DIFFERENTIAL GEOMETRY OF STRONGLY INTEGRABLE SYSTEMS OF HYDRODYNAMIC TYPE

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The integrability of one-dimensional systems of hydrodynamic type (SHT)

$$u_t^i = \sum_j v_j^i(u)\, u_x^j, \quad i = 1, \ldots, N, \tag{1}$$

is assured, as was shown by S. P. Novikov [1], by their diagonalizability, i.e., the reducibility by changes of the u-coordinates of the matrix $(v_j{}^i(u))$ to diagonal form, and the Hamiltonian property:

$$u_t^i = \{u^i(x), \hat{H}\}, \quad \hat{H} = \int H(u)\, dx, \tag{2}$$

where $\{\cdot,\ \cdot\}$ have the form [2]

$$\{u^i(x),\ u^j(y)\} = g^{ij}(u(x))\, \delta'(x-y) - g^{is}\Gamma_{sk}^j u_x^k \delta(x-y). \tag{3}$$

Here the matrix (g^{ij}) (assumed nondegenerate) defines a pseudo-Riemannian metric (with upper indices) of zero curvature on the u-space, $\Gamma_{jk}{}^i = \Gamma_{jk}{}^i(u)$ being the corresponding Levi-Civita connection. Thus, the integrability condition can be formulated in terms of the differential geometry of SHT. For such integrable systems S. P. Tsarev [3] found a generalization (for $N \geq 3$) of the hodograph method which lets one "linearize" the system and thus in some sense "integrate" it.

More precisely, let the system (1) in the coordinates $u^1, \ldots, u^N$ be already diagonal:

$$u_t^i = v_i(u)\, u_x^i, \quad i = 1, \ldots, N, \tag{4}$$

where $v_i \neq v_j$ for $i \neq j$. Then the metric (g^{ij}) in (3) is also diagonal in these coordinates. Hence, $u^1, \ldots, u^N$ is an orthogonal curvilinear coordinate system in N-dimensional Euclidean or pseudo-Euclidean space. Any such coordinate system generates a family of commuting Hamiltonian flows of the form (4)

$$u_s^i = \{u^i(x),\ \hat{P}\} = w_i(u)\, u_x^i, \quad i = 1, \ldots, N, \tag{5}$$

$$\hat{P} = \int P(u)\, dx. \tag{6}$$

One seeks these flows from the linear system

$$\partial_i w_j = \Gamma_{ji}^j (w_i - w_j), \quad i \neq j, \quad \partial_i = \partial/\partial u^i, \tag{7}$$

and their Hamiltonians [defining the family of conservation laws for the systems (4), (5)] from the linear system

$$\partial_i \partial_j P - \Gamma_{ji}^j \partial_j P - \Gamma_{ij}^i \partial_i P = 0, \quad i \neq j. \tag{8}$$

Each flow (5) commuting with (4) defines a solution of the system (4) implicitly:

$$w_i(u) = v_i(u)\, t + x, \quad i = 1, \ldots, N, \quad u = u(x, t). \tag{9}$$

In this way one gets all solutions locally. Also defined are the semi-Hamiltonians of diagonal SHT (there are more of them than Hamiltonians) to which the generalized hodograph method is applicable. All these results are due to Tsarev [3] (cf. also [4]).

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For each class of integrable SHT, the so-called averaged systems of the theory of solitons describing the hydrodynamics of weak deformations of soliton lattices [4], the linear system (7), as shown by I. M. Krichever [5], has "increased integrability" and can be solved by the methods of algebraic geometry. Some other examples of SHT for which one can effectively find commuting flows are indicated in [6].

How should one single out the class of "strongly integrable" Hamiltonian SHT in the framework of their differential geometry? In the present note we propose a solution (possibly one of several possible ones) of this problem.

1. Let

$$ds^2 = \sum_{i=1}^{N} \varepsilon_i h_i^2 (u)\, du^{i2}, \quad N \geqslant 3, \tag{10}$$

be a diagonal (pseudo-)Riemannian metric, the numbers $\varepsilon_i = \pm 1$ define its signature. The quantities

$$\gamma_{ij} = \partial_j h_i / h_j, \quad i \neq j, \tag{11}$$

are called the rotation coefficients of the metric (10) [7]. We say that (10) is a metric of Egorov type (cf. [7, 8]), if the following relations hold for it:

$$\gamma_{ji} = \varepsilon_i \varepsilon_j \gamma_{ij}. \tag{12}$$

Riemannian metrics of this form arose in the theory of orthogonal curvilinear coordinate systems of a number of 19-th century geometers. Darboux proposed to call them Egorov systems in honor of D. F. Egorov who gave the most complete account [8] of the theory of such metrics.

