BV SOLUTIONS OF THE SEMIDISCRETE UPWIND SCHEME

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ABSTRACT. We consider the semidiscrete upwind scheme

$$u(t,x)_t + \frac{1}{\epsilon}\Big(f\big(u(t,x)\big) - f\big(u(t,x-\epsilon)\big)\Big) = 0. \tag{0.1}$$

We prove that if the initial data $\bar{u}$ of (0.1) has small total variation, then the solution $u^\epsilon(t)$ has uniformly bounded BV norm, independent of t, ϵ. Moreover by studying the equation for a perturbation of (0.1) we prove the Lipschitz continuous dependence of $u^\epsilon(t)$ on the initial data.

Using a technique similar to the vanishing viscosity case, we show that as $\epsilon \to 0$ the solution $u^\epsilon(t)$ converges to a weak solution of the corresponding hyperbolic system,

$$u_t + f(u)_x = 0. \tag{0.2}$$

Moreover this weak solution coincides with the trajectory of a Riemann Semigroup, which is uniquely determined by the extension of Liu's Riemann solver to general hyperbolic systems.

1. Introduction

Consider the strictly hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \tag{1.1}$$

where $u \in \mathbb{R}^N$, $f : \mathbb{R}^N \mapsto \mathbb{R}^N$ smooth. The system is said to be strictly hyperbolic provided that each Jacobian matrix $A(u) \doteq Df(u)$ has N distinct eigenvalues, $\lambda_1(u) < \ldots < \lambda_n(u)$.

In [16] weak solutions to (1.1) are constructed under the assumption that the initial data

$$u(0,x) = \bar{u}(x), \tag{1.2}$$

has small total variation and that for each $m \in \{1,\ldots,N\}$, the m-th characteristic field is either *linearly degenerate* or else it is *genuinely nonlinear*.

The idea in Glimm's proof is to obtain an a priori estimate on the total variation of the approximate solutions by introducing a *wave interaction potential*. In turn, the control of the total variation yields the compactness of the family of approximate solutions, and hence the existence of a strongly convergent subsequence in $L^1(\mathbb{R},\mathbb{R}^N)$. Alternative constructions of approximate solutions, based on front tracking approximations, were subsequently developed in [9], [15].

The well posedness of the Cauchy problem was established in a series of papers [10], [12], [13]. For a comprehensive account of the recent uniqueness and stability theory we refer to [11].

Recently in [4] it is proved that the solution u^ϵ of the parabolic system

$$u_t + f(u)_x - \epsilon u_{xx} = 0 \tag{1.3}$$

with initial data (1.2) converges to the unique entropy weak solution of (1.1) as $\epsilon \to 0$. This solution depends Lipschitz continuously on the initial data $\bar{u}$ in the L^1 norm and can be characterized by defining a Riemann solver for (1.1). It turns out that this Riemann solver is the extension of Liu's Riemann solver for general strictly hyperbolic systems, see [2], [4], [18].

In this paper we consider a semidiscrete approximation to (1.1), obtained by discretizing only the space variable x:

$$\frac{\partial}{\partial t}u(t,x) + \frac{1}{\epsilon}\Big(f\big(u(t,x)\big) - f\big(u(t,x-\epsilon)\big)\Big) = 0. \tag{1.4}$$

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For linear stability, we assume that $\lambda_1(u) > 0$.

The solution of (1.4) is defined on the lattice $\mathbb{R} \times \epsilon\mathbb{Z}$. In the following we shall denote by $u^j(t)$ the value of the solution in $j\epsilon$,

$$u^j(t) \doteq u(t, j\epsilon).$$

Our aim is to prove that if $\bar{u}$ has sufficiently small total variation, then the solution $u^j(t)$ to (1.4) has bounded BV norm uniformly in t and ϵ. As in the parabolic case, it is clear that the rescaling

$$t \mapsto \frac{t}{\epsilon}, \quad x \mapsto \frac{x}{\epsilon}, \tag{1.5}$$

leaves the total variation of the initial data $\bar{u}$ and of the solution $u^j(t)$ unchanged, and (1.4) becomes

$$u^j_t + f(u^j) - f(u^{j-1}) = 0. \tag{1.6}$$

The main result of this paper is the following theorem:

Theorem 1.1. *Consider the semidiscrete upwind scheme* (1.6) *with initial data* $u^j(0) = \bar{u}^j$. *Assume that*

$$\text{Tot.Var.}(u^j) = \sum_j \big|u^j - u^{j-1}\big| \leq \delta_0/4,$$

for some constant δ_0 *sufficiently small. Then, for some constant* N, L, L', *the solution to* (1.6) *exists for all* $t \in \mathbb{R}$ *and has uniformly bounded total variation:*

$$\text{Tot.Var.}\big(u^j(t)\big) \leq 4N\delta_0, \quad \forall t \in \mathbb{R}^+. \tag{1.7}$$

Moreover, the semigroup $\mathcal{S}_t\bar{u}^j = u^j(t)$ *generated by* (1.6) *is Lipschitz continuous w.r.t. the* ℓ^1*-norm: there exist constant* L, L' *such that if* $u^j(t)$, $z^j(t)$ *are two solutions, then*

$$\sum_j \big|u^j(t) - z^j(s)\big| \leq L \sum_j \big|u^j(0) - z^j(0)\big| + L'\big|t - s\big|. \tag{1.8}$$

Note that the continuous dependence w.r.t time follows trivially from (1.6).

