant bilinear form (3.10.83). The monodromy does not depend on the point of the Frobenius manifold. Therefore every term

$$\partial_x^{k-1}\partial_1 p_\alpha \, G^{\alpha\beta}\, \partial_x^{r-k+1}\partial^\gamma p_\beta$$

in the expression (3.10.87) is a single valued function in λ with regular singularities at the same points. Hence this is a rational function with poles at $u_1, \ldots, u_n, \infty$. From the expansion (3.6.66), (3.6.67) it easily follows that the pole at infinity disappears. To compute the pole of the highest order it suffices to derive from (3.10.79) that

$$\partial_x^m \phi_i = \frac{(2m-1)!!}{2^m}\frac{(u_i')^m}{(\lambda - u_i)^m}\phi_i + \ldots \quad (3.10.89)$$

where dots denote the terms with poles of the lower order. Using the orthogonality we obtain the highest order pole at $\lambda = u_i$ in

$$\partial_x^{k-1}(\sum_i \psi_{i1}(u)\phi_{i\alpha})\, G^{\alpha\beta}\partial_x^{r-k+1}(\sum_j \psi_j{}^\gamma(u)\phi_{j\beta})$$
$$= -\frac{(2k-3)!!(2r-2k+1)!!}{2^r}\frac{(u_i')^r}{(\lambda-u_i)^{r+1}}\psi_{i1}(u)\psi_i{}^\gamma(u) + \ldots \quad (3.10.90)$$

where dots denote poles of lower order. To finish the proof of the lemma it remains to observe that

$$\left(\frac{1}{E-\lambda}\right)^\gamma = \sum_{i=1}^n \frac{\psi_{i1}(u)\psi_i{}^\gamma(u)}{u_i - \lambda}. \quad (3.10.91)$$

So, the highest order pole at $\lambda = u_i$ in the first term in (3.10.87) equals

$$-\frac{\psi_{i1}(u)\,\psi_i^\gamma(u)\,r!\,(u_i')^r}{(\lambda-u_i)^{r+1}}.$$

The lemma is proved. □

Theorem 3.10.20 *The solution of the system of Virasoro constraints (3.10.64) for a semisimple Frobenius manifold is unique.*

Proof It suffices to prove that, for any N the linear homogeneous equation

$$\sum_{r=0}^{N}\frac{\partial\mathcal{F}}{\partial v^{\gamma,r}}\,\partial_x^r\left(\frac{1}{E-\lambda}\right)^\gamma+\sum_{r=1}^{N}\frac{\partial\mathcal{F}}{\partial v^{\gamma,r}}\sum_{k=1}^{r}\begin{pmatrix}r\\k\end{pmatrix}\partial_x^{k-1}\partial_1 p_\alpha\,G^{\alpha\beta}\,\partial_x^{r-k+1}\partial^\gamma p_\beta=0 \quad (3.10.92)$$

has only trivial solution $\mathcal{F}=\mathcal{F}(v;v_x,\ldots,v^{(N)})$. According to the previous Lemma the l.h.s. of the equation is a rational function in λ with poles of the order $N+1$ at the points $\lambda=u_1,\ \ldots,\ \lambda=u_n$. The highest order pole at the point $\lambda=u_i$ reads

$$-\sum_\gamma\frac{\partial\mathcal{F}}{\partial v^{\gamma,N}}\frac{\psi_{i1}(u)\,\psi_i^\gamma(u)\,(u_i')^N}{(\lambda-u_i)^{N+1}}\left[N!+2^{-N}\sum_{k=1}^{N}\begin{pmatrix}N\\k\end{pmatrix}(2k-3)!!(2N-2k+1)!!\right].$$

Due to nondegeneracy of the matrix $({\psi_i}^\gamma(u))$ we derive that

$$\frac{\partial\mathcal{F}}{\partial v^{\gamma,N}}=0,\quad \gamma=1,\ldots,n.$$

The theorem is proved. □

Corollary 3.10.21 *If the quasitriviality transformation satisfies the assumption of linearization of the Virasoro symmetries, then it is equivalent, up to an action of the Miura group, to the transformation of the form*

$$v_\alpha\mapsto w_\alpha=v_\alpha+\partial_x\partial_{t^{\alpha,0}}\sum_{g=1}^{\infty}\epsilon^{2\,g}\mathcal{F}_g(v;v_x,\ldots,v^{(3g-2)}) \quad (3.10.93)$$

where the functions $\mathcal{F}_g$ are uniquely determined from the system of Virasoro constraints (3.10.64).

Proof Using Theorem 3.9.5, we can kill the first term $\mathcal{F}^{[1]}$ in the quasitriviality transformation (3.10.49). All subsequent trems $\mathcal{F}^{[k]}$ for odd k must vanish due to the uniqueness theorem. Theorem 3.9.5 implies that the highest order of jets in $\mathcal{F}_g$ is equal to $3\,g-2$. The corollary is proved. □

In the framework of the theory of Gromov - Witten invariants, the generating function of the genus g Gromov-Witten invariants is conjectured to have the form $\mathcal{F}_g(v, v_x, \dots, v^{3g-2})$ where v^α are some special two point correlation functions, it makes part of the so-called *Virasoro conjecture* of T.Eguchi *et. al.* [56]. For the case where Frobenius manifold M coincides with the quantum cohomology of a smooth projective variety X with $H^{\rm odd}(X) = 0$ the Virasoro conjecture suggests a system of differential equations for the generating function of Gromov - Witten invarants of X and their descendents that coincides, in this particular case, with our Virasoro constraints (see details in [52]).

Corollary 3.10.22 *The partial derivatives $\partial\mathcal{F}_g/\partial u^{i,k}$ with respect to the jet coordinates $u^{i,k} := \partial_x^k u^i$, $k = 1, \dots, 3g-2$, are rational functions of the jets with the denominator containing only powers of $u_x^1, \dots, u_x^n$.*

Corollary 3.10.23 *Any monotone solution $w = w(x, \mathbf{t}, \epsilon)$ satisfying $w(0, \mathbf{0}, 0) \in M_{s\,s}$ is admissible. The quasitriviality transformation (3.10.93) establishes a one-to-one correspondence between monotone solutions $v(x, \mathbf{t}, \epsilon)$ to the Principal Hierarchy satisfying and admissible solutions to the full hierarchy.*

Let us now derive the generating function of the remaing part of the system of Virasoro constraints (3.10.66). The following statement will be convenient for such a derivation.

Lemma 3.10.24 *Let $b'_{\alpha,p}$ and $b''_{\beta,q}$ be two sets of elements of a commutative algebra, $\alpha, \beta = 1, \dots, n$, $p, q = 0, 1, \dots$. Denote*

$$b'_p = (b'_{1,p}, \dots, b'_{n,p}), \quad b''_p = (b''_{1,p}, \dots, b''_{n,p})$$

and put

$$\sigma'_\alpha(\lambda) := \left(\int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda\, z} \left(\sum_{p=0}^\infty b'_p z^p \right) z^{\hat\mu} z^R \right)_\alpha ,$$

$$\sigma''_\beta(\lambda) := \left(\int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda\, z} \left(\sum_{p=0}^\infty b''_p z^p \right) z^{\hat\mu} z^R \right)_\beta .$$

Then

$$-\frac{1}{2} \frac{\partial \sigma'_\alpha(\lambda)}{\partial\lambda} G^{\alpha\beta} \frac{\partial\sigma''_\beta(\lambda)}{\partial\lambda} = \sum_{m=1}^\infty \sum_{p+q=m-1} \frac{a_m^{\alpha,p;\beta,q}}{\lambda^{m+2}} b'_{\alpha,p} b''_{\beta,q}. \qquad (3.10.94)$$

Proof follows immediately from the free field realization of the Virasoro operators.

Let us denote

$$\Delta\mathcal{F}(v; v_x, \dots; \epsilon) := \sum_{k=1}^\infty \epsilon^{k-2} \mathcal{F}^{[k]}.$$

Lemma 3.10.25 *The loop equation (3.10.66) is equivalent to the following differential equations for the function $\Delta\mathcal{F}$ on the jet space*

$$\begin{aligned}
&\frac{\partial\Delta\mathcal{F}}{\partial v^{\gamma,r}}\,\partial_x^r\left(\frac{1}{E-\lambda}\right)^{\gamma}+\sum_{r\geq 1}\frac{\partial\Delta\mathcal{F}}{\partial v^{\gamma,r}}\sum_{k=1}^{r}\binom{r}{k}\partial_x^{k-1}\partial_1 p_\alpha\, G^{\alpha\beta}\,\partial_x^{r-k+1}\partial^\gamma p_\beta\\
&=\frac{1}{2}\frac{\partial p_\alpha(v;\lambda)}{\partial\lambda}*\frac{\partial p_\beta(v;\lambda)}{\partial\lambda}G^{\alpha\beta}-\frac{\kappa_0}{\lambda^2}\\
&+\frac{\epsilon^2}{2}\sum\left(\frac{\partial^2\Delta\mathcal{F}}{\partial v^{\gamma,k}\partial v^{\rho,l}}+\frac{\partial\Delta\mathcal{F}}{\partial v^{\gamma,k}}\frac{\partial\Delta\mathcal{F}}{\partial v^{\rho,l}}\right)\partial_x^{k+1}\partial^\gamma p_\alpha G^{\alpha\beta}\partial_x^{l+1}\partial^\rho p_\beta\\
&+\frac{\epsilon^2}{2}\sum\frac{\partial\Delta\mathcal{F}}{\partial v^{\gamma,k}}\partial_x^{k+1}\left[\nabla\frac{\partial p_\alpha(v;\lambda)}{\partial\lambda}\cdot\nabla\frac{\partial p_\beta(v;\lambda)}{\partial\lambda}\cdot v_x\right]^{\gamma}G^{\alpha\beta}.
\end{aligned}\tag{3.10.95}$$

Proof Direct substitution of $\exp(\epsilon^{-2}\mathcal{F}_0+\Delta\mathcal{F})$ into the system of Virasoro constraints gives

$$\begin{aligned}
&\frac{\partial\Delta\mathcal{F}}{\partial v^{\gamma,r}}\,\partial_x^r\left(\frac{1}{E-\lambda}\right)^{\gamma}+\sum_{r\geq 1}\frac{\partial\Delta\mathcal{F}}{\partial v^{\gamma,r}}\sum_{k=1}^{r}\binom{r}{k}\partial_x^{k-1}\partial_1 p_\alpha\, G^{\alpha\beta}\,\partial_x^{r-k+1}\partial^\gamma p_\beta\\
&+\epsilon^2\sum\frac{a_m^{\alpha,p;\beta,q}}{\lambda^{m+2}}\left(\frac{\partial^2\Delta\mathcal{F}}{\partial t^{\alpha,p}\partial t^{\beta,q}}+\frac{\partial\Delta\mathcal{F}}{\partial t^{\alpha,p}}\frac{\partial\Delta\mathcal{F}}{\partial t^{\beta,q}}\right)\\
&+\sum\frac{a_m^{\alpha,p;\beta,q}}{\lambda^{m+2}}\Omega_{\alpha,p;\beta,q}(v)+\frac{\kappa_0}{\lambda^2}=0.
\end{aligned}\tag{3.10.96}$$

Applying the formula (3.10.94) to

$$b'_{\alpha,p}=b''_{\alpha,p}=\frac{\partial\Delta\mathcal{F}}{\partial t^{\alpha,p}}$$

we obtain

$$\sigma'_\alpha(\lambda)=\sigma''_\alpha(\lambda)=\{\Delta\mathcal{F}(v(x);\ldots;\epsilon),\left[\int_0^\infty\frac{dz}{z^{3/2}}e^{-\lambda z}\left(\sum_{p=0}^\infty\bar\theta_p z^p\right)z^{\hat\mu}z^R\right]_\alpha\}_1.$$

So

$$\begin{aligned}
\frac{\partial\sigma_\alpha(\lambda)}{\partial\lambda}&=-\{\Delta\mathcal{F}(v(x);\ldots;\epsilon),\int_0^\infty\frac{dz}{\sqrt{z}}e^{-\lambda z}\left(\int\tilde v_\alpha(v(x);z)dx\right)\}_1\\
&=-\{\Delta\mathcal{F}(v(x);\ldots;\epsilon),\bar p_\alpha(\lambda)\}_1.
\end{aligned}$$

Therefore

$$\sum\frac{a_m^{\alpha,p;\beta,q}}{\lambda^{m+2}}\frac{\partial\Delta\mathcal{F}}{\partial t^{\alpha,p}}\frac{\partial\Delta\mathcal{F}}{\partial t^{\beta,q}}=-\frac{1}{2}\{\Delta\mathcal{F}(v(x);\ldots;\epsilon),\bar p_\alpha(\lambda)\}_1G^{\alpha\beta}\{\Delta\mathcal{F}(v(x);\ldots;\epsilon),\bar p_\beta(\lambda)\}_1.$$

This gives the second term in the third line in the equation (3.10.95). Similarly, to calculate the two terms of the expression

$$\sum\frac{a_m^{\alpha,p;\beta,q}}{\lambda^{m+2}}\frac{\partial^2\Delta\mathcal{F}}{\partial t^{\alpha,p}\partial t^{\beta,q}}$$

$$= \sum \frac{a_m^{\alpha,p;\beta,q}}{\lambda^{m+2}} \frac{\partial^2 \Delta\mathcal{F}}{\partial v^{\gamma,k}\partial v^{\rho,l}} \partial_x^k \frac{\partial v^\gamma}{\partial t^{\alpha,p}} \partial_x^l \frac{\partial v^\rho}{\partial t^{\beta,q}} + \sum \frac{a_m^{\alpha,p;\beta,q}}{\lambda^{m+2}} \frac{\partial \Delta\mathcal{F}}{\partial v^{\gamma,k}} \partial_x^{k+1} \frac{\partial \Omega_{\alpha,p;\beta,q}(v)}{\partial t_{\gamma,0}}$$

we are to use the same trick and also the formula

$$\frac{\partial \Omega_{\alpha,p;\beta,q}(v)}{\partial t_{\gamma,0}} = [\nabla\theta_{\alpha,p} \cdot \nabla\theta_{\beta,q} \cdot v_x]^\gamma .$$

This will give the first term of the third line and also the last line of the equation (3.10.95). Finally, applying the same trick to the calculation of

$$\sum \frac{a_m^{\alpha,p;\beta,q}}{\lambda^{m+2}} \frac{\partial^2 \mathcal{F}_0}{\partial t^{\alpha,p}\partial t^{\beta,q}} = \sum \frac{a_m^{\alpha,p;\beta,q}}{\lambda^{m+2}} \theta_{\alpha,p} * \theta_{\beta,q}$$

we obtain the first term of the second line of the equation. The lemma is proved. □

