AN ESTIMATE ON THE FLOW GENERATED BY MONOTONE OPERATORS

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Abstract. In this paper we study the transport equation

$$\mu_t + \operatorname{div}(a(t,x)\mu) = 0, \quad u_t + a(t,x) \cdot \nabla u = 0,$$

in the case where the vector field $a(t,x)$ is monotone in space. The main result is a stability result w.r.t. weak convergence of $a(t,x)$ of the corresponding flow.

1. Introduction

In this paper, we study the conservative transport equation

$$\mu_t + \operatorname{div}(a(t,x)\mu) = 0, \quad \mu(0) = \bar{\mu}, \tag{1.1}$$

and the advective transport equation,

$$u_t + a(t,x) \cdot \nabla u = 0, \tag{1.2}$$

in the case where the vector field $a(t,x)$ is monotone in space.

In [6], uniqueness for the conservative equation is obtained in the class of reversible solutions, which are defined by means of a generalized flow. This flow however is not unique, though its jacobian determinant is. The problem is that, given a maximal monotone operator $A(t,x)$, the vector field $a(t,x) \in A(t,x)$ is uniquely determined $\mathcal{L}^d$-a.e. When the measure μ, solution to (1.1) becomes singular w.r.t. $\mathcal{L}^d$, then depending on the selection $a(t,x)$, one can have multiple solutions, or no solution at all.

In the present paper, we give an explicit expression for the solution to the transport equation by means of the Filippov flow $X(t,x)$ of the differential inclusion

$$\dot{x} \in -A(t,x), \tag{1.3}$$

where $A(t,x)$ is a maximal monotone operator, of which $a(t,x)$ will be a selection. This selection yields a unique solution to (1.1), stable with respect to perturbations of the maximal monotone operator $A(t,x)$.

In Section 4, in fact, we prove a quantitative estimate for the stability of the Filippov flow with respect to strong convergence of A in $L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$. (Remeber that $A(t,x) = \{a(t.x)\}$ $\mathcal{L}^d$-a.e.)

Theorem 1.1. *Let $A_i(t)$, $i = 1,2$, be monotone operators in $L^1((0,T); L^\infty_{\mathrm{loc}}(\mathbb{R}^d))$ with the topology inherited from $L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$, and $x_i(t)$, $i = 1,2$, the solutions to (1.3) with the same initial data $x_i(0) = \bar{x} \in B(0,r)$. Then the following estimate holds:*

$$|x_1(t) - x_2(t)|^2 \leq C \int_0^t \Big(\|A_1(s)\|_{L^\infty(B(0,2R))} + \|A_2(s)\|_{L^\infty(B(0,2R))} \Big)^{1-1/d} \|A_1(s) - A_2(s)\|^{1/d}_{L^1(B(0,2R))} \mathrm{d}s, \tag{1.4}$$

for some constant C which depends only on the dimension d, and

$$R = r + \int_0^T \max\{|A_1(s,0)|, |A_2(s,0)|\} \mathrm{d}s.$$

We next prove that also weak stability holds for the flow.

Theorem 1.2. *Assume that the monotone functions A_n converge weakly in $L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$ to A. Then the flows X_n generated by A_n converge locally in $C^0((0,T) \times \mathbb{R}^d)$ to the flow X generated by A.*

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In Section 5, we then deduce by a uniqueness theorem from [3] that the unique solution to the conservative transport equation is given by the formula

$$\mu(t) = X(t)\sharp\mu(0),$$

where $X(t) = X(t, \cdot)$ is the flow of the maximal monotone extension $A(t, x)$ of the vector field $a(t, x)$. The solution to the advective equation is the obtained by the duality formulation

$$(u\mu)_t + \operatorname{div}(au\mu) = 0.$$

It is given by the formula

$$u(t)(X(t)\sharp\mu) = X(t)\sharp(u(0)\mu).$$

Finally, we obtain the explicit formula

$$u(t, X(t, x)) = u(0, x),$$

by which the duality solution is uniquely determined $\mathcal{L}^d$-a.e. We thus have the following theorem, which extends the result of [6]:

Theorem 1.3. *The solution (in duality sense) of the advective transport equation is uniquely determined as an L^∞ function. Moreover it depends continuously in the L^1_{loc}-norm w. r. t. the weak convergence of $A(t)$ in $L^1((0, T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$.*

2. Settings

We consider the differential inclusion

$$\dot{x}(t) \in -A(t, x(t)), \tag{2.1}$$

where $A(t, x)$ is a time-dependent quasi-monotone operator, i. e. it is a $\mathcal{L}^{d+1}$ measurable set-valued function from $[0, T] \times \mathbb{R}^d$ into $\mathbb{R}^d$ which for $\mathcal{L}^1$-a. e. t fulfills the condition

$$\langle x_1 - x_2, y_1 - y_2\rangle \geq -\alpha(t)|x_1 - x_2|^2 \quad \text{for all } x_i \in \mathbb{R}^d,\ y_i \in A(t, x_i),\ i = 1, 2, \tag{2.2}$$

for some $\alpha(t) \geq 0$. We assume that

$$\int_0^T \alpha(t)\,\mathrm{d}t < +\infty. \tag{2.3}$$

For $\mathcal{L}^1$-a. e. t the operator

$$A(t, x) + \alpha(t)x$$

is assumed to be maximal monotone and inclusion (2.1) holds for $\mathcal{L}^1$-a. e. t.

For $\mathcal{L}^1$-a. e. $t \in [0, T]$, $A(t) = A(t, \cdot)$ is single-valued $\mathcal{L}^d$-a. e. in its domain (cf. Theorem 2.2 in [4]). In the following, we will therefore use the same notation both for the set-valued function and for the $\mathcal{L}^d$-a. e. defined single-valued function. I denotes the identity function on $\mathbb{R}^d$, and $|A(t, x)| = \max\{|y| : y \in A(t, x)\}$.