Using the above estimates, it is now possible to prove that, as $\epsilon \to 0$, the solutions $u^{j,\epsilon}$ of (1.4) yield a weak solution to (1.1). In fact, define the function u^ϵ by

$$u^\epsilon(t,x) = u^{j,\epsilon}(t) \qquad (j-1)\epsilon < x \leq j\epsilon. \tag{1.9}$$

Since u^ϵ solves (1.4) with initial data

$$u^\epsilon(0,x) = \frac{1}{\epsilon}\int_{(j-1)\epsilon}^{j\epsilon} \bar{u}(x)dx \qquad (j-1)\epsilon < x \leq j\epsilon,$$

we can write in weak form

$$\begin{aligned} \iint_{\mathbb{R}^+\times\mathbb{R}} \bigg\{ u^\epsilon(t,x)\varphi_t(t,x) + f\big(u^\epsilon(t,x)\big)\frac{\varphi(t,x+\epsilon) - \varphi(t,x)}{\epsilon} \bigg\} dtdx \\ + \sum_j \int_{(j-1)\epsilon}^{j\epsilon} \bar{u}(x) \left(\int_{(j-1)\epsilon}^{j\epsilon} \varphi(0,y)dy \right) dx = 0, \end{aligned} \tag{1.10}$$

where φ is a smooth function with compact support. From (1.7) and the fact that by construction

$$\lim_{j\to-\infty} u^j(t) = \lim_{j\to-\infty} \bar{u}^j,$$

it follows that, up to a subsequence, u^ϵ converges in L^1_{loc} to a BV function $u(t,x)$, so that passing to the limit in (1.10) we obtain

$$\iint_{\mathbb{R}^+\times\mathbb{R}} \Big\{ u(t,x)\varphi_t(t,x) + f\big(u(t,x)\big)\varphi_x(t,x) \Big\} dtdx + \int_{\mathbb{R}} \bar{u}(0,x)\varphi(0,x)dx = 0.$$

Since it is easy to check that

$$\big\|u^\epsilon\big\|_{L^1} = \epsilon\big\|u^{j,\epsilon}\big\|_{\ell^1} = \epsilon\big\|u^j\big\|_{\ell^1},$$

from (1.8), a simple argument shows that up to a subsequence the limit solutions of (1.1) satisfy

$$\big\|u(t) - z(s)\big\|_{L^1} \leq L\big\|u(0) - z(0)\big\|_{L^1} + L'\big|t - s\big|.$$

In particular they form a Lipschitz continuous semigroup in L^1.

At this stage we must rule out the possibility that different subsequences may give rise to different semigroups. To characterize the limit, we follow the approach of [4], i.e., we first identify the Riemann solver compatible with the semidiscrete approximation (1.4).

Consider an initial data of the form

$$u(0,x) = \begin{cases} u^- & x < 0 \\ u^+ & x \geq 0 \end{cases} \tag{1.11}$$

with $u^+ - u^-$ sufficiently small. We will prove that when $\epsilon \to 0$, the solution $u^\epsilon(t)$ of (1.4) tends to a particular self-similar function, which can be precisely described by means of the dynamics on the center manifold of the retarded functional equation

$$\begin{cases} -\sigma\phi'(\xi) + f\big(\phi(\xi)\big) - f\big(\phi(\xi-1)\big) & = & 0 \\ \sigma'(\xi) & = & 0 \end{cases} \tag{1.12}$$

In [2] it is shown how to construct a Riemann Solver compatible with the semidiscrete scheme (1.4). In particular one can show that this Riemann Solver coincides with the Riemann Solver obtained by means of the vanishing viscosity approximations, and can be identified by saying that all jumps in the solution $u(t)$ satisfy Liu's stability condition. Due to the particular choice of the initial data (1.11), one can show using the same arguments of [4] that the whole sequence $u^\epsilon(t)$ converges to $u(t)$.

At this point the limiting solution $u(t)$ is identified by proving that it is a viscosity solution to

$$u_t + Df(u)u_x = 0. \tag{1.13}$$

The definition of viscosity solution is given in [10], and relies on local integral estimates: in the first estimate one compares $u(t)$ with the solution of a local Riemann problem, while in the second estimate one compares the solution $u(t)$ with the with the solution to the linear system

$$u_t + Df\big(u(\tau,\xi)\big)u_x = 0,$$

obtained by freezing the coefficients at some point $u(\tau,\xi)$. Note that, while the weak solution to a linear system is uniquely defined, different Riemann Solvers define different viscosity solutions. The viscosity solution we are considering here is the one corresponding to the Riemann Solver described above.

As in [10], one can show that the semigroup trajectories corresponding to a given Riemann solver are precisely the viscosity solution. Since we show that each limiting solution $u(t)$ of (1.13) obtained by the semidiscrete approximations $u^\epsilon(t)$ is a viscosity solution to (1.13), the uniqueness of the limit follows.

We prove then the following theorem:

Theorem 1.2. *As $\epsilon \to 0$, the solution $u^\epsilon(t)$ defined in* (1.9) *converges in L^1 to a unique limit $u(t)$. This limit coincides with the viscosity solution characterized by the Riemann solver defined in* [2, 4]*, i.e., with the unique Riemann solver such that each shock satisfies Liu's stability condition. In particular, $u(t)$ coincides with the vanishing viscosity limit of* (1.3).

The paper is organized as follows.

In Section 2 we prove some regularity estimates for the solution to (1.6). In particular, assuming that u^j has small total variation, we prove that the ℓ^∞-norm and the total variation of the time derivative u^j_t are of second order w.r.t. Tot.Var.(u^j). Note that a trivial estimate is that the ℓ^∞-norm of u^j_t is less than or equal to the total variation of $f(u^j)$: our results shows that if Tot.Var.(u^j) is small, then a better estimate is

$$\big\|u^j_t\big\|_{\ell^1} = \mathcal{O}(1)\text{Tot.Var.}(u^j)^2.$$

Of course the above estimate is meaningless when Tot.Var.(u^j) is large.

In Section 3 we study the properties of the center manifold of travelling profiles. In [1], using the results for general Retarded Functional Differential Equations proved in [17], it is shown that there exists locally an invariant manifold in $C^0([-1,0];\mathbb{R}^{N+1})$ for the equation of travelling profiles, namely

$$\begin{cases} -\sigma\phi'(\xi) + f\big(\phi(\xi)\big) - f\big(\phi(\xi-1)\big) & = & 0 \\ \sigma'(\xi) & = & 0 \end{cases}$$

This manifold is $n+2$-dimensional and contains all the long term dynamics near a fixed point $u \equiv u_0$, $\sigma \equiv \lambda_m(u_0)$, where $\lambda_m(u)$ is the m-th eigenvalue of $Df(u)$. In particular it contains all the small bounded travelling profiles with speed close to that eigenvalue. We can parametrize the manifold by

$\phi(0)$, $-\sigma_m \phi_m'(0)$, σ, i.e., the value of the profile, the m-th component of the time derivative at $\xi = 0$, and the speed σ:

$$\big(u, v_m, \sigma\big) \quad \mapsto \quad \phi\big(\xi; u, v_m, \sigma\big),$$

where $\xi \in [-1, 0]$. In particular we obtain the functions

$$-\sigma\phi'(0) = \omega_m\big(u, v_m, \sigma\big), \qquad -\sigma\phi_m'(-1) = \pi_{mm}\big(u, v_m, \sigma\big),$$

i.e., the derivative at $\xi = 0$ (not just the m-th component) and the m-th component at $\xi = -1$. We show that the knowledge of the functions ω_m, π_{mm} is equivalent to the knowledge of the center manifold. Using a Taylor expansion, we prove some properties of the functions ω, π_{mm} which will be used in the rest of the paper. We will give two examples where we can compute this center manifold explicitly.