Example 3.10.26 *For $n = 1$ we have*

$$p = \sqrt{v - \lambda}, \quad G = 4,$$

$$\partial_\lambda p * \partial_\lambda p = \frac{1}{32\,\lambda^2} - \frac{1}{32(v-\lambda)^2}.$$

So the spelling of the loop equation (3.10.66) reads

$$\sum_r \frac{\partial \Delta\mathcal{F}}{\partial v^{(r)}} \partial_x^r \frac{1}{v-\lambda} + \sum_{r\geq 1} \frac{\partial \Delta\mathcal{F}}{\partial v^{(r)}} \sum_{k=1}^r \binom{r}{k} \partial_x^{k-1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{r-k+1} \frac{1}{\sqrt{v-\lambda}}$$
$$= \frac{1}{16\,\lambda^2} - \frac{1}{16(v-\lambda)^2} - \frac{\kappa_0}{\lambda^2}$$
$$+ \frac{\epsilon^2}{2} \sum \left[\frac{\partial^2 \Delta\mathcal{F}}{\partial v^{(k)}\partial v^{(l)}} + \frac{\partial \Delta\mathcal{F}}{\partial v^{(k)}} \frac{\partial \Delta\mathcal{F}}{\partial v^{(l)}} \right] \partial_x^{k+1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{l+1} \frac{1}{\sqrt{v-\lambda}}$$
$$- \frac{\epsilon^2}{16} \sum \frac{\partial \Delta\mathcal{F}}{\partial v^{(k)}} \partial_x^{k+2} \frac{1}{(v-\lambda)^2}. \tag{3.10.97}$$

Now we can determine recursively each term of the expansion

$$\Delta\mathcal{F} = \mathcal{F}_1(v; v_x) + \epsilon^2 \mathcal{F}_2(v; v_x, v_{xx}, v_{xxx}, v_{xxxx}) + \ldots$$

substituting it into equation (3.10.97). For $\mathcal{F}_1$ we obtain

$$\frac{1}{v-\lambda} \frac{\partial \mathcal{F}}{\partial v} - \frac{3}{2} \frac{v'}{(v-\lambda)^2} \frac{\partial \mathcal{F}}{\partial v'} = \frac{1}{16\,\lambda^2} - \frac{1}{16(v-\lambda)^2} - \frac{\kappa_0}{\lambda^2}.$$

This implies that

$$\kappa_0 = \frac{1}{16}, \quad \mathcal{F}_1 = \frac{1}{24} \log v'. \tag{3.10.98}$$

For the next term

$$\mathcal{F} := \mathcal{F}_2(v; v', v'', v''', v^{IV})$$

we obtain from (3.10.97)

$$\frac{1}{(v-\lambda)^5}\left(\frac{105}{2048}v'^2-\frac{945}{16}v'^4\frac{\partial\mathcal{F}}{\partial v^{IV}}\right)$$
$$+\frac{1}{(v-\lambda)^4}\left(-\frac{49}{1536}v''+\frac{735}{8}v'^2v''\frac{\partial\mathcal{F}}{\partial v^{IV}}+\frac{105}{8}v'^3\frac{\partial\mathcal{F}}{\partial v'''}\right)$$
$$+\frac{1}{(v-\lambda)^3}\left[\frac{1}{192}\frac{v'''}{v'}-\frac{23}{4608}\frac{v''^2}{v'^2}-\left(16v''^2+\frac{87}{4}v'v'''\right)\frac{\partial\mathcal{F}}{\partial v^{IV}}-\frac{55}{4}v'v''\frac{\partial\mathcal{F}}{\partial v'''}\right.$$
$$\left.-\frac{15}{4}v'^2\frac{\partial\mathcal{F}}{\partial v''}\right]+\frac{1}{(v-\lambda)^2}\left(3v^{IV}\frac{\partial\mathcal{F}}{\partial v^{IV}}+\frac{5}{2}v'''\frac{\partial\mathcal{F}}{\partial v'''}+2v''\frac{\partial\mathcal{F}}{\partial v''}+\frac{3}{2}v'\frac{\partial\mathcal{F}}{\partial v'}\right)$$
$$-\frac{1}{v-\lambda}\frac{\partial\mathcal{F}}{\partial v}=0. \tag{3.10.99}$$

Solving this system we easily obtain

$$\mathcal{F}_2=\frac{v^{IV}}{1152\,v'^2}-\frac{7\,v''v'''}{1920\,v'^3}+\frac{v''^3}{360\,v'^4}. \tag{3.10.100}$$

Example 3.10.27 *For the two-dimensional Frobenius manifold (3.6.57) (i.e., for the* $\mathbf{CP}^1$ *sigma-model) one can choose the following system of independent periods*

$$p_1=v_2-2\,\log\left(v_1-\lambda+\sqrt{(v_1-\lambda)^2-4\,\exp v_2}\right)$$
$$p_2=v_2. \tag{3.10.101}$$

The Gram matrix is equal to

$$\left(G^{\alpha\beta}\right)=\frac{1}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

A simple calculation gives

$$\partial_\lambda p_\alpha*\partial_\lambda p_\beta G^{\alpha\beta}=\frac{2e^{v_2}(4\,e^{v_2}+(v_1-\lambda)^2)}{(4\,e^{v_2}-(v_1-\lambda)^2)^3}.$$

So the loop equation (3.10.66) reads

$$\sum_{r\ge0}\left(\frac{\partial\Delta\mathcal{F}}{\partial v_1^{(r)}}\partial_x^r\frac{v_1-\lambda}{D}-2\frac{\partial\Delta\mathcal{F}}{\partial v_2^{(r)}}\partial_x^r\frac{1}{D}\right)$$
$$+\sum_{r\ge1}\sum_{k=1}^{r}\binom{r}{k}\partial_x^{k-1}\frac{1}{\sqrt{D}}\left(\frac{\partial\Delta\mathcal{F}}{\partial v_1^{(r)}}\partial_x^{r-k+1}\frac{v_1-\lambda}{\sqrt{D}}-2\frac{\partial\Delta\mathcal{F}}{\partial v_2^{(r)}}\partial_x^{r-k+1}\frac{1}{\sqrt{D}}\right)$$
$$=D^{-3}e^{v_2}\left(4\,e^{v_2}+(v_1-\lambda)^2\right)-\frac{\kappa_0^2}{\lambda^2}$$

$$
\begin{aligned}
&+\sum_{k,l}\frac{\epsilon^2}{4}\left[-\left(\frac{\partial^2\Delta\mathcal{F}}{\partial v_1^{(k)}\partial v_1^{(l)}}+\frac{\partial\Delta\mathcal{F}}{\partial v_1^{(k)}}\frac{\partial\Delta\mathcal{F}}{\partial v_1^{(l)}}\right)\partial_x^{k+1}\frac{v_1-\lambda}{\sqrt{D}}\partial_x^{l+1}\frac{v_1-\lambda}{\sqrt{D}}\right.\\
&+4\left(\frac{\partial^2\Delta\mathcal{F}}{\partial v_1^{(k)}\partial v_2^{(l)}}+\frac{\partial\Delta\mathcal{F}}{\partial v_1^{(k)}}\frac{\partial\Delta\mathcal{F}}{\partial v_2^{(l)}}\right)\partial_x^{k+1}\frac{v_1-\lambda}{\sqrt{D}}\partial_x^{l+1}\frac{1}{\sqrt{D}}\\
&\left.-4\left(\frac{\partial^2\Delta\mathcal{F}}{\partial v_2^{(k)}\partial v_2^{(l)}}+\frac{\partial\Delta\mathcal{F}}{\partial v_2^{(k)}}\frac{\partial\Delta\mathcal{F}}{\partial v_2^{(l)}}\right)\partial_x^{k+1}\frac{1}{\sqrt{D}}\partial_x^{l+1}\frac{1}{\sqrt{D}}\right]\\
&-\frac{\epsilon^2}{2}\sum_k\left\{\frac{\partial\Delta\mathcal{F}}{\partial v_1^{(k)}}\partial_x^{k+1}e^{v_2}\frac{-[(v_1-\lambda)^2+4\,e^{v_2}]\,v_1'+4\,e^{v_2}(v_1-\lambda)v_2'}{D^3}\right.\\
&\left.+\frac{\partial\Delta\mathcal{F}}{\partial v_2^{(k)}}\partial_x^{k+1}e^{v_2}\frac{4\,(v_1-\lambda)\,v_1'-[(v_1-\lambda)^2+4\,e^{v_2}]\,v_2'}{D^3}\right\}
\end{aligned}
\tag{3.10.102}
$$

where

$$D=(v_1-\lambda)^2-4\,e^{v_2}.$$

In the next sections we will solve the system of Virasoro constraints for low genera $g\leq 2$ and compare this solution with the topological one.

3.10.7 Genus one case and the final form of the loop equation

Lemma 3.10.28 *For an arbitrary semisimple Frobenius manifold and an arbitrary system of independent periods $p_\alpha(v;\lambda)$, $\alpha=1,\dots,n$, $G^{\alpha\beta}=(\partial/\partial p_\alpha,\partial/\partial p_\beta)_\lambda$, the following identity holds true*

$$
\begin{aligned}
&\partial_\lambda p_\alpha(v(u);\lambda)*\partial_\lambda p_\beta(v(u);\lambda)G^{\alpha\beta}\\
&=-\frac{1}{8}\sum_{i=1}^n\frac{1}{(\lambda-u_i)^2}+\sum_{i<j}\frac{V_{ij}^2}{(\lambda-u_i)(\lambda-u_j)}+\frac{1}{2\,\lambda^2}\mathrm{tr}\left(\frac{1}{4}-\hat\mu^2\right).
\end{aligned}
\tag{3.10.103}
$$

Proof Introducing, as above, the functions $\phi_{i\,\alpha}(v;\lambda)$ by

$$\partial_i p_\alpha(v(u);\lambda)=\psi_{i\,1}(u)\phi_{i\,\alpha}(v(u);\lambda)$$

we obtain

$$\partial_i(\partial_\lambda p_\alpha*\partial_\lambda p_\beta G^{\alpha\beta})=\partial_\lambda\phi_{i\,\alpha}\partial_\lambda\phi_{i\,\beta}G^{\alpha\beta}.$$

Using differential equations (3.10.81) we rewrite the r.h.s. as

$$=\frac{1}{(u_i-\lambda)^2}\left[\frac{1}{2}\phi_{i\,\alpha}+\sum_k V_{ik}\phi_{k\,\alpha}\right]G^{\alpha\beta}\left[\frac{1}{2}\phi_{i\,\beta}+\sum_l V_{il}\phi_{l\,\beta}\right]$$

$$= \frac{1}{4(u_i - \lambda)^3} + \sum_j \frac{V_{ij}^2}{(u_i - \lambda)^2 (u_j - \lambda)}.$$

To derive the last formula we have used the orthogonality condition (3.10.83). Using the differential equations for the matrix V_{ij} [41, 42] we integrate the last formula to obtain

$$\partial_\lambda p_\alpha * \partial_\lambda p_\beta G^{\alpha\beta} = -\frac{1}{8} \sum_i \frac{1}{(u_i - \lambda)^2} + \sum_{i<j} \frac{V_{ij}^2}{(u_i - \lambda)(u_j - \lambda)} + c(\lambda) \qquad (3.10.104)$$

where the rational function $c(\lambda)$ is an integration constant. It can have poles only at $\lambda = \infty$. To determine this integration constant we will use the basis of the regularized periods (see Section 3.6.3 above) as follows

$$\lim_{\nu \to 0} \partial_\lambda p_\alpha^{(\nu)} G^{\alpha\beta}(\nu) * \partial_\lambda p_\beta^{(-\nu)} = -\frac{1}{\pi} \times$$

$$\times \sum_{p,q \geq 1} \sum_{r \geq 0} \frac{\theta_{p-1} \left[e^{R\,\partial_\nu} \right]_r \left(\Gamma(\hat\mu + \nu + p - r + \frac{1}{2}) \cos \pi(\hat\mu + \nu) \Gamma(-\hat\mu - \nu + q + \frac{1}{2}) \right) * \theta^{q-1}}{\lambda^{p+q+r+1}}.$$

Here

$$\theta_k = (\theta_{1,k}, \ldots, \theta_{n,k}), \quad \theta^k = (\theta^{1,k}, \ldots, \theta^{n,k})^t, \quad \theta^{\alpha,k} := \eta^{\alpha\beta} \theta_{\beta,k}.$$

Therefore the integration constant $c(\lambda)$ must be chosen in such a way that the r.h.s. of (3.10.104) $= O(1/\lambda^3)$. Hence

$$c(\lambda) = \frac{n}{8\,\lambda^2} - \sum_{i<j} \frac{V_{ij}^2}{\lambda^2} = \frac{1}{2\,\lambda^2} \mathrm{tr} \left(\frac{1}{4} - V^2 \right).$$

To complete the proof it remains to observe that

$$\mathrm{tr}\, V^2 = \mathrm{tr}\,(\hat\mu + R_0)^2 = \mathrm{tr}\,\hat\mu^2$$

due to nilpotency of R_0 and commutativity $[R_0, \hat\mu] = 0$. □

Theorem 3.10.29 *For an arbitrary semisimple Frobenius manifold the system (3.10.95) implies*

$$\mathcal{F}_1 = \log \frac{\tau_I(u)}{J^{1/24}(u)} + \frac{1}{24} \sum_{i=1}^n \log u_i', \qquad (3.10.105)$$

$$\kappa_0 = \frac{1}{4} \mathrm{tr} \left(\frac{1}{4} - \hat\mu^2 \right). \qquad (3.10.106)$$

Here the isomonodromic tau-function $\tau_I(u)$ of the Frobenius manifold is defined by (3.10.76), $J(u)$ is the Jacobian of the transformation from canonical to the flat coordinates

$$J(u) = \det \left(\frac{\partial v^\alpha}{\partial u_i} \right) = \pm \prod_{i=1}^n \psi_{i\,1}(u). \qquad (3.10.107)$$

We proved the formula (3.10.105) in [51] using results [72] on topology of the moduli space $\bar{\mathcal{M}}_{1,4}$. Remarkably, the very same formula follows from our axioms of integrable PDEs!

