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REAL THETA-FUNCTION SOLUTIONS OF THE KADOMTSEV-PETVIASHVILI EQUATION

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ABSTRACT. A complete description of smooth, real, finite-zone solutions of the Kadomtsev-Petviashvili equation is obtained.

Bibliography: 19 titles.

Introduction

The problem of realness of the method of "finite-zone integration" posed by S. P. Novikov (see the introduction to [3]) has presently been solved mainly for 1 + 1-systems (one space variable) of the theory of solitons (for a bibliography see [4]). The Kadomtsev-Petviashvili equation is the first example of a system with two space variables for which it has been possible to completely solve this problem. This solution is presented below.

As we know, the Kadomtsev-Petviashvili (KP) equation is a generalization of the Korteweg-de Vries (KdV) equation to the two-dimensional case and has the same degree of universality in the theory of nonlinear waves as the KdV equation. Two versions of this equation (more precisely, of a system of equations) have the following form:

a) the *stable version* (also called the KP2 equation)

$$\begin{aligned} \tfrac{3}{4}u_y &= w_x, \\ w_y &= u_t - \tfrac{1}{4}(6uu_x + u_{xxx}), \end{aligned} \tag{0.1}$$

b) the *unstable version* (the KP1 equation)

$$\begin{aligned} \tfrac{3}{4}u_y &= w_x, \\ w_y &= u_t - \tfrac{1}{4}(6uu_x - u_{xxx}) \end{aligned} \tag{0.2}$$

(this version is formally obtained from (0.1) by the change $x \mapsto ix$, $y \mapsto iy$, $t \mapsto it$).

The KP equation was the first physically important example of a (2 + 1)-system admitting the application of the method of the inverse problem [8], [1]. For KP2 the commutation representation for equations (0.1), (0.2) has the form

$$[-\partial_y + L, -\partial_t + A] = 0, \tag{0.3}$$

$$L = \partial_x^2 + u, \tag{0.4}$$

$$A = \partial_x^3 + \tfrac{3}{4}(u\partial_x + \partial_x u) + w; \tag{0.5}$$

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for KP1 the commutation representation is obtained from (0.3)–(0.5) by the change $(x, y, t) \mapsto (ix, iy, it)$.

A broad class of exact, rapidly decreasing solutions of the KP equation was constructed by Zakharov and Shabat [8]. It is only recently that methods have been developed which make it possible to obtain a complete description of all rapidly decreasing solutions of this equation (see [12], [13] and [19]).

A method of constructing exact periodic and quasiperiodic solutions of the KP equations was created by Krichever in [9]. These solutions are constructed according to the following scheme.

Let Γ be a compact Riemann surface of genus g, let P_∞ be a point on Γ, and let k^{-1} be a local parameter on Γ defined in a neighborhood of P_∞ with $k^{-1}(P_\infty) = 0$. The triple (Γ, P_∞, k) defines a family of exact solutions of the KP equation parametrized by the divisors D of degree g on the surface $\Gamma \backslash P_\infty$. Namely, let $\psi = \psi(x, y, t; P)$ be the Baker-Akhiezer function on the surface Γ which is meromorphic on $\Gamma \backslash P_\infty$ with poles at the points of the divisor D and which at P_∞ has exponential asymptotics of the form

$$\psi = e^{kx+k^2y+k^3t}\left(1 + \frac{\xi_1}{k} + \frac{\xi_2}{k^2} + \cdots\right), \qquad \xi_i = \xi_i(x, y, t), \quad i = 1, 2, \ldots. \tag{0.6}$$

Then ψ is an eigenfunction for certain linear differential operators, i.e.,

$$\frac{\partial \psi}{\partial y} = L\psi, \tag{0.7}$$

$$\frac{\partial \psi}{\partial t} = A\psi, \tag{0.8}$$

where the operators L and A have the form (0.4) and (0.5) respectively, and their coefficients u and w can be expressed in terms of the coefficients ξ_i of (0.6) as follows:

$$u = -2\frac{\partial \xi_1}{\partial x}, \qquad w = 3\xi_1\frac{\partial \xi_1}{\partial x} - \frac{3}{2}\cdot\frac{\partial^2 \xi_1}{\partial x^2} - 3\frac{\partial \xi_2}{\partial x}. \tag{0.9}$$

Since the consistency condition for equations (0.7) and (0.8) has the form (0.3), the coefficients u and w satisfy the KP equation (0.1).

We note that the change of the local parameter

$$k \mapsto \lambda k + a + \frac{b}{k} + O\left(\frac{1}{k^2}\right) \tag{0.10}$$

(λ, a, and b are arbitrary complex numbers, $\lambda \neq 0$) leads to another family of solutions of this same KP equation. These other solutions are obtained by means of the transformations

$$\begin{gathered} x \mapsto \lambda x + 2\lambda a y + (3\lambda a^2 + 3\lambda^2 b)t, \\ y \mapsto \lambda^2 y + 3\lambda^2 a t, \\ t \mapsto \lambda^3 t, \\ u \mapsto \lambda^{-2}u - 2\lambda^{-1}b. \end{gathered} \tag{0.11}$$

From this it follows that the dependence of the solution (0.9) of the KP equation on the local parameter reduces to a dependence only on its germ of third order.

The solutions constructed can be expressed in terms of the theta function of the Riemann surface Γ after fixing an arbitrary canonical basis of cycles $a_1, \ldots, a_g$, $b_1, \ldots, b_g$:

$$u(x, y, t) = 2\partial_x^2 \ln \theta(xU + yV + tW + z_0) + c, \tag{0.12}$$

$$w(x, y, t) = \tfrac{3}{2}\partial_x\partial_y \ln \theta(xU + yV + tW + z_0) + c_1. \tag{0.13}$$

Here θ is the theta function of the Riemann surface Γ, i.e.,

$$\theta(x) = \sum_{N_1,\dots,N_g\in\mathbf{Z}} \exp\left[\frac{1}{2}\sum_{i,j=1}^{g} B_{ij}N_iN_j + \sum_{i=1}^{g} N_i z_i\right], \tag{0.14}$$

and $B = (B_{ij})$ is the matrix of periods of holomorphic differentials $\omega_1, \dots, \omega_g$ on the surface Γ:

$$\oint_{a_j} \omega_k = 2\pi i\delta_{kj}, \quad \oint_{b_j} \omega_k = B_{kj}, \qquad k, j = 1, \dots, g. \tag{0.15}$$

We further define the vectors $U = (U_1, \dots, U_g)$, $V = (V_1, \dots, V_g)$, and $W = (W_1, \dots, W_g)$. Let Ω_1, Ω_2, and Ω_3 be differentials of second kind on Γ with zero a-periods which are holomorphic away from the point P_∞ and have principal parts at this point of the form

$$\Omega_1 = dk + \cdots, \quad \Omega_2 = d(k^2) + \cdots, \quad \Omega_3 = d(k^3) + \cdots \tag{0.16}$$

(the dots denote the correction terms). Then

$$U_j = \oint_{b_j} \Omega_1, \quad V_j = \oint_{b_j} \Omega_2, \quad W_j = \oint_{b_j} \Omega_3, \qquad j = 1, \dots, g. \tag{0.17}$$

Finally, the vector z_0 is defined on the basis of the divisor D; it assumes arbitrary values as D runs through all possible divisors of degree g. The form of the constants c and c_1 is inconsequential for our purposes, and we shall not present them.

Generally speaking, the solutions (0.12) and (0.13) are quasiperiodic, complex, meromorphic functions. This paper is devoted to the problem of distinguishing among them smooth real solutions. We formulate our basic result.

THEOREM. *For smoothness and realness of the solutions* (0.12), (0.13) *of the KP*1 *equation (for which in* (0.12) *and* (0.13) *it is necessary to make the change* $(x, y, t) \mapsto (ix, iy, it)$) *and of the KP2 equation it is necessary and sufficient that for the triple* (Γ, P_∞, k) *and the vector* z_0 *the following conditions be satisfied*:

1°. *The Riemann surface* Γ *admits an antiholomorphic involution* $\sigma\colon \Gamma \to \Gamma$, $\sigma^2 = 1$, *where* $\sigma(P_\infty) = P_\infty$ *and* $\sigma^*(k) = \overline{k}$.