We assume that

$$\int_0^T \|A(t)\|_{L^\infty(K)}\,\mathrm{d}t < \infty \tag{2.4}$$

for all compact subsets K of $\mathbb{R}^d$, where

$$\|A(t)\|_{L^\infty(K)} = \text{ess sup}\big\{|A(t, x)| : x \in K\big\}.$$

Note that since for any monotone function A

$$\|A\|_{L^\infty(B(0,R))} \leq C\|A\|_{L^1(B(0,2R))}, \quad \|A\|_{\mathrm{BV}_{\mathrm{loc}}(B(0,R))} \leq C\|A\|_{L^1(B(0,2R))}, \tag{2.5}$$

for some constant C independent of B (see [1], section 5), we can require equivalently that

$$\int_0^T \|A(t)\|_{L^1(K)}\,\mathrm{d}t < \infty,$$

i.e. $A \in L^1((0, T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$. Clearly the above L^∞ condition is equivalent to a uniform bound of the form

$$\int_0^T \sup\big\{|y|, y \in A(t, x), x \in B(0, R)\big\}\,\mathrm{d}t < \infty.$$

for all $R > 0$. For a survey on monotone functions see [1].

We can reduce the problem to the monotone case $\alpha(t) = 0$ by the following transformation: by setting

$$\tilde{x}(t) = x(t)\exp\left(-\int_0^t \alpha(s)\,\mathrm{d}s\right),$$

we have that the variable $\tilde{x}$ fulfills the differential inclusion

$$\dot{\tilde{x}}(t) \in -\tilde{A}(t, \tilde{x}(t)),$$

with $\tilde{A}(t, \tilde{x})$ given by

$$\tilde{A}(t, \tilde{x}) = A\left(t, \tilde{x}\exp\left(\int_0^t \alpha(s)\,\mathrm{d}s\right)\right)\exp\left(-\int_0^t \alpha(s)\,\mathrm{d}s\right) + \alpha(t)\tilde{x}.$$

Now let $\tilde{y}_i \in \tilde{A}(t, \tilde{x}_i)$, $i = 1, 2$, then

$$\tilde{y}_i = y_i \exp\left(-\int_0^t \alpha(s)\,\mathrm{d}s\right) + \alpha(t)\tilde{x}_i \quad \text{for some} \quad y_i \in A\left(t, \tilde{x}_i \exp\left(\int_0^t \alpha(s)\,\mathrm{d}s\right)\right),$$

thus by (2.2)

$$\langle \tilde{x}_1 - \tilde{x}_2, \tilde{y}_1 - \tilde{y}_2\rangle = \langle x_1 - x_2, y_1 - y_2\rangle \exp\left(-2\int_0^t \alpha(s)\,\mathrm{d}s\right) + \alpha(t)|\tilde{x}_1 - \tilde{x}_2|^2 \geq 0.$$

Thus $\tilde{A}(t, \tilde{x})$ is a monotone operator. Furthermore, for $\tilde{x} \in B(0, r)$ we have $x = \tilde{x}\exp\left(\int_0^t \alpha(s)\,\mathrm{d}s\right) \in B(0, R)$ with $R = r\exp(\|\alpha\|_{L^1(0,T)}) < +\infty$, and therefore

$$\int_0^T \|\tilde{A}(t, \cdot)\|_{L^\infty(B(0,r))}\,\mathrm{d}t \leq \int_0^T \|A(t, \cdot)\|_{L^\infty(B(0,R))}\,\mathrm{d}t + \|\alpha\|_{L^1(0,T)} r.$$

Thus condition (2.4) holds also for $\tilde{A}(t, \tilde{x})$. In the following we will therefore assume that $A(t)$ is maximal monotone for a. e. $t \in [0, T]$, i. e. it is monotone and its graph is maximal with respect to inclusion in $\mathbb{R}^d \times \mathbb{R}^d$. Under these assumptions, a classical result shows the existence of a Filippov flow: for completeness, we repeat the proof below.

The conditions (2.3) and (2.4) ensure that the trajectories do not blow up in finite time. Indeed, by (2.1) and the monotonicity of A for $\mathcal{L}^1$-a. e. $t \in [0, T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 = 2\langle x(t), \dot{x}(t)\rangle \leq -2\langle x(t), y\rangle \quad \text{for any } y \in A(t, 0),$$

and thus in particular

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)| \leq |A(t, 0)|,$$

which yields

$$|x(t)| \leq |x(0)| + \int_0^t |A(s, 0)|\,\mathrm{d}s. \tag{2.6}$$

Without additional assumptions, the trajectories of (2.1) could collapse to a single point in finite time. To avoid this, we assume

$$|A(t, x)| \leq C(1 + |x|) \tag{2.7}$$

for a. e. $t \in [0, T]$, all $x \in \mathbb{R}^d$. With this bound, the flow will be surjective in $\mathbb{R}^d$ at all times $t \in [0, T]$. We note that many of the following results do not depend on this assumption.

3. Existence of a solution

A solution of the differential inclusion (2.1) can be constructed by means of the *Yosida approximations.* In the following, we generalize the construction in [4], Chapter 3, to the non-autonomous case.

Definition 3.1. We define the Yosida approximation $A_\lambda(t, x)$ of $A(t, x)$ by setting for $\mathcal{L}^1$-a. e. $t \in [0, T]$

$$A_\lambda(t) = \lambda^{-1}(I - (I + \lambda A(t))^{-1}).$$

Lemma 3.2. *The following holds:*

(1) *The map* $(I+\lambda A(t))^{-1}$ *is non-expansive (and thus in particular single-valued), and* $A_\lambda(t)$ *is Lipschitz continuous with Lipschitz constant bounded by* λ^{-1} *([4], p. 146, Theorem 2), and monotone.*

(2) *The graph of* $A_\lambda(t)$ *is obtained by shearing the graph of* $A(t)$ *in* $\mathbb{R}^d \times \mathbb{R}^d$ *in the following sense: for all* $x \in \mathbb{R}^d$,

$$y = A_\lambda(t, x+\lambda y) \quad \Longleftrightarrow \quad y \in A(t,x). \tag{3.1}$$

(3) *The Yosida approximations form a semigroup with respect to the parameter* λ*:*

$$(A_\lambda)_\mu = A_{\lambda+\mu}. \tag{3.2}$$

(4) $|A_\lambda(t,x)|$ *is monotone in* λ*:*

$$|A_{\lambda+\mu}(t,x)| \leq |A_\lambda(t,x)| \quad \text{for all } x \in \mathbb{R}^d,\ \lambda,\mu \geq 0, \tag{3.3}$$

where we denote $A_0 = A$.