Proof We are to solve the following equation for the function $\mathcal{F}_1 = \mathcal{F}_1(v; v_x)$

$$\sum_{i=1}^n \frac{\partial \mathcal{F}_1}{\partial u_i} \frac{1}{u_i - \lambda} - \sum_{i=1}^n \frac{\partial \mathcal{F}_1}{\partial u_i'} \frac{u_i'}{(u_i - \lambda)^2} + \sum \frac{\partial \mathcal{F}_1}{\partial v_x^\gamma} \partial_1 p_\alpha G^{\alpha\beta} \partial_x \partial^\gamma p_\beta$$
$$= -\frac{1}{16} \sum_{i=1}^n \frac{1}{(\lambda - u_i)^2} + \frac{1}{2} \sum_{i<j} \frac{V_{ij}^2}{(\lambda - u_i)(\lambda - u_j)} + \frac{1}{4\,\lambda^2} \mathrm{tr} \left(\frac{1}{4} - \hat{\mu}^2 \right) - \frac{\kappa_0}{\lambda^2}. \tag{3.10.108}$$

Using the formulae

$$\frac{\partial \mathcal{F}_1}{\partial v_x^\gamma} = \sum_i \frac{\psi_{i\,\gamma}(u)}{\psi_{i\,1}(u)} \frac{\partial \mathcal{F}_1}{\partial u_i'}$$
$$\partial_\sigma p_\alpha = \sum \psi_{k\,\sigma}(u) \phi_{k\,\alpha}(v(u); \lambda)$$

we obtain

$$\sum \frac{\partial \mathcal{F}_1}{\partial v_x^\gamma} \partial_1 p_\alpha G^{\alpha\beta} \partial_x \partial^\gamma p_\beta = \sum \frac{\partial \mathcal{F}_1}{\partial u_i'} \left(\frac{\psi_{j\,1}}{\psi_{i\,1}} \frac{\psi_{i\,\gamma} \partial_x {\psi_j}^\gamma}{u_j - \lambda} + \frac{\psi_{k\,1}}{\psi_{i\,1}} \phi_{k\,\alpha} G^{\alpha\beta} \partial_x \phi_{i\,\beta} \right).$$

With the help of the differential equations for the functions $\psi_{i\,\sigma}$ and $\phi_{k\,\alpha}$ and the orthogonality conditions (3.10.83) and

$$\psi_{i\,\gamma} {\psi_j}^\gamma = \delta_{ij}$$

we rewrite the equation (3.10.108) in the form

$$\sum_i \frac{\partial \mathcal{F}_1}{\partial u_i} \frac{1}{u_i - \lambda} - \frac{3}{2} \sum \frac{\partial \mathcal{F}_1}{\partial u_i'} \frac{u_i'}{(u_i - \lambda)^2} - \sum \frac{\partial \mathcal{F}_1}{\partial u_i'} \frac{\psi_{j\,1}}{\psi_{i\,1}} \frac{V_{ij}}{(u_i - \lambda)(u_j - \lambda)}$$

$$= -\frac{1}{16} \sum_{i=1}^n \frac{1}{(\lambda - u_i)^2} + \frac{1}{2} \sum_{i<j} \frac{V_{ij}^2}{(\lambda - u_i)(\lambda - u_j)} + \frac{1}{4\,\lambda^2} \mathrm{tr} \left(\frac{1}{4} - \hat{\mu}^2 \right) - \frac{\kappa_0}{\lambda^2}.$$

The formulae (3.10.105) and (3.10.76) easily follow from the last equation. □

As in (3.6.91) we define “genus one correlators”

$$\langle\langle \tau_{p_1}(\phi_{\alpha_1}) \tau_{p_2}(\phi_{\alpha_2}) \ldots \tau_{p_k}(\phi_{\alpha_k}) \rangle\rangle_1 := \epsilon^k \frac{\partial^k \mathcal{F}_1(v; v_x)}{\partial t^{\alpha_1, p_1} \partial t^{\alpha_2, p_2} \ldots \partial t^{\alpha_k, p_k}} \tag{3.10.109}$$

where instead of v, v_x one is to substitute the topological solution (3.6.89), (3.6.90) and its x-derivative.

Theorem 3.10.30 *For an arbitrary semisimple Frobenius manifold the genus 1 solution (3.10.105) of the loop equation evaluated on the topological solution (3.6.89), (3.6.90) satisfies the following identities.*

1). The genus one topological recursion relations

$$\langle\langle\tau_p(\phi_\alpha)\rangle\rangle_1 = \langle\langle\tau_{p-1}(\phi_\alpha)\tau_0(\phi_\nu)\rangle\rangle_0\eta^{\nu\mu}\langle\langle\tau_0(\phi_\mu)\rangle\rangle_1 + \frac{1}{24}\eta^{\nu\mu}\langle\langle\tau_{p-1}(\phi_\alpha)\tau_0(\phi_\nu)\tau_0(\phi_\mu)\rangle\rangle_0. \tag{3.10.110}$$

2). The restriction

$$G(v) := \mathcal{F}_1(v; v_x)$$

onto the small phase space $t^{\alpha,p} = 0$ for $p > 0$ evaluated onto the topological solution satisfies

$$\sum_{1\le\alpha_1,\alpha_2,\alpha_3,\alpha_4\le n} z_{\alpha_1}z_{\alpha_2}z_{\alpha_3}z_{\alpha_4}\left(3\,c^\mu_{\alpha_1\alpha_2}\,c^\nu_{\alpha_3\alpha_4}\frac{\partial^2 G}{\partial v^\mu\partial v^\nu} - 4\,c^\mu_{\alpha_1\alpha_2}\,c^\nu_{\alpha_3\mu}\frac{\partial^2 G}{\partial v^{\alpha_4}\partial v^\nu}\right.$$
$$-c^\mu_{\alpha_1\alpha_2}\,c^\nu_{\alpha_3\alpha_4\mu}\frac{\partial G}{\partial v^\nu} + 2\,c^\mu_{\alpha_1\alpha_2\alpha_3}\,c^\nu_{\alpha_4\mu}\frac{\partial G}{\partial v^\nu} + \frac{1}{6}c^\mu_{\alpha_1\alpha_2\alpha_3}\,c^\nu_{\alpha_4\mu\nu}$$
$$\left.+\frac{1}{24}c^\mu_{\alpha_1\alpha_2\alpha_3\alpha_4}\,c^\nu_{\mu\nu} - \frac{1}{4}c^\mu_{\alpha_1\alpha_2\nu}\,c^\nu_{\alpha_3\alpha_4\mu}\right) = 0. \tag{3.10.111}$$

Here the coefficients $c^\alpha_{\beta\delta} = c^\alpha_{\beta\delta}(v)$ are structure functions of the Frobenius multiplication on T_vM, $c^{\alpha\beta\gamma\delta} = c^{\alpha\beta\gamma\delta}(v)$ and $c^\mu_{\alpha\beta\gamma\delta} = c^\mu_{\alpha\beta\gamma\delta}(v)$ are linear combinations of the 4th and 5th order derivatives of the potential $F(v)$,

$$c^{\alpha\beta}_{\gamma\mu} = \eta^{\alpha\alpha'}\,\eta^{\beta\beta'}\,\frac{\partial^4 F(v)}{\partial v^{\alpha'}\,\partial v^{\beta'}\,\partial v^\gamma\,\partial v^\mu},\quad c^\mu_{\alpha\beta\gamma\delta} = \eta^{\mu\nu}\frac{\partial^5 F(v)}{\partial v^\nu\partial v^\alpha\partial v^\beta\partial v^\gamma\partial v^\delta}. \tag{3.10.112}$$

It is understood that all the coefficients of the degree 4 polynomial in $z_1, \ldots, z_n$ in (3.10.111) are equal to zero.

Proof of the Theorem can be obtained just inverting the arguments of [51] where, vice versa, the formula (3.10.105) has been derived starting from the "topological" equations (3.10.110), (3.10.111).

Similarly to the genus zero relations (3.6.92) the above equations (3.10.110), (3.10.111) can be spelled out as an infinite system of identities for the genus one and genus zero Gromov - Witten and Mumford - Morita - Miller classes in the case when the Frobenius manifold comes from quantum cohomology. For this case the genus one topological recursion relations (3.10.110) were derived by E. Witten in [146], the equations (3.10.111) were derived by E. Getzler [72]. They are known to be the defining relations for the genus one classes (see details in [72]).

We arrive at the main

Theorem 3.10.31 *The tau-symmetric quasitrivial (0,n) Poisson pencil with the leading term (3.5.24) corresponding to a semisimple Frobenius manifold and satisfying the axiom of linearization is obtained from (3.5.24) by the transformation*

$$v_\alpha \mapsto w_\alpha v_\alpha + \epsilon^2 \partial_x \partial_{t^{\alpha,0}} \Delta\mathcal{F}(v; v_x, v_{xx}, \dots; \epsilon^2), \quad \alpha = 1, \dots, n$$

where the function

$$\Delta\mathcal{F} = \sum_{g\geq 1} \epsilon^{2g-2} \mathcal{F}_g(v; v_x, \dots, v^{(3g-2)})$$

is uniquely determined from the following universal loop equation

$$\begin{aligned}
&\frac{\partial \Delta\mathcal{F}}{\partial v^{\gamma,r}} \partial_x^r \left(\frac{1}{E-\lambda}\right)^\gamma + \sum_{r\geq 1} \frac{\partial \Delta\mathcal{F}}{\partial v^{\gamma,r}} \sum_{k=1}^r \binom{r}{k} \partial_x^{k-1} \partial_1 p_\alpha \, G^{\alpha\beta} \, \partial_x^{r-k+1} \partial^\gamma p_\beta \\
&= -\frac{1}{16} \mathrm{tr}\, (\mathcal{U}-\lambda)^{-2} - \frac{1}{4} \mathrm{tr}\left[(\mathcal{U}-\lambda)^{-1} \mathcal{V}\right]^2 \\
&+ \frac{\epsilon^2}{2} \sum \left(\frac{\partial^2 \Delta\mathcal{F}}{\partial v^{\gamma,k} \partial v^{\rho,l}} + \frac{\partial \Delta\mathcal{F}}{\partial v^{\gamma,k}} \frac{\partial \Delta\mathcal{F}}{\partial v^{\rho,l}}\right) \partial_x^{k+1} \partial^\gamma p_\alpha G^{\alpha\beta} \partial_x^{l+1} \partial^\rho p_\beta \\
&+ \frac{\epsilon^2}{2} \sum \frac{\partial \Delta\mathcal{F}}{\partial v^{\gamma,k}} \partial_x^{k+1} \left[\nabla \frac{\partial p_\alpha(v;\lambda)}{\partial \lambda} \cdot \nabla \frac{\partial p_\beta(v;\lambda)}{\partial \lambda} \cdot v_x\right]^\gamma G^{\alpha\beta}. \qquad (3.10.113)
\end{aligned}$$

It remains an open problem to prove existence of solution to the loop equation (3.10.113) – this sounds plausible since the number of equations is equal to the number of unknowns – and also to prove polynomiality of the resulting Poisson pencil in every order g. For $g = 1$ this follows from the results of [51]. In the next section we will consider the $g = 2$ terms.

3.10.8 Genus two

Let us now proceed to the genus two case. After long but straightforward calculations we obtain the following formula for $\mathcal{F}_2(u; u_x, \dots, u_{xxxx})$. Denote

$$u_{ij} = u_i - u_j, \; i \neq j,$$

$$u_i' := u_{i,x}, \quad u_i'' := u_{i,xx} \;\; \text{etc.}$$

$$h_i = h_i(u) := \psi_{i1}(u), \; i = 1, \dots, n.$$

Theorem 3.10.32 *For an arbitrary semisimple Frobenius manifold the following formula holds true*

$$\begin{aligned}
\mathcal{F}_2 &= \frac{1}{1152} \frac{u_i^{IV}}{u_i'^2 h_i^2} - \frac{7}{1920} \frac{u_i'' u_i'''}{u_i'^3 h_i^2} + \frac{1}{360} \frac{u_i''^3}{u_i'^4 h_i^2} + \frac{1}{40} \frac{V_{ij}^2 u_i'''}{u_{ij} u_i' h_i^2} \\
&+ \frac{1}{640} \frac{V_{ij} h_j u_j' u_i'''}{u_{ij} u_i'^2 h_i^3} - \frac{19}{2880} \frac{V_{ij} u_i''' h_j}{u_{ij} u_i' h_i^3} + \frac{1}{1152} \frac{V_{ij} u_i''' h_i}{u_{ij} u_j' h_j^3}
\end{aligned}$$

$$+\frac{7}{40}\frac{V_{ij}^2\,V_{ik}^2\,u_i''}{u_{ij}\,u_{ik}\,h_i^2}-\frac{1}{240}\frac{V_{ij}^2\,V_{ik}\,u_i''\,h_k\,(32\,u_i'-7\,u_k')}{u_{ij}\,u_{ik}\,u_i'\,h_i^3}$$

$$+\frac{1}{40}\frac{V_{ij}\,V_{jk}^2\,u_i''\,h_i}{u_{ij}\,u_{jk}\,h_j^3}-\frac{1}{48}\frac{V_{ij}\,V_{jk}^2\,u_j'\,u_i''}{u_{ij}\,u_{jk}\,u_i'\,h_i\,h_j}-\frac{3}{64}\frac{V_{ij}^2\,u_i''}{u_{ij}^2\,h_i^2}$$

$$-\frac{11}{480}\frac{V_{ij}^2\,u_i''^2}{u_{ij}\,u_i'^2\,h_i^2}+\frac{29}{5760}\frac{V_{ij}\,V_{jk}\,u_i''\,h_i\,h_k\left(u_k'-2\,u_j'\right)}{u_{ij}\,u_{jk}\,u_j'\,h_j^4}$$

$$+\frac{1}{1920}\frac{V_{ij}\,V_{ik}\,u_i''\,h_j\,h_k\left(54\,u_i'^2-25\,u_i'\,u_j'-u_j'\,u_k'\right)}{u_{ij}\,u_{ik}\,u_i'^2\,h_i^4}$$

$$+\frac{1}{384}\frac{V_{ij}\,V_{ik}\,u_i''\,h_k\,(u_i'-u_k')}{u_{ij}\,u_{ik}\,u_j'\,h_j^3}-\frac{1}{384}\frac{V_{ik}\,V_{jk}\,u_k'\,u_i''\,h_i}{u_{ik}\,u_{jk}\,u_j'\,h_j^3}$$

$$+\frac{1}{576}\frac{V_{ij}\,V_{jk}\,u_i''\,h_k\left(2\,u_j'-u_k'\right)}{u_{ij}\,u_{jk}\,u_i'\,h_i\,h_j^2}$$

$$-\frac{1}{5760}\frac{V_{ij}\,V_{jk}\,u_k'\,u_i''\,h_k\,(27\,u_i'+u_k')}{u_{jk}\,u_{ik}\,u_i'^2\,h_i^3}-\frac{19}{1920}\frac{V_{ij}\,V_{jk}\,u_i''\,h_k}{u_{ij}\,u_{ik}\,h_i^3}$$

