CHEBYSHEV MEASURES AND THE VECTOR MEASURES WHOSE RANGE IS STRICTLY CONVEX

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In this paper we resume the most important results that we obtained in our papers [1,2,5,6,7] concerning a broad class of measures that we defined in dealing with a bang–bang control problem.
Let $\mathcal{M}$ be the $\sigma-$algebra of the Lebesgue measurable subsets of $[0,1]$ and $\mu : \mathcal{M} \to \mathbb{R}^n$ be a non–atomic vector measure.
A well known Theorem of Lyapunov (see [11]) states that the range of μ, defined by $\mathcal{R}(\mu) = \{\mu(E) : E \in \mathcal{M}\}$, is closed and convex or, equivalently, that given a measurable function ρ with values in $[0,1]$ there exists a set E in $\mathcal{M}$ such that

$$(*) \qquad \int_X \rho \, d\mu = \mu(E).$$

Lyapunov's Theorem has been widely applied in bang–bang control theory [10] and, more recently, in some non–convex problems of the Calculus of Variations [3].
As an example we mention the following bang–bang existence result:

Theorem 1. *Let $Lx = x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_1(t)x' + a_0(t)x$ be a linear differential operator of order n and let x in $W^{n,1}([0,1])$ and ρ in $L^\infty([0,1])$ with values in $[0,1]$ be such that $Lx = \rho$ a.e. in $[0,1]$. There exists a function y in $W^{n,1}([0,1])$ such that $Ly \in \{0,1\}$ a.e. and $y^{(k)}(0) = x^{(k)}(0)$, $y^{(k)}(1) = x^{(k)}(1)$ for $k = 0, \dots, n-1$.*

In dealing with some minimization problems involving a function of the state x it may be useful to have some more informations on the behaviour of the new function y with respect to the prescribed x. However Lyapunov's Theorem doesn't give any information on the above set E involved on the construction of y: such a set may be as bad as one wishes. For instance let T be a measurable subset of $[0,1]$, μ be the bidimensional measure defined by $\mu(A) = (\lambda(A), \lambda(A \cap T))$, λ being the Lebesgue measure on $\mathbb{R}$, and ρ be the characteristic function of T: it is easy to see that if a set E satisfies $(*)$ then necessarily $E = T$. One of the main results in [5,7] is a condition on the measure under which its values can be obtained restricting the measure on the finite union of intervals.

Let ν be a positive non–atomic finite measure on $X = [0,1]$ and assume that $\nu(I) \neq 0$ and that $|\mu|(I) \neq 0$ for every non–trivial interval I, $|\mu|$ being the total variation of μ. We shall denote by $\mathcal{S}_n$ the symmetric group of order n and, for σ in $\mathcal{S}_n$, by $\epsilon(\sigma)$ its sign. To the vector measure $\mu = (\mu_1, \ldots, \mu_n)$ we associate a measure $\det \mu$.

Definition. (determinant measure)
The measure $\det \mu$ is the measure defined on the product space $(X^n, \mathcal{M}^{\otimes n})$ by

$$\det \mu \;=\; \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma)\, \mu_{\sigma(1)} \otimes \cdots \otimes \mu_{\sigma(n)}.$$

This is the only measure whose restrictions to the product sets $A_1 \times \cdots \times A_n$ satisfy $\det \mu(A_1 \times \cdots \times A_n) = \det[\mu(A_1), \cdots, \mu(A_n)]$.

Let Γ_n be the set defined by $\Gamma_n = \{(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n : 0 \le \gamma_1 \le \cdots \le \gamma_n \le 1\}$.

Remark. [7] Let μ be a T_ν–measure and A in $\mathcal{M}$ be such that $\nu(A) > 0$. Then $\mu(A) \neq O$; in particular ν is absolutely continuous with respect to $|\mu|$. In fact assume that $\nu(A) > 0$. If μ is a T_ν–measure then the continuity of the map $\alpha \mapsto \nu([0,\alpha])$ allows to decompose A as a disjoint union of some non ν–negligible sets $A_1, \ldots, A_n$ such that their product $A_1 \times \cdots \times A_n$ is contained in Γ_n. It follows that $\det[\mu(A_1), \ldots, \mu(A_n)] = \det \mu(A_1 \times \cdots \times A_n) > 0$ so that the vectors $\mu(A_1), \ldots, \mu(A_n)$ are linearly independent. If $\mu(A) = O$ then $O = \mu(A) = \mu(A_1) + \cdots + \mu(A_n)$, a contradiction.

It may happen however that $|\mu|$ is not absolutely continuous with respect to ν.