2°. *The set of all fixed ovals of the involution* σ *decomposes the surface* Γ *into two pieces* Γ^+ *and* Γ^- *(a so-called involution of decomposing type).*

3°. *Suppose* $\Gamma_1, \dots, \Gamma_{k+1}$ *are fixed ovals of the involution* σ, $k \geq 0$, *and* $P_\infty \in \Gamma_{k+1}$. *Set* $\rho = (g - k)/2$ *(a natural number). On* Γ *construct the basis of cycles (see* [3])

$$a_1, b_1, \dots, a_\rho, b_\rho; \quad a_{\rho+1}, b_{\rho+1}, \dots, a_{\rho+k}, b_{\rho+k}; \quad a'_1, b'_1, \dots, a'_\rho, b'_\rho \tag{0.18}$$

so that $a_{\rho+j} = \Gamma_j$, $j = 1, \dots, k$, *and*

$$\begin{gathered} a_i, b_i \in \Gamma^+, \quad \sigma(a_i) = a'_i, \quad \sigma(b_i) = -b'_i, \qquad i = 1, \dots, \rho, \\ \sigma(a_{\rho+j}) = a_{\rho+j}, \quad \sigma(b_{\rho+j}) = -b_{\rho+j}, \qquad j = 1, \dots, k. \end{gathered} \tag{0.19}$$

Then the vector z_0 *for the KP*1 *equation is an arbitrary vector of the form*

$$z_0 = (\xi; \eta; \overline{\xi}), \qquad \xi \in \mathbf{C}^\rho, \ \eta \in \mathbf{R}^k. \tag{0.20}$$

4°. *For the KP2 equation there is an additional topological condition on the surface* Γ: *on* Γ *the involution* σ *must have a maximum number of ovals (equal to* $g+1$). *If a* *basis of cycles of the form* (0.19) *is chosen* ($k = g$ *and* $\rho = 0$), *then* z_0 *is an arbitrary vector with purely imaginary coordinates.*

Sufficiency of the conditions of the theorem was proved by one of the authors in [3] and [15]. In this paper we prove the necessity of these conditions([1]) under an additional assumption on solutions $u(x, y, t)$ and $w(x, y, t)$ of the form (0.12) and (0.13). We assume not only smoothness of these solutions but also of all solutions of the same form constructed on the basis of the same triple (Γ, P_∞, k) and obtained from u and w by variation of the vector z_0 in (0.12) and (0.13) which preserves realness of u and w. If all the periods of the quasiperiodic functions u and w are independent, then this assumption is not restrictive; on the contrary, in the case of periodicity in x it is possible to give up this assumption for the KP1 equation (for more details, see the end of §3).

In our proof the greatest difficulty is caused by the proof of "realness" for the Riemann surface (i.e., the existence on it of an antiholomorphic involution). In other words, we prove that realness of the Abelian functions $\partial_x^2 \ln \theta(Ux + Vy + z_0)$ and $\partial_x \partial_y \ln \theta(Ux + Vy + z_0)$ (for real x and y, and some z_0) implies realness of the Riemann surface.

We note that our proof also goes through for those solutions of the KP equation which are constructed according to Krichever's scheme on the basis of singular algebraic curves (these are solitons, rational solutions, and their superpositions with one another and with quasiperiodic solutions).

§1. Proof of realness of the Riemann surface

We begin the proof of the necessity of the conditions of the theorem by proving 1°, i.e., by proving "realness" of the Riemann surface Γ relative to some antiholomorphic involution σ. For this we use the fact that the theta functions of arbitrary Riemann surfaces, aside from the KP equation, satisfy, according to [9], a further infinite series of differential relations—the so-called *KP hierarchy*. All these equations admit a commutation representation of zero curvature of the form

$$[\partial_{x_i} - L_i, \partial_{x_j} - L_j] = 0, \qquad i, j = 1, 2, \dots, \tag{1.1}$$

where the operators L_l have the form

$$L_i = \partial_x^i + \sum_{k=0}^{i-2} u_{ik} \partial_x^k, \qquad x = x_1. \tag{1.2}$$

For $i = 1$ we have $L_1 = \partial_x$; for $i = 2$ and $j = 3$ we obtain the first nontrivial equation of the hierarchy—the KP equation itself, where $x_2 = y$, $x_3 = t$, $L_2 = L$, and $L_3 = A$. The coefficients $u_{ik} = u_{ik}(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, \dots)$, of all the operators L_i can be algorithmically expressed in terms of the theta function and its derivatives. Equations (1.1) are the consistency conditions for linear equations of the form

$$\frac{\partial \psi}{\partial x_i} = L_i \psi, \qquad i = 1, 2, \dots, \tag{1.3}$$

([1])As I. M. Krichever has communicated to the authors, he has very recently found an approach to the proof of necessity of the conditions of our theorem for the KP1 equation based on the use of the spectral theory of the time-dependent Schrödinger operator $i\partial_y + \partial_x^2 + u$ with periodic coefficients (see [10]). The approach of [10] is also applicable to the description of conditions of realness for finite-zone two-dimensional Schrödinger operators with periodic coefficients.

where $\psi = \psi(\boldsymbol{x}; P)$, $P \in \Gamma$, is the Baker-Akhiezer function on the Riemann surface Γ which has g poles there and an essential singularity at the point P_∞ of the form

$$\psi(\boldsymbol{x}; P) = e^{\sum x_i k^i}\left(1 + \frac{\xi_1(\boldsymbol{x})}{k} + \frac{\xi_2(\boldsymbol{x})}{k^2} + \cdots\right), \qquad k = k(P). \tag{1.4}$$

Following [14], we show how to reduce the system of equations for the function $u_{ik}(\boldsymbol{x})$ to a system of equations for only one function. It turns out that there exists a formal pseudodifferential operator L of the form

$$L = \partial_x + \sum_{j=1}^{\infty} u_j(\boldsymbol{x})\partial_x^{-j}, \qquad x = x_1, \tag{1.5}$$

such that

$$L_i = [L^i]_+, \qquad i = 1, 2, \ldots, \tag{1.6}$$

where $[\ ]_+$ denotes the positive (differential) part of the pseudodifferential operator. (We recall the rules for computing the superposition of a pseudodifferential operator with an operator of multiplication by a function. First of all,

$$\partial_x^{-1} f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots; \tag{1.7}$$

and the remaining rules are deduced from this.) The dependence of the operator L on the variables $x = x_1, x_2, \ldots$ is determined from equations of Lax type

$$\frac{\partial L}{\partial x_i} = [L_i, L], \qquad i = 1, 2, \ldots. \tag{1.8}$$

The operator L has an eigenfunction $\psi = \psi(\boldsymbol{x}; k)$,

$$L\psi = k\psi, \tag{1.9}$$

which has the form (1.4), where the series are understood as formal series. This function also satisfies (1.3). If we introduce the pseudodifferential operator P (in $x = x_1$) by setting

$$P = 1 + \xi_1(\boldsymbol{x})\partial_x^{-1} + \xi_2(\boldsymbol{x})\partial_x^{-2} + \cdots, \tag{1.10}$$

then we have

$$Pe^{\sum x_j k^j} = \psi(\boldsymbol{x}; k), \tag{1.11}$$

$$L = P\partial_x P^{-1}. \tag{1.12}$$

Further, the function $\psi(\boldsymbol{x}; k)$ can be represented in the form

$$\psi(\boldsymbol{x}; k) = e^{\sum x_j k^j}\frac{(x_1 - k^{-1}, x_2 - \frac{1}{2}k^{-2}, x_3 - \frac{1}{3}k^{-3}, \ldots)}{\tau(x_1, x_2, x_3, \ldots)}, \tag{1.13}$$

where the τ-function $\tau(\boldsymbol{x})$ is defined, generally speaking, on finite sequences $\boldsymbol{x} = (x_1, x_2, \ldots)$ which are the same as those on which the coefficients of the operators L_i and L are defined; (1.13) is understood in the sense of equality of formal series. In particular, this makes it possible to express solutions of the KP hierarchy in terms of the τ-function and its derivatives. For the solutions u, w of the KP equation itself we obtain

$$u = 2\partial_x^2 \ln\tau, \qquad w = \tfrac{3}{2}\partial_x\partial_y \ln\tau. \tag{1.14}$$

Equations (1.1) of the KP hierarchy can be written in the form of a set equations for the single function $\tau(\boldsymbol{x})$. All these equations can be written simply by means of the "bilinear Hirota operators". We recall the definition of them. If $f(x)$ is a function of one variable, then for any polynomial (or power series) Q the action of the Hirota operator $Q(D_x)f(x)\cdot f(x)$ is defined by

$$Q(D_x)f(x)\cdot f(x) = Q(\partial_y)[f(x+y)f(x-y)]_{y=0}. \tag{1.15}$$

For functions of several variables the definition is similar. The generating function for the equations of the KP hierarchy thus has the form

$$\sum_{j=0}^{\infty} p_j(-2y)p_{j+1}(\tilde{D})\exp\left(\sum_{i=1}^{\infty} y_i D_i\right)\tau\cdot\tau = 0, \tag{1.16}$$

where $y=(y_1, y_2, \dots)$ are auxiliary independent variables, $\tilde{D} = (D_1, 2^{-1}D_2, 3^{-1}D_3, \dots)$, D_j is the Hirota operator in the variable x_j, $j = 1, 2, \dots$, and p_j are the Schur polynomials defined from the following expansion:

$$\exp\left(\sum_{j=1}^{\infty} x_j k^j\right) = \sum_{j=0}^{\infty} k^j p_j(x_1, \dots, x_j). \tag{1.17}$$