Proof. We prove only points (2) through (4).

(2) Since $(I+\lambda A(t))^{-1}$ is single-valued, we have that

$$(I+\lambda A(t))^{-1}(x+\lambda A(t,x)) = \{x\}.$$

(3.1) then follows from Definition 3.1.

(3) By (3.1), we have that

$$y \in A(t,x) \quad \Longleftrightarrow \quad y = A_\lambda(t,x+\lambda y) \quad \Longleftrightarrow \quad y = A_{\lambda+\mu}(t, x+(\lambda+\mu)y).$$

Applying (3.1) to A_λ yields that

$$y = A_\lambda(t,x+\lambda y) \quad \Longleftrightarrow \quad y = (A_\lambda)_\mu(t, x+\lambda y+\mu y).$$

Hence (3.2) follows.

(4) Take any $x \in \mathbb{R}^d$, $y \in A_\lambda(t,x)$, $z \in A_\lambda(t,x+\mu y)$. By the monotonicity of $A_\lambda(t,x)$, we have

$$\langle y, z-y\rangle \geq 0$$

and therefore

$$|z| \geq |y|.$$

Using (3.1) and (3.2), the claimed inequality (3.3) follows, since $(I+\mu A_\lambda(t))$ is surjective by Minty's Theorem ([4], p. 142, Theorem 1).

□

Proposition 3.3. *For any initial datum* $x(0) = x_0 \in \mathbb{R}^d$ *there exists a unique solution* $x(t)$ *of* (2.1) *on* $[0,T]$*, with* 1*-Lipschitz dependence on the initial datum.*

Proof. Due to the Lipschitz continuity of $A_\lambda(t)$,

$$\dot{x}_\lambda = -A_\lambda(t,x_\lambda), \quad x_\lambda(0) = x_0$$

has a unique absolutely continuous solution $x_\lambda(t)$, with Lipschitz continuous dependence on the initial datum x_0.

In order to show the convergence of $x_\lambda(t)$ as $\lambda \to 0$, we first note that by (3.1), $A_\lambda(t,x_\lambda) \in A(t,(I+\lambda A(t))^{-1}(x_\lambda))$, and therefore

$$\Big\langle (I+\lambda A(t))^{-1}(x_\lambda) - (I+\mu A(t))^{-1}(x_\mu), A_\lambda(t,x_\lambda) - A_\mu(t,x_\mu)\Big\rangle \geq 0.$$

Thus we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|x_\lambda - x_\mu|^2 &= -2\langle x_\lambda - x_\mu, A_\lambda(t,x_\lambda) - A_\mu(t,x_\mu)\rangle \\
&\leq -2\Big\langle x_\lambda - (I+\lambda A(t))^{-1}(x_\lambda) - x_\mu + (I+\mu A(t))^{-1}(x_\mu), A_\lambda(t,x_\lambda) - A_\mu(t,x_\mu)\Big\rangle \\
&= -2\Big\langle \lambda A_\lambda(t,x_\lambda) - \mu A_\mu(t,x_\mu), A_\lambda(t,x_\lambda) - A_\mu(t,x_\mu)\Big\rangle \\
&\leq 2(\lambda+\mu)|A_\lambda(t,x_\lambda)||A_\mu(t,x_\mu)| \leq 2(\lambda+\mu)\|A(t)\|^2_{L^\infty(B(0,R))},
\end{aligned}$$

with $R = |x_0| + \int_0^T |A(s,0)|\,\mathrm{d}s$. Here we have used (2.6) and (3.3). We thus obtain

$$|x_\lambda - x_\mu|^2 \le (\lambda+\mu)C \quad \text{where} \quad C = 2\int_0^T \|A(t)\|^2_{L^\infty(B(0,R))}\,\mathrm{d}t < \infty$$

by the boundedness assumption (2.7). Thus there exists a uniform limit $x_\lambda \to x$ on $[0,T]$.

It remains to show that x is a solution to (2.1). Since the maximal monotone operator $A(t)$ is upper semicontinuous ([1], Corollary 1.3), it follows from $x_\lambda \to x$ and (3.1) that for a sequence λ_i,

$$A_{\lambda_i}(t, x_{\lambda_i}(t)) \to y(t) \in A(t,x(t)) \quad \text{for } \mathcal{L}^1\text{-a. e. } t \in [0,T].$$

More generally, since $A_\lambda(t,x)$ is convex, we have that if $x_i(t) \to x(t)$, $\lambda_i \to 0$, for all convex combinations

$$\alpha_{ij} \ge 0, \quad \sum_{i\ge j} \alpha_{ij} = 1,$$

the following holds:

$$\sum_{i\ge j} \alpha_{ij} A_{\lambda_i}(t, x_i(t)) \ni y_j(t) \to y(t) \in A(t,x(t)). \tag{3.4}$$

Using the bound (3.3), we can pass to the limit $\lambda \to 0$ in

$$x_\lambda(t) = x_0 + \int_0^t A_\lambda(s, x_\lambda(s))\,\mathrm{d}s$$

to obtain

$$x(t) = x_0 + \int_0^t y(s)\,\mathrm{d}s.$$

Hence there exists a linear combination

$$\sum_{i\ge j} \alpha_{ij}\dot{x}_i(t), \quad \alpha_{ij} \ge 0, \ \sum_{i\ge j} \alpha_{ij} = 1,$$

converging strongly to $y(t)$. Using (3.4) and dominated convergence, it follows that

$$\dot{x}(t) = y(t) \in A(t,x(t)) \quad \text{for } \mathcal{L}^1\text{-a. e. } t \in [0,T],$$

therefore x is a solution to (2.1).