In Section 4 we consider the problem of finding a Glimm type functional for a semidiscrete scalar equation. The main difficulty here is that the semidiscrete scheme presents dispersion. We first prove a general proposition which allows us to identify locally a travelling profile. The proof is heavily based on the results of Section 3. Next we construct the Glimm type functional. As it is suggested by the results of an example where we can compute explicitly the functional, we prove that, due the dispersion, the weight that we assign to each wave in the interaction functional is a nonlinear function of the speed, and not the speed itself. This nonlinear function is obtained directly from the dispersion relation.

In Section 5 we work out the decomposition results in the general case. The main difference from the parabolic or hyperbolic case is that, to identify a wave, we must look at the solution at three consecutive points, i.e., u^j, u^{j-1}, u^{j-2}, and not at the local behavior of u. This implies that we cannot decompose the solution u^j to (1.6) locally, but we must decompose it as a whole. A consequence of this fact is that, when we write the equations satisfied by the scalar components of our decomposition, we will get a source term which depends on all the previous waves. This means that the source in j depends on all interaction at points $k \leq j$. Remember in fact that the kernel of the linearized equation moves only forward, i.e., the solution in $k > j$ has no influence at the point j.

We can arrange the source terms in 3 categories:

(1) terms corresponding to the interaction among waves of different families. Following [5], we will refer to these terms as *transversal terms*.
(2) terms corresponding to the interaction of waves of the same family. We will call them *non transversal terms*.
(3) terms due to the fact that the center manifold is defined for speed close to $\lambda_m(u_0)$, so that in some cases we cannot give the right speed. This means that the numerical diffusion of (1.6) is greater than the drift. These terms will be called the *energy terms*.

In Section 6 we study the source terms arising in the decomposition, proving that they are integrable in $\mathbb{R}^+ \times \mathbb{Z}$ and that their integral is of the order of the total variation of u^j squared. To achieve this result, we will introduce four functionals. The first functional is related to the transversal terms, and was studied in [3]: this is the semidiscrete analog of the Glimm functional for waves of different families. The other three are the semidiscrete analogs of the functionals introduced in [6], [7, 4] for studying interaction among waves of the same family. The construction of these functionals follows the analysis of Section 4. Using these functionals we will prove that the source terms of total variation are of the order of the total variation squared, hence the total variation remains uniformly bounded if it is sufficiently small at $t = 0$:

$$\text{Tot.Var.}\big(u^j(t)\big) \leq 16N \text{Tot.Var.}(\bar{u}^j).$$

Again we observe that, differently from the vanishing viscosity case, we have to consider the non-transversal and energy functionals together, i.e., the time derivative of each functional contains the derivative of the others, multiplied by a small constant. This implies that we cannot show the decrease of a single functional without considering it together with the other two.

In Section 7 we prove an analogous result for a perturbation of (1.6): the ℓ^1 norm of a perturbation ζ satisfies

$$\big\|\zeta(t)\big\|_{\ell^1} \leq L\big\|\zeta(0)\big\|_{\ell^1}$$

for some constant L depending only on the total variation of u^j. A simple homotopy argument then shows that the solutions to the semidiscrete upwind scheme (1.6) form a Lipschitz continuous semigroup: if $u(t)$, $z(t)$ are two solutions of (1.6), then

$$\big\|u(t) - z(t)\big\|_{\ell^1} \leq L\big\|u(0) - z(0)\big\|_{\ell^1}.$$

This concludes the proof of Theorem 1.1.

Finally in Section (8) we prove that the limit of u^ϵ defined in (1.9) coincides with the vanishing viscosity solution. We prove that in the limit $\epsilon \to 0$ the solution u has finite speed of propagation. Next, using the results of [2], [4], we can identify the Riemann Solver corresponding to the limiting solution $u(t)$. Finally we prove that $u(t)$ can be identified as a viscosity solution to (1.13).

This will yield a proof of Theorem 1.2.

2. Regularity estimates for semidiscrete upwind schemes

Consider the semidiscrete upwind scheme

$$u^j_t + f\big(u^j\big) - f\big(u^{j-1}\big) = 0, \tag{2.1}$$

where $u \in \mathbb{R}^N$ and $f : \mathbb{R}^N \mapsto \mathbb{R}^N$ is a smooth vector function. We assume that the Jacobian matrix $A(u) \doteq Df(u)$ is strictly hyperbolic, and we order its eigenvalues λ_m, $m = 1, \dots, N$, as

$$0 < \lambda_1(u) < \lambda_2(u) < \dots < \lambda_N(u).$$

We denote by $r_m(u)$, $l_m(u)$, $m = 1, \dots, N$, the right and left eigenvectors of $A(u)$, respectively, normalized by

$$\big|r_m(u)\big| = 1, \qquad \big\langle l_n(u), r_m(u)\big\rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{2.2}$$

Let K_0 be a compact set in $\mathbb{R}^N$, and for $\delta > 0$ define K_1 as

$$K_1 \doteq \Big\{ \big|u - z\big| \leq \delta, z \in K_0 \Big\} \subset \mathbb{R}^N. \tag{2.3}$$