All these equations are graded-homogeneous if gradation i is assigned to the operators D_i. The first several equations of the KP hierarchy have the form

$$[4D_1D_3 - 3D_2^2 - D_1^4]\tau\cdot\tau = 0 \tag{1.18}$$

(this is the KP2 equation itself, gradation 4),

$$[3D_1D_4 - 2D_2D_3 - D_1^3D_2]\tau\cdot\tau = 0 \tag{1.19}$$

etc. According to [18], the equations of the KP hierarchy can be written in the form

$$\det\begin{vmatrix} p_{f_1+1}(-\tilde{D}/2) & p_{f_1+1}(\tilde{D}/2) & \dots & p_{f_1+m+1}(\tilde{D}/2) \\ p_{f_2}(-\tilde{D}/2) & p_{f_2}(\tilde{D}/2) & \dots & p_{f_2+m-1}(\tilde{D}/2) \\ \dots & \dots & \dots & \dots \\ p_{f_m-m+2}(-\tilde{D}/2) & p_{f_m-m+2}(\tilde{D}/2) & \dots & p_{f_m}(\tilde{D}/2) \end{vmatrix}\tau\cdot\tau = 0, \tag{1.20}$$

where $f_1 \geq \dots \geq f_m \geq 1$ are natural numbers, $m \geq 2$. The gradation of such an equation is equal to $f_1 + \dots + f_m + 1$.

REMARK. If $Q(D)\tau\cdot\tau = 0$ is one of the equations of the KP hierarchy, then it may be assumed that in the polynomial $Q(D)$ all monomials have even degree in the variables $D_1, D_2, \dots$, since the monomials of odd degree give trivial Hirota operators.

The change of the formal parameter k of the form

$$k = f(k') = \lambda_{-1}k' + \lambda_0 + \sum_{j=1}^{\infty}\lambda_j k'^{-j}, \qquad \lambda_{-1} \neq 0, \tag{1.21}$$

leads to a certain transformation preserving the form of the equations of the KP hierarchy. First of all, there occurs a triangular transformation of the variables $x_1, x_2, \dots$ according to the law

$$x_i' = \sum_{j=1}^{\infty}\mu_{ij}x_j, \qquad i = 1, 2, \dots, \tag{1.22}$$

where the triangular matrix (μ_{ij}) is defined from the conditions

$$[f(k')]^j = \sum_{i=0}^{\infty}\mu_{ij}k'^i + O(k'^{-1}), \qquad j = 1, 2, \dots. \tag{1.23}$$

Moreover, the operators L_i also change. According to (1.9), their transformation is defined by the transformation of the operator $L \mapsto L'$, where $L = f(L')$. The new ψ-function is defined from the condition

$$\psi'(\mathbf{x}'; k') = \exp\left(-\sum_{j=1}^{\infty} \mu_{0j} x_j\right) \psi(\mathbf{x}; f(k')), \tag{1.24}$$

where the variables $\mathbf{x}$ and $\mathbf{x}'$ are connected by (1.22). It can also be expressed in terms of the new τ-function $\tau'(\mathbf{x}')$ according to formulas of the form (1.13). We say that the function $\tau'(\mathbf{x}')$ is *equivalent* to $\tau(\mathbf{x})$. (Below we indicate the explicit form of the transformation $\tau(\mathbf{x}) \mapsto \tau'(\mathbf{x}')$ for some special changes $k \mapsto k'$.) The system (1.20) is invariant with respect to the transformations $\tau(\mathbf{x}) \mapsto \tau'(\mathbf{x}')$. Moreover, this system is invariant with respect to "gauge transformations" of the form

$$\tau(\mathbf{x}) \mapsto e^{\sum \alpha_i x_i + \beta} \tau(\mathbf{x}). \tag{1.25}$$

This follows from the fact that for any polynomial Q the corresponding Hirota operator $Q(D)\tau \cdot \tau$ possesses the property

$$Q(D)\tau' \cdot \tau' = e^{2(\sum \alpha_i x_i + \beta)} Q(D)\tau \cdot \tau, \qquad \tau' = e^{\sum \alpha_i x_i + \beta}\tau. \tag{1.26}$$

According to [9], the algebraic-geometric solutions of the KP hierarchy are defined by the triple (Γ, P_∞, k) and the divisor D (see the Introduction). Fixing a canonical basis of cycles makes it possible to express these solutions in terms of the theta function of the surface Γ in the form

$$\tau(\mathbf{x}) = e^{Q(x)} \theta(x_1 U_1 + x_2 U_2 + \cdots + z_0), \tag{1.27}$$

where z_0 is an arbitrary g-dimensional vector determined by the divisor D; the g-dimensional vectors $U_1, U_2, \ldots$ are defined in terms of the expansions of the basis holomorphic differentials $\omega_1, \ldots, \omega_g$ for $P \to P_\infty$:

$$\omega_j = [(U_1)_j + (U_2)_j z + (U_3)_j z^2 + \cdots]\, dz, \qquad z = k^{-1},\ j = 1, 2, \ldots, g. \tag{1.28}$$

In particular, $U_1 = U$, $U_2 = V$, and $U_3 = W$. Further, if the Ω_n are normalized holomorphic differentials of second kind with principal part at P_∞ of the form

$$\Omega_n = d(k^n) + \text{correction terms}, \qquad n = 1, 2, \ldots, \tag{1.29}$$

and the coefficients q_{ij} are defined from the expansions of these differentials according to the formulas

$$\int \Omega_n = k^n + 2\sum_{m=1}^{\infty} \frac{q_{mn}}{m k^m}, \tag{1.30}$$

then the quadratic form $Q(\mathbf{x})$ has the form

$$Q(\mathbf{x}) = \sum_{i,j=1}^{\infty} q_{ij} x_i x_j. \tag{1.31}$$

This follows immediately from a comparison of the Krichever formula for the Baker-Akhiezer function ψ with the definition of the τ-function (1.13) (see [14]). All the series in (1.27) converge with suitable analytic conditions on the infinite vector $\mathbf{x}$.

We proceed to the proof of the theorem. First of all, it will be convenient for us to go over from the equations of the KP hierarchy written in the form of quadratic equations for the τ-function to equations for the logarithmic derivative of the τ-function. For this it is necessary to divide all these equations by τ^2 and use the following assertion.

LEMMA 1. *The following assertions hold for the Hirota operators:*

$$\tau^{-2}D_{i_1}\cdots D_{i_n}\tau\cdot\tau = 2(\ln\tau)_{i_1\cdots i_n} + \sum_{q=1}^{n}\sum_{(i')}\lambda_{(i)}^{(i')}(\ln\tau)_{i'_1\cdots i'_{k_1}}(\ln\tau)_{i'_{k_1+1}\cdots i'_{k_2}}\cdots(\ln\tau)_{i'_{k_q+1}\cdots i'_n}, \tag{1.32}$$

where $n = 2k$, the inner sum goes over all permutations $i'_1, \dots, i'_n$ of the indices $i_1, \dots, i_n$, the $\lambda_{(i)}^{(i')}$ are certain universal rational coefficients, $(\ln\tau)_{i_1\cdots i_n} = \ln(\tau)_{x_{i_1}\cdots x_{i_n}} = \ln(\tau)_{x_{i_1}\cdots x_{i_n}}$, *and the right side does not contain logarithmic derivatives of first order.*

PROOF. The left side of (1.32) can be expressed only in terms of the logarithmic derivatives of the function τ in view of its invariance relative to transformations $\tau \mapsto c\tau$, c a constant. The leading term on the right side can be computed directly from the definition of the Hirota operators. There are no logarithmic derivatives of first order on the right side due to the invariance of the left side relative to the gauge transformations (1.25) (see (1.26)). The lemma is proved.

In particular,

$$D_iD_j\tau\cdot\tau = 2\tau^2(\ln\tau)_{ij}, \tag{1.33}$$

$$D_1^4\tau\cdot\tau = 2\tau^2(\ln\tau)_{1111} - 4\tau^2((\ln\tau)_{11})^2. \tag{1.34}$$

We set $v = \ln\tau$ and

$$v_{i_1\cdots i_n} = \frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_n}}v \tag{1.35}$$

We also denote derivatives with respect to the variable $x = x_1$ as follows:

$$v^{(k)} = \frac{\partial^k v}{\partial x_1^k}. \tag{1.36}$$

If the τ-function has the form (1.27), then all these logarithmic derivatives are Abelian functions (meromorphic functions on the Jacobian of the surface Γ) for $n \geq 2$ or $k \geq 2$. On restricting them, for example, to the complex axis x they become meromorphic quasiperiodic functions.