Example. [2] Let $n = 1$ and μ, ν be two non–atomic (non–trivial) positive measures on $[0,1]$ which are positive on every non trivial interval. If ν is absolutely continuous with respect to μ then μ is a T_ν–measure: in fact if $\nu(A) > 0$ for some measurable set A then trivially $\det \mu(A) = \mu(A) > 0$. Let, for instance, μ be the Lebesgue measure on $[0,1]$, E be a set such that $0 < \mu(E \cap I) < \mu(I)$ for every non trivial interval I and ν be the measure defined by $\nu(A) = \mu(A \cap E)$ for every measurable set A. Then ν is absolutely continuous w.r. to μ so that μ is a T_ν–measure; however μ is not absolutely continuous with respect to ν ($\nu([0,1] \setminus E) = 0$).

Definition. The vector measure μ is a Chebyshev measure with respect to ν (or T_ν–measure) if the measure $\det \mu$ satisfies

$$\forall A \in \mathcal{M}^{\otimes n}, \quad A \subset \Gamma_n, \qquad \nu^{\otimes n}(A) > 0 \quad \Longrightarrow \quad \det \mu(A) > 0.$$

If $\nu = |\mu|$ then μ is called a Chebyshev measure (or T–measure).

Remark. A more general definition for Chebyshev measures defined in arbitrary measurable spaces is given in [7].

Definition. Let $f = (f_1, \cdots, f_n)$ be a measurable vector–valued function defined on X. We say that $f = (f_1, \cdots, f_n)$ is a Chebyshev system with respect to ν (or a T_ν–system) if the determinant $\det[f(x_1), \cdots, f(x_n)]$ is positive for $\nu^{\otimes n}$–almost all $(x_1, \cdots, x_n)$ in Γ_n.

Chebyshev systems for continuous functions have been widely studied in Approximation Theory [9]; our interest in them relies in the fact that they generate Chebyshev measures.

Theorem 2. *Suppose μ is absolutely continuous with respect to ν. Let $f = (f_1, \cdots, f_n)$ be its density function. Then μ is a T_ν–measure if and only if f is a T_ν–system.*

Proof. For any measurable subset A of X^n the application of Fubini–Tonelli's Theorem yields

$$\begin{aligned}\det \mu(A) &= \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) \big(\mu_{\sigma(1)} \otimes \cdots \otimes \mu_{\sigma(n)}\big)(A) \\ &= \int_A \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) f_{\sigma(1)}(x_1) \cdots f_{\sigma(n)}(x_n)\, d(\nu(x_1) \otimes \cdots \otimes \nu(x_n)) \\ &= \int_A \det[f(x_1), \ldots, f(x_n)]\, d\nu^{\otimes n}(x_1, \ldots, x_n). \quad \square\end{aligned}$$

Chebyshev systems arise naturally from linear differential equations.

Example. [5] Let $h \in \mathcal{C}^n(\mathbb{R})$ be such that $h^{(i)}(0) = 0$ for $i = 0, \ldots, n-2$ and $h^{(n-1)}(0) = 1$. Then there exists $\delta > 0$ such that if we set $f = (h^{(n-1)}, h^{(n-2)}, \ldots, h', h)$ then $\det[f(t_1), \ldots, f(t_n)] > 0$ for every $-\delta < t_1 < \cdots < t_n < \delta$.

The main result in [7] is the following representation Theorem which allows, given a function ρ, to choose a set E verifying $(*)$ among the finite union of intervals.

Theorem 3. *Suppose μ is a T_ν–measure and let ρ be a measurable function with values in $[0, 1]$. There exists $\alpha = (\alpha_1, \ldots, \alpha_n)$ in Γ_n satisfying*

$$\mu(E_\alpha) = \int_X \rho\, d\mu \quad \textit{where} \quad E_\alpha = \bigcup_{\substack{0 \le i \le n \\ i \text{ odd}}} \big[\alpha_i, \alpha_{i+1}\big] \quad (\alpha_{n+1} = 1).$$

If $0 < \rho < 1$ on a ν–non negligible set then α is unique and belongs to the interior of Γ_n; moreover the point $\int_X \rho\, d\mu$ belongs to the interior of $\mathcal{R}(\mu)$.

Actually we prove that the map $\theta : \Gamma_n \to \mathcal{R}(\mu)$ defined by $\theta(\alpha) = \mu(E_\alpha)$ for α in Γ_n is a homeomorphism from Γ_n onto $\mathcal{R}(\mu)$.
The fact that the characteristic function of E_α has at most n discontinuity points and that $\alpha_1 > 0$ if $0 < \rho < 1$ allows in [5] to apply an extension of Rolle's theorem yielding the following improvement of Theorem 1.