$$+\frac{1}{5760}\frac{V_{ij}\,V_{jk}\,h_k\left(27\,u_i'\,u_k'-u_j'^2+2\,u_j'\,u_k'\right)\,u_i''}{u_{ij}\,u_{jk}\,u_i'^2\,h_i^3}$$

$$+\frac{1}{288}\frac{V_{ij}\,V_{jk}\,u_i''\,h_i}{u_{jk}\,u_{ik}\,h_k^3}+\frac{1}{384}\frac{V_{ij}\,V_{jk}\,u_i'\,u_i''\,h_i}{u_{ij}\,u_{ik}\,u_k'\,h_k^3}-\frac{1}{576}\frac{V_{ij}\,V_{jk}\,u_k'\,u_i''}{u_{jk}\,u_{ik}\,u_i'\,h_i\,h_k}$$

$$+\frac{1}{1920}\frac{V_{ij}\,u_i''^2\,h_j\left(11\,u_i'-5\,u_j'\right)}{u_{ij}\,u_i'^3\,h_i^3}-\frac{1}{5760}\frac{V_{ij}\,u_i''\,u_j''\,h_j}{u_{ij}\,u_i'^2\,h_i^3}$$

$$+\frac{1}{5760}\frac{V_{ij}\,u_i''\,h_j\left(57\,u_i'^2-27\,u_i'\,u_j'-u_j'^2\right)}{u_i'^2\,h_i^3}$$

$$+\frac{1}{1152}\frac{V_{ij}\,u_i''\,h_i\left(4\,u_j'-3\,u_i'\right)}{u_j'\,h_j^3}-\frac{1}{576}\frac{V_{ij}\,u_j'\,u_i''}{u_i'\,h_i\,h_j}$$

$$-\frac{1}{1152}\frac{V_{ij}\,u_i''\,u_j''}{u_i'\,u_j'\,h_i\,h_j}+\frac{1}{10}\frac{V_{ij}^2\,V_{ik}^2\,V_{il}^2\,u_i'^2}{u_{ij}\,u_{ik}\,u_{il}\,h_i^2}$$

$$-\frac{7}{20}\frac{V_{ij}^2\,V_{ik}^2\,V_{il}\,h_l\,u_i'^2}{u_{ij}u_{ik}u_{il}\,h_i^3}+\frac{7}{40}\frac{V_{ij}^2\,V_{ik}^2\,V_{il}\,h_l\,u_i'\,u_l'}{u_{ij}\,u_{ik}\,u_{il}\,h_i^3}-\frac{1}{8}\frac{V_{ij}^2\,V_{ik}\,V_{kl}^2\,u_i'\,u_k'}{u_{ij}\,u_{ik}\,u_{kl}\,h_i\,h_k}$$

$$+\frac{1}{40}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,h_l\left(u_k'^2-3\,u_i'^2-2\,u_k'\,u_l'\right)}{u_{ij}\,u_{ik}\,u_{kl}\,h_i^3}+\frac{3}{40}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,u_i'\,u_l'\,h_l}{u_{ij}\,u_{ik}\,u_{il}\,h_i^3}$$

$$+\frac{1}{40}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,h_l\left(3\,u_i'^2+u_l'^2\right)}{u_{ij}\,u_{kl}\,u_{il}\,h_i^3}+\frac{1}{48}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,h_l\,u_i'\,(2\,u_k'-u_l')}{u_{ij}\,u_{ik}\,u_{kl}\,h_i\,h_k^2}$$

$$+\frac{5}{96}\frac{V_{ij}^2\,V_{ik}\,V_{il}\,h_k\,h_l\,\left(4\,u_i'^2-4\,u_i'\,u_k'+u_k'u_l'\right)}{u_{ij}u_{ik}u_{il}\,h_i^4}-\frac{83}{480}\frac{V_{ij}^2\,V_{ik}^2\,u_i'^2}{u_{ij}\,u_{ik}^2\,h_i^2}$$

$$+\frac{1}{144}\frac{V_{ij}\,V_{ik}\,V_{jl}\,V_{kl}\,u_i'^2}{u_{ik}\,u_{jl}\,u_{il}\,h_i^2}-\frac{1}{144}\frac{V_{ij}\,V_{ik}\,V_{jl}\,V_{kl}\,u_i'^2}{u_{ij}\,u_{ik}\,u_{kl}\,h_i^2}-\frac{1}{48}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,u_i'\,u_l'}{u_{ij}\,u_{kl}\,u_{il}\,h_i\,h_l}$$

$$+\frac{29}{1920}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\,\left(u'_k\,u'_l-u'_i\,u'_k+2\,{u'_i}^2-u'_i\,u'_l\right)}{u_{ij}\,u_{ik}\,u_{il}\,h_i^4}$$
$$-\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\,{u'_l}^2\,(2\,u'_i-u'_k)}{u_{ik}\,u_{jl}\,u_{il}\,h_i^4\,u'_i}$$
$$-\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\,u'_j\,\left(2\,u'_k\,u'_l+2\,u'_i\,u'_j-u'_j\,u'_k-4\,u'_i\,u'_l\right)}{u_{ij}\,u_{ik}\,u_{jl}\,h_i^4\,u'_i}$$
$$-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\,\left(4\,u'_i\,u'_j-4\,u'_i\,u'_l+u'_k\,u'_l\right)}{u_{ij}\,u_{ik}\,u_{jl}\,h_i^2\,h_j^2}$$
$$-\frac{1}{384}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,\left(u'_i\,{u'_j}^2-2\,u'_j\,u'_i\,u'_l\right)}{u_{ij}\,u_{ik}\,u_{jl}\,u'_k\,h_k^3}$$
$$+\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u'_i}^2\,(u'_i-3\,u'_l)}{u_{ij}\,u_{ik}\,u_{il}\,u'_k\,h_k^3}-\frac{1}{384}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,u'_i\,{u'_l}^2}{u_{ik}\,u_{jl}\,u_{il}\,u'_k\,h_k^3}$$
$$-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u'_j}^2\,\left(3\,u'_l-2\,u'_j\right)}{u_{ij}\,u_{jl}\,u_{jk}\,u'_k\,h_k^3}-\frac{1}{288}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,u'_j\,\left(u'_j-2\,u'_l\right)}{u_{ik}\,u_{jl}\,u_{jk}\,h_k^3}$$
$$+\frac{1}{576}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,u'_k\,(2\,u'_k-3\,u'_l)}{u_{ik}\,u_{jk}\,u_{kl}\,h_k^3}-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u'_l}^3}{u_{jl}\,u_{kl}\,u_{il}\,u'_k\,h_k^3}$$
$$+\frac{1}{288}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u'_l}^2}{u_{ik}\,u_{jl}\,u_{kl}\,h_k^3}-\frac{1}{576}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,u'_l\,(u'_k-2\,u'_i)}{u_{ik}\,u_{jl}\,u_{il}\,h_i^2\,h_l}$$
$$-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,u'_k\,u'_l}{u_{ik}\,u_{jl}\,u_{kl}\,h_k\,h_l}$$
$$-\frac{7}{1440}\frac{V_{ij}\,V_{ik}\,V_{il}\,h_j\,h_k\,h_l\,\left(8\,{u'_i}^3-12\,{u'_i}^2\,u'_j-u'_j\,u'_k\,u'_l+6\,u'_i\,u'_j\,u'_k\right)}{u_{ij}\,u_{ik}\,u_{il}\,h_i^5\,u'_i}$$
$$-\frac{29}{1152}\frac{V_{ij}\,V_{ik}\,V_{jk}\,{u'_i}^2}{u_{ij}\,u_{ik}^2\,h_i^2}-\frac{1}{320}\frac{V_{ij}^2\,V_{ik}\,h_k\,\left(3\,{u'_i}^2-8\,{u'_k}^2\right)}{u_{ij}\,u_{ik}^2\,h_i^3}-\frac{53}{1920}\frac{V_{ij}^2\,V_{ik}\,h_k\,u'_i\,u'_k}{u_{ij}\,u_{ik}\,u_{jk}\,h_i^3}$$
$$-\frac{V_{ij}^2\,V_{ik}\,u'_i\,h_k}{u_{ij}^2\,u_{jk}\,h_i^3}\left(\frac{27}{640}\,u'_k-\frac{233}{2880}\,u'_i\right)$$
$$-\frac{V_{ij}^2\,V_{ik}\,u'_i\,h_k}{u_{ik}^2\,u_{jk}\,h_i^3}\left(\frac{233}{2880}\,u'_i-\frac{67}{960}\,u'_k\right)+\frac{1}{1152}\frac{V_{ij}^2\,V_{ik}\,h_i\,{u'_i}^3}{u_{ij}\,u_{ik}^2\,u'_k\,h_k^3}$$
$$-\frac{1}{576}\frac{V_{ij}^2\,V_{ik}\,h_i\,{u'_i}^3}{u_{ij}^2\,u_{ik}\,u'_k\,h_k^3}-\frac{1}{48}\frac{V_{ij}^2\,V_{ik}\,u'_i\,u'_k}{u_{ij}\,u_{ik}^2\,h_i\,h_k}$$
$$+\frac{233}{1440}\frac{V_{ij}^3\,h_j\,{u'_i}^2}{u_{ij}^3\,h_i^3}-\frac{43}{384}\frac{V_{ij}^3\,h_j\,u'_i\,u'_j}{u_{ij}^3\,h_i^3}-\frac{1}{12}\frac{V_{ij}^3\,u'_i\,u'_j}{u_{ij}^3\,h_i\,h_j}$$
$$+\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,u'_j\,u'_k\,h_j\,h_k\,(u'_k-6\,u'_i)}{u_{ij}\,u_{ik}^2\,u'_i\,h_i^4}$$
$$+\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,h_j\,h_k\,\left(3\,u'_i\,u'_k+3\,u'_j\,u'_k+6\,u'_i\,u'_j-6\,{u'_i}^2-2\,{u'_j}^2\right)}{u_{ij}^2\,u_{ik}\,h_i^4}$$

$$+\frac{1}{576}\frac{V_{ij}\,V_{ik}\,u'_j\,h_k\,(2\,u'_i-u'_k)}{u_{ij}^2\,u_{ik}\,h_i^2\,h_j}$$
$$+\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,u_{ij}\,h_k\left(3\,{u'_i}^2\,u'_k-3\,u'_i\,{u'_k}^2+{u'_k}^3-{u'_i}^3\right)}{u_{ik}^2\,u_{jk}^2\,u'_j\,h_j^3}$$
$$+\frac{1}{576}\frac{V_{ij}\,V_{ik}\,u_{ik}\,h_k\left(-{u'_i}^3+3\,{u'_j}^2\,u'_k-4\,u'_i\,u'_j\,u'_k+2\,{u'_i}^2\,u'_j-2\,{u'_j}^3\right)}{u_{ij}^2\,u_{jk}^2\,u'_j\,h_j^3}$$
$$+\frac{1}{384}\frac{V_{ij}\,V_{ik}\,h_k\left(-u'_i\,{u'_k}^2+{u'_i}^3-6\,{u'_j}^2\,u'_k\right)}{u_{ij}\,u_{jk}^2\,u'_j\,h_j^3}$$
$$+\frac{1}{288}\frac{V_{ij}\,V_{ik}\,h_k\left(4\,u'_i\,u'_j\,u'_k+u'_j\,{u'_k}^2-2\,{u'_i}^2\,u'_j+3\,{u'_j}^3\right)}{u_{ij}\,u_{jk}^2\,u'_j\,h_j^3}$$
$$+\frac{1}{384}\frac{V_{ij}\,V_{ik}\,h_k\left(2\,u'_i\,{u'_k}^2-{u'_i}^2\,u'_k-{u'_k}^3\right)}{u_{ik}\,u_{jk}^2\,u'_j\,h_j^3}$$
$$+\frac{1}{288}\frac{V_{ij}\,V_{ik}\,h_k\left(u'_j\,{u'_k}^2-2\,u'_i\,u'_j\,u'_k+{u'_i}^2\,u'_j\right)}{u_{ik}\,u_{jk}^2\,u'_j\,h_j^3}$$
$$+\frac{1}{384}\frac{V_{ij}\,V_{ik}\,h_k\,{u'_i}^2\,u'_k}{u_{ij}^2\,u_{jk}\,u'_j\,h_j^3}-\frac{1}{576}\frac{V_{ij}\,V_{ik}\,u'_j\,u'_k}{u_{ik}\,u_{jk}^2\,h_j\,h_k}$$
$$+\frac{1}{1152}\frac{V_{ij}^2\,u'_i\left(37\,u'_i\,u'_j\,h_j^2+10\,u'_i\,u'_j\,h_i^2-3\,{u'_i}^2\,h_i^2+11\,{u'_j}^2\,h_j^2\right)}{u_{ij}\,u'_j\,h_i^2\,h_j^2}$$
$$-\frac{1}{576}\frac{V_{ij}\,h_j\left(4\,{u'_i}^3+4\,u'_i\,{u'_j}^2-6\,{u'_i}^2\,u'_j-{u'_j}^3\right)}{u_{ij}^3\,u'_i\,h_i^3}+\frac{1}{576}\frac{V_{ij}\,u'_i\,u'_j}{u_{ij}^3\,h_i\,h_j}. \tag{3.10.114}$$

A summation over repeated indices is assumed in each term of the formula provided the denominators do not vanish.