LEMMA 2. *Suppose the function τ satisfies the KP hierarchy. Then the function $v = \ln\tau$ satisfies equations of the form*

$$v_{ij} = \sum_{m=1}^{\infty}\sum_{t_1+\cdots+t_m+s_1+\cdots+s_m=i+j} R_{t_1\cdots t_m}^{s_1\cdots s_m}(i,j)v_{t_1}^{(s_1)}\cdots v_{t_m}^{(s_m)}, \tag{1.37}$$

where $i, j = 2, 3, \dots$, and the $R_{t_1\cdots t_m}^{s_1\cdots s_m}(i,j)$ are certain universal rational coefficients.

Here and henceforth in this section all summation indices are natural numbers.

PROOF. We shall carry out induction on the sum $i+j$, beginning with $i+j=4$. For $i=j=2$ by the KP equation (1.18) and formulas (1.33) and (1.34) we have

$$v_{22} = \tfrac{4}{3}v_3^{(1)} - \tfrac{1}{3}[v^{(4)} - 2(v^{(2)})^2]. \tag{1.38}$$

This is one of the nontrivial relations (1.37), since $v^{(4)} = v_1^{(3)}$, etc.

We suppose that the lemma has been proved for all $i+j \leq N$. We note, first of all, that from this assumption it follows that

$$v_{i_1\cdots i_n} = \sum_{m=1}^{\infty}\sum_{t_1+\cdots+t_m+s_1+\cdots+s_m=i_1+\cdots+i_n} R_{t_1\cdots t_m}^{s_1\cdots s_m}(i_1,\dots,i_n)v_{t_1}^{(s_1)}\cdots v_{t_m}^{(s_m)} \tag{1.39}$$

for $n \geq 2$ and $i_1 + \cdots + i_n \leq N + n - 2$, where the $R^{s_1 \cdots s_m}_{t_1 \cdots t_m}(i_1, \ldots, i_n)$ are universal rational coefficients (as in (1.37), all summation indices are natural numbers). This can be proved immediately by differentiation of (1.37). We now make the induction step on $i + j$. Suppose $i + j = N + 1$ and $i \geq j \geq 1$. In this case we shall prove (1.37) by induction on j. For $j = 1$ this is a tautology. To move into the region of larger values of j we use the KP hierarchy. We take equation (1.20) with $m = 3$, $f_1 = i - 1$, $f_2 = j - 1$, and $f_3 = 1$. Since $p_0 = 1$ and $p_i(x_1, \ldots, x_i) = x_i + \cdots$, where the dots denote nonlinear terms, it follows that

$$\det\begin{pmatrix} p_i\left(-\frac{\tilde{D}}{2}\right) & p_i\left(\frac{\tilde{D}}{2}\right) & p_{i+1}\left(\frac{\tilde{D}}{2}\right) \\ p_{j-1}\left(-\frac{\tilde{D}}{2}\right) & p_{j-1}\left(\frac{\tilde{D}}{2}\right) & p_j\left(\frac{\tilde{D}}{2}\right) \\ 1 & 1 & \frac{1}{2}D_1 \end{pmatrix}$$

$$= 2^{-3}\det\begin{pmatrix} -\frac{1}{i}D_i + \cdots & \frac{1}{i}D_i + \cdots & \frac{1}{i+1}D_{i+1} + \cdots \\ -\frac{1}{j-1}D_{j-1} + \cdots & \frac{1}{j-1}D_{j-1} + \cdots & \frac{1}{j}D_j + \cdots \\ 2 & 2 & D_1 \end{pmatrix}$$

$$= \frac{1}{2}\left\{\frac{1}{ij}D_iD_j - \frac{1}{(i+1)(j-1)}D_{i+1}D_{j-1} - \sum_{q=4}^{\infty}\sum_{r_1+\cdots+r_q=N+1} P^{ij}_{r_1\ldots r_q}D_{r_1}\cdots D_{r_q}\right\}\tau\cdot\tau = 0.$$

Here the $P^{ij}_{r_1 \cdots r_q}$ are universal rational coefficients. The conditions on the summation indices are obtained automatically by virtue of the fact that this equation has gradation $f_1 + f_2 + f_3 + 1 = i + j = N + 1$. We multiply both sides of this equation by τ^{-2}. By Lemma 1 we obtain

$$\frac{1}{ij}v_{ij} = \frac{1}{(i+1)(j-1)}v_{(i+1),(j-1)} + \sum_{q=4}^{\infty}\sum_{r_1+\cdots+r_q=N+1} P^{ij}_{r_1\cdots r_q}\left[v_{r_1\cdots r_q} + \frac{1}{2}\sum_{p=1}^{q}\sum_{(r')}\lambda^{(r')}_{(r)}v_{r'_1\cdots r'_{k_1}}\cdots v_{r'_{k_p+1}\cdots r'_q}\right]. \tag{1.40}$$

The derivative $v_{(i+1),(j-1)}$ can be represented in the form (1.37) by the induction hypothesis on j. The expressions in square brackets can be represented as a polynomial in $v^{(n)}_m$ by (1.39). The lemma is proved.

We shall especially study the form of some terms linear in $v_t(s)$ in (1.37) and (1.39). They could have been computed during the proof of Lemma 2, but in order not to encumber the exposition we present this computation as a separate lemma.

LEMMA 3. *For the coefficients $R^{s_1}_{t_1}(i_1, \ldots, i_n)$ in the terms linear in $v^{(s)}_t$ in* (1.37) *and* (1.39), *the following relations are satisfied*:

$$R^1_{i+j-1}(i, j) = \frac{ij}{i+j-1}, \tag{1.41}$$

$$R^{s_1}_{t_1}(i_1, \ldots, i_n) = 0 \quad \textit{for } s_1 \leq n-2, \tag{1.42}$$

$$R^{n-1}_{i_1+\cdots+i_n-n+1}(i_1, \ldots, i_n) = \frac{i_1 \cdots i_n}{i_1 + \cdots + i_n - n + 1}. \tag{1.43}$$

PROOF. As in Lemma 2 we carry out induction on $i_1 + \cdots + i_n$. For $i_1 + \cdots + i_n = 4$ (the only nontrivial case here is $n = 2$, $i_1 = i_2 = 2$) everything follows from (1.18).

We suppose that (1.41) has already been proved from $i + j \le N$. We first show that this implies (1.42) and (1.43) for $i_1 + \cdots + i_n \le N + n - 2$. Indeed, differentiating the equality

$$v_{i_1 i_2} = \frac{i_1 i_2}{i_1 + i_2 - 1} v^{(1)}_{i_1+i_2-1} + \sum_{s=2}^{i_1+i_2-1} R^{s_1}_{i_1+i_2-s_1}(i_1, i_2) v^{(s_1)}_{i_1+i_2-s_1} + \sum_{(m=2)}^{\infty} \sum_{t_1+\cdots+t_m+s_1+\cdots+s_m=i_1+i_2} R^{s_1\cdots s_m}_{t_1\cdots t_m}(i_1, i_2) v^{(s_1)}_{t_1} \cdots v^{(s_m)}_{t_m}$$

with respect to x_{i_3}, we obtain

$$v_{i_1 i_2 i_3} = \frac{i_1 i_2}{i_1 + i_2 - 1} v^{(1)}_{i_1+i_2-1, i_3} + \sum_{s_1=2}^{i_1+i_2-1} R^{s_1}_{i_1+i_2-s_1}(i_1, i_2) v^{(s_1)}_{i_1+i_2-s_1, i_3} + \sum_{m=2}^{\infty} \sum_{(s),(t)} R^{s_1\cdots s_m}_{t_1\cdots t_m}(i_1, i_2)[v^{(s_1)}_{t_1} \cdots v^{(s_m)}_{t_m}]_{i_3}.$$

It is clear that differentiation of the nonlinear terms and subsequent transformation of them according to (1.37) does not affect the linear terms. Further, after transformation of the linear terms according to (1.37) it is possible by the induction hypothesis to use (1.41) for $i_1 + i_2 - 1 + i_3 \le N$, i.e., for $i_1 + i_2 + i_3 \le N + 1$. We obtain

$$v_{i_1 i_2 i_3} = \frac{i_1 i_2 i_3}{i_1 + i_2 + i_3 - 2} v^{(2)}_{i_1+i_2+i_3-2} + \sum_{s_1=3}^{i_1+i_2+i_3-1} R^{s_1}_{i_1+i_2+i_3-s_1}(i_1, i_2, i_3) v_{i_1+\cdots+i_3-s_1} + \cdots,$$

where the dots denote the nonlinear terms. This implies the validity of (1.42) and (1.43) for $n = 3$. We then differentiate the last equality with respect to i_4, etc.