Theorem 4. *Under the assumptions of Theorem 1 the function y can be chosen in such a way that $y(t) \leq x(t)$ for every t in* $[0, 1]$.

A new existence result for a class of non–convex optimization problems then follows ([5]). Chebyshev measures provide a broad class of measures whose range is strictly convex.

Theorem 5. [7] *The range $\mathcal{R}(\mu)$ of a T_ν–measure is strictly convex. A point $\mu(E)$ belongs to the boundary of $\mathcal{R}(\mu)$ if and only if there exists γ in the boundary of Γ_n such that $\mu(E\Delta E_\gamma) = 0$. In particular $O = \mu(E_{(1,\ldots,1)})$ belongs to the boundary of $\mathcal{R}(\mu)$.*

Proof of strict convexity. Let E, F in Γ be such that $\mu(E) \neq \mu(F)$: then $|\mu|(E\Delta F) \neq 0$; let for instance $|\mu|(E \setminus F) > 0$. Then for λ in $]0, 1[$ the function $\rho = \lambda\chi_E + (1 - \lambda)\chi_F$ is such that $0 < \rho < 1$ a.e. on $E \setminus F$. Theorem 3 implies that $\int \rho \, d\mu = \lambda\mu(E) + (1 - \lambda)\mu(F)$ belongs to the interior of $\mathcal{R}(\mu)$. $\square$

We point out that $\gamma = (\gamma_1, \ldots, \gamma_n)$ belongs to the boundary of Γ_n if and only if the characteristic function of the set E_γ has at most $n - 1$ discontinuity points.

Let us recall that the range of a non–atomic vector measure is closed, convex, contains the origin ($\mu(\emptyset) = O$) and is centrally symmetric ($\mu(X \setminus A) + \mu(A) = \mu(X)$). Conversely, if $n = 2$, Herz's theorem [8] shows that a non empty, compact, centrally symmetric subset of $\mathbb{R}^2$ containing the origin is the range of a bidimensional measure. We show that if such a set is strictly convex then it is a translate of the range of a Chebyshev measure.

Theorem 6. [1] *Let K be a non empty, compact, centrally symmetric, strictly convex subset of $\mathbb{R}^2$ such that O belongs to the boundary of K. Then there exists a Chebyshev measure μ on the Borelians of* $[0, 1]$ *such that $K = \mathcal{R}(\mu)$.*

Sketch of the proof. Since O is an exposed point of $\mathcal{R}(\mu)$ we may assume, modulo a rotation, that $K \subset \{(x, y) : x \geq 0\}$ and that the y–axis is a supporting line to K.
Let $L = \max\{x : (x, y) \in K\}$, $y : [0, L] \to \mathbb{R}$ be the function defined by

$$\forall x \in [0, L] \qquad y(x) = \min\{y : (x, y) \in K\}$$

and set $G(x) = (x, y(x))$. It is easy to prove that, the set K being symmetric,

$$K = \{G(x_2) - G(x_1) : 0 \leq x_1 \leq x_2 \leq 1\}. \tag{$\circ$}$$

Moreover the map y is strictly convex: therefore there exists a strictly monotonic function g such that $y(x) = \int_0^x g(t) \, dt$ for $x \leq L$. Let μ be the measure whose density function with respect to the Lebesgue measure λ is $f = (1, g)$; clearly f is a T_λ–system so that μ is a Chebyshev measure with respect to λ. Moreover by $(\circ)$ we have

$$K = \{\mu(x_1, x_2) : 0 \leq x_1 \leq x_2 \leq 1\}$$

and thus $K \subset \mathcal{R}(\mu)$. Actually it turns out easily that $K = \mathcal{R}(\mu)$. $\square$

The above argument also yield a new and constructive proof of Herz's theorem.

Remark. With the notations of Theorem 6, in [2] we show that if K is the range of a bidimensional measure μ then necessarily μ is a Chebyshev measure (in the generalized sense defined in [7]).

When $n > 2$ there is no analogue of Herz's theorem: for instance an octahedron is not the range of a measure. R. Schneider recently gave the following characterization of the measures whose range is strictly convex.

Theorem 7. [12] *$\mathcal{R}(\mu)$ is strictly convex if and only if for every A such that $\mu(A) \neq O$ there exist $A_1, \ldots, A_n$ in A such that $\mu(A_1), \ldots, \mu(A_n)$ are linearly independent.*

However it seems difficult to check whether a measure satisfies this condition. In [2] we present two characterizations in terms of the Radon–Nikodym derivative f of μ with respect to the total variation $|\mu|$. We recall that $f \in L^1_{|\mu|}(X, \mathbb{R}^n)$.