For the semidiscrete scheme (2.1), we consider the initial condition

$$u^j(0) = \bar u^j, \qquad j \in \mathbb{Z}. \tag{2.4}$$

It is well known that if the initial datum is bounded, then (2.1) defines a continuous flow $\mathcal{S}_t$ on $\ell^\infty(\mathbb{Z}, \mathbb{R}^n)$, at least for some interval $[0, T]$. In the rest of the paper we will consider solutions with small total variation:

$$\sum_j \big|u^j - u^{j-1}\big| \leq \delta. \tag{2.5}$$

Is it clear that the assumption $\lambda_1(u) > 0$ for all $u \in K$ implies that f is locally invertible, so that for δ sufficiently small we have

$$L_1 \sum_j \big|u^j(t) - u^{j-1}(t)\big| \leq \sum_j \big|u^j_t(t)\big| \leq L_2 \sum_j \big|u^j(t) - u^{j-1}(t)\big|. \tag{2.6}$$

Note that L_1, L_2 can be chosen such that the above equation holds uniformly for any sequence $u^j \in K_1$ satisfying (2.5). Defining $v^j \doteq u^j_t = f(u^{j-1}) - f(u^j)$, in the following as a measure of the total variation we will use the quantity

$$\mathcal{V}\big(u(t)\big) \doteq \sum_j \big|v^j(t)\big| = \sum_j \big|u^j_t(t)\big| = \sum_j \big|f\big(u^{j-1}(t)\big) - f\big(u^j(t)\big)\big|. \tag{2.7}$$

By (2.6), $\mathcal{V}(u)$ is equivalent to the sum in (2.5). In particular $\bar u$ has bounded total variation, so that the limit for $j \to -\infty$ exists: we assume that it belongs to K_0, i.e.,

$$u_0 \doteq \lim_{j \to -\infty} \bar u^j \in K_0. \tag{2.8}$$

By choosing δ_0 sufficiently small, we set

$$\delta > 4N\delta_0.$$

The aim of this section is to prove a regularity estimate on the solution $\zeta^j(t)$ of the linearized equation describing the evolution of a first order perturbation to (2.1),

$$\begin{aligned} \zeta^j_t &= A\big(u^{j-1}\big)\zeta^{j-1} - A\big(u^j\big)\zeta^j \\ &= A\big(u_0\big)\big(\zeta^{j-1} - \zeta^j\big) + \Big(A\big(u^{j-1}\big) - A\big(u_0\big)\Big)\zeta^{j-1} - \Big(A\big(u^j\big) - A\big(u_0\big)\Big)\zeta^j, \end{aligned} \tag{2.9}$$

assuming that $\mathcal{V}(u)$ is small. Note that a particular solution of (2.9) is u^j_t, so that the estimates we obtain will be valid also for v^j. The idea is that, if $\|v^j(t)\|_{\ell^1}$, $\|\zeta^j(0)\|_{\ell^1}$ are bounded in $[0,T]$, then $\|\zeta^j\|_{\ell^\infty}$ and $\|\zeta^j - \zeta^{j-1}\|_{\ell^1}$ are bounded and small after a small time of regularization $\tilde{t}$. These estimates correspond to the parabolic estimates obtained in [4]. The approach is essentially the same, the only difference being that here we use the Green kernel of

$$\zeta^j_t + A(u_0)\big(\zeta^j - \zeta^{j-1}\big) = 0, \tag{2.10}$$

instead of the standard Gaussian heat kernel.

Define the constants

$$\bar\lambda \doteq \max_{u\in K_1} \lambda_N(u), \qquad \underline{\lambda} \doteq \min_{u\in K_1} \lambda_1(u) > 0, \tag{2.11}$$

and let c be the following quantity:

$$c \doteq \min\Big\{\lambda_{m+1}(u) - \lambda_m(z);\ u,z\in K_1,\ |u-z|\le\delta,\ m=1,\dots,N-1\Big\} > 0. \tag{2.12}$$

In the following we will need to consider derivatives of smooth functions defined in K_1: we assume that all these derivatives are bounded by a sufficiently big constant C_0.

For $u_0 \in K_0$, denote by G the kernel of the linear equation (2.10), which can be easily computed by Fourier transform (see [3]):

$$G^j(t) = \begin{cases} \displaystyle\sum_m \frac{\big(\lambda_m(u_0)t\big)^j}{j!} e^{-\lambda_m(u_0)t} r_m(u_0)\otimes l_m(u_0) & j\ge 0 \\ 0 & j<0 \end{cases} \tag{2.13}$$

Using Stirling's formula, the following estimates follow:

$$\max_j \big|G^j(t)\big| \le \min_m \left\{\frac{(\lambda_m t)^{\lambda_m t} e^{-\lambda_m t}}{\Gamma(\lambda_m t)}\right\} \le \frac{1}{\sqrt{\underline{\lambda} t + 1}}, \tag{2.14}$$

$$\sum_j \Big|G^j(t) - G^{j-1}(t)\Big| \le 2\max_j \big|G^j(t)\big| \le \frac{2}{\sqrt{\underline{\lambda} t+1}}. \tag{2.15}$$

We have the following proposition:

Proposition 2.1. *Assume that for $t\in[0,T]$*

$$\mathcal{V}(u) \doteq \sum_j \big|u^j_t\big| \le 4N\delta_0 < \delta, \tag{2.16}$$

with δ_0 sufficiently small. Then we have the following regularity estimates:

$$\big\|\zeta^j(t)\big\|_{\ell^\infty} \le \frac{2}{\sqrt{t+1}}\big\|\zeta(0)\big\|_{\ell^1}, \qquad \sum_j\big|\zeta^j(t) - \zeta^{j-1}(t)\big| \le \frac{4}{\sqrt{t+1}}\big\|\zeta(0)\big\|_{\ell^1}, \tag{2.17}$$

for $0\le t<\min\{T,\tilde t\}$, where

$$\sqrt{\underline{\lambda}\tilde t + 1} \doteq \frac{\underline{\lambda}}{16N\delta_0\pi}\frac{L_1}{L_2} > 1. \tag{2.18}$$

Proof. We will only prove the first inequality of (2.17), since the proof of the second inequality is the same. Using Duhamel's principle we can write the solution to (2.9) as

(2.19)

$$\begin{aligned} \zeta^j(t) &= \sum_k G^{j-k}(t)\zeta^k(0) + \sum_k \int_0^t G^{j-k}(t-s)\Big\{\Big(A\big(u^{k-1}\big) - A\big(u_0\big)\Big)\zeta^{k-1} - \Big(A\big(u^k\big) - A\big(u_0\big)\Big)\zeta^k\Big\}dt \\ &= \sum_k G^{j-k}(t)\zeta^k(0) + \sum_k \int_0^t \Big(G^{j-k-1}(t-s) - G^{j-k}(t-s)\Big)\Big(A\big(u^k\big) - A\big(u_0\big)\Big)\zeta^k dt. \end{aligned}$$