Thus, (1.42) and (1.43) have been derived from (1.41) under the conditions indicated on $i_1 + \cdots + i_n$. We now make the induction step on $i + j$ in (1.41). For this, as in Lemma 2, we use equation (1.40) of the KP hierarchy. Suppose that $i + j = N + 1$ in this equation. We observe that the terms in (1.40) with $q \ge 4$ make no contribution to the terms linear in $v^{(t)}_s$. This follows immediately from (1.42). Therefore, from (1.40) we obtain

$$\frac{1}{ij} R^1_{i+j-1}(i, j) = \frac{1}{(i+1)(j-1)} R^1_{i+j-1}(i+1, j-1).$$

From this we obtain (1.41), since by definition $R^1_{i+j-1}(i + j - 1, 1) = 1$. The lemma is proved.

COROLLARY. *For $s_1 + \cdots + s_m < m + n - 2$ and $t_1 + \cdots + t_m + s_1 + \cdots + s_m = i_1 + \cdots + i_n$*

$$R^{s_1\cdots s_m}_{t_1\cdots t_m}(i_1, \ldots, i_n) = 0. \tag{1.44}$$

The proof is obvious.

LEMMA 4. *For changes of the parameter k of the form*

$$k = k' + \frac{a}{(k')^q} + O((k')^{-1-l}) \tag{1.45}$$

the logarithmic derivatives of the τ-function transform according to the law

$$v_l^{(1)} \mapsto v_l^{(1)} \quad \text{for } l < q, \qquad v_q^{(1)} \mapsto v_q^{(1)} + qa. \tag{1.46}$$

PROOF. We define the function $\eta_s(\boldsymbol{x})$ by the equality

$$\ln \psi = \sum_{j=1}^{\infty} x_j k^j + \sum_{s=1}^{\infty} \eta_s k^{-s}, \tag{1.47}$$

where $\psi = \psi(\boldsymbol{x}; k)$ is the eigenfunction for operators of the KP hierarchy. By (1.13) we have $\eta_1 = \partial_{x_1} \ln \tau$. Substituting (1.45) into (1.47), we obtain

$$\ln \psi = \sum_{j=1}^{\infty} x_j k^j + \sum_{s=1}^{\infty} \eta_s k^{-s}$$
$$= \sum_{j=1}^{q} x_j k'^j + O(k')^{q+1} + (qax_q + \eta_1)(k')^{-1} + O(k')^{-2}.$$

From this we find $x_l \mapsto x_l$ for $l \le q$. Differentiating with respect to these x_l, we obtain (1.46). The lemma is proved.

We now proceed to the lemma which is basic for the proof of realness of the Riemann surface—a type of "uniqueness theorem" for the KP hierarchy.

LEMMA 5. *Suppose τ and $\tilde{\tau}$ are two solutions of the KP hierarchy of the form* (1.27) *such that the corresponding (see* (1.14)) *functions u, w and $\tilde{u}, \tilde{w}$ coincide as functions of x and y. Then after a suitable (formal) change of the parameter k of the form*

$$k = k' + \sum_{j=3}^{\infty} c_j (k')^{-j} \tag{1.48}$$

the function τ goes over into the function $\tilde{\tau}$ (up to the gauging (1.25)).

PROOF. We choose a change $k \mapsto k'$ such that for the logarithmic derivatives of the function $\tau(\boldsymbol{x}')$ equivalent to $\tau(\boldsymbol{x})$ we have

$$v_j'^{(1)}|_x = \tilde{v}_j^{(1)}|_x, \qquad j = 2, 3, \dots. \tag{1.49}$$

It may here be assumed that the change has the form (1.48), since $v_{11}|_{x,y} = \tilde{v}_{11}|_{x,y}$ and $v_2^{(1)}|_{x,y} = \tilde{v}_2^{(1)}|_{x,y}$, by the hypothesis of the lemma. We construct this change $k \mapsto k'$ inductively step by step. We suppose that a change has already been selected so that equalities (1.49) are satisfied for $j \le N$. We have (1.39) for the logarithmic derivatives of the function τ'. In particular,

$$\begin{aligned} v'^{(1)}_{\underbrace{2\cdots 2}_{N}} = {} & \frac{2^N}{2N+1} v'^{(N)}_{N+1} + \sum_{q=1}^{N} c_{q,N} v'^{(N+q)}_{N+1-q} \\ & + \sum_{m=2}^{\infty} \sum_{\substack{t_1+\cdots+t_m+s_1+\cdots+s_m=2N+1 \\ s_1+\cdots+s_m \ge m+N-1}} R^{s_1\cdots s_m}_{t_1\cdots t_m}(2, 2, \dots, 2, 1) v'^{(s_1)}_{t_1} \cdots v'^{(s_m)}_{t_m}. \end{aligned} \tag{1.50}$$

Here the $c_{q,N}$ are rational coefficients. By Lemma 3 and its corollary all terms in (1.50) except the first are polynomials in $v_t'^{(s)}$ for $t \le N$. There is also a analogous equation for $\tilde{v}$. By the hypothesis of the lemma and the induction hypothesis we have

$$v'^{(N)}_{N+1}|_x = \tilde{v}^{(N)}_{N+1}|_x. \tag{1.51}$$

Since $v'^{(1)}_{N+1}|_x$ and $\tilde{v}^{(1)}_{N+1}|_x$ are quasiperiodic meromorphic functions, it follows from (1.51) that

$$v'^{(1)}_{N+1}|_x = \tilde{v}^{(1)}_{N+1}|_x + c, \tag{1.52}$$

where c is a constant. We now change the parameter k' by setting

$$k' = k'' + \frac{c}{(N+2)(k'')^{N+1}}.$$

After this change by Lemma 4 equalities (1.49) are preserved for $j \le N$, and $v''^{(1)}_{N+1}|_x$ is equal to $\tilde{v}^{(1)}_{N+1}|_x$. This completes the induction step.

It is now not hard to complete the proof of the lemma. From (1.39) and (1.49) it follows that

$$v'_{i_1\cdots i_n}|_x = \tilde{v}_{i_1\cdots i_n}|_x, \qquad n \ge 2, \tag{1.53}$$

for all $i_1, \dots, i_n$. Therefore, all these logarithmic derivatives also coincide on all vectors $\boldsymbol{x}$ with finitely many nonzero components. Hence, the functions τ' and $\tilde{\tau}$ are gauge-equivalent. The lemma is proved.

REMARK 1. If one is interested only in the restriction of the τ-functions to a finite number of variables $x_1, \dots, x_N$, where N is any fixed number, then in the formulation of the lemma it is possible to restrict attention only to polynomial changes of the parameter.

REMARK 2. An obvious modification of Lemma 5 also holds: it is possible to replace by an equivalent function not only the function τ (by means of the reparametrization (1.48)) but also the function $\tilde{\tau}$ by means of an analogous change of the corresponding parameter k,

$$\tilde{k} = \tilde{k}' + \sum_{j=3}^{\infty} \tilde{c}_j (\tilde{k}')^{-j}. \tag{1.54}$$

After the changes (1.48) and (1.54) we obtain coincidence of the τ-functions, $\tau' = \tilde{\tau}'$.

We now proceed directly to the proof of the realness of the Riemann surface Γ. We begin from KP2. We thus know that functions u and w of the form (1.14) are real as functions of x, y, and t, where the τ-function in (1.14) is constructed on the basis of Γ. We shall first show that after a suitable choice of the local parameter k^{-1} the function $\tau(\boldsymbol{x})$ can be made real. We set $\tilde{\tau}(\boldsymbol{x}) = \overline{\tau(\overline{\boldsymbol{x}})}$. The function $\tilde{\tau}(\boldsymbol{x})$ can be expressed in terms of the theta functions and Abelian integrals of the dual Riemann surface $\overline{\Gamma}$ with respect to the local parameter $\tilde{k} = \overline{k}$. All the conditions of Lemma 5 are satisfied for τ and $\tilde{\tau}$. This makes it possible to choose local parameters k' (on Γ) and $\tilde{k}'$ (on $\overline{\Gamma}$) so that $\tau|_{x_1\cdots x_N} = \tilde{\tau}|_{x_1\cdots x_N}$ up to the gauge (1.25) for any fixed value of N. In the changes (1.48) and (1.54) the coefficients c_j and $\tilde{c}_j$ can be chosen to be complex conjugates, $\tilde{c}_j = \overline{c}_j$, $j = 3, 4, \dots$, since at each step of the algorithm of Lemma 5 the constant $c = (\ln\tau)_{1,N+1} - (\ln\tilde{\tau})_{1,N+1}$ in (1.52) is imaginary. Thus, the function $\tau(\boldsymbol{x})$ constructed on the basis of the triple (Γ, P_∞, k') and the function $\overline{\tau(\overline{\boldsymbol{x}})}$, coinciding with it, constructed on the basis of the triple $(\overline{\Gamma}, \overline{P}_\infty, \overline{k'})$ are both real on the N variables indicated (we shall deal with the choice of N later). We henceforth omit the prime on the new local parameter.