Example 3.10.33 *For the* $\mathbf{CP}^1$ *model defined by the potential (3.6.57) the canonical coordinates are*

$$u_1=v_1+2\,\exp(\frac{v_2}{2}),\quad u_2=v_1-2\,\exp(\frac{v_2}{2}). \tag{3.10.115}$$

The functions h_1, h_2 *and the matrix* V *have the form*

$$h_1=\frac{\sqrt{2}}{\sqrt{u_1-u_2}},\quad h_2=-\frac{\sqrt{2}\,i}{\sqrt{u_1-u_2}},$$
$$V=\frac{i}{2}\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

By substituting the above formulae for the functions h_i, V_{ij} *into the formula of genus two free energy for general semisimple Frobenius manifolds, we get the following expression* *for the genus two free energy of the* $\mathbf{CP}^1$ *model:*

$$
\begin{aligned}
24^2\,\mathcal{F}_2 = &\frac{4\,u_1''^3\,u_{12}}{5\,{u_1'}^4} - \frac{4\,u_2''^3\,u_{12}}{5\,{u_2'}^4} - \frac{u_1''\,u_2''}{4\,u_1'\,u_2'}\\
&+\frac{3\,u_1''}{4\,{u_1'}^3}\left(\frac{1}{2}\,u_1''\,u_2' - \frac{7}{5}\,u_1'''\,u_{12}\right) + \frac{3\,u_2''}{4\,{u_2'}^3}\left(\frac{1}{2}\,u_2''\,u_1' + \frac{7}{5}\,u_2'''\,u_{12}\right)\\
&+\frac{1}{4\,{u_1'}^2}\left(\frac{33}{10}\,u_1''^2 - \frac{9}{10}\,u_1'''\,u_2' + \frac{1}{10}\,u_1''\,u_2'' + u_1^{IV}\,u_{12}\right)\\
&+\frac{1}{4\,{u_2'}^2}\left(\frac{33}{10}\,u_2''^2 - \frac{9}{10}\,u_2'''\,u_1' + \frac{1}{10}\,u_1''\,u_2'' - u_2^{IV}\,u_{12}\right)\\
&-\frac{1}{4\,u_1'}\left(\frac{17}{5}\,u_1''' + \frac{1}{2}\,u_2'''\right) - \frac{1}{4\,u_2'}\left(\frac{17}{5}\,u_2''' + \frac{1}{2}\,u_1'''\right)\\
&-\frac{1}{10\,u_{12}^2}\left(\frac{{u_1'}^3}{u_2'} + \frac{{u_2'}^3}{u_1'}\right) - \frac{1}{u_{12}^2}\left({u_1'}^2 - \frac{11}{5}\,u_1'\,u_2' + {u_2'}^2\right)\\
&+\frac{u_1'' - u_2''}{u_{12}}\left(\frac{u_2'}{5\,u_1'} + \frac{u_1'}{5\,u_2'} + 1\right).
\end{aligned}
\tag{3.10.116}
$$

Here we denote $u_{12} = u_1 - u_2$ *as above.*

Let us now explain how to use the above formula for computation of the genus 2 Gromov - Witten invariants and their descendents in the $\mathbf{CP}^1$ *model. We must substitute into the formula for* $\mathcal{F}_2$ *the genus zero two point correlation functions*

$$
v_\alpha = \frac{\partial^2}{\partial t^{1,0}\partial t^{\alpha,0}}\langle \exp(\sum \tau_p(\phi_\beta)t^{\beta,p})\rangle_0, \quad \alpha = 1, 2,
\tag{3.10.117}
$$

where $\tau_p(\phi_\alpha), p \geq 0, \alpha = 1, 2$ *are the gravitational descendents of the primary fields* $\phi_1 = 1, \phi_2$ *of the* $\mathbf{CP}^1$ *model. The expansion of these two point correlation functions in power serieses of* $t^{\alpha,p}$ *can be obtained by solving the following equations (see Section 3.6.4)*

$$
v_\alpha(t) = \sum_{p\geq 0}\sum_{\beta=1}^{2} t^{\beta,q}\frac{\partial\theta_{\beta,q}}{\partial v_\gamma}, \quad \alpha = 1, 2.
\tag{3.10.118}
$$

The generating function for $\theta_{\beta,q}$ *is given in (3.6.60), (3.6.61), we list few of them*

$$
\begin{aligned}
&\theta_{1,0} = v_2, \quad \theta_{2,0} = v_1,\\
&\theta_{1,1} = v_1\,v_2, \quad \theta_{2,1} = e^{v_2} + \frac{1}{2}\,v_1^2,\\
&\theta_{1,2} = \frac{1}{2}\,v_1^2\,v_2 + v_2\,e^{v_2} - 2\,e^{v_2},\\
&\theta_{2,2} = \frac{1}{6}\,v_1^3 + v_1\,e^{v_2},\\
&\theta_{1,3} = \frac{1}{6}\,v_1^3\,v_2 + v_1\,v_2\,e^{v_2} - 2\,v_1\,e^{v_2},\\
&\theta_{2,3} = \frac{1}{24}\,v_1^4 + \frac{1}{2}\,v_1^2\,e^{v_2} + \frac{1}{4}\,e^{2\,v_2}.
\end{aligned}
\tag{3.10.119}
$$

The expansion of the correlation functions of (3.10.117) is the unique solution of the form

$$v_\alpha(t) = t^{\alpha,0} + \sum_{k\geq 1,\, p_i \geq 1} A^\alpha_{\beta_1,q_1;\ldots;\beta_k,q_k}(t^{1,0},t^{2,0})\, t^{\beta_1,q_1}\ldots t^{\beta_k,q_k}$$

of the equations given in (3.10.118), the coefficients are determined recursively by (3.10.118). For example, we have

$$A_{\beta,q} = \left.\frac{\partial\theta_{\beta,q}}{\partial v_\alpha}\right|_{v_\gamma=t^{\gamma,0}}, \quad A_{\beta_1,q_1;\beta_2,q_2} = \left.\frac{1}{2}\frac{\partial^2\theta_{\beta_1,q_1}}{\partial v_\alpha \partial v_\gamma}\frac{\partial\theta_{\beta_2,q_2}}{\partial v_\gamma}\right|_{v_\xi=t^{\xi,0}}.$$

Taking the expansions of v_α up to the 6-th order of $t^{\alpha,p}$, we obtain the following expansions for u_1, u_2 defined in (3.10.115):

$$\begin{aligned}
u_1' &= 1 + t^{1,1} + (t^{1,1})^2 + t^{1,0}\,t^{1,2} - (t^{1,2})^2 - t^{1,3} - 2\,t^{1,1}\,t^{1,3} - 2\,t^{1,2}\,t^{1,3} + (t^{1,3})^2 \\
&\quad + t^{1,2}\,t^{2,0} + t^{2,1} + 2\,t^{1,1}\,t^{2,1} + t^{1,2}\,t^{2,1} - 3\,t^{1,3}\,t^{2,1} + \frac{t^{2,0}\,t^{2,1}}{2} + (t^{2,1})^2 + t^{2,2} \\
&\quad + t^{1,0}\,t^{2,2} + 2\,t^{1,1}\,t^{2,2} - 4\,t^{1,3}\,t^{2,2} + t^{2,0}\,t^{2,2} + \frac{7\,t^{2,1}\,t^{2,2}}{2} + 2\,(t^{2,2})^2 + t^{2,3} + t^{1,0}\,t^{2,3} \\
&\quad + 2\,t^{1,1}\,t^{2,3} - \frac{t^{1,2}\,t^{2,3}}{2} - 5\,t^{1,3}\,t^{2,3} + \frac{3\,t^{2,0}\,t^{2,3}}{2} + 3\,t^{2,1}\,t^{2,3} + 4\,t^{2,2}\,t^{2,3} \\
&\quad + \frac{3\,(t^{2,3})^2}{2} + \mathcal{O}(t^3), \\
u_1'' &= t^{1,2} + 3\,t^{1,1}\,t^{1,2} + t^{1,0}\,t^{1,3} - 4\,t^{1,2}\,t^{1,3} - 2\,(t^{1,3})^2 + t^{1,3}\,t^{2,0} + 3\,t^{1,2}\,t^{2,1} \\
&\quad + t^{1,3}\,t^{2,1} + \frac{(t^{2,1})^2}{2} + t^{2,2} + 3\,t^{1,1}\,t^{2,2} + 3\,t^{1,2}\,t^{2,2} - 3\,t^{1,3}\,t^{2,2} + \frac{t^{2,0}\,t^{2,2}}{2} + 3\,t^{2,1}\,t^{2,2} \\
&\quad + \frac{7\,(t^{2,2})^2}{2} + t^{2,3} + t^{1,0}\,t^{2,3} + 3\,t^{1,1}\,t^{2,3} + 2\,t^{1,2}\,t^{2,3} - \frac{9\,t^{1,3}\,t^{2,3}}{2} + t^{2,0}\,t^{2,3} + 5\,t^{2,1}\,t^{2,3} \\
&\quad + 7\,t^{2,2}\,t^{2,3} + 4\,(t^{2,3})^2 + \mathcal{O}(t^3), \\
u_1''' &= 3\,(t^{1,2})^2 + t^{1,3} + 4\,t^{1,1}\,t^{1,3} - 4\,(t^{1,3})^2 + 4\,t^{1,3}\,t^{2,1} + 6\,t^{1,2}\,t^{2,2} + 4\,t^{1,3}\,t^{2,2} \\
&\quad + \frac{3\,t^{2,1}\,t^{2,2}}{2} + 3\,(t^{2,2})^2 + t^{2,3} + 4\,t^{1,1}\,t^{2,3} + 6\,t^{1,2}\,t^{2,3} - t^{1,3}\,t^{2,3} + \frac{t^{2,0}\,t^{2,3}}{2} + 4\,t^{2,1}\,t^{2,3} \\
&\quad + 12\,t^{2,2}\,t^{2,3} + 7\,(t^{2,3})^2 + \mathcal{O}(t^3), \\
u_1^{IV} &= 10\,t^{1,2}\,t^{1,3} + 10\,t^{1,3}\,t^{2,2} + \frac{3\,(t^{2,2})^2}{2} + 10\,t^{1,2}\,t^{2,3} + 10\,t^{1,3}\,t^{2,3} + 2\,t^{2,1}\,t^{2,3} \\
&\quad + 10\,t^{2,2}\,t^{2,3} + 12\,(t^{2,3})^2 + \mathcal{O}(t^3), \\
u_2' &= 1 + t^{1,1} + (t^{1,1})^2 + t^{1,0}\,t^{1,2} - (t^{1,2})^2 - t^{1,3} - 2\,t^{1,1}\,t^{1,3} + 2\,t^{1,2}\,t^{1,3} + (t^{1,3})^2 \\
&\quad - t^{1,2}\,t^{2,0} - t^{2,1} - 2\,t^{1,1}\,t^{2,1} + t^{1,2}\,t^{2,1} + 3\,t^{1,3}\,t^{2,1} - \frac{t^{2,0}\,t^{2,1}}{2} + (t^{2,1})^2 + t^{2,2} \\
&\quad - t^{1,0}\,t^{2,2} + 2\,t^{1,1}\,t^{2,2} - 4\,t^{1,3}\,t^{2,2} + t^{2,0}\,t^{2,2} - \frac{7\,t^{2,1}\,t^{2,2}}{2} + 2\,(t^{2,2})^2 - t^{2,3} + t^{1,0}\,t^{2,3} \\
&\quad - 2\,t^{1,1}\,t^{2,3} - \frac{t^{1,2}\,t^{2,3}}{2} + 5\,t^{1,3}\,t^{2,3} - \frac{3\,t^{2,0}\,t^{2,3}}{2} + 3\,t^{2,1}\,t^{2,3}
\end{aligned}$$

$$-4\,t^{2,2}\,t^{2,3}+\frac{3\,(t^{2,3})^2}{2}+\mathcal{O}(t^3),$$

$$u_2''=t^{1,2}+3\,t^{1,1}\,t^{1,2}+t^{1,0}\,t^{1,3}-4\,t^{1,2}\,t^{1,3}+2\,(t^{1,3})^2-t^{1,3}\,t^{2,0}-3\,t^{1,2}\,t^{2,1}$$
$$+t^{1,3}\,t^{2,1}-\frac{(t^{2,1})^2}{2}-t^{2,2}-3\,t^{1,1}\,t^{2,2}+3\,t^{1,2}\,t^{2,2}+3\,t^{1,3}\,t^{2,2}-\frac{t^{2,0}\,t^{2,2}}{2}+3\,t^{2,1}\,t^{2,2}$$
$$-\frac{7\,(t^{2,2})^2}{2}+t^{2,3}-t^{1,0}\,t^{2,3}+3\,t^{1,1}\,t^{2,3}-2\,t^{1,2}\,t^{2,3}-\frac{9\,t^{1,3}\,t^{2,3}}{2}+t^{2,0}\,t^{2,3}-5\,t^{2,1}\,t^{2,3}$$
$$+7\,t^{2,2}\,t^{2,3}-4\,(t^{2,3})^2+\mathcal{O}(t^3),$$

$$u_2'''=3\,(t^{1,2})^2+t^{1,3}+4\,t^{1,1}\,t^{1,3}-4\,(t^{1,3})^2-4\,t^{1,3}\,t^{2,1}-6\,t^{1,2}\,t^{2,2}+4\,t^{1,3}\,t^{2,2}$$
$$-\frac{3\,t^{2,1}\,t^{2,2}}{2}+3\,(t^{2,2})^2-t^{2,3}-4\,t^{1,1}\,t^{2,3}+6\,t^{1,2}\,t^{2,3}+t^{1,3}\,t^{2,3}-\frac{t^{2,0}\,t^{2,3}}{2}+4\,t^{2,1}\,t^{2,3}$$
$$-12\,t^{2,2}\,t^{2,3}+7\,(t^{2,3})^2+\mathcal{O}(t^3),$$

$$u_2^{IV}=10\,t^{1,2}\,t^{1,3}-10\,t^{1,3}\,t^{2,2}-\frac{3\,(t^{2,2})^2}{2}-10\,t^{1,2}\,t^{2,3}+10\,t^{1,3}\,t^{2,3}$$
$$-2\,t^{2,1}\,t^{2,3}+10\,t^{2,2}\,t^{2,3}-12\,(t^{2,3})^2+\mathcal{O}(t^3),$$

$$u_{12}=t^{1,2}+3\,t^{1,1}\,t^{1,2}+t^{1,0}\,t^{1,3}-4\,t^{1,2}\,t^{1,3}-2\,(t^{1,3})^2+t^{1,3}\,t^{2,0}+3\,t^{1,2}\,t^{2,1}+t^{1,3}\,t^{2,1}$$
$$+\frac{(t^{2,1})^2}{2}+t^{2,2}+3\,t^{1,1}\,t^{2,2}+3\,t^{1,2}\,t^{2,2}-3\,t^{1,3}\,t^{2,2}+\frac{t^{2,0}\,t^{2,2}}{2}+3\,t^{2,1}\,t^{2,2}+\frac{7\,(t^{2,2})^2}{2}$$
$$+t^{2,3}+t^{1,0}\,t^{2,3}+3\,t^{1,1}\,t^{2,3}+2\,t^{1,2}\,t^{2,3}-\frac{9\,t^{1,3}\,t^{2,3}}{2}+t^{2,0}\,t^{2,3}+5\,t^{2,1}\,t^{2,3}$$
$$+7\,t^{2,2}\,t^{2,3}+4\,(t^{2,3})^2+\mathcal{O}(t^3).$$