Now let $x_i(t)$, $i = 1,2$ be two solutions to (2.1). By the monotonicity of $A(t,x)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|x_1(t) - x_2(t)|^2 = 2\langle x_1(t) - x_2(t), \dot{x}_1(t) - \dot{x}_2(t)\rangle \le 0,$$

and thus

$$|x_1(t) - x_2(t)| \le |x_1(0) - x_2(0)|.$$

Thus the solution depends 1-Lipschitz continuously on the initial datum x_0. In particular, for any initial datum the solution is unique. □

We denote by $X(t,s,x)$, $t \ge s$, the flow of the inclusion,

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t,s,x) \in -A(t, X(t,s,x)), \quad X(s,s,x) = x,$$

and by $X(t,x) = X(t,0,x)$ the flow restricted to initial time $s = 0$.

With the bound (2.7), we have that $X(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is surjective for all $t \in [0,T]$. In fact, from

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}X(t,x)\right| \le C(1 + |X(t,x)|)$$

it follows that

$$|X(t,x)| \ge (1+|x|)e^{-Ct} - 1.$$

Therefore, given $p \in \mathbb{R}^d$, we have that for $R > (|p|+1)e^{Ct} - 1$

$$p \notin X(s, \partial B_R(0)), \quad \text{for all } s \in [0,T],$$

so we get the topological degree

$$\deg(X(t,\cdot), B_R(0), p) = \deg(I, B_R(0), p) = 1,$$

since $p \in B_R(0)$. Thus there is an $x \in \mathbb{R}^d$ such that $X(t,x) = p$. In particular, the flow $X(t,s,x)$ is completely determined by its values $X(t,x)$ for initial time $s = 0$ by the semigroup property:

$$X(t,s,x) = X(t,y) \quad \text{for } y \in [X(s,\cdot)]^{-1}(x).$$

In the following, we will therefore refer to both $X(t,s,x)$ and $X(t,x)$ as the flow of the differential inclusion (2.1).

Remark 3.4. In the autonomous case, one can recover an ODE by selecting the element with minimal norm in $A(t,x)$ ([4], p. 147, Theorem 2). This, however, is not true in general in the non-autonomous case, as can be seen from the example of a *"sliding motion"*:

$$A(t,x) = \begin{cases} -1 & x < t/2 \\ [-1,0] & x = t/2 \\ 0 & x > t/2 \end{cases}$$

where any flow line entering the singularity $x = t/2$ at time $t = t_0$ will remain on it for all $t > t_0$, and then $\dot{x}(t) = 1/2$.

Since an a. e. defined monotone function has a unique maximal monotone extension, each operator $A(t,x) \in L^1((0,T); L^\infty_{\text{loc}}(\mathbb{R}^d))$ generates a unique flow $X(t,x)$ solving (2.1). In fact, there is a one-to-one correspondence, as the following proposition shows.

Proposition 3.5. *Assume $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ with $X(0,x) = x$ is absolutely continuous in t,*

$$\frac{\mathrm{d}}{\mathrm{d}t}|X(t,x_1) - X(t,x_2)| \le 0 \quad \textit{for all } x_1, x_2 \in \mathbb{R}^d, \ \textit{a. e. } t \in [0,T], \tag{3.5}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}X(t,x)\right| \le C(1 + |X(t,x)|) \quad \textit{for all } x \in \mathbb{R}^d, \ \textit{a. e. } t \in [0,T]. \tag{3.6}$$

Then there exists a unique maximal monotone operator $A(t,x)$ in $L^1((0,T); L^\infty_{\text{loc}}(\mathbb{R}^d))$ with the bound (2.7) *generating X.*

Proof. By assumption, $\frac{\mathrm{d}}{\mathrm{d}t}X(t,x)$ is defined for all $x \in \mathbb{R}^d$, a. e. $t \in [0,T]$. Using (3.5), we can define

$$a(t, X(t,x)) = -\frac{\mathrm{d}}{\mathrm{d}t}X(t,x).$$

Then $a(t,X(t,x))$ is defined for $\mathcal{L}^{d+1}$-a. e. $(t,x) \in [0,T] \times \mathbb{R}^d$. As we have seen above, the bound (3.6) implies that $X(t,\cdot)$ is surjective. Since by the assumptions $X(t,x)$ is absolutely continuous in t and Lipschitz continuous in x, it follows that $\mathcal{L}^d(X(t,A)) = 0$ if $\mathcal{L}^d(A) = 0$. Then $a(t,x)$ is defined for $\mathcal{L}^{d+1}$-a. e. $(t,x) \in [0,T] \times \mathbb{R}^d$. It remains to show that $a(t,x)$ is monotone in x. This follows again from (3.5):

$$\begin{aligned} &\langle a(t,X(t,x_1)) - a(t,X(t,x_2)), X(t,x_1) - X(t,x_2)\rangle \\ &= -\left\langle \frac{\mathrm{d}}{\mathrm{d}t}X(t,x_1) - \frac{\mathrm{d}}{\mathrm{d}t}X(t,x_2), X(t,x_1) - X(t,x_2)\right\rangle \\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|X(t,x_1) - X(t,x_2)|^2 \ge 0. \end{aligned}$$

□

Remark 3.6. Note that $a(t,x)$ is defined for $\mathcal{L}^{d+1}$-a. e. $(t,x) \in [0,T] \times \mathbb{R}^d$: we will show that it is actually defined for $\mu(t)$-a. e. (t,x) if $\mu(t)$ is a solution of the conservative transport equation (1.1).

4. Continuous dependence

In this section we study the dependence of the solutions to (2.1) on the monotone operator A.