Assertion 1 (cf. [8]). A metric of Egorov type has zero curvature if and only if

$$\partial_k \gamma_{ij} = \gamma_{ik} \gamma_{kj}, \quad i, j, k \tag{13}$$

$$\partial \gamma_{ij} = 0 \tag{14}$$

for all $i \neq j$, where $\partial = \sum_{i=1}^{N} \partial_i$.

Remark. Egorov metrics are potential, i.e., can be represented in the form

$$g_{ii} (u) = \partial_i V (u), \tag{15}$$

where $V(u)$ is a (potential) function. Conversely, being Egorov follows from the curvature being zero and potentiality. Another equivalent definition of Egorov metrics: these are diagonal metrics of zero curvature, invariant with respect to a one-parameter group of transformations acting nonidentically along each axis. The symmetry (12) corresponds to a choice of coordinates $u^1, \ldots, u^N$, for which the action of the indicated group can be written in the form

$$u^i \mapsto u^i + \tau, \quad i = 1, \ldots, N.$$

Assertion 2 (cf. [9]). The commuting flows (5) and conservation laws (6) corresponding to a Egorov metric of zero curvature can be found from the following linear systems:

$$\partial_j \psi_i = \gamma_{ij} \psi_j, \quad j \neq i, \quad \text{for} \quad \psi_i = h_i w_i, \tag{16}$$

$$\partial_j \varphi_i = \varphi_j \gamma_{ji}, \quad j \neq i, \quad \text{for } \partial_i P = h_i \varphi_i. \tag{17}$$

Both assertions can be verified by easy calculations.

We note that the Lame coefficients $h_i = \psi_i$ themselves can be sought from (16).

Remark 1. If the curvature of a Egorov metric is nonzero but (13) holds, then the operation (3) does not define the Poisson brackets. Nevertheless, the system (16) defines a family of commuting diagonal flows (5) which will be semi-Hamiltonian systems [9] and the system (17) defines the family of their general conservation laws. Actually, such semi-Hamiltonian systems were studied in [10] (although the author of this paper was apparently not acquainted with Tsarev [9]). The symmetry (12) is inessential for semi-Hamiltonianness.

Remark 2. If an Egorov metric satisfies the following condition

$$\partial h_i = 0, \quad i = 1, \ldots, N, \tag{18}$$

which is stronger than (14), then the correspondence between conservation laws and commuting Hamiltonian flows acquires the particularly simple form

$$u_s^i = \{u^i(x), \hat{P}\} = g^{ii}\partial_i \partial P u_x^i; \quad i = 1, \ldots, N, \tag{19}$$

where the density P of the conservation law $\hat{P}$ satisfies (17). Hence, one seeks the corresponding solutions (9) of the original system (4) from the following "enveloping equations."

$$\partial_i(\partial P(u) - t\partial H(u) - x\partial I(u)) = 0, \quad i = 1, \ldots, N, \tag{20}$$

where H is the density of the Hamiltonian system (4), I(u) is the density of the momentum [the generator of translations in relation to the bracket (3)]. [By the way, the function $V(u) = \partial I(u)$ is in this case the potential (15) of the metric g_{ii}].

2. Our basic observation is the fact that the system of equations (13), (14) for the rotation coefficients of Egorov metrics of zero curvature is an integrable system of the theory of solitons (it was given, for example, in [11]), and (12) is the reduction relation for this system. This system actually reduces to the following (1 + 1)-dimensional one: the restriction of the variables $\gamma_{ij}(u)$ to any plane $u^i = a^i x + b^i t$ satisfies the following system of equations of the problem of N-waves [12] (where one usually substitutes $x, t \mapsto ix, it$)

$$[A, \Gamma_t] - [B, \Gamma_x] = [[A, \Gamma], [B, \Gamma]],$$
$$A = \operatorname{diag}(a^1, \ldots, a^N), \quad B = \operatorname{diag}(b^1, \ldots, b^N), \quad \Gamma = (\gamma_{ij}) \tag{21}$$

with the additional reduction

$$\operatorname{Im}\Gamma = 0, \quad \Gamma^T = J\Gamma J, \quad J = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_N). \tag{22}$$

The system (13), (14) can be represented in the form of compatibility conditions of linear equations

$$\partial_j \psi_i = \gamma_{ij}\psi_j, \quad j \neq i, \tag{23}$$
$$\partial \psi_i = \lambda \psi_i, \quad i = 1, \ldots, N, \tag{23'}$$

(λ being the spectral parameter). The first of these equations coincides with (16). The system (17) for the conservation laws is adjoint to (23). The system (13) is also integrable [even without the symmetry condition (12)]. It is obtained by multidimensionalization a la Zakharov–Shabat of equations (13), (14).