In the r.h.s. of the above formulae we only keep the terms up to the quadratic ones in $t^{\alpha,p}$ with $p\le 3$. The function $\mathcal{F}_2$ thus has the following expansion:

$$\mathcal{F}_2=-\frac{t^{1,3}}{240}+\frac{7\,t^{2,2}}{5760}-\frac{5\,(t^{1,2})^2}{576}-\frac{t^{1,1}\,t^{1,3}}{80}+\frac{29\,(t^{1,3})^2}{2880}+\frac{7\,t^{1,3}\,t^{2,0}}{5760}+\frac{7\,t^{1,2}\,t^{2,1}}{1920}$$
$$+\frac{7\,t^{1,1}\,t^{2,2}}{1920}+\frac{t^{1,3}\,t^{2,2}}{192}+\frac{(t^{2,2})^2}{1152}+\frac{7\,t^{1,0}\,t^{2,3}}{5760}+\frac{t^{1,2}\,t^{2,3}}{192}+\frac{25\,(t^{2,3})^2}{2304}+\mathcal{O}(t^3).$$

Again the above formula for F_2 is at the approximation up to quadratic terms in $t^{\alpha,p}$ with $p\le 3$. The above expansion of the function $\mathcal{F}_2$ coincides, to the extend of the above mentioned approximation, to the generating function of the genus two Gromov-Witten invariants for the $\mathbf{CP}^1$ model defined by

$$\begin{aligned}\tilde{\mathcal{F}}_2&=\sum_{n\ge 0}\frac{1}{k!}\langle(\sum_{p\ge 0}\sum_{\alpha=1}^{2}\tau_p(\phi_\alpha)t^{\alpha,p})^k\rangle_2\\&=1+\sum_{p\ge 0}\sum_{\alpha=1}^{2}\langle\tau_p(\phi_\alpha)\rangle_2 t^{\alpha,p}\\&\quad+\frac{1}{2}\sum_{p,q\ge 0}\sum_{\alpha,\beta=1}^{2}\langle\tau_p(\phi_\alpha)\tau_q(\phi_\beta)\rangle_2 t^{\alpha,p}\,t^{\beta,q}+\mathcal{O}(t^3).\end{aligned}\qquad(3.10.120)$$

A list of some of the invariants can be found in [135], the numbers $\langle \tau_{k,i} \rangle_2$ *and* $\langle \tau_{k,i}\, \tau_{m,j} \rangle_2$ *there correspond to* $\langle \tau_k(\phi_{i+1}) \rangle_2$ *and* $\langle \tau_k(\phi_{i+1}) \tau_m(\phi_{j+1}) \rangle_2$ *respectively. For example,*

$$
\begin{aligned}
&\langle \tau_3(\phi_1) \rangle_2 = -\frac{1}{240}, \quad \langle \tau_2(\phi_2) \rangle_2 = \frac{7}{5760}, \\
&\langle \tau_2(\phi_1) \tau_2(\phi_1) \rangle_2 = -\frac{5}{288}, \quad \langle \tau_3(\phi_1) \tau_3(\phi_1) \rangle_2 = \frac{29}{1440}, \\
&\langle \tau_2(\phi_1) \tau_1(\phi_2) \rangle_2 = \frac{7}{1920}, \quad \langle \tau_3(\phi_1) \tau_2(\phi_2) \rangle_2 = \langle \tau_2(\phi_1) \tau_3(\phi_2) \rangle_2 = \frac{1}{192}, \\
&\langle \tau_2(\phi_2) \tau_2(\phi_2) \rangle_2 = \frac{1}{576}, \quad \langle \tau_3(\phi_2) \tau_3(\phi_2) \rangle_2 = \frac{25}{1152}.
\end{aligned} \tag{3.10.121}
$$

Example 3.10.34 *The expansion of the function* $\mathcal{F}_2$ *for the* $\mathbf{CP}^2$ *model can be obtained in a similar way as we did for the* $\mathbf{CP}^1$ *model. The canonical coordinates* u_i *are the roots of the characteristic polynomial of the matrix correspondent to the operation of multiplication by the Euler vector field, express* u_i *in the form*

$$
u_i = v_1 + v_3^{-1}\, z_i(y), \quad i = 1, 2, 3,
$$

where $y = v_2 + 3 \log v_3$. *Then the functions* $z_i(y)$ *are roots of the following cubic polynomial:*

$$
\begin{aligned}
&z^3 - f''\, z^2 + (6\, f - 15\, f' - 9\, f'')\, z - 54\, f + 243\, f' \\
&+4\, f'^2 - 243\, f'' - 6\, f\, f'' + 3\, f'\, f'' + 9\, f''^2 = 0.
\end{aligned} \tag{3.10.122}
$$

Here $f = f(y)$ *is the generating function of the genus zero Gromov - Witten invariants of* $\mathbf{CP}^2$, *i.e.*

$$
f(y) = \sum_{k \geq 1} \frac{N_k^{(0)}}{(3k-1)!} e^{ky}
$$

with $N_k^{(0)}$ *being the number of rational curves of degree* k *on* $\mathbf{CP}^2$ *passing through* $3k-1$ *generic points. The matrix* $\Psi = (\psi_{i,\alpha})$ *is given by*

$$
\begin{aligned}
\psi_{i,1} &= a_i\, v_3 \sqrt{f'' + 9 - z_i}, \\
\psi_{i,2} &= \frac{a_i\, (3 z_i + 2 f' - 6 f'')}{\sqrt{f'' + 9 - z_i}}, \\
\psi_{i,3} &= \frac{a_i\, (z_i^2 - f''\, z_i + 6 f' - 18 f'')}{v_3 \sqrt{f'' + 9 - z_i}},
\end{aligned} \tag{3.10.123}
$$

where the coefficents a_i *are defined by*

$$
a_i = \frac{1}{\sqrt{\prod_{k \neq i} (z_i - z_k)}}
$$

The matrix $V = (V_{ij})$ has the following expression

$$V = \Psi\,\mu\,\Psi^{-1}, \quad \mu = diag(-1, 0, 1).$$

The genus zero two point functions v_1, v_2, v_3 can be obtained by solving (3.6.78). Thus we have all the data to compute the expansion of the function $\mathcal{F}_2$ for the $\mathbf{CP}^2$ model.

Let us now consider the deformations of the Principal Hierarchy induced by the quasitriviality transformation (3.10.49) satisfying the loop equation (3.10.66). It remains an open problem to prove that, at every order in the ϵ^2-expansion the coefficients of the bihamiltonian structure are *polynomials in the derivatives*. For the first term in the ϵ^2-expansion this was proved in [51]. At the next approximation the calculations are more complicated. We consider here bihamiltonian structures corresponding to two-dimensional Frobenius manifolds (3.6.55).

As we already know from Example 3.6.9 of Section 3.6.2 for generic two-dimensional Frobenius manifold

$$F(v_1, v_2) = \frac{1}{2} v_1^2 v_2 + \frac{v_2^{\kappa+1}}{\kappa^2 - 1}$$

the $t = -t^{1,1}$-flow

$$v_t + v \cdot v_x = 0$$

of the Principal Hierarchy coincides with the equations of motion of 1D polytropic gas with the equation of state of the form

$$p = \frac{\kappa\,\rho^{k+1}}{\kappa + 1}$$

(we redenote as above $v_1 = u$, $v_2 = \rho$).

Our construction produces the following deformation of the polytropic gas equations

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \partial_x \Bigg\{ \frac{u^2}{2} + \rho^\kappa \\
&+\epsilon^2 \left[\frac{\kappa - 2}{8} \rho^{\kappa-3}\,\rho_x^2 + \frac{\kappa}{12}\,\rho^{\kappa-2} \rho_{xx} \right] \\
&+\epsilon^4 (\kappa - 2)(\kappa - 3) \big[a_1\,\rho^{-4}\,u_x^2\,\rho_x^2 + a_2\,\rho^{\kappa-6}\,\rho_x^4 \\
&\quad + a_3\,\rho^{-3}\,u_{xx}\,u_x\,\rho_x + a_4\,\rho^{-2}\,u_{xx}^2 + a_5\,\rho^{-3}\,u_x^2\,\rho_{xx} \\
&\quad + a_6\,\rho^{\kappa-5}\,\rho_x^2\,\rho_{xx} + a_7\,\rho^{\kappa-4}\,\rho_{xx}^2 + a_8\,\rho^{-2}\,u_x\,u_{xxx} \\
&\quad + a_9\,\rho^{\kappa-4}\,\rho_x\,\rho_{xxx} \big] + \epsilon^4 \frac{\kappa\,(\kappa^2 - 1)\,(\kappa^2 - 4)}{360} \rho^{\kappa-3}\,\rho_{xxxx} \Bigg\} = O(\epsilon^6),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \partial_x \Bigg\{ \rho\,u + \epsilon^2 \left(\frac{(2-\kappa)(\kappa - 3)}{12\,\kappa\,\rho}\,u_x\,\rho_x + \frac{1}{6}\,u_{xx} \right) \\
&+\epsilon^4 (\kappa - 2)(\kappa - 3) \big[b_1\,\rho^{-4}\,u_x\,\rho_x^3 + b_2\,\rho^{-3}\,\rho_x^2\,u_{xx} + b_3\,\rho^{-3}\,u_x\,\rho_x\,\rho_{xx} \\
&+ b_4\,\rho^{-2}\,u_{xx}\,\rho_{xx} + b_5\,\rho^{-2}\,u_{xxx}\,\rho_x + b_6\,\rho^{-2}\,u_x\,\rho_{xxx} + b_7\,\rho^{-1}\,u_{xxxx} \big] \Bigg\} \\
&= O(\epsilon^6).
\end{aligned}$$

integrable up to corrections of the order $O(\epsilon^6)$. The coefficients are given below

$$a_1 = \frac{36 + 144\,\kappa - 59\,\kappa^2 + 19\,\kappa^3}{5760\,\kappa^3}, \quad a_2 = \frac{60 + 176\,\kappa + 433\,\kappa^2 - 182\,\kappa^3 + 17\,\kappa^4}{5760\,\kappa^3}$$

$$a_3 = \frac{6 - 19\,\kappa - 11\,\kappa^2 - 4\,\kappa^3}{1440\,\kappa^3}, \quad a_4 = \frac{-6 - 5\,\kappa + 13\,\kappa^2}{1440\,\kappa^3}, \quad a_5 = \frac{-42 + 13\,\kappa - 7\,\kappa^2}{2880\,\kappa^2}$$

$$a_6 = \frac{-36 - 72\,\kappa - 245\,\kappa^2 - 61\,\kappa^3 + 30\,\kappa^4}{2880\,\kappa^2}, \quad a_7 = \frac{6 + 5\,\kappa + 15\,\kappa^2 + 5\,\kappa^3 + 5\,\kappa^4}{1440\,\kappa^2}$$

$$a_8 = \frac{1}{120\,\kappa}, \quad a_9 = \frac{2 + 5\,\kappa}{240}$$

$$b_1 = \frac{108 + 192\,\kappa - 97\,\kappa^2 + 17\,\kappa^3}{2880\,\kappa^3}, \quad b_2 = \frac{-18 - 75\,\kappa + 47\,\kappa^2 - 10\,\kappa^3}{1440\,\kappa^3}$$

$$b_3 = -\frac{6 + 17\,\kappa - 5\,\kappa^2 + 2\,\kappa^3}{288\,\kappa^3}, \; b_4 = \frac{6 - 4\,\kappa + \kappa^2}{180\,\kappa^2}, \; b_5 = \frac{6 + \kappa + \kappa^2}{720\,\kappa^2}$$

$$b_6 = \frac{6 + \kappa + \kappa^2}{720\,\kappa^2}, \; b_7 = -\frac{1}{360\,\kappa} \tag{3.10.124}$$

The polytropic gas equations can be considered as the dispersionless limit of the above integrable system.

The bihamiltonian structure of the above system will be written in the coordinates where the first Poisson bracket has the canonical form (3.10.124) up to $O(\epsilon^6)$. These coordinates $\bar{v}_1$, $\bar{v}_2$ are given by the following Miura-type transformation (recall $v_1 = u$, $v_2 = \rho$; we do not distinguish between upper and lower indices in these formulae)

$$\bar{v}_\alpha = v_\alpha + \epsilon^2\, A_\alpha(v, v', v'') + \epsilon^4\, B_\alpha(v, v', \ldots, v^{IV}), \; \alpha = 1,\, 2 \tag{3.10.125}$$

where

$$A_1 = \frac{(\kappa-2)(\kappa-3)}{24\,\kappa}\,\frac{\partial}{\partial x}\left(\frac{v_1'}{v_2}\right), \quad A^2 = \frac{(\kappa-2)(\kappa-3)}{24\,\kappa}\,\frac{\partial}{\partial x}\left(\frac{v_2'}{v_2}\right)$$

$$\begin{aligned} B_1 = \frac{(\kappa-2)(\kappa-3)}{240}\,\partial_x \Bigg[& \frac{\kappa^2 - 13\,\kappa + 6}{\kappa^2}\left(\frac{1}{4}\,\frac{v_1'''}{v_2^2} - \frac{1}{2}\,\frac{v_1''\,v_2'}{v_2^3} - \frac{7}{12}\,\frac{v_1'\,v_2''}{v_2^3}\right) \\ & + \frac{1}{6}\,\frac{5\,\kappa^3 - 54\,\kappa^2 + 25\,\kappa - 6}{\kappa^3}\,\frac{v_1'\,{v_2'}^2}{v_2^4} - \frac{1}{18}\,\frac{(\kappa+3)(\kappa+2)(\kappa+1)}{(\kappa-1)\,\kappa^4}\,\frac{{v_1'}^3}{v_2^{\kappa+2}} \Bigg]. \\ B_2 = 0 \end{aligned}$$