We now use the construction of the dual Baker-Akhiezer function. Let

$$P^+ = 1 + (-\partial^{-1})\xi_1 + (-\partial)^{-2}\xi_2 + \cdots \tag{1.55}$$

be the operator formally adjoint to (1.10). We define the dual function $\psi^+(\boldsymbol{x}; k)$ by

$$\psi^+(\boldsymbol{x}; k) = P^{+^{-1}} e^{-\sum x_i k^i}. \tag{1.56}$$

This function ψ^+ is an eigenfunction for the formally adjoint operators

$$L^+\psi^+ = k\psi^+, \qquad L^+ = -\partial + (-\partial)^{-1}u_1 + (-\partial)^2 u_2 + \cdots = -(P^+)^{-1}\partial P^+, \tag{1.57}$$

$$\frac{\partial\psi^+}{\partial x_n} = L_n^+, \qquad L_n^+ = [L^{+^n}]_+, \quad n = 1, 2, \ldots. \tag{1.58}$$

For algebraic-geometric solutions, where $\psi(\boldsymbol{x}; k)$ is the expansion of the Baker-Akhiezer function constructed on the basis of the triple (Γ, P_∞, k) and some divisor of poles D, $\psi^+(\boldsymbol{x}; k)$ is also the expansion of the Baker-Akhiezer function for the same triple (Γ, P_∞, k) with divisor of poles D^+, where

$$D^+ + D - 2P_\infty \sim K, \tag{1.59}$$

K denotes the canonical class of the surface Γ, and the tilde denotes linear equivalence (see [14] and [11]). The function $\psi^+(\boldsymbol{x}; k)$ can be expressed in terms of τ as follows

$$\psi^+(\boldsymbol{x}; k) = e^{-\sum x_i k^i} \frac{\tau(x_1 + k^{-1}, x_2 + \frac{1}{2}k^{-2}, x_3 + \frac{1}{3}k^{-3}, \ldots)}{\tau(x_1, x_2, \ldots)}. \tag{1.60}$$

Their product $\psi(\boldsymbol{x}; k)\psi^+(\boldsymbol{x}; k)$ is the expansion of a meromorphic function on Γ. This expansion has the form

$$\begin{aligned}
\psi(\boldsymbol{x}; k)\psi^+(\boldsymbol{x}; k) &= \frac{\tau(x_1 - k^{-1}, x_2 - \frac{1}{2}k^{-2}, \ldots)\tau(x_1 + k^{-1}, x_2 + \frac{1}{2}k^{-2}, \ldots)}{\tau^2(x_1, x_2, \ldots)} \\
&= \tau^{-2}[\tau(x_1 - y_1 - k^{-1}, x_2 - y_2 - \tfrac{1}{2}k^{-2}, \ldots) \\
&\qquad \times \tau(x_1 + y_1 + k^{-1}, x_2 + y_2 + \tfrac{1}{2}k^{-2}, \ldots)]_{\mathbf{y}=0} \\
&= \tau^{-2}\exp\left\{\sum_{i=1}^{\infty} \frac{1}{ik^i} y_i \partial_{y_i}\right\}[\tau(\boldsymbol{x} - \boldsymbol{y})\tau(\boldsymbol{x} + \boldsymbol{y})]_{\boldsymbol{y}=0} \\
&= \tau^{-2}\exp\left\{\sum_{i=1}^{\infty} \frac{1}{ik^i} D_i\right\}\tau \cdot \tau \\
&= \sum_{j=0}^{\infty} \frac{k^{-j} p_j(\tilde{D})\tau \cdot \tau}{\tau^2} \equiv \sum_{j=0}^{\infty} \varphi_j(\boldsymbol{x})k^{-j},
\end{aligned}$$

where all the coefficients $\varphi_j = \tau^{-2}p_j(\tilde{D})\tau \cdot \tau$ (for $j > 0$) can be expressed in the form of polynomials in $y_{i_1 \cdots i_n}$ for $n \geq 2$. In particular, $\varphi_1 = 0$ and $\varphi_2 = u/2$. We note that the coefficients φ_i are not sensitive to the gauge arbitrariness in the τ-function. They are thus real for real $x_1, \ldots, x_N$.

Suppose a and b are two real numbers such that the values $u_1 = u|_{x_1=a}$ and $u_2 = u|_{x_1=b}$ are defined and distinct ($x_i = 0$ for $i \geq 2$). We introduce meromorphic functions z and w of degree $2g$ on Γ by setting

$$z = \psi\psi^+|_{x_1=a}, \qquad w = \psi\psi^+|_{x_1=b}. \tag{1.61}$$

These functions satisfy an algebraic equation of the form

$$F(z, w) \equiv \sum a_{ij} z^i w^j = 0, \tag{1.62}$$

where $F(z, w)$ is a polynomial of degree $2g$ in each variable which in $\mathbf{C}^2$ with coordinates z, w defines the affine part of the Riemann surface Γ. We shall show that all the coefficients a_{ij} of the polynomial F can be chosen to be real. Indeed, they can be defined from linear homogeneous systems whose coefficients can be expressed in a polynomial manner in terms of φ_k $(x_1 = a)$ and φ_l $(x_1 = b)$. (For the proof it suffices to expand (1.62) in powers of k^{-1} in a neighborhood of P_∞.) Because φ_k and φ_l are real for $k, l \leq N$, it is possible to choose all the a_{ij} also to be real (here N is chosen so that the system of equations for a_{ij} has a unique solution up to a factor).

Once the coefficients a_{ij} of (1.62) are real, the surface Γ is invariant relative to an involution σ of the form

$$\sigma(z, w) = (\overline{z}, \overline{w}). \tag{1.63}$$

The point P_∞, having coordinates $z(P_\infty) = w(P_\infty) = 1$, is fixed relative to σ.

It remains to show that the local parameter k with respect to which the τ-function is real is invariant with respect to the involution σ. Indeed, the function

$$\tilde{k} = \frac{u_1}{2(z-1)} = k + O(1) \tag{1.64}$$

gives a real local parameter in a neighborhood of P_∞. Therefore, the parameter k is also real. The proof of part 1° of the main theorem for the KP2 equation has been completed.

For the KP1 equation in the equations of the KP hierarchy it is necessary to make the change $\boldsymbol{x} \mapsto i\boldsymbol{x}$ after which all arguments can be repeated word for word. We note that all equations (1.20) of the KP hierarchy after such a change, as before, have real coefficients (signs are changed in some places). The same applies to (1.37) and (1.39) (in the latter for odd n it will be necessary to cancel by i).

REMARK. From all the equations (1.20) of the KP hierarchy we have used in the proof only a minor part—equations with $m = 3$ and $f_3 = 1$. This circumstance is not accidental. The following assertion is one explanation of the importance of precisely these equations of the hierarchy.

ASSERTION 1. *Equations* (1.20) *with* $m = 3$, $f_3 = 1$, *and* $f_1 \geq f_2 \geq 1$ *arbitrary differentially generate all the remaining equations of the KP hierarchy.*

PROOF. From equations (1.37) and (1.39), which are differential consequences of equations (1.20) indicated in the formulation of the assertion, it is possible to uniquely recover (up to the gauge transformations (1.25)) the function $\tau(\boldsymbol{x})$ on the basis of the "Cauchy data" $v_i^{(1)}|_x$, $i = 1, 2, \ldots$. We shall show that this function is a solution of all the remaining equations of the KP hierarchy. For this it obviously suffices to prove that for the "Cauchy data" $v_i^{(1)}|_x$, $i = 1, 2, \ldots$, it is possible to take arbitrary functions of x. To prove the last proposition we note that the coefficients $u_1(x), u_2(x), \ldots$ of the operator L of (1.5) are independent Cauchy data for the KP hierarchy written in the form (1.8). The independence of the Cauchy data $v_i^{(1)}$, $i = 1, 2, \ldots$, follows from this by virtue of the next lemma.

LEMMA 6. *The coefficients* $u_1(x), u_2(x), \ldots$ *of the operator* L *are connected with the functions* $v_1^{(1)}(x), v_2^{(1)}(x), \ldots$ *by an invertible transformation of the form*

$$u_j(x) = U_j(v_1^{(1)}(x), \ldots, v_j^{(1)}(x)), \qquad v_j^{(1)}(x) = V_j(u_1(x), \ldots, u_j(x)), \tag{1.65}$$

where $j = 1, 2, \ldots$, *and* U_j *and* V_j *are polynomials in their arguments and their derivatives with respect to* x.