Using (2.6) we have

$$\big|u^j - u_0\big| \le \sum_{k\le j}\big|u^k - u^{k-1}\big| \le \frac{1}{L_1}\sum_{j\le k}\big|v^k\big| \le \frac{4N\delta_0}{L_1},$$

so that, if L_2 is sufficiently large, we conclude that

$$\left|\Big(A(u^{k-1}) - A(u_0)\Big)\zeta^{k-1}\right| \le \frac{L_2}{L_1}\|v^j\|_{\ell^1}\|\zeta^j\|_{\ell^\infty} \le \frac{4N\delta_0 L_2}{L_1}\|\zeta^j\|_{\ell^\infty}.$$

Using the above equation in (2.19), we obtain

$$\|\zeta^j(t)\|_{\ell^\infty} \le \frac{1}{\sqrt{\underline{\lambda}t+1}}\|\zeta^j(0)\|_{\ell^1} + \frac{8N\delta_0 L_2}{L_1}\int_0^t \frac{1}{\sqrt{\underline{\lambda}(t-s)+1}}\|\zeta^j(s)\|_{\ell^\infty}ds. \tag{2.20}$$

By direct substitution, one check that

$$\|\zeta^j(t)\|_{\ell^\infty} \le \frac{2}{\sqrt{\underline{\lambda}t+1}}\|\zeta^j(0)\|_{\ell^1} \quad \text{if} \quad 0 \le \sqrt{\underline{\lambda}t+1} < \frac{\underline{\lambda}}{16N\delta_0\pi}\frac{L_1}{L_2}.$$

Note that the above inequality is meaningful only if $\delta_0 < \underline{\lambda}L_1/(16N\pi L_2)$. □

A consequence of Proposition 2.1 is the following

Corollary 2.2. *Assume that $\mathcal{V}(u) \le 4N\delta_0$ for $0 \le t \le T$, with $T \ge \tilde{t}$. Then*

$$\|z^j(t)\|_{\ell^\infty} \le \frac{32N\pi L_2}{L_1}\delta_0\|z^j(0)\|_{\ell^1}, \qquad \sum_j \big|z^j(t) - z^{j-1}(t)\big| \le \frac{64\pi L_2}{L_1}\delta_0\|z^j(0)\|_{\ell^1}, \tag{2.21}$$

for $\tilde{t} \le t \le T$.

In particular, if we consider v^j instead of ζ^j, we obtain the estimate

$$\|v^j(t)\|_{\ell^\infty} = \mathcal{O}(1)\mathcal{V}(u)^2 = \mathcal{O}(1)\delta_0^2, \qquad t \ge \tilde{t}. \tag{2.22}$$

Note that a trivial estimate is of course

$$\|v^j(t)\|_{\ell^\infty} \le \|v^j(t)\|_{\ell^1} \le 4N\delta_0,$$

but (2.22) shows that $\|v^j(t)\|_{\ell^\infty}$ is of the order of $\|v^j(t)\|_{\ell^1}^2$ if δ_0 is small.

The regularity estimates of Corollary 2.2 are needed in the next section to prove that the total variation of u remain sufficiently small, i.e., less than $4N\delta_0$. These estimates are valid for $t \ge \tilde{t}$, so that we have to consider an initial layer $0 \le t \le \tilde{t}$ where the total variation of the solution can increase. In the next proposition we show that during this initial interval the total variation remains bounded if δ_0 is sufficiently small.

Proposition 2.3. *Assume that at $t = 0$ the total variation of u^j is less than or equal to $\delta_0/4$. Then for $0 \le t \le \tilde{t}$*

$$\|\zeta^j(t)\|_{\ell^1} \le 2\|\zeta^j(0)\|_{\ell^1}. \tag{2.23}$$

Proof. Using again the representation (2.19) we can estimate

$$\|\zeta^j(t)\|_{\ell^1} \le \|\zeta^j(0)\|_{\ell^1} + \frac{L_2}{L_1}\int_0^t \frac{2}{\sqrt{\underline{\lambda}(t-s)+1}}\|v^j(s)\|_{L^\infty}\|\zeta^j(s)\|_{\ell^1}ds.$$

Therefore, using the a priori estimate (2.17), it follows that

$$\|\zeta^j(t)\|_{\ell^1} \le 2\|\zeta^j(0)\|_{\ell^1} \qquad \text{if} \qquad 0 \le t < \tilde{t}.$$

In particular the above estimate holds for $v^j(t)$, so that the a priori assumption (2.16) is satisfied in $[0, \tilde{t}]$. □

The above results implies that, without any loss of generality, we can set $\tilde{t} = 0$ and assume that $u^j(0)$ satisfies

$$\mathcal{V}\big(u^j(0)\big) \le \frac{\delta_0}{2}, \qquad \|v^j(0) - v^{j-1}(0)\|_{\ell^\infty} \le \frac{16\pi L_2}{L_1}\delta_0^2 \le C_0\delta_0^2, \tag{2.24}$$

where C_0 is a big constant.

3. An invariant manifold of travelling profiles

In [1] is proved the existence of a smooth center manifold for the retarded functional differential equation (RFDE)

$$\left\{ \begin{array}{rcl} -\sigma u_x & = & f\big(u(x-1)\big) - f\big(u(x)\big) \\ \sigma_t & = & 0 \end{array} \right. \tag{3.1}$$

where $u \in \mathbb{R}^N$ and Df is strictly hyperbolic with positive eigenvalues. This manifold is defined in a small neighborhood of the fixed point $u \equiv u_0 \in K_1$ and $\sigma \equiv \lambda_m(u_0)$ in the functional space $C^0([-1,0]\times\mathbb{R}^+;\mathbb{R}^N)$, and contains all the small bounded travelling profiles of the semidiscrete scheme (2.1), taking values close to u_0: we will denote it by $\mathcal{C}_m$, with $m \in \{1,\dots,N\}$. The aim of this section is to prove some properties of this invariant manifold.