In the new coordinates our deformed bihamiltonian structure reads

$$\begin{aligned} \{\bar{v}^\alpha(x), \bar{v}^\beta(y)\}_1 &= \{\bar{v}^\alpha(x), \bar{v}^\beta(y)\}_1^{[0]} + \mathcal{O}(\epsilon^6), \\ \{\bar{v}^\alpha(x), \bar{v}^\beta(y)\}_2 &= \{\bar{v}^\alpha(x), \bar{v}^\beta(y)\}_2^{[0]} + \epsilon^2\,(d_1 X)^{\alpha\beta} + \epsilon^4\,(d_1 Y)^{\alpha\beta} + \mathcal{O}(\epsilon^6). \end{aligned}$$

with the vector fields X and Y defined by

$$X^1 = \frac{1}{24}\left[(3\,\kappa^2 - 7\,\kappa + 2)\,v_2^{\kappa-3}\,{v_2'}^2 + (4\kappa - 2)v_2^{\kappa-2}v_2''\right]$$

$$X^2 = \frac{\kappa+1}{12\,\kappa}v_1''$$

and

$$Y^1 = \frac{(\kappa-2)(\kappa-3)}{240}\left(\frac{g_1 v_1''' v_1'}{{v_2}^2} + \frac{g_2 {v_1''}^2}{{v_2}^2} + \frac{g_3 {v_1'}^2\, v_2''}{{v_2}^3} + \frac{g_4 {v_1'}^2 {v_2'}^2}{{v_2}^4} + \frac{g_5 {v_1'}^4}{{v_2}^{\kappa+2}} \right.$$
$$\left. + \frac{g_6\, {v_2}^\kappa\, v_2^{IV}}{{v_2}^3} + \frac{g_7\, {v_2}^\kappa {v_2''}^2}{{v_2}^4} + \frac{g_8\, {v_2}^\kappa\, v_2''\, {v_2'}^2}{{v_2}^5} + \frac{g_9\, {v_2}^\kappa\, {v_2'}^4}{{v_2}^6}\right)$$

$$Y^2 = \frac{(\kappa-2)(\kappa-3)}{240}\left(\frac{f_1\, v_1^{IV}}{v_2} + \frac{f_2\, v_1'''\, v_2'}{{v_2}^2} + \frac{f_3 v_1''\, v_2''}{{v_2}^2} - 2\, f_3\,\frac{v_1''\, {v_2'}^2}{{v_2}^3} + \frac{f_4 v_1'\, v_2'''}{{v_2}^2} \right.$$
$$\left. -6\, f_4\,\frac{v_1'\, v_2''\, v_2'}{{v_2}^3} + 6\, f_4\,\frac{v_1'\, {v_2'}^3}{{v_2}^4}\right),$$

where

$$g_1 = -\frac{1}{24}\,\frac{9\,\kappa^3 - 120\,\kappa^2 + 101\,\kappa + 30}{\kappa^3}$$

$$g_2 = -\frac{1}{12}\,\frac{6\,\kappa^3 - 81\,\kappa^2 + 79\,\kappa + 6}{\kappa^3}$$

$$g_3 = -\frac{1}{12}\,\frac{8\,\kappa^3 - 109\,\kappa^2 + 117\,\kappa - 6}{\kappa^3}$$

$$g_4 = \frac{1}{24}\,\frac{37\,\kappa^4 - 478\,\kappa^3 + 461\,\kappa^2 + 28\,\kappa + 12}{\kappa^4}$$

$$g_5 = -\frac{1}{72}\,\frac{(\kappa+3)(\kappa+2)(\kappa^2-1)}{\kappa^5}$$

$$g_6 = \frac{1}{4}\,\frac{9\,\kappa^3 - 12\,\kappa^2 + 41\,\kappa - 18}{\kappa^2\,(\kappa-3)}$$

$$g_7 = -\frac{1}{24}\,\frac{43\,\kappa^3 - 30\,\kappa^2 + 197\,\kappa - 90}{\kappa^2}$$

$$g_8 = -\frac{1}{12}\,\frac{63\,\kappa^4 - 287\,\kappa^3 + 492\,\kappa^2 - 1258\,\kappa + 540}{\kappa^2}$$

$$g_9 = -\frac{1}{72}\,\frac{78\,\kappa^6 - 732\,\kappa^5 + 2268\,\kappa^4 - 4393\,\kappa^3 + 8340\,\kappa^2 - 3395\,\kappa - 6}{\kappa^3}$$

$$f_1 = -\frac{7\,\kappa^2 - 35\,\kappa + 18}{12\,\kappa^3}$$

$$f_2 = -\frac{21\,\kappa^3 - 312\,\kappa^2 + 529\,\kappa - 138}{24\,\kappa^3}$$

$$f_3 = -\frac{21\,\kappa^3 - 291\,\kappa^2 + 424\,\kappa - 84}{12\,\kappa^3}$$

$$f_4 = -\frac{21\,\kappa^3 - 284\,\kappa^2 + 389\,\kappa - 66}{24\,\kappa^3}$$

For the particular value $\kappa = 2$ the above formulae truncate at the order ϵ^2. We obtain a system of two uncoupled KdVs

$$\dot{u}_\pm + u_\pm u'_\pm \pm \frac{\epsilon^2}{12\sqrt{2}} u'''_\pm = 0$$

for

$$u_\pm = u \pm \rho\sqrt{2}.$$

For $\kappa = 3$ the formulae (3.10.126) define a Poisson pencil. That is, the Jacobi identity becomes exact but not just modulo $O(\epsilon^6)$. One obtains the bihamiltonian structure for the Boussinesq hierarchy (see details in [51]). The Boussinesq equation itself for the unknown function $v_2 = v_2(x,t)$ is the spelling of the $t^{2,0}$-flow of the full hierarchy.

The last interesting particular value of the parameter κ is $\kappa = 1$. The formula for the potential of the Frobenius manifold must be modified

$$F = \frac{1}{2} v_1^2 v_2 + \frac{1}{2} v_2^2 \log v_2. \tag{3.10.126}$$

However, the formula for the $-t^{1,1}$-flow of the hierarchy and the bihamiltonian structure are well defined for this particular value of the parameter. The PDE (3.10.124) coinicides, in the leading approximation in ϵ, with the nonlinear Schrödinger equation (NLS)

$$i\,\psi_t = -\frac{1}{2}\psi_{xx} + |\psi|^2\psi \tag{3.10.127}$$

when using

$$\begin{aligned} u &= \frac{1}{2i}\left(\log \psi/\bar{\psi}\right)_x \\ \rho &= |\psi|^2 \end{aligned} \tag{3.10.128}$$

as the new dependent variables

$$\begin{aligned} &u_t + \partial_x\left[\frac{u^2}{2} + \rho + \epsilon^2\left(\frac{\rho'^2}{8\,\rho^2} - \frac{\rho''}{4\,\rho}\right)\right] = 0 \\ &\rho_t + \partial_x(\rho\, u) = 0. \end{aligned} \tag{3.10.129}$$

The relationship of the bihamiltonian structure of NLS to topological field theory was discussed in [9] (see also recent paper [14, 15]). To reduce the bihamiltonian structure of NLS to our normal form one is to perform a Miura-type transformation that contains an infinite number of terms in the ϵ-expansion. We will consider this transformation in a subsequent publication.

The above formulae work also for the exceptional value $\kappa = -1$ where the potential of the Frobenius manifold reads

$$F = \frac{1}{2} v_1^2 v_2 - \frac{1}{2} \log v_2.$$

It remains to consider the Frobenius manifold (3.6.57) of the $\mathbf{CP}^1$ model. The transformation reducing the first Poisson structure to the canonical form in this case becomes very simple

$$v_\alpha \mapsto \bar{v}_\alpha = v_\alpha + \frac{\epsilon^2}{24} v_\alpha'' + \frac{\epsilon^4}{1920} v_\alpha^{IV} + O(\epsilon^6), \quad \alpha = 1, 2. \tag{3.10.130}$$

The Poisson pencil has the form (3.10.126) with

$$X^1 = \left(\frac{1}{6} v_2'' + \frac{1}{8} {v_2'}^2\right) e^{v_2}, \quad X^2 = \frac{1}{12} v_1''$$
$$Y^1 = \left(\frac{1}{120} v_2^{IV} + \frac{1}{96} v_2' v_2''' + \frac{1}{288} {v_2''}^2 - \frac{1}{1152} {v_2'}^4\right) e^{v_2}$$
$$Y^2 = -\frac{1}{720} v_1^{IV}.$$

The resulting Poisson pencil is equivalent, within the order ϵ^4 approximation, to the bihamiltonian structure of Toda lattice [153].

4 Conclusions

In this paper we presented a new approach to the problem of classification of hierarchies of 1+1-dimensional integrable PDEs of certain class. We have formulated a system of simple axioms that can be used as defining relations in the theory of such hierarchies. We developed an efficient tools for a perturbative reconstruction of the hierarchy starting from a given semisimple Frobenius manifold. We computed the first few terms of the perturbative expansion and showed that, for low genera the identities for the tau-function of a particular solution to the hierarchy correctly reproduce the universal identities among Gromov - Witten classes and their descendents in the cohomology of the moduli spaces of stable algebraic curves.

Of course, our system of axioms can probably be improved (in particular, independence of the quasitriviality axiom is questionable). However, as the reader may agree, it is something more than just putting Gromov - Witten invariants into the game in the beginning in order to get them back at the very end.

Some important problems of our classification programme remain to be fixed. We did not prove polynomiality in the derivatives in every term of the perturbative expansion of the integrable hierarchy corresponding to an arbitrary semisimple Frobenius manifold. For $g \leq 1$ this follows from the results of [51]. We are to also check whether main examples of integrable hierarchies (e.g., Drinfeld - Sokolov hierarchies [32] of the ADE type, Toda lattice etc.) satisfy our four axioms. In the present paper we did it only for the illustrative example of KdV.

A more challenging problem is to find a “non-perturbative construction” of the integrable hierarchies in question performing a summation over all genera. We hope that the structure theory of semisimple Frobenius manifolds based on a Riemann - Hilbert problem [41] will be important for such a summation. A related problem is to study relationships between our classification approach and topological results of the very recent paper of A. Givental [79]. It might happen that a reasonable non-perturbative construction of the hierarchy can be obtained only for certain particularly "nice" Frobenius manifolds. Recall [41], that generic n-dimensional Frobenius manifold depends on $\frac{n(n-1)}{2}$ complex parameters. Mathematically or physically interesting integrable hierarchies might correspond to special points in the space of parameters. For example, from results of C. Hertling [83] it can probably be derived that integrable hierarchies satisfying our axioms that are polynomial also in u must correspond to the Frobenius structures on the orbit space of Weyl groups of the ADE type. However, just the universality of our approach suggests to apply it to other ingredients of the theory of integrable systems, first of all to the theory of classical and quantum W-algebras [151, 61, 21] and, more generally, of vertex Poisson algebras and vertex algebras [63] in order to unravel the eventual role of the topology of moduli spaces of stable algebraic curves in the theory of these algebraic structures.

From the point of view of applications to the theory of dispersive waves the integrable systems we are constructing may look ugly. Indeed, they typically contain infinite number of terms of the dispersion in the r.h.s. However, one can obtain an *approximately integrable system* just truncating the expansion at some order in ϵ. We expect that solutions to such an approximately integrable system exhibit an integrable behaviour within a certain range of physical parameters. For $n = 1$ such approximately integrable systems were studied in [103] (only the bihamiltonian property has been used). In numerical experiments elastic scattering of solitons was observed in such systems. It could be of interest to study numerically our approximately integrable deformation of the one-dimensional gas dynamic equations constructed in Section 3.10.8.

Finally, the problem of classification of Poisson pencils of (p, q)-brackets with $(p, q) \neq (0, n)$ and study of the properties of associated bihamiltonian hierarchies is still completely open. The $(n, 0)$ case was recently studied by P. Lorenzoni (unpublished) where preliminary results were obtained. Such a classification is needed to extend our general classification strategy onto other classes of integrable systems not admitting dispersionless limit (e.g., integrable systems of the Sine-Gordon type).

We plan to study these problems in subsequent publications.

Note added. After finishing this work we have received the paper of S. Barannikov, Semi-infinite variations of Hodge structures and integrable hierarchies of KdV-type, math.AG/0108148. The author also presents a construction of a bihamiltonian hierarchy associated to an arbitrary semisimple Frobenius manifold. Virasoro symmetries of this hierarchy were not discussed. Relations of the Barannikov's hierarchy to Gromov - Witten invariants is still to be investigated.

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Appendix: Degenerate Frobenius manifolds, spectral curves, and Poisson pencils

Let M be a degenerate Frobenius manifold, $F(v)$ the corresponding solution to the equations of associativity, $e = \partial_1$ and $E = E^\alpha \partial_\alpha$ the unity and the Euler vector fields on M resp. satisfying

$$\partial_E F(v) = k\, F(v) + \text{quadratic}, \quad [e, E] = 0. \tag{A.1}$$

We introduce the following linear operator on TM

$$\mathcal{V} := \frac{k}{2} - \nabla E \tag{A.2}$$

(cf. (3.6.5) above). This (1,1)-tensor is constant in the flat coordinates on M.

Lemma A.1 *The following equations hold true*

$$Lie_E < \, , \, >= k < \, , \, > \tag{A.3}$$

$$Lie_E(a \cdot b) = Lie_E a \cdot b + a \cdot Lie_E b. \tag{A.4}$$

Proof Differentiating (A.1) along ∂_1, ∂_α, ∂_β one obtains

$$< \mathcal{V}x, y > + < x, \mathcal{V}y >= 0 \tag{A.5}$$

for any two vectors x, y. This implies (A.3). Differentiating the same equation along ∂_α, ∂_β, ∂_γ we obtain

$$\partial_E c_{\alpha\beta\gamma} = -\frac{k}{2}\, c_{\alpha\beta\gamma} + \mathcal{V}_\alpha^\epsilon c_{\epsilon\beta\gamma} + \mathcal{V}_\beta^\epsilon c_{\alpha\epsilon\gamma} + \mathcal{V}_\gamma^\epsilon c_{\alpha\beta\epsilon}.$$

Raising the index γ and using (A.5) we arrive at

$$Lie_E\, c_{\alpha\beta}^\gamma = 0.$$

This coincides with (A.4). □

Denote

$$L(v; z) := z\,\mathcal{U} + \mathcal{V}. \tag{A.6}$$

Lemma A.2 *The operator $L(v; z)$ is horizontal w.r.t. to the deformed flat connection*

$$\tilde\nabla(z)\, L(v; z) = 0. \tag{A.7}$$

The proof easily follows from Lemma A.1.

Corollary A.3 *The spectral curve does not depend on the point* $v \in M$ *of the degenerate Frobenius manifold. The differentials of the deformed flat connection* $\xi = d\tilde{v}$, $\tilde{v} = \tilde{v}(v; z)$ *are meromorphic sections of the vector bundle of eigenvectors of* $L(v; z)$ *over the spectral curve*

$$\mathcal{C}: \ \det(L(v; z) - w) = 0. \tag{A.8}$$

Proof For the covectors ξ we obtain the compatible system

$$\partial_\alpha \xi = z\, C_\alpha(v)\, \xi \tag{A.9}$$

$$L(v; z)\, \xi = w\, \xi. \tag{A.10}$$

This proves the corollary. □

Let us assume that the spectral curve has no singularities at $|z| < \infty$ and that (A.10) is a line bundle on $\mathcal{C}$. We will see below that this is the case under the assumption of semisimplicity. In this case one can choose a meromorphic trivialization of the line bundle. Denote $\tilde{v}(v; P)$ the corresponding flat function for the deformed connection $\tilde{\nabla}$ considered as a meromorphic function on the spectral curve depending on the point $v \in M$. Let $z = 0$ be not a ramification point of the spectral curve. Denote

$$P_\alpha := (z = 0, w = \mu_\alpha), \quad \alpha = 1, \ldots, n$$

the points of the spectral curve over $z = 0$. Expanding the branches of the function $\tilde{v}(v; z)$ near $P \to P_\alpha$ we obtain, after a multiplication by an appropriate normalization factor depending only on z, the basis of the deformed flat coordinates $\tilde{v}_\alpha(v; z)$, $\alpha = 1, \ldots, n$. Recall that the coefficients of expansion of these functions in z are Hamiltonian densities of the analogue of the Principal Hierarchy for the case of degenerate Frobenius manifold. This procedure can be modified in an obvious way in the case when $z = 0$ is a ramification point.