Theorem 8. [2] *The following equivalence holds:*

1) *$\mathcal{R}(\mu)$ is strictly convex;*
2) *For every p in $\mathbb{R}^n \setminus \{O\}$ the set $\{x \in X : p \cdot f(x) = 0\}$ is negligible;*
3) *$\det[f(x_1), \ldots, f(x_n)] \neq 0$ a.e. on X^n.*

Sketch of the proof. 1) $\Leftrightarrow$ 2): a closed convex set K is strictly convex if and only if each of its exposed faces is reduced to a point. If p is a non–zero vector we define

$$D^+(p) = \{x : p \cdot f(x) > 0\}, \quad D^-(p) = \{x : p \cdot f(x) < 0\}, \quad D^0(p) = \{x : p \cdot f(x) = 0\}.$$

The support function $h(\mathcal{R}(\mu), p)$ of $\mathcal{R}(\mu)$ at p is given by

$$\begin{aligned} h(\mathcal{R}(\mu), p) &= \max_{E \in \mathcal{M}} p \cdot \mu(E) = \max_{E \in \mathcal{M}} \int_E p \cdot f(x)\, d|\mu| = \\ &= \int_{E \cap D^-(p)} p \cdot f(x)\, d|\mu| + \int_{E \cap D^+(p)} p \cdot f(x)\, d|\mu| \\ &\leq \int_{E \cap D^+(p)} p \cdot f(x)\, d|\mu| \leq \int_{D^+(p)} p \cdot f(x)\, d|\mu|. \end{aligned}$$

Moreover the above inequalities show that, for E in $\mathcal{M}$, the equality $p\cdot\mu(E) = p\cdot\mu(D^+(p))$ holds if and only if $D^+(p) \subset E \subset D^+(p) \cup D^0(p)$. It follows that the exposed face $F(\mathcal{R}(\mu), p)$ of outer normal vector p is given by

$$F(\mathcal{R}(\mu), p) = \{\mu(E) : D^+(p) \subset E \subset D^+(p) \cup D^0(p)\};$$

therefore each exposed face is reduced to a point if and only if $|\mu|(D^0(p)) = 0$ for every p.
2) $\Leftrightarrow$ 3) (for $n = 2$): Let $\Delta = \{(x, y) : \det[f(x), f(y)] = 0\}$. If $\mathcal{R}(\mu)$ is not strictly convex there exists p in $\mathbb{R}^2 \setminus \{O\}$ such that $|\mu|(D^0(p)) > 0$: clearly

$$D^0(p) \times D^0(p) \subset \Delta$$

so that $|\mu|^{\otimes 2}(\Delta) > 0$. Conversely let $\mathcal{R}(\mu)$ be strictly convex. If $|\mu|^{\otimes 2}(\Delta) > 0$ by Fubini–Tonelli's Theorem there exists x such that $f(x)$ does not vanish and the $x-$ section of Δ given by $\Delta_x = \{y : \det[f(x), f(y)] = 0\}$ is non negligible. Let p be orthogonal to $f(x)$; clearly $\Delta_x = D^0(p)$, a contradiction. □

Theorem 8 yields an alternative and simple proof to Schneider's Theorem 7; it has also several applications to Chebyshev measures.

Assume that μ is a T_ν–measure. If μ is absolutely continuous with respect to ν then, trivially, μ is a $T_{|\mu|}$–measure. If $|\mu|$ is not absolutely continuous w.r. to ν there exists a subset A in $\mathcal{M}^{\otimes n}$ such that $|\mu|^{\otimes n}(A) > 0$ and $\nu^{\otimes n}(A) = 0$ and therefore it is not clear whether $\det \mu(A) > 0$. Surprisingly Theorem 8 implies that the above inequality is satisfied.

Theorem 9. [2] *Let μ be a measure on $[0, 1]$. The following equivalence hold:*

1) μ is a Chebyshev measure with respect to some measure ν;

2) $\det[f(x_1), \dots, f(x_n)] > 0 \quad |\mu|^{\otimes n}$– a.e. on Γ_n;

3) μ is a $T_{|\mu|}$–measure.