We consider two monotone operators $A_i(t)$, $i = 1,2$, in $L^1((0,T); L^\infty_{\text{loc}}(\mathbb{R}^d))$ with the topology inherited from $L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^d))$. Let $x_i(t)$ be the solutions to

$$\begin{cases} \dot{x}_i(t) \in -A_i(t, x_i(t)), \\ x_i(0) = \bar{x}, \end{cases} \qquad i = 1,2. \tag{4.1}$$

By (2.6), we have that for initial data $\bar{x} \in B(0,r)$, the solutions are bounded by

$$|x_i(t)| \le R, \qquad R = r + \int_0^T \max\{|A_1(s,0)|, |A_2(s,0)|\}\mathrm{d}s.$$

Theorem 4.1. *Let $A_i(t)$, $i = 1,2$, be monotone operators in $L^1((0,T); L^\infty_{\mathrm{loc}}(\mathbb{R}^d))$ with the topology inherited from $L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$, and $x_i(t)$, $i = 1,2$, the solutions to (4.1) with the same initial data $\bar{x} \in B(0,r)$. Then the following estimate holds:*

(4.2)

$$|x_1(t) - x_2(t)|^2 \le C \int_0^t \Big(\|A_1(s)\|_{L^\infty(B(0,2R))} + \|A_2(s)\|_{L^\infty(B(0,2R))} \Big)^{1-1/d} \|A_1(s) - A_2(s)\|^{1/d}_{L^1(B(0,2R))} \mathrm{d}s,$$

for some constant C which depends only on the dimension d.

Proof. The idea of the proof is that by the monotonicity of $A_i(t)$, we obtain that whenever the solutions are moving apart, i. e. $\frac{\mathrm{d}}{\mathrm{d}t}|x_1(t) - x_2(t)| > 0$, then $A_1(t)$ and $A_2(t)$ differ on a set of positive Lebesgue measure in $\mathbb{R}^d$.

First note that we can assume $A_i(t,x)$ to be Lipschitz continuous in x (and therefore single-valued). In fact, if $x_{i,\lambda}$ is the solution to (4.1), with the Yosida approximation $A_{i,\lambda}$ instead of A_i, then by Proposition 3.3 the solutions converge to x_i for $\lambda \to 0$, and (3.3) yields the estimate.

For the two solutions $\dot{x}_i = -A_i(t, x_i)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|x_1 - x_2|^2 = -2\langle x_1 - x_2, A_1(t,x_1) - A_2(t,x_2)\rangle,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}|x_1 - x_2| = \left\langle \frac{x_1 - x_2}{|x_1 - x_2|}, A_1(t,x_1) - A_2(t,x_2) \right\rangle. \tag{4.3}$$

For any fixed $t \in [0,T]$, consider a point $x = \alpha x_1 + (1-\alpha)x_2 + z \in \mathbb{R}^d$. By the monotonicity,

$$\langle z - (1-\alpha)(x_1 - x_2), A_1(t,x) - A_1(t,x_1)\rangle \ge 0,$$
$$\langle z + \alpha(x_1 - x_2), A_2(t,x) - A_2(t,x_2)\rangle \ge 0,$$

and therefore

$$\langle x_1 - x_2, A_1(t,x)\rangle \le \langle x_1 - x_2, A_1(t,x_1)\rangle + \frac{1}{1-\alpha}\langle z, A_1(t,x) - A_1(t,x_1)\rangle,$$
$$\langle x_1 - x_2, A_2(t,x)\rangle \ge \langle x_1 - x_2, A_2(t,x_2)\rangle - \frac{1}{\alpha}\langle z, A_2(t,x) - A_2(t,x_2)\rangle.$$

Since $x_1, x_2 \in B(0,R)$, taking $z \in B(0,R)$, $\alpha \in [0,1]$, we have that $x \in B(0,2R)$. Using (4.3), we then obtain for $\alpha \in [\frac{1}{4}, \frac{3}{4}]$

$$\left\langle \frac{x_1 - x_2}{|x_1 - x_2|}, A_2(t,x) - A_1(t,x) \right\rangle \ge \frac{\mathrm{d}}{\mathrm{d}t}|x_1 - x_2| - 8K(t)\frac{|z|}{|x_1 - x_2|},$$

where $K(t)$ is the $L^1(0,T)$ function defined as

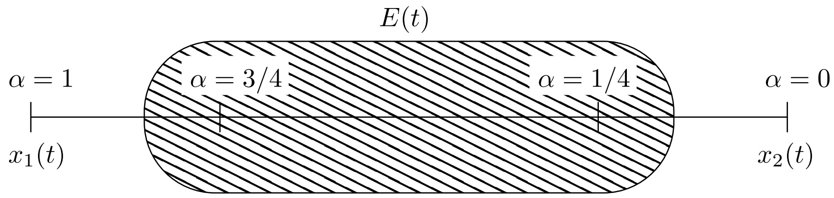
$$K(t) = \|A_1(t)\|_{L^\infty(B(0,2R))} + \|A_2(t)\|_{L^\infty(B(0,2R))}.$$

It follows that

$$|A_1(t,x) - A_2(t,x)| \ge \max\left\{0, \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x_1 - x_2|\right\}$$

in the set

$$E(t) = \left\{ x = \alpha x_1 + (1-\alpha)x_2 + z : \frac{1}{4} \le \alpha \le \frac{3}{4}, |z| \le \frac{|x_1 - x_2|}{16K(t)} \max\left\{0, \frac{\mathrm{d}}{\mathrm{d}t}|x_1 - x_2|\right\} \right\}.$$