Due to symmetry of $\mathcal{U}$ and antisymmetry of $\mathcal{V}$ w.r.t. the bilinear form $< \, , \, >$ the spectral curve admits a holomorphic involution

$$\sigma: \ (z, w) \mapsto (-z, -w). \tag{A.11}$$

The orthogonality condition in this case reads as follows: the inner product $< \nabla\tilde{v}(v; P), \nabla\tilde{v}(v; \sigma(P)) >$ does not depend on $v \in M$. Considering the function on the Cartesian square of the spectral curve

$$\frac{< \nabla\tilde{v}(v; P), \nabla\tilde{v}(v; \sigma(P')) > - < \nabla\tilde{v}(v; P), \nabla\tilde{v}(v; \sigma(P)) >}{z(P) + z(P')}$$

and expanding it into a power series in $z = z(P)$, $z' = z(P')$ near $P \to P_\alpha$, $P' \to P_\beta$ we obtain, after an appropriate renormalization, the matrix $\Omega(v; z, z')$.

The system for the differentials of periods $p(v; \lambda)$ (i.e., of the flat coordinates of the flat pencils of metrics $(\, , \,)_\lambda$) reads

$$(\mathcal{U} - \lambda)^{-1} \partial_\alpha \phi + C_\alpha \mu \phi = 0, \quad \phi = \nabla p(v; \lambda).$$

From here it follows that

$$\partial_{E-\lambda e}\phi = 0$$

and, more important, that the operator $(\mathcal{U}-\lambda)^{-1}\mathcal{V}$ is horizontal w.r.t. the connection $\nabla^* - \lambda\,\nabla$. Therefore ϕ is meromorphic on the spectral curve

$$\det[\nu(\mathcal{U}-\lambda)+\mu]=0, \quad (\mathcal{U}-\lambda)^{-1}\mu\phi+\nu\phi=0$$

The new spectral curve is birationally equivalent to the old one:

$$w=\lambda\nu, \;\; z=\nu. \tag{A.12}$$

Therefore the system of independent periods $p_1(v;\lambda), \ldots, p_n(v;\lambda)$ is obtained as follows

$$p_\alpha(v;\lambda)=\tilde{v}(v;P_\alpha(\lambda)) \tag{A.13}$$

where $P_\alpha(\lambda)$ are the intersection points of the line (A.12) with the spectral curve. For large λ they can be ordered in such a form that

$$P_\alpha(\infty)=P_\alpha, \quad \alpha=1,\ldots,n.$$

Example A.4 *An arbitrary two-dimensional degenerate Frobenius manifold has the potential*

$$F=\frac{1}{2}v_1^2 v_2 + a^2 v_2\log v_2, \quad E=v_2\partial_2 \tag{A.14}$$

where a is a parameter. Then

$$\mathcal{V}=\operatorname{diag}(1/2,-1/2),$$

$$L(v;z)=\begin{pmatrix} 1/2 & -a^2 z/v_2 \\ v_2 z & -1/2 \end{pmatrix}.$$

This gives the spectral curve

$$w^2+a^2z^2=\frac{1}{4}.$$

Uniformizing it

$$z=\frac{1}{a}\frac{s}{1+s^2}, \qquad w=\frac{1}{2}\frac{1-s^2}{1+s^2}$$

we obtain the eigenvector in the form

$$\xi=\phi\begin{pmatrix} a \\ s v_2 \end{pmatrix}$$

where ϕ is a normalizing factor. The dependence of it on v_1 and v_2 is completely determined by the above differential equations. This gives

$$\tilde{v}=e^{\frac{s\,v_1}{a\,(1+s^2)}}\,v_2^{\frac{1}{1+s^2}}.$$

Therefore the basis of the deformed flat coordinates reads

$$\tilde{v}_1 = z^{-1}\left(e^{z\,v_1} v_2^{\frac{1-\sqrt{1-4a^2z^2}}{2}} - 1\right)$$

$$\tilde{v}_2 = e^{z\,v_1} v_2^{\frac{1+\sqrt{1-4a^2z^2}}{2}}.$$

The coefficients of expansion of these functions in powers of z *are hamiltonians of the hierarchy related to the degenerate polytropic gas equations*

$$\begin{aligned} &\rho_t + (\rho\, u)_x = 0 \\ &u_t + u\, u_x - a^2(\log\rho)_x = 0. \end{aligned} \tag{A.15}$$

Here $u = v_1$, $\rho = v_2$, $t = -t^{2,1}$.

The above involution becomes $s \mapsto -1/s$. *The orthogonality gives*

$$< \nabla\tilde{v}(s), \nabla\tilde{v}(-\frac{1}{s}) >= -2zw.$$

Using this one can easily compute the tau-fucntion of the hierarchy.

Let us now add the assumption of semisimplicity. Recall that this means that the eigenvalues of the operator of multiplication by E are pairwise distinct. One can locally introduce a system of canonical coordinates $u_1, \ldots, u_n$ on M such that the vector fields

$$\pi_i := \frac{\partial}{\partial u_i}, \quad i = 1, \ldots, n$$

are the basic idempotents of the Frobenius algebra on T_uM at each point $(u_1, \ldots, u_n) = u \in M$,

$$\pi_i \cdot \pi_j = \delta_{ij}\pi_i.$$

Lemma A.5 *The Euler vector field in the canonical coordinates equals*

$$E = \sum_{i=1}^{n} c_i \frac{\partial}{\partial u_i} \tag{A.16}$$

where $c_1, \ldots, c_n$ *are pairwise distinct constants.*

Proof Define the matrix (b_{ij}) by

$$Lie_E\, \pi_i = \sum_{j=1}^{n} b_{ij}\pi_j.$$

Using (A.4) one obtains, for $i \neq j$

$$0 = Lie_E(\pi_i \cdot \pi_j) = b_{ij}\pi_j + b_{ji}\pi_i.$$

Hence $b_{ij} = b_{ji} = 0$. Using the same trick for the square of π_i we have

$$b_{ii}\pi_i = Lie_E(\pi_i \cdot \pi_i) = 2b_{ii}\pi_i.$$

Hence

$$Lie_E \pi_i = 0, \quad i = 1, \dots, n. \tag{A.17}$$

This means that E has the form (A.16) with some constants c_i. These constants clearly are the eigenvalues of the operator of multiplication by E,

$$E \cdot \pi_i = c_i \pi_i.$$

Therefore

$$c_i \neq c_j \text{ for } i \neq j.$$

Introduce, as usual, the basis of normalized idempotents

$$f_i = \frac{\pi_i}{\sqrt{<\pi_i, \pi_i>}}$$

and the transition matrix

$$\Psi = (\psi_{i\alpha}(u)), \quad \partial_\alpha = \sum_{i=1}^{n} \psi_{i\alpha} f_i.$$

The matrix Ψ satisfies the linear system

$$\partial_i \Psi = V_i \Psi, \quad i = 1, \dots, n \tag{A.18}$$

where the antisymmetric matrix V_i is defined by this equation. The matrices V_i satisfy

$$[E_i, V_j] = [E_j, V_i], \quad \partial_j V_i - \partial_j V_j + [V_i, V_j] = 0, \quad i \neq j.$$

The last system coincides with the equations of flateness of the deformed connection $\tilde{\nabla}$

$$[\partial_i - (zE_i + V_i), \partial_j - (zE_j + V_j)] = 0 \tag{A.19}$$

depending on the parameter z. Here E_i is the i-th matrix unity, i.e.

$$(E_i)_{pq} = 0 \text{ for } (p, q) \neq (i, i), \quad (E_i)_{ii} = 1.$$

The flat coordinates and the structure constants can be reconstructed from an arbitrary solution to (A.18), (A.19) by the formulae of [37]

$$v_\alpha = \Omega_{\alpha,0;1,0}, \quad dv_\alpha = \sum_{i=1}^{n} \psi_{i1}\psi_{i\alpha} du_i \tag{A.20}$$

$$\eta_{\alpha\beta} = \sum_{i=1}^{n} \psi_{i\alpha}\psi_{i\beta} \tag{A.21}$$

$$F = \frac{1}{2}\Omega_{1,1;1,1} - v^\alpha \Omega_{\alpha,0;1,1} + \frac{1}{2} v^\alpha v^\beta \Omega_{\alpha,0;\beta,0} \tag{A.22}$$

$$e = \sum_{i=1}^{n} \frac{\partial}{\partial u_i}. \tag{A.23}$$

All these statements were proved in [37] for an arbitrary solution to the associativity equations (the quasihomogeneity has not been used in their derivation).

Let us now use the Euler vector field of the form (A.16).

Lemma A.6 *The matrices* Ψ, V_i *satisfy*

$$\partial_E \Psi = \Psi\, \mathcal{V} \tag{A.24}$$

$$\partial_E V_i = 0, \quad i = 1, \ldots, n. \tag{A.25}$$

Proof Using (A.3) and also

$$< \, , \, >= \sum_{i=1}^{n} \psi_{i1}^2 \, du_i^2$$

we obtain

$$\partial_E \psi_{i1} = -\frac{k}{2}\psi_{i1}.$$

From this using commutativity (A.17) we obtain (A.24). Therefore

$$\partial_E V_i = \partial_E \left[\partial_i \Psi \cdot \Psi^{-1}\right] = 0.$$

□

Lemma A.7 *The operators* $\partial_i - (zE_i + V_i)$ *commute with the operators of multiplication by the matrix*

$$L(z) := zC + V, \quad C = \mathrm{diag}(c_1, \ldots, c_n), \quad V = \sum_{i=1}^{n} c_i V_i = -\Psi \mathcal{V} \Psi^{-1}. \tag{15}$$

Proof This is just the spelling of the statement of Lemma A.6. □

We arrive at

Theorem A.8 *For a semisimple degenerate Frobenius manifold the antisymmetric matrix $V \in so(n)$ satisfies the Euler equations of free rotations of n-dimensional rigid body*

$$\partial_i V = [V_i, V], \quad V_i = \mathrm{ad}_{E_i}\mathrm{ad}_C^{-1}(V), \quad i = 1, \ldots, n. \tag{A.26}$$

Conversely, let $V = V(u_1, \ldots, u_n)$ be any solution to (A.26), and $\Psi = \Psi(u_1, \ldots, u_n)$ be a solution to the linear system (A.18). Then the formulae (A.20) - (A.23), (A.16) determine a structure of a semisimple degenerate Frobenius manifold on an open subset in M consisting of the points $u = (u_1, \ldots, u_n)$ where

$$\prod_{i=1}^{n} \psi_{i1}(u) \neq 0.$$

The nonlinear system (A.26) can be integrated [34] in terms of Prym theta-functions of the spectral curve (A.8) with the involution (A.11). The solution Ψ to the linear system (A.18) and also all the ingredients of the degenerate Frobenius structure can be given in terms of Baker - Akhiezer functions on the spectral curve. In the generic situation the spectral curve is a smooth plane algebraic curve of the degree n. It has the genus equal

$$g = \frac{(n-1)(n-2)}{2}.$$

The fixed points of the involution σ are

$$\infty_k = (z,\, w \to \infty, \frac{w}{z} \to c_k), \quad k = 1, \ldots, n$$

and, for odd n, also the point

$$P_0 = (z = 0, w = 0).$$

Baker-Akhiezer function is a vector function $Y = Y(u, P)$, $u = (u_1, \ldots, u_n)$, meromorphic in $P = (z, w) \in \mathcal{C} \setminus (\infty_1, \ldots, \infty_n)$ such that

$$Y = e^{z\, u_k}(e_k + O(\frac{1}{z}), \quad P \to \infty_k, \quad k = 1, \ldots, n \tag{A.27}$$

with a nonspecial divisor of poles D, $\deg D = \frac{n(n-1)}{2}$ that must belong to the odd part of the generalized Jacobian $J(\mathcal{C}, \infty_1, \ldots, \infty_n)$ of the spectral curve with identified infinite points. Here

$$(e_k)_i = \delta_{ik}.$$

We will give elsewhere the explicit formulae in terms of Prym theta functions for the corresponding degenerate Frobenius manifold and for the analog of the Principal Hierarchy for this manifold. We also postpone for a subsequent publication the study of the perturbations of the hierarchy.

Remark A.9 *The method of constructing solutions to the associativity equations using theta-functions has been suggested in [37]. The explicit solutions for an arbitrary n-sheeted Riemann surface with an involution were obtained by I.Krichever [94]. The relationship of these solutions to bihamiltonian structures of hydrodynamic type was not observed before. This relationship takes place only for the case where the spectral curve is a plane algebraic curve of the degree* n *or its degeneration.*

Example A.10 $n = 3$. *According to the above theory, three-dimensional semisimple degenerate Frobenius manifolds are expressed via solutions to the classical Euler equations of free rotations of a rigid body. The potential of the degenerate Frobenius manifold reads*

$$F = \frac{1}{2}v_1^2 v_3 + \frac{1}{2}v_1 v_2^2 + f(v_2, v_3) \tag{A.28}$$

where

$$f(x, y) = x^2 g(y\, x^{-2}) + a\, x^2 \log x + b\, y\, \log y. \tag{A.29}$$

Here a, b *are arbitrary constants,* $g = g(s)$ *is given by an elliptic quadrature*

$$s^3 g'' = \frac{1}{8}\left(1 - 4\, a\, s + \sqrt{1 - 8\, a\, s + 16(a^2 + b)\, s^2 + 8(c - 8\, a\, b)\, s^3}\right) \tag{A.30}$$

where c *is another constant. The spectral curve is a plane cubic*

$$w^3 - 2a\, w^2 z + 4b\, w\, z^2 - c\, z^3 - w + 2a\, z = 0. \tag{A.31}$$

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E-mail addresses: dubrovin@sissa.it, yzhang@math.tsinghua.edu.cn