Proof. The equivalence 2) $\Leftrightarrow$ 3) follows from Theorem 2 applied for $\nu = |\mu|$ and the implication 3) $\Rightarrow$ 1) is trivial. It remains to show that 1) $\Rightarrow$ 3). Let μ be a T_ν measure. By Theorem 5 the range of μ is strictly convex; Theorem 8 then implies that

$$\det[f(x_1), \dots, f(x_n)] \neq 0 \qquad |\mu|^{\otimes n} - \text{ a.e.}$$

Let

$$P^- = \{(x_1, \dots, x_n) \in \Gamma_n : \det[f(x_1), \dots, f(x_n)] < 0\}$$

and assume that $|\mu|^{\otimes n}(P^-) > 0$. Since, by definition

$$\det \mu(P^-) = \int_{P^-} \det[f(x_1), \dots, f(x_n)] d|\mu|^{\otimes n},$$

then by the continuity of $|\mu|$ with respect to the sets $[0,\alpha]$, there exists a $(2n)$–uple $\left(\alpha_1^1,\alpha_1^2,\cdots,\alpha_n^1,\alpha_n^2\right)$ in $\mathbb{R}^{2n}$ such that $0<\alpha_1^1<\alpha_1^2<\cdots<\alpha_n^1<\alpha_n^2<1$ and

$$\det \mu(A\cup P^-)<0, \qquad A=[\alpha_1^1,\alpha_1^2]\cup\cdots\cup[\alpha_n^1,\alpha_n^2].$$

However, ν being a positive measure, we have $\nu^{\otimes n}(A\cup P^-)\geq \nu^{\otimes n}(A)>0$ so that $\det\mu(A\cup P^-)>0$, a contradiction. Therefore f is a $T_{|\mu|}$–system; Theorem 2 yields the conclusion. □

Theorem 8 allows to give a topological characterization of bidimensional Chebyshev measures. For μ being a two dimensional measure on $[0,1]$ let

$$\theta:\Gamma_2=\{(x,y):0\leq x\leq y\leq 1\}\longrightarrow \mathcal{R}(\mu)$$

be the map defined by $\theta(\alpha,\beta)=\mu([\alpha,\beta])$. We have the following converse of Theorem 3.

Theorem 10. [1] *If θ induces a homeomorphism from the interior of Γ_2 onto the interior of $\mathcal{R}(\mu)$ then μ is a Chebyshev measure.*

The proof of Theorem 10 is based on the fact that the assumption on θ implies the strict convexity of $\mathcal{R}(\mu)$ so that Theorem 8 gives $\det[f(x),f(y)]\neq 0$ a.e.; a degree theory argument yields the conclusion.

Finally, we give a geometrical property of the range of Chebyshev measures.

Proposition. *Let $\mu=(\mu_1,\ldots,\mu_n)$ be a Chebyshev measure and $n\geq 3$. Then the boundary $\partial\mathcal{R}(\mu)$ of $\mathcal{R}(\mu)$ is not a $(n-1)$ dimensional $\mathcal{C}^1$–manifold.*

Sketch of the proof. Assume for simplicity that $|\mu|$ is the Lebesgue measure and that $n=3$; it is not restrictive to assume that μ is a Chebyshev measure in $[0,1]$ with respect to the intervals. For every α in $[0,1[$ let $\lambda_\alpha:[0,1-\alpha]\to\mathbb{R}^3$ be the curve defined by

$$\forall t\in[0,1-\alpha]: \qquad \lambda_\alpha(t)=\mu([\alpha,\alpha+t]).$$

The characteristic function of $[\alpha,\alpha+t]$ has at most two discontinuity points; since $n>2$ Theorem 5 implies that $\mu([\alpha,\alpha+t])=\lambda_\alpha(t)$ belongs to $\partial\mathcal{R}(\mu)$ for every t.
Remark that $\lambda_\alpha(0)=0$: if we assume that $\partial\mathcal{R}(\mu)$ is regular then at every Lebesgue point α of f the vector $f(\alpha)=\displaystyle\lim_{t\to 0^+}\frac{\mu([\alpha,\alpha+t])}{|\mu|([\alpha,\alpha+t])}$ belongs to the tangent plane to $\partial\mathcal{R}(\mu)$ at O; therefore if α,β,γ are Lebesgue points of f we have $\det[f(\alpha),f(\beta),f(\gamma)]=0$. However μ is a Chebyshev measure and thus $\det[f(\alpha),f(\beta),f(\gamma)]\neq 0$ a.e.; a contradiction. □

It follows that Theorem 6 cannot be generalized in higher dimensions: for instance it is well known that a ball in $\mathbb{R}^n$ containing the origin is the range of a non–atomic measure; however the latter Proposition shows that, if $n>2$, it is not the range of some Chebyshev measure.

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