Since the volume of this set is equal to

$$\frac{1}{2}|x_1 - x_2|\,\omega(d-1)\left(\frac{|x_1-x_2|}{16K(t)}\max\left\{0, \frac{\mathrm{d}}{\mathrm{d}t}|x_1-x_2|\right\}\right)^{d-1},$$

where $\omega(d-1)$ is the volume of the unit sphere in $\mathbb{R}^{d-1}$, it follows that

$$\int_{B(0,2R)} |A_1(t,x) - A_2(t,x)|\,\mathrm{d}x \geq \frac{\omega(d-1)}{2^{4d-3}}\frac{1}{K(t)^{d-1}}\left(\max\left\{0, \frac{\mathrm{d}}{\mathrm{d}t}|x_1-x_2|\right\}\right)^d |x_1-x_2|^d$$

Thus with some constant $C = C(d)$ we have the estimate

$$\begin{aligned}|x_1(t) - x_2(t)|^2 &= 2\int_0^t |x_1(s)-x_2(s)|\frac{\mathrm{d}}{\mathrm{d}t}|x_1(s)-x_2(s)|\,\mathrm{d}s\\ &\leq 2\int_0^t |x_1(s)-x_2(s)|\max\left\{0, \frac{\mathrm{d}}{\mathrm{d}t}|x_1(s)-x_2(s)|\right\}\mathrm{d}s\\ &\leq C\int_0^t K(s)^{1-1/d}\left(\int_{B(0,2R)}|A_1(s,x)-A_2(s,x)|\,\mathrm{d}x\right)^{1/d}\mathrm{d}s.\end{aligned}$$

This proves (4.2). □

Note that the exponent in (4.2) is sharp. This can be seen for example by taking A_i to be the monotone functions obtained as the subdifferentials of the convex functions

$$\phi_1(x) = |x|, \qquad \phi_2(x) = \max\{0, |x| - c\}, \quad c > 0.$$

Taking an initial datum $x_i(0) = \bar{x}$ with $|\bar{x}| = c$ one has on the one hand $x_1(t) = 0$ for all $t \geq c$ and $x_2(t) = \bar{x}$ for all $t \geq 0$, so that

$$|x_1(t) - x_2(t)|^2 = c^2 \quad \text{for all } t \geq c.$$

On the other hand,

$$\|A_i\|_{L^\infty(B(0,2R))} = 1, \qquad \|A_1 - A_2\|_{L^1(B(0,2R))} = \omega(d)c^d,$$

where $\omega(d)$ is the volume of the unit ball in $\mathbb{R}^d$. Evaluating (4.2) at time $t = c$, one has therefore that both sides of the inequality are proportional to c^2.

Remark 4.2. Applying Hölder's inequality with exponent d to (4.2) yields that in the set

$$\left\{A(t,x) : \|A(t)\|_{L^\infty(B(0,2R))} \leq H(t)\right\}, \qquad H(t) \in L^1(0,T),$$

we have the estimate

$$\|X_1(t) - X_2(t)\|_{C^0(B(0,r))} \leq \tilde{C}\|A_1 - A_2\|^{1/d}_{L^1((0,T)\times B(0,2R))}, \qquad \tilde{C} = 2C\left(\int_0^T H(s)\mathrm{d}s\right)^{1-1/d}. \tag{4.4}$$

Note that for $d \geq 2$, the constant $\tilde{C}$ cannot be chosen independently of $H(t)$, as can be seen from the following example. For simplicity of notation, we consider the case $d = 2$; clearly, this can easily be extended to higher dimensions. We denote $x = (x_1, x_2) \in \mathbb{R}^2$, and define for $|x_1| \leq 1$, $c > 0$:

$$a^{(1)}(x) = (0, cx_2),$$

$$a^{(2)}(x) = (cb(x), cx_2), \quad \text{where} \quad b(x) = -\frac{(\sqrt{2/c} - |x_2|)^2}{2(x_1+2)} \text{ for } |x_2| < \sqrt{2/c}, \text{ and } b(x) = 0 \text{ otherwise.}$$

One can check that $a^{(1)}$, $a^{(2)}$ are monotone functions on $[-1,1] \times \mathbb{R}$, which we can extend to maximal monotone operators $A^{(1)}$, $A^{(2)}$ on $\mathbb{R}^2$, coinciding with $a^{(1)}$ and $a^{(2)}$ respectively on $(-1,1)\times\mathbb{R}$. Let $R = 1/2$, $T = 1$, and $x^{(1)}$, $x^{(2)}$ the solutions to

$$\begin{cases}\dot{x}^{(i)}(t) = -a^{(i)}(t, x^{(i)}(t)),\\ x^{(i)}(0) = -1/2,\end{cases} \qquad i = 1,2.$$

Then on the one hand,

$$|x^{(1)}(1) - x^{(2)}(1)| > 1/2$$

does not depend on c. On the other hand, $|a^{(1)}(x) - a^{(2)}(x)| \leq 1$ for $x \in B(0, 2R)$, and since $a^{(1)}(x) = a^{(2)}(x)$ for $|x_2| \geq \sqrt{2/c}$, it follows that

$$\|A^{(1)} - A^{(2)}\|_{L^1(B(0,2R))} \to 0$$

for $c \to +\infty$.

The previous result is a quantitative estimate of the distance of two trajectories with the same initial data (for different initial data one can use the 1-Lipschitz estimate). In the following theorem we weaken the assumptions, requiring only weak convergence.

Theorem 4.3. *Assume that the monotone functions A_n converge weakly in $L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$ to A. Then the flows X_n generated by A_n converge locally in $C^0((0,T) \times \mathbb{R}^d)$ to the flow X generated by A.*

Proof. The weak convergence implies the equi-integrability of the sequence A_n. In particular, using (2.5), we have that for any $R > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_{t_1}^{t_2} \|A_n(t)\|_{L^\infty(B(0,R))} \, \mathrm{d}t < \epsilon \quad \text{for all } n \in \mathbb{N}, [t_1, t_2] \subset [0, T] \text{ such that } |t_1 - t_2| < \delta.$$

Using (2.6), it follows that the sequence X_n is locally uniformly bounded. It then follows from (2.7) that the sequence X_n is locally uniformly Lipschitz. By passing to a subsequence, we can therefore assume that X_n converges locally uniformly to some flow $\tilde{X}$.