PROOF. According to [14], we have relations of the form

$$v_n^{(1)} = nu_n + \cdots, \qquad n = 1, 2, \ldots,$$

where the dots denote a polynomial in the functions $u_1, \dots, u_{n-1}$ and their derivatives with respect to x. The validity of the lemma, and together with it Assertion 1, follows from this.

It is clear that the equations (1.20) listed in this assertion are the minimal set of equations which differentially generate the entire KP hierarchy.

§2. Termination of the proof of the main theorem for the KP2 equation

First of all, we use the realness of the τ-function proved in the preceding section (up to the gauge transformation (1.25)) for real values of the arguments:

$$\overline{\tau(\boldsymbol{x})} = e^{\sum \alpha_i x_i + \beta} \tau(\boldsymbol{x}). \tag{2.1}$$

For the Baker-Akhiezer function (1.13) it follows from this that

$$\overline{\psi(\boldsymbol{x}; \sigma(P))} = \exp\left[-\sum_{i=1}^{\infty} \frac{\alpha_i k^{-i}}{i}\right] \psi(\boldsymbol{x}; P), \qquad k = k(P) \to \infty.$$

Thus, the coefficients of the expansion of the function $\overline{\psi(\boldsymbol{x}; \sigma(P))}/\psi(\boldsymbol{x}; P)$, meromorphic everywhere on Γ, in a series in powers of k^{-1} (in a neighborhood of P_∞) do not depend on $\boldsymbol{x}$. Computing them for $\boldsymbol{x} = 0$, we find that all $\alpha_i = 0$, $i = 1, 2, \dots$. Thus, the Baker-Akhiezer function corresponding to real solutions of KP2 possesses the following property of realness with respect to the involution σ:

$$\overline{\psi(\boldsymbol{x}; \sigma(P))} = \psi(\boldsymbol{x}; P). \tag{2.2}$$

Hence, the divisor D of poles of $\psi(\boldsymbol{x}; P)$ is invariant with respect to the involution σ.

Until now we have not used the smoothness of the solutions $u(x, y, t), w(x, y, t)$. We shall show that part 4° of the theorem follows from their smoothness (part 2° is in this case a trivial corollary of 4°, since every real Riemann surface with a maximal number of ovals is of decomposing type).

The divisor $D - gP_\infty$ of degree zero is invariant with respect to σ, i.e., it lies on the real component of the Jacobian $J(\Gamma)$. In other words, if we take P_∞ as the initial point of the Abel mapping, i.e.,

$$A(\Gamma) = \left(\int_{P_\infty}^{P} \omega_1, \dots, \int_{P_\infty}^{P} \omega_g\right) \in J(\Gamma), \tag{2.3}$$

then a vector z_0 having the form

$$-z_0 = A(D) + K \tag{2.4}$$

(K is the vector of Riemann constants) satisfies the condition of realness on $J(\Gamma)$:

$$\sigma(z_0) \equiv z_0. \tag{2.5}$$

Here the antiholomorphic involution induced on $J(\Gamma)$ is denoted by the same letter σ; the symbol $\equiv$ is used to denote equality of points on the Jacobian (equivalence of vectors modulo the lattice of periods; see [16]). We assume that the basis of cycles on Γ has been chosen so that the plane spanned by $a_1, \dots, a_g$ is invariant with respect to σ. In this case the vectors U, V, and W are tangent to the real components of (2.5). The vector $z = xU + yV + tW + z_0$, which is the argument of the theta function in the formulas for $u(x, y, t)$ and $w(x, y, t)$ as x, y, and t vary then runs through the real component on which z_0 lies. To values x, y, and t for which $\theta(z) = 0$ there correspond poles of the solutions $u(x, y, t), w(x, y, t)$. We suppose that on the entire real component of the Jacobian passing through a point z_0 of the form (2.4) the theta function does not vanish. (We recall that the zeros of the theta function have codimension one.) We shall show that from this it already follows that on a surface Γ of genus g there must be $g + 1$ ovals.

We suppose that the number of ovals on the surface Γ is equal to n; $n \geq 1$, since $\sigma(P_\infty) = P_\infty$. We consider the values of the theta function on vectors of the form (2.4), where the vector z_0 runs over one of the real components of the Jacobian. Since the point P_∞ has been selected as the initial point of the Abel mapping, it follows that $\theta(z_0) = 0$ if and only if the divisor D contains the point D_∞ (see [16] with regard to the zeros of the theta function). We shall show that for $n \leq g$ the divisor D can be deformed with preservation of the conditions $\sigma(D) = D$ in such a way that it contains the point P_∞. We denote the ovals by $\Gamma_1, \dots, \Gamma_n$. Suppose that P_∞ lies on Γ_n. The divisor D, which is invariant relative to σ, can be represented in the form

$$D = \sum_{i=1}^{g-2m} Q_i + \sum_{j=1}^{m} [Q'_j + \sigma(Q'_j)], \tag{2.6}$$

where the points Q_i are fixed relative to σ, i.e., they lie on real ovals, while the points Q'_j are "nonreal". If at least one of the points Q_i lies on Γ_n, then without changing the remaining points, it can be shifted over Γ_n to the point P_∞. We therefore assume that none of the points Q_i lies on Γ_n. There are then two possible variants:

a) $m \neq 0$. We then choose a path γ going from the point Q'_1 to P_∞ and draw the point Q'_1 to P_∞ along the path γ, while we draw the point $\sigma(Q'_1)$ to P_∞ in symmetrical fashion along the path $\sigma(\gamma)$. After this deformation, which preserves the symmetry $\sigma(D) = D$, we obtain a divisor containing P_∞.

b) $m = 0$. In this case all g points of the divisor D are real, and, since none of them lies on Γ_n, on at least one of the ovals there is a pair of points Q_i, Q_j. In this case it is possible to coalesce them and then symmetrically draw them into the "imaginary" domain, i.e., deform them into the pair $Q'_1, \tau(Q'_1)$. The rest of the deformation is constructed as above. Thus, the case $n \leq g$ contradicts smoothness.

We shall now prove that if we choose a basis of cycles on the surface Γ with real ovals $\Gamma_1, \dots, \Gamma_{g+1}, P_\infty \in \Gamma_{g+1}$, so that $a_i = \Gamma_i$, $i = 1, \dots, g$, then a vector z_0 of the form (2.4) will have purely imaginary components. Indeed, from the arguments presented above it is evident that in the case $n = g+1$ there is precisely one connected component of the Jacobian on which $\theta(z)$ has no zeros. It is formed by divisors D of degree g where on each oval $\Gamma_1, \dots, \Gamma_g$ there is exactly one point of the divisor (cf. [16]). The image (2.4) of such divisors on the Jacobian consists precisely of all purely imaginary vectors.

The theorem has been proved for the KP2 equation.

§3. Termination of the proof of the main theorem for the KP1 equation

In the case of KP1, (2.1) for the τ-function is also satisfied. However, the Baker-Akhiezer function $\psi(\boldsymbol{x}; P)$ is now expressed in terms of the τ-function by

$$\psi(\boldsymbol{x}; P) = e^{i\sum_{j=1}^{\infty} x_j k^j} \frac{\tau(x_1 - ik^{-1}, x_2 - \frac{i}{2}k^{-2}, \dots)}{\tau(x_1, x_2, \dots)} \tag{3.1}$$

(in (1.13) we have made the change $\boldsymbol{x} \mapsto i\boldsymbol{x}$). In a similar way it is possible to rewrite (1.60) for the dual Baker-Akhiezer function $\psi^+(\boldsymbol{x}; P)$ whose divisor of poles D^+ is connected with the divisor of poles of $\psi(\boldsymbol{x}; P)$ by (1.59). From (2.1) we have

$$\overline{\psi(\boldsymbol{x}; \sigma(P))} = \exp\left[-i\sum_{j=1}^{\infty}\frac{\alpha_j k^{-j}}{j}\right]\psi^{+}(\boldsymbol{x}; P).$$

As in the preceding section, it can be proved that all $\alpha_i = 0$. In other words,

$$\overline{\psi(\boldsymbol{x}; \sigma(P))} = \psi^{+}(\boldsymbol{x}; P). \tag{3.2}$$

Hence, $D^{+} = \sigma(D)$, i.e., the divisor D satisfies

$$D + \sigma(D) \sim K + 2P_{\infty}. \tag{3.3}$$

Divisors D of degree g satisfying (3.3) cover the imaginary (relative to σ) components of the Jacobian under the Abel mapping (2.4) (see [16]). As in §2, the entire matter reduces to an investigation of the zeros of the theta function. Namely, if the Riemann surface Γ is nondecomposing, then on the imaginary components of the Jacobian $J(\Gamma)$ covered by divisors D with condition (3.3) the function $\theta(z)$ has zeros (see the Appendix to [6]). Hence, the surface Γ is decomposing. On such surfaces the theta function has no zeros only on an imaginary component of the form (0.20) (see [16]). The theorem is proved.