In [1] it is shown that the manifold $\mathcal{C}_m$ has dimension $N+2$, and is tangent to the following linear subspace of C^0:

$$M_m \doteq \left\{ \phi(\xi) = u - \xi \frac{v}{\lambda_m(u_0)} r_m(u_0), \sigma(\xi) = \sigma; \xi \in [-1,0], (u,v,\sigma) \in \mathbb{R}^{N+1} \times \mathbb{R}^+ \right\}. \tag{3.2}$$

Note that, by (3.2), M_m is parametrized by the $N+2$ variables

$$u = \phi(0), \quad v_m = \big\langle l_m(u_0), -\lambda_m(u_0)\phi'(0)\big\rangle, \quad \sigma_m, \tag{3.3}$$

respectively in $\mathbb{R}^n$, $\mathbb{R}$ and $\mathbb{R}$. As a consequence of this tangency, the manifold $\mathcal{C}_m$ can be parametrized by $u(0)$ and the scalar quantities

$$v_m(0) = \langle l_m(u_0), u_t(0)\rangle = -\sigma_m \langle l_m(u_0), u_x\rangle \quad \text{and} \quad \sigma_m > 0.$$

This means that, given the vector u and the scalars v_m, σ_m with

$$\big|u - u_0\big| \le 3\delta_1, \quad \big|v_m\big| \le 3\delta_1, \quad \big|\sigma_m - \lambda_m\big(u_0\big)\big| \le 3\delta_1,$$

where δ_1 is sufficiently small, there exists a trajectory

$$\phi(\xi) = \phi\big(\xi; u, v_m, \sigma_m\big) \in C^1\big([-1,0],\mathbb{R}^N\big) \tag{3.4}$$

such that

$$\phi(0) = 0, \quad \langle l_m(u_0), \phi'(0)\rangle = -\frac{v_m}{\sigma_m},$$

and ϕ is a solution of (3.1) with $\sigma \equiv \sigma_m$. Note that $u^j(t) = \phi(j - \sigma_m t)$ is then a solution of (2.1).

From the map (3.4), we can obtain two important functions: first of all, we get all the other components of the vector $u_t = -\sigma_m \phi'(0)$. Moreover, the scalar quantity $\phi'_m(-1)$ gives the value $u_{t,m}(-1)$, which allows us to transform (3.1) into an ODE on the manifold $\mathcal{C}_m$, see (3.11) below.

Let u be a travelling wave on the center manifold $\mathcal{C}_m$. Denoting by v, $v^{(-1)}$ the time derivative of u, and $\xi = -1$ respectively, define the two maps:

$$v_n = \big\langle l_n\big(u_0\big), -\sigma_m\phi'(0)\big\rangle = \omega_{nm}\big(u, v_m, \sigma_m\big), \quad v_n^{(-1)} = \big\langle l_n\big(u_0\big), -\sigma_m\phi'(-1)\big\rangle = \pi_{nm}\big(u, v_m, \sigma_m\big). \tag{3.5}$$

Note that all the equilibria $u(x) \equiv u$, σ_m for $|u - u_0| \le 3\delta_1$, $\sigma_m > 0$ belong to $\mathcal{C}_m$, and on the manifold $\mathcal{C}_m$ they correspond to $v_m = 0$: as a consequence we know that $v_m = 0$ implies $v = v^{(-1)} = 0$, and thus

$$v = v_m \tilde{r}_m\big(u, v_m, \sigma_m\big), \qquad v_m^{(-1)} = v_m \tilde{p}_m\big(u, v_m, \sigma_m\big), \tag{3.6}$$

for some smooth functions $\tilde{r}_m$, $\tilde{p}_m$. The parametrization (3.3) implies that the "generalized eigenvectors" $\tilde{r}_m$ are normalized by the relation

$$\big\langle l_m(u_0), \tilde{r}_m\big(u, v_m, \sigma_m\big)\big\rangle = 1. \tag{3.7}$$

In general, given u^j, v_m^j and a speed σ_m^j in the point j, we have determined the travelling profile ϕ, so that we know all the quantities u^{j+k} for all $k \in \mathbb{Z}$. To distinguish these quantities for the real solution $u^j(t)$, for any given function $f(u^{j-k},\dots,u^{j+l})$, we denote by $\hat{f}$ the function evaluated on these quantities, i.e.,

$$\hat{f} \doteq f\big(\phi(-k),\dots,\phi(l)\big).$$

Note that in particular also $v^{j\pm k}$, $k \in \mathbb{Z}$ is determined, because of (2.1). Given a sequence u^j, we will also denote by f^j the quantity

$$f^j \doteq f\big(u^{j-k-l},\dots,u^j\big),$$

i.e., the function f evaluated on the points $u^{j-k-l}, \dots, u^j$.

Differentiating (2.1) w.r.t. t, and defining as before $v^j = u^j_t$, we obtain

$$v^j_t + A\big(u^j\big)v^j - A\big(u^{j-1}\big)\hat{v}^{j-1} = 0,$$

so that on a travelling profile $u^j(t) = \phi(j - \sigma_m t)$ we have

$$\begin{aligned} &- \sigma_m v^j_x + A\big(u^j\big)v^j - A\big(u^{j-1}\big)\hat{v}^{j-1} = \\ &\qquad - \sigma_m v^j_{m,x}\tilde{r}^j_m - \sigma v^j_m \tilde{r}^j_{m,x} + A\big(u^j\big)v^j_m \tilde{r}^j_m - A\big(u^{j-1}\big)\hat{v}^{j-1}_m \widehat{\tilde{r}^{j-1}_m} = 0. \end{aligned} \tag{3.8}$$

Projecting along $l_m(u_0)$, since by (3.7) one has that $\langle l_m(u_0), \tilde{r}^j_{m,x}\rangle = 0$, we obtain the scalar reduced equation

$$-\sigma_m v^j_{m,x} + \tilde{\lambda}^j_m v^j_m - \widehat{\tilde{\lambda}^{j-1}_m}\hat{v}^{j-1}_m = 0, \tag{3.9}$$

where we define

$$\tilde{\lambda}_m = \tilde{\lambda}_m\big(u, v_m, \sigma_m\big) \doteq \big\langle l_m\big(u_0\big), A(u)\tilde{r}_m\big(u, v_m, \sigma_m\big)\big\rangle, \tag{3.10}$$

and, following the above notation,

$$\widehat{\tilde{\lambda}^{j-1}_m} \doteq \tilde{\lambda}_m\big(u^{j-1}, \hat{v}^{j-1}_m, \sigma_m\big) = \tilde{\lambda}_m\big(u^{j-1}, v^j_m \tilde{p}(u^j, v^j_m, \sigma_m), \sigma_m\big).$$