Assume now by contradiction that $\tilde{X} \neq X$. By continuity, we find $\epsilon > 0$, $x_0 \in \mathbb{R}^d$ and $t_1 < t_2 \in [0, T]$ such that

$$|\tilde{X}(t_1, x_0) - X(t_1, x_0)| = \epsilon,$$
$$|\tilde{X}(t_2, x_0) - X(t_2, x_0)| = 2\epsilon.$$

Let $R \in (0, +\infty)$ be large enough for $B(0, R)$ to contain $X(t, x_0)$ and $X_n(t, x_0)$ for all $n \in \mathbb{N}$, $t \in [t_1, t_2]$. By the same computation as in the proof of Theorem 4.1, we get for $\xi(\alpha, z) = \alpha X_n(t, x_0) + (1-\alpha) X(t, x_0) + z$, $\alpha \in [1/4, 3/4]$,

$$-\langle v_n(t), A_n(t, \xi(\alpha, z)) - A(t, \xi(\alpha, z)) \rangle \geq \frac{\mathrm{d}}{\mathrm{d}t} |X_n(t, x_0) - X(t, x_0)| - 8K_n(t) \frac{|z|}{|X_n(t, x_0) - X(t, x_0)|}, \tag{4.5}$$

where we denote

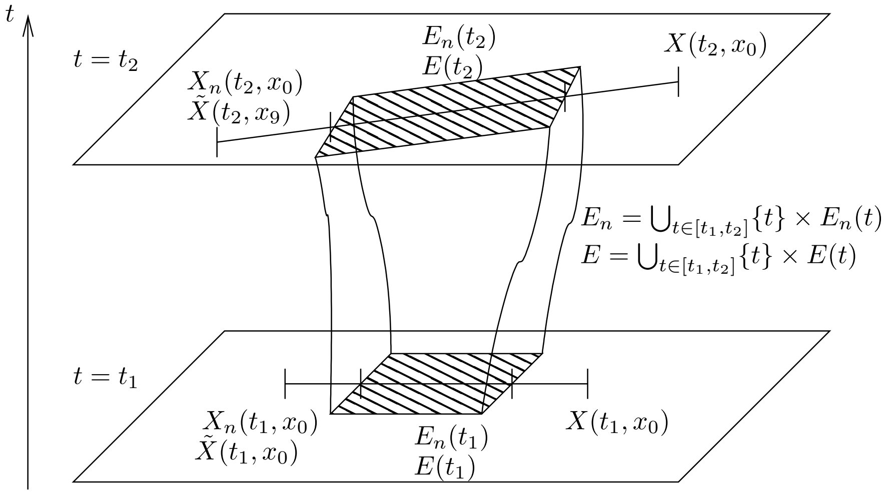
$$v_n(t) = \frac{X_n(t, x_0) - X(t, x_0)}{|X_n(t, x_0) - X(t, x_0)|}$$

and

$$K_n(t) = \|A_n(t)\|_{L^\infty(B(0,R))} + \|A(t)\|_{L^\infty(B(0,R))}.$$

In this case, we cannot pass to the modulus on the left-hand side, therefore we cannot integrate over a fixed set as in the strong convergence case. We define the sets

$$E_n(t) = \Big\{ \xi(\alpha, z) : \alpha \in [1/4, 3/4], \ z \perp (X_n(t, x_0) - X(t, x_0)), \ |z| < \delta \Big\},$$
$$E(t) = \Big\{ \xi(\alpha, z) : \alpha \in [1/4, 3/4], \ z \perp (\tilde{X}(t, x_0) - X(t, x_0)), \ |z| < \delta \Big\},$$
$$E_n = \bigcup_{t \in [t_1, t_2]} \{t\} \times E_n(t), \qquad E = \bigcup_{t \in [t_1, t_2]} \{t\} \times E(t).$$



We have that the Lebesgue measure of $E_n(t)$ is

$$\mathcal{L}^d(E_n(t)) = \frac{1}{2}\omega(d-1)|X_n(t,x_0) - X(t,x_0)|\delta^{d-1}.$$

Moreover, it follows from the uniform convergence of X_n to $\tilde{X}$ that $\mathcal{L}^d(E_n \triangle E) \to 0$. Integrating the inequality (4.5) in ξ over $E_n(t)$ gives

$$-\int_{E_n(t)} \langle v_n(t), A_n(t,\xi) - A(t,\xi)\rangle \,\mathrm{d}\xi \geq \delta^{d-1}\omega(d-1)\frac{\mathrm{d}}{\mathrm{d}t}|X_n(t,x_0) - X(t,x_0)|^2 - 4K_n(t)\omega(d-1)\delta^d.$$

By the equi-integrability, $\bar{K} = \sup_n \int_{t_1}^{t_2} K_n(t)\,\mathrm{d}t < +\infty$. Moreover, for n sufficiently large we have $|X_n(t,x_0) - \tilde{X}(t,x_0)| \leq \epsilon/4$. Therefore, integrating the inequality in t from t_1 to t_2 gives

$$\begin{aligned}
-&\int_{t_1}^{t_2}\int_{E_n(t)} \langle v_n, A_n(t,\xi) - A(t,\xi)\rangle \,\mathrm{d}\xi\,\mathrm{d}t \\
&\geq \delta^{d-1}\omega(d-1)\Big(|X_n(t_2,x_0) - X(t_2,x_0)|^2 - |X_n(t_1,x_0) - X(t_1,x_0)|^2\Big) - 4\bar{K}\omega(d-1)\delta^d \\
&\geq \left(\frac{3}{2}\epsilon^2 - 4\bar{K}\delta\right)\omega(d-1)\delta^{d-1}.
\end{aligned}$$

Since v_n converges strongly in $L^1(t_1,t_2)$, it follows that the left hand side converges to 0, for any fixed choice of δ. For sufficiently small positive δ however, the right hand side is a positive constant, therefore we have a contradiction. □

5. Transport equations and monotone operators

In order to apply the preceding results to the transport equation, we first estimate the Lebesgue measure of the set where the inverse image of the flow at a fixed time $t > 0$ contains more than one point,

$$S = \Big\{x \in \mathbb{R}^d : \exists y_1 \neq y_2\ x = X(t,y_1) = X(t,y_2)\Big\}.$$

Since the inverse image of a point $[X(t,\cdot)]^{-1}(y)$ is a connected set ([7], Theorem 6.6), we have that S is contained in the image $X(t,S')$ of

$$S' = \Big\{x \in \mathbb{R}^d : \nabla_x X(t,x) \text{ not defined or } \det(\nabla_x X(t,x)) = 0\Big\}.$$

The Lebesgue measure of $X(t,S')$ follows easily from the area formula.