REMARK. In the case where the functions $u(x)$ and $w(x)$ are quasiperiodic and all g of their frequencies are "maximally incommensurate" (the group of frequencies has rank g over $\mathbf{Z}$), the assumption of the theorem regarding the absence of zeros of the theta function on the entire real (or imaginary) component of the Jacobian $J(\Gamma)$ corresponding to a given solution is obviously not restrictive, since the x-winding is everywhere dense on this component. In the other extreme case where the functions $u(x)$ and $w(x)$ are periodic in x with period T for the KP1 equation it is possible to give up this additional assumption using only the smoothness of the functions $u(x)$ and $w(x)$. We present the argument, following basically [5].

We first prove that the operator

$$L = i\partial_x + \sum_{k=1}^{\infty} u_k (i\partial_x)^{-k} \tag{3.4}$$

and all the operators $L_n = [L^n]_+$ of the KP hierarchy are selfadjoint, i.e.,

$$L^{*} \equiv \overline{L}^{+} = L, \qquad L_n^{*} = L_n, \quad n = 1, 2, \ldots \tag{3.5}$$

It suffices to prove the selfadjointness of L. From (1.57) for $k \mapsto \overline{k}$ we have $L^{+}\psi^{+}(\boldsymbol{x}; \overline{k}) = \overline{k}\psi^{+}(\boldsymbol{x}; k)$. Acting on this equality with complex conjugation and using (3.2), we obtain $L^{*}\psi(\boldsymbol{x}; k) = k\psi(\boldsymbol{x}; k)$. Hence $L^{*} = L$, since L is uniquely determined from the ψ-function. We note also that the coefficients of L and L_n can be expressed in terms of the logarithmic derivatives of the function $\theta(z)$ for $z = ixU + iyV + itW + z_0$ (of not lower than second order by Lemma 6). They are therefore all smooth functions of $\boldsymbol{x}$. From the periodicity of the functions $u(x)$ and $w(x)$ by Lemma 5 we obtain

$$\tau(x_1 + T, x_2, x_3, \ldots) = e^{i\sum \alpha_j x_j}\tau(x_1, x_2, x_3, \ldots). \tag{3.6}$$

Hence, all the coefficients of L and L_n are periodic in x (by Lemma 6). The Baker-Akhiezer function is a Bloch function, i.e.,

$$\psi(x + T; P) = e^{ipT}\psi(x; P), \tag{3.7}$$

where for the quantity $p = p(P)$ (the quasimomentum) as $P \to P_\infty$ we have an expansion of the form

$$p(P) = k - \sum_{j=1}^{\infty} \frac{\alpha_j k^{-j}}{j}, \qquad k = k(P) \to \infty. \tag{3.8}$$

The function $\exp(ip(P)T)$ defined by (3.7) is holomorphic on $\Gamma \backslash P_\infty$. Therefore, $p(P)$ is an Abelian integral on Γ, and its differential dp is an Abelian differential with a double pole at P_∞ all of whose periods are integral multiples of $2\pi/T$,

$$\oint_\gamma dp = 2\pi n_\gamma T^{-1}, \qquad \gamma \in H_1(\Gamma; \mathbf{Z}),\ n_\gamma \in \mathbf{Z}. \tag{3.9}$$

Further, we recall [9] that each meromorphic function $\lambda = \lambda(P)$ on the surface Γ with a single pole of order n at P_∞ defines an ordinary differential operator M (in the variable x) of order n such that

$$M\psi(x; P) = \lambda(P)\psi(x; P). \tag{3.10}$$

If the Laurent expansion of $\lambda(P)$ for $P \to P_\infty$ has the form

$$\lambda(P) = c_0 k^n + c_1 k^{n-1} + \cdots + c_n + O(k^{-1}), \tag{3.11}$$

then the operator M can be expressed in terms of the operators L_i of the KP hierarchy by

$$M = c_0 L_n + \cdots + c_n = [\lambda(L)]_+. \tag{3.12}$$

If $\lambda(P)$ is real relative to σ, $\lambda(\sigma(P)) = \overline{\lambda(P)}$, then all the coefficients $c_0, \dots, c_n$ are real, and hence the operator M is selfadjoint, $M^* = M$.

We shall show that the surface Γ with involution σ is decomposing. From (3.2) and (3.7) it follows that $\sigma^*[\exp(ipT)] = \exp(-i\overline{p}T)$. By (3.9) we find that the function $\operatorname{Im} p(P)$ is single-valued on Γ. It vanishes on the fixed ovals of Γ. Since M is selfadjoint, $\operatorname{Im} p(P) = 0$ implies that $\operatorname{Im} \lambda(P) = 0$, i.e., the zeros of the function $\operatorname{Im} p(P)$ coincide precisely with the real ovals. Hence, the real ovals decompose Γ into two halves: $\Gamma^+ = \{\operatorname{Im} p(P) < 0\}$ and $\Gamma^- = \{\operatorname{Im} p(P) > 0\}$. From this it also follows that the differential dp is positive on real ovals oriented as the boundary of Γ^+.

We now derive the conditions on the divisor D. Condition (3.3) implies that $D + \sigma(D)$ is the divisor of zeros of a differential Ω of second kind with a double pole at P_∞. Condition (0.20) of the theorem implies that this differential is positive on real ovals oriented as the boundary of Γ^+ (see [16]). To prove positivity of Ω we construct it explicitly, using methods developed in [4] for matrix differential operators.

We realize the surface Γ as an n-sheeted covering over the λ-plane by means of the mapping λ of (3.11). Let $(\lambda, 1), \dots, (\lambda, n)$ be points on Γ ordered in an arbitrary manner corresponding to the same value of λ. They may be assumed to be distinct. We construct the Wronskian matrix

$$\psi_j^i(x; \lambda) = \psi^{(i-1)}(x; (\lambda, j)), \qquad i, j = 1, \dots, n. \tag{3.13}$$

The matrix $\psi_j^i(x; \lambda)$ is nondegenerate. We denote the inverse matrix by $\varphi_j^i(x; \lambda)$. We define the differentials $\Omega_j^i(x; P)$ by setting

$$\Omega_j^i(x; P) = \psi_m^i(x; \lambda)\varphi_j^m(x; \lambda)\, d\lambda, \qquad P = (\lambda, m). \tag{3.14}$$

It is easy to verify that this definition is independent of the initial enumeration of the points. These differentials are regular for $|\lambda| < \infty$ and can have poles only at P_∞. We shall determine the form of these poles. We have

$$\psi_j^i(x;\lambda) = (\varepsilon_j\kappa)^{i-1}e^{\varepsilon_j\kappa x}(1+O(\kappa^{-1})), \qquad \kappa = \sqrt[n]{\lambda c_0^{-1}},\ \varepsilon_j = \exp(2\pi ij/n). \tag{3.15}$$

Hence,

$$\varphi_j^i(x;\lambda) = \frac{1}{n}(\varepsilon_j\kappa)^{-j+1}e^{-\varepsilon_i\kappa x}(1+O(\kappa^{-1})). \tag{3.16}$$

We obtain

$$\Omega_j^i(x;(\lambda,m)) = \frac{1}{n}(\varepsilon_m\kappa)^{i-j}\,d\lambda(1+O(\kappa^{-1})). \tag{3.17}$$

For the differential $\Omega_n^1(x;P)$ from (3.17) we obtain a principal part of the form

$$\Omega_n^1(x;P) = dk(1+O(k^{-2})). \tag{3.18}$$

We set

$$\Omega(P) = \Omega_n^1(0;P). \tag{3.19}$$

We note that the function

$$\varphi(x;P) = \varphi_n^m(x;\lambda), \qquad P = (\lambda, m), \tag{3.20}$$

is an eigenfunction for the adjoint operator M^+ with eigenvalue λ. The function $\psi^+(x;P)$ can therefore differ from it only by normalization:

$$\psi^+(x;P) = \frac{\varphi(x;P)}{\varphi(0;P)} = \frac{\varphi(x;P)}{\Omega(P)}\,d\lambda. \tag{3.21}$$

From this and (3.2) we obtain

$$\psi(x;P)\overline{\psi(x;\sigma(P))}\Omega(P) = \psi(x;P)\varphi(x;P)\,d\lambda = \Omega(x;P). \tag{3.22}$$

We have thus found that $D+\sigma(D)$ is the divisor of zeros of the differential $\Omega(P)$.

We show that the differential $\Omega(P)$ (or $\Omega(x;P)$) is positive on the ovals. First of all, $\Omega(x;P)$ preserves sign on each oval, since its real zeros and poles are of even multiplicity. For its mean over a period we have (see [2], formula (40))

$$\frac{1}{T}\int_0^T \Omega(x;P)\,dx = dp. \tag{3.23}$$

Since dp is positive, $\Omega(x;P)$ is also positive on each oval for all x, as required.

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