Based on these observations we make the following

Definition. A diagonal Hamiltonian SHT is said to be strongly integrable if the metric (10) associated with its Hamiltonian structure is a metric of Egorov type.

It is clear by virtue of what was said above that in the theory of strongly integrable SHT powerful methods of the theory of solitons such as the method of the Riemann problem (applied to the system (12)-(14) cf. [12]), the algebro-geometric method (for the system (12)-(14) cf. [13-15]), etc., can be applied.

Example. Let the metric (10) of Egorov type of zero curvature have the self-similarity property

$$g_{ii}(ku) = k^{-2s} g_{ii}(u), \quad k > 0, \quad i = 1, \ldots, N. \tag{24}$$

We show that such metrics are defined (for fixed s) by giving a finite number of parameters [equal to $N^2(N + 1)/2$]. Indeed the rotation coefficients γ_{ij} of such metrics define a self-similar solution of the system (12)-(14)

$$\gamma_{ij}(ku) = k^{-1}\gamma_{ij}(u). \tag{25}$$

For the construction of such solutions (in the sector $u^1 < \ldots < u^N$; in other sectors analogously) we use the method of the Riemann problem.

Assertion 3. Let $u^1 < u^2 < \ldots < u^N$. Let $\Psi(u, \lambda) = (\Psi_{ij}(u, \lambda))$ – $(N \times N)$-matrix-valued function, analytic in λ for $\mathrm{Re}\,\lambda \neq 0$, which is a solution of the following Riemann problem:

$$\begin{gathered}\Psi(u, i\rho + 0) = \Psi(u, i\rho - 0)\, G,\\ \Psi(u, -i\rho + 0) = \Psi(u, -i\rho - 0)\, JG^T J,\end{gathered} \tag{26}$$

$\rho > 0$, G be a constant, lower-triangular complex matrix with ones on the main diagonal, satisfying

$$\bar{G}G = E \tag{27}$$

(the dash denotes the complex conjugate, E is the identity matrix), normalized by asymptotics of the form

$$\Psi(u, \lambda) \exp(-\lambda U) = E + \Gamma/\lambda + o(\lambda^{-1}), \quad \lambda \to \infty, \quad U = \mathrm{diag}(u^1, \ldots, u^N). \tag{28}$$

Then the matrix $\Gamma = (\gamma_{ij})$ defined by (28) is real and satisfies the system (12)-(14). The metric g_{ii} with condition (24) can be reconstructed in terms of the solution of the Riemann problem with the help of the Mellin transform

$$h_i(u) = \frac{1}{2\pi i}\sum_{j=1}^{N} c_j \int_{-i\infty}^{i\infty} \lambda^{s-1}\Psi_{ij}(u, \lambda)\, d\lambda, \quad g_{ii} = \varepsilon_i h_i^2, \tag{29}$$

where $c_1, \ldots, c_N$ are arbitrary real constants. In this way one gets all metrics of Egorov type curvature zero satisfying (24).

The proof of this assertion is obtained by analogy with [16].

We note that all self-similar solutions corresponding to strongly integrable SHT with any exponent of self-similarity can also be constructed (and studied) with the help of the formulas of assertion 3: if for the original system (4)

$$v_i(ku) = k^{-p} v_i(u), \tag{30}$$

then its self-similar solutions with exponent of self-similarity γ,

$$u^i(x, t) = t^{\gamma p^{-1}} U_i(x t^{-1-\gamma}), \tag{31}$$

have the form

$$\sum_{j=1}^{N} \int_{-i\infty}^{i\infty} (a_j\lambda^{q-1} - t b_j \lambda^{r-1} - x c_j \lambda^{s-1})\, \Psi_{ij}(u, \lambda)\, d\lambda = 0, \quad i = 1, \ldots, N, \tag{32}$$

where

$$r = s + p, \quad q = s + p(1 + \gamma)/\gamma, \tag{33}$$

the real constants $b_1, \ldots, b_N$ are defined by the system (4), (30) analogously to (29), the real constants $a_1, \ldots, a_N$ are arbitrary. Cf. [12, 4] on applications of self-similar solutions of SHT.