We can thus reduce (3.1) on the manifold $\mathcal{C}_m$. In fact, since f is locally invertible, we can obtain $u^{(-1)}$ by

$$v + f(u) - f(u^{(-1)}) = 0,$$

so that we obtain the system

$$\left\{\begin{array}{ccc} u_x & = & v_m \tilde{r}_m\big(u, v_m, \sigma_m\big) \\ -\sigma_m v_{m,x} & = & v_m\Big(\tilde{\lambda}_m\big(u^{(-1)}, v_m\tilde{p}_m, \sigma_m\big)\tilde{p}_m - \tilde{\lambda}_m\big(u, v_m, \sigma_m\big)\Big) \\ \sigma_{m,x} & = & 0 \end{array}\right. \tag{3.11}$$

Since this is a system of ODE, we can solve it and thus reconstruct the profile. This proves that the knowledge of the functions $\tilde{r}_m$, $\tilde{p}_m$ is equivalent to the knowledge of the center manifold $\mathcal{C}_m$.

Remark 3.1. The choice of the normalization (3.7) is due to the fact that we want (3.9) to be in conservation form, i.e.,

$$\frac{d}{dt}\sum_i v^j_m = 0.$$

Together with the functionals introduced in the next sections, this will be an important tool for proving the BV estimate.

In general, with other normalizations, the above inequality is not valid. If, for example, we normalize the vectors $\tilde{r}_m$ as unit length vectors as in [7, 4, 5], i.e.,

$$\big|\tilde{r}_m\big(u, v_m, \sigma_m\big)\big| = 1,$$

then the reduced scalar equation (3.9) is not in conservation form. This is a consequence of the fact that the length of the piecewise linear curve $\gamma(t)$, whose nodes are the points $u^j(t) = \phi(j - \sigma_m t)$, is not constant in time (fig. 1). Using instead (3.7), the sum of the scalar v^j_m is always equal to

$$\sum_j v^j_m(t) = \sum_j f_m(u^{j-1}) - f_m(u^j) = \lim_{j\to-\infty} f_m(u^j) - \lim_{j\to+\infty} f_m(u^j) = f_m(u^-) - f_m(u^+),$$

i.e., it does not depend on time.

Since the vectors $\tilde{r}_m$ are tangent to trajectories of (3.1), their derivatives satisfy a vector identity. Using (3.9) and denoting by I the $N \times N$ identity matrix, (3.8) becomes

$$\begin{aligned} 0 = {} & v^j_m D\tilde{r}^j_m\big(-\sigma_m u^j_x\big) + v^j_m\big(-\sigma_m v^j_{m,x}\big)\tilde{r}^j_{m,v} + \big(A^j - \tilde{\lambda}^j_m I\big)v^j_m\tilde{r}^j_m - \hat{v}^{j-1}_m\big(A^{j-1}\widehat{\tilde{r}^{j-1}_m} - \widehat{\tilde{\lambda}^{j-1}_m}\tilde{r}^j_m\big) = \\ & \big(v^j_m\big)^2 D\tilde{r}^j_m\tilde{r}^j_m - v^j_m\big(\tilde{\lambda}^j_m v^j_m - \widehat{\tilde{\lambda}^{j-1}_m}\hat{v}^{j-1}_m\big)\tilde{r}^j_{m,v} + \big(A^j - \tilde{\lambda}^j_m I\big)v^j_m\tilde{r}^j_m - \hat{v}^{j-1}_m\big(A^{j-1}\widehat{\tilde{r}^{j-1}_m} - \widehat{\tilde{\lambda}^{j-1}_m}\tilde{r}^j_m\big). \end{aligned}$$

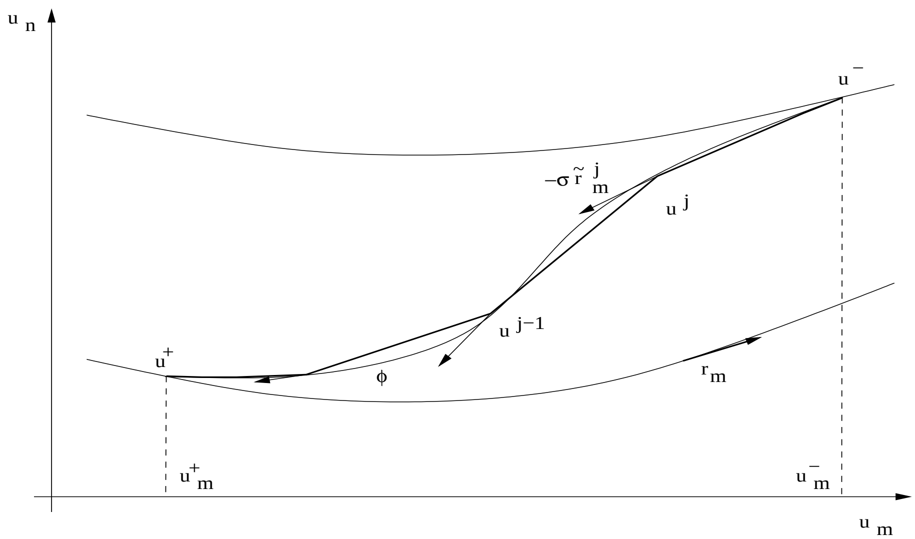

FIGURE 1. Motion of the line γ corresponding to a travelling profile ψ

We arrive finally at the fundamental relation satisfied by the generalized eigenvalues $\tilde{r}_m$:

$$(3.12)\qquad \big(v^j_m\big)^2 D\tilde{r}^j_m\tilde{r}^j_m + v^j_m\Big(A^j\tilde{r}^j_m - \tilde{\lambda}^j_m\big(\tilde{r}^j_m + v^j_m\tilde{r}^j_{m,v}\big)\Big) = \hat{v}^{j-1}_m\Big(A^{j-1}\widehat{\tilde{r}^{j-1}_m} - \widehat{\tilde{\lambda}^{j-1}_m}\big(\tilde{r}^j_m + v^j_m\tilde{r}^j_{m,v}\big)\Big),$$