Lemma 5.1. *The set $X(t,S')$ has Lebesgue measure 0.*

Proof. Since $\chi_{X(t,S')}(y) \leq \mathcal{H}^0(S' \cap [X(t,\cdot)]^{-1}(y))$ for all $y \in \mathbb{R}^d$, we have by the area formula that

$$\mathcal{L}^d(X(t,S')) = \int_{\mathbb{R}^d} \chi_{X(t,S')}(y)\,\mathrm{d}y \leq \int_{\mathbb{R}^d} \mathcal{H}^0(S' \cap [X(t,\cdot)]^{-1}(y))\,\mathrm{d}y = \int_{S'} \det(\nabla_x X(t,y))\,\mathrm{d}y = 0.$$

□

Remark 5.2. It is clear that in general, the assumption that $Y : \mathbb{R}^d \mapsto \mathbb{R}^d$ is 1-Lipschitz and $\det(\nabla Y) \geq 0$ does not imply that there exists $A(t)$ monotone in space generating a flow $X(t)$ such that $Y = X(1)$, apart from the 1-dimensional case where the operators can be given by

$$A(t, x) = \{z - Y(z), x = (1 - t)z + tY(z)\}.$$

Notice that in particular by taking Y to be

$$Y = (x + C(x))^{-1},$$

where C is the Cantor-Vitali function, we have that the set $X(t, S')$ can have any Hausdorff dimension in $[0, 1)$.

Definition 5.3. For a maximal monotone operator $A(t, x)$, define a single-valued, everywhere defined selection $a(t, x)$ by first setting

$$a(t, X(t, x)) = -\frac{\mathrm{d}}{\mathrm{d}t} X(t, x)$$

as in the proof of Proposition 3.5, and then extending it by 0 on the remaining null set.

By the equivalence theorem for uniqueness of the solutions to transport equations and ODE, [3] Theorem 9, we have following proposition:

Proposition 5.4. *Let $A(t, x)$ be a maximal monotone operator, and $\bar{\mu}$ a non-negative measure. Then the solution to the conservative transport equation with coefficient $a(t, x)$ as defined above and initial datum $\bar{\mu}$,*

$$\mu_t + \operatorname{div}(a(t, x)\mu) = 0, \quad \mu(0) = \bar{\mu},$$

is given by the formula

$$\mu(t) = X(t)\sharp\mu(0),$$

where $X(t) = X(t, \cdot)$ is the flow of the differential inclusion (2.1).

To construct a solution to the advective transport equation,

$$u_t + a(t, x) \cdot \nabla u = 0, \tag{5.1}$$

we can use the duality formulation

$$(u\mu)_t + \operatorname{div}(au\mu) = 0, \tag{5.2}$$

for some measure μ. It turns out that the solution depends on the test measure μ chosen, as the following example shows:

Example 5.5. For $d = 1$, consider the vector field

$$a(t, x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

and the following two solutions of the conservative equation:

$$\mu_1(t) = \mathcal{L}^1 + 2t\delta(x), \quad \mu_2(t) = (1 + \chi_{(0,\infty)})\mathcal{L}^1 + 3t\delta(x).$$

Then the solutions with initial datum $u(0, x) = \chi_{(0,\infty)}(x)$ with test measures μ_1 and μ_2 are given by

$$u(t, x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0 \end{cases} \qquad u(t, x) = \begin{cases} 0 & x < 0 \\ 2/3 & x = 0 \\ 1 & x > 0 \end{cases}$$

For the general case, we have the following proposition, valid not only in the monotone vector field case.

Proposition 5.6. *The duality solution is given by the formula*

$$u(t)(X(t)\sharp\mu) = X(t)\sharp(u(0)\mu). \tag{5.3}$$

The proof is just an application of the definition of duality solution (5.2) and Proposition 5.4. Using the definition of pushforward, we can rewrite (5.3) as

$$\begin{aligned}\int_E u(t,x)\,\mathrm{d}[X(t)\sharp\mu](x) &= \int_{[X(t)]^{-1}(E)} u(t,X(t,x))\,\mathrm{d}\mu(x)\\ &= \int_{[X(t)]^{-1}(E)} u(0,x)\,\mathrm{d}\mu(x) = \int_E \mathrm{d}[X(t)\sharp(u(0)\mu)](x).\end{aligned}$$

Assume the inverse image of y under $X(t,\cdot)$ consists of a single point, $[X(t)]^{-1}(y) = \{x\}$. Using the regularity and the bound (2.7), we have that $[X(t)]^{-1}(B(y,r)) \subset B(\rho,x)$, with $\rho \to 0$ as $r \to 0$. Thus we get

$$u(t,y) = u(0,x).$$

Since this holds $\mathcal{L}^d$-a. e., the L^∞-norm of u is preserved and the duality solutions differ only on negligible sets. Using the results on the continuous dependence of the flow on $A(t,x)$, we conclude with the following theorem:

Theorem 5.7. *The solution (in duality sense) of* (5.1) *is uniquely determined as an L^∞ function. Moreover it depends continuously in the L^1_{loc}-norm w. r. t. the weak convergence of $A(t)$ in $L^1((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^d))$.*

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