Remark. For $N = 3$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (+, -, +)$ self-similar solutions of the system (12)-(14) are found from the system

$$\begin{aligned}(zp_1)' &= -p_2p_3,\\ p_2' &= -p_3p_1,\\ ((z-1)\,p_3)' &= p_1p_2,\end{aligned} \tag{34}$$

where

$$\gamma_{23} = p_1(z)/(u^3 - u^1), \quad \gamma_{31} = -p_2(z)/(u^3 - u^1),$$
$$\gamma_{12} = p_3(z)/(u^3 - u^1), \quad z = (u^2 - u^3)/(u^1 - u^3). \tag{35}$$

The order of this system is lowered to two with the help of the integral

$$(zp_1)^2 - p_2^2 + [(z-1)\,p_3]^2 = R^2. \tag{36}$$

As shown in [17], the second order system obtained reduces to the sixth Painleve equation (PVI) with parameters depending on R.

In particular, the Hamiltonian formalism of the Wheezem equations (cf. [4]) describing the hydrodynamics of weak deformations of cnoidal waves for the Korteweg–de Vries (KdV) equation obtained by averaging (according to [2]) the Gardner–Zakharov–Faddeev brackets in diagonal variables $u^1 < u^2 < u^3$ (Riemannian invariants for the Wheezem equations) is described by a metric (10) of Egorov type of signature (+, −, +) satisfying (18) and (24) with s = 0 (cf. [9]). It has the form

$$h_1 = f(z)/\sqrt{2(1-z)}, \quad h_2 = (f(z) + z - 1)/\sqrt{2z(1-z)},$$
$$h_3 = (f - 1)/\sqrt{2z}, \tag{37}$$

where z is as in (35) and the function f can be expressed in terms of complete elliptic integrals

$$f(z) = \mathrm{E}(\sqrt{z})/\mathrm{K}(\sqrt{z}). \tag{38}$$

The corresponding solution of (34) has the form

$$p_1 = -h_1/z\sqrt{2}, \quad p_2 = -h_2/\sqrt{2}, \quad p_3 = -h_3/(z-1)\sqrt{2} \tag{39}$$

and satisfies (36) with R = 1/2. The formulas (39) define a solution of the PVI equation by virtue of [17] which can be extracted from [18]. In this case a solution of the Riemann problem (26)-(28) can be constructed with the help of the results of Krichever [5]. Some particular solutions of (23) with $\lambda = 0$ are constructed in [9] also using the Egorov properties of the metric (37).

3. The KdV equation for n > 1 averaged over n-zoned solutions is also strongly integrable in our sense. The diagonal variables $u^1 < u^2 < \ldots < u^{2n+1}$ (denoted by $r_1, \ldots, r_{2n+1}$ in [4]) are branch points of the corresponding Riemann surface of genus n (cf., e.g., [4] for the form of the system in these variables).

Assertion 4. The metric which is diagonal in the variables $u^1, \ldots, u^{2n+1}$ giving the Gardner–Zakharov–Faddeev bracket averaged over n-zoned solutions of the KdV is a Egorov metric of curvature zero of signature (+, −, +, ..., −, +).

Proof. For the metric indicated it is easy to get

$$g_{ii} = \operatorname*{Res}_{\lambda = u^i} \left[\frac{dp}{d\lambda}\right]^2 d\lambda, \quad i = 1, \ldots, 2n+1. \tag{40}$$

Here $p = p(\lambda;\, u^1, \ldots, u^{2n+1})$ is the "quasimomentum," i.e., an Abelian integral of the second kind on a two-sheeted Riemann surface with branch points $\lambda = u^1, \ldots, \lambda = u^{2n+1}$ having a unique pole at the point $\lambda = \infty$ with principal part

$$p = \sqrt{\lambda} - u^0/2\sqrt{\lambda} + O(\lambda^{-3/2}) \tag{41}$$

and real periods over all cycles (the function $p(\lambda;\, u^1, \ldots, u^{2n+1})$ is uniquely determined by these conditions; in particular, $u^0 = u^0(u^1, \ldots, u^{2n+1})$). It follows from (40) that the metric $\Sigma g_{ii} du^{i2}$ has the indicated signature and potential:

$$g_{ii} = \partial_i u^0. \tag{42}$$

It has zero curvature by virtue of [2]. Hence, it is Egorov.

Apparently the strong integrability and self-similarity of systems obtained by averaging equations of the theory of solitons should arise as a result of "averaging" groups of symmetries (of the type of Galilean transformations and scale transformations) of the original equations. If this conjecture is valid, then such averaged systems have universality by virtue of Assertion 3 (above), i.e., form the corresponding finite-parameter family. We analyze this question in a subsequent paper.

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