which, using (3.6), can be written as

$$(3.13)\qquad v^j_m D\tilde{r}^j_m\tilde{r}^j_m + \Big(A^j\tilde{r}^j_m - \tilde{\lambda}^j_m\big(\tilde{r}^j_m + v^j_m\tilde{r}^j_{m,v}\big)\Big) = \tilde{p}^j_m\Big(A^{j-1}\widehat{\tilde{r}^{j-1}_m} - \widehat{\tilde{\lambda}^{j-1}_m}\big(\tilde{r}^j_m + v^j_m\tilde{r}^j_{m,v}\big)\Big),$$

where we define

$$\tilde{p}^j_m \doteq \tilde{p}_m\big(u^j, v^j_m, \sigma_m\big).$$

Remark 3.2. Note that the left-hand side is defined by the local values u^j, v^j_m, σ^j_m, while the right-hand side contains quantities, for example $\hat{v}^{j-1}_m$, which are computed using the travelling profile ϕ. Thus, when we consider a solution u^j, in general these quantities winll be different from the corresponding quantities for the real solution, i.e., $v^{j-1}_m \neq \hat{v}^{j-1}_m$.

Example 3.3. As an example where we can construct explicitly a center manifold of (unbounded) travelling profiles, we consider the following system:

$$(3.14)\qquad \left\{\begin{array}{rcl} u^j_{1,t} + \lambda_1\big(u^j_1 - u^{j-1}_1\big) & = & 0 \\ u^j_{2,t} + \Big(\lambda_2 u^j_2 - \big(u^j_1\big)^2/2\Big) - \Big(\lambda_2 u^{j-1}_2 - \big(u^{j-1}_1\big)^2/2\Big) & = & 0 \end{array}\right.$$

It is easy to check that the travelling profiles with speed σ_1 of the first equations are exponentials:

$$(3.15)\qquad u_1(\xi) = a + b\frac{e^{\beta\xi} - 1}{\beta},$$

where β is related to σ_1 by the dispersion relation

$$(3.16)\qquad \frac{\sigma_1}{\lambda_1} = \frac{1 - e^{-\beta}}{\beta}.$$

In fact, for each a, b, equation (3.15) is a solution to the RFDE

$$-\sigma_1 u_{1,x}(\xi) + u_1(\xi) - u_1(\xi - 1) = 0.$$

Moreover, if we consider (3.15) as a map from $\mathbb{R}^3$ into $C^1([-1,0],\mathbb{R})$, then this map defines a manifold tangent for $\sigma_1 \to \lambda_1$ to the null eigenspace of the linearized system, eigenspace which is made by the functions

$$u_1(\xi) = a + b\xi, \qquad a, b \in \mathbb{R}.$$

Substituting (3.15) into the second equation we obtain

$$-\sigma_1 u_{2,\xi} + \lambda_2\big(u_2(\xi) - u_2(\xi-1)\big) = \big(a - b/\beta\big) b e^{\beta\xi}\big(1-e^{-\beta}\big)/\beta + \frac{1}{2} b^2 e^{2\beta\xi}\big(1-e^{-2\beta}\big)/\beta^2.$$

One can check that a solution is given by the function

$$u_2(\xi) = c + \frac{ab}{\lambda_2-\lambda_1}\frac{e^{\beta\xi}-1}{\beta} + \frac{b^2}{\beta^2}\left(\frac{1+e^{-\beta}}{2\big(\lambda_2(1+e^{-\beta})-2\lambda_1\big)}\big(e^{2\beta\xi}-1\big) - \frac{1}{\lambda_2-\lambda_1}\big(e^{\beta\xi}-1\big)\right). \tag{3.17}$$

The form (3.17) is quite complicated, because there are many constants which could be collected in c, but we can take the limit of (3.17) when $\beta \to 0$, i.e., $\sigma_1 \to \lambda_1$. This means that, among all the invariant manifold of the form

$$u_2(\xi) = c_0 + c_1 e^{\beta\xi} + c_2 e^{2\beta\xi},$$

we choose the one tangent to the null eigenspace, i.e., a center manifold of (3.14).

We now can write explicitly the generalized eigenvectors $\tilde r_1$: with easy computations in fact we obtain

$$v_2(0) = -\sigma_1 u_{2,\xi}(0) = -\sigma_1 b\frac{a}{\lambda_2-\lambda_1} - \sigma b\frac{b\lambda_1(1-e^{-\beta})}{\beta}\frac{1}{(\lambda_2-\lambda_1)\big(\lambda_2(1+e^{-\beta})-2\lambda_1\big)},$$

and noting that $u_1(0) = a$, $v_1(0) = -\sigma_1 b$ and using (3.16), we conclude that

$$\tilde r_1\big(u, v_1, \sigma_1\big) = \left(1, \frac{u_1}{\lambda_2-\lambda_1} - v_1\frac{1}{(\lambda_2-\lambda_1)\big(\lambda_2(1+e^{-\beta})-2\lambda_1\big)}\right),$$

where β can be determined by (3.16) in terms of σ_1. Note that we have the relations

$$\tilde r_1\Big|_{v_1=0} = r_1(u), \qquad \tilde r_{1,\sigma} = \mathcal{O}(1)v_1.$$

Note also that the above equations are a consequence of the fact that the center manifold is tangent to the null space of the linearized equation. For $v_1 \to 0$, in fact, the travelling profile reduces to

$$\phi(\xi) = u + \xi r_1(u),$$

with no dependence on σ_1.

In the next sections we will need a precise estimate of the functions $\tilde r_m$, $\tilde p_m$ for $v^j_m \to 0$, i.e., an estimate of the Taylor expansion of $\mathcal{C}_m$. We have the following proposition:

Proposition 3.4. *With fixed u^j, v^j_m and σ_m, consider the function $u \in C^0([-1,0],\mathbb{R}^n)$ given by*

(3.18)

$$\begin{aligned} u(\xi) = {}& u^j - \frac{v^j_m}{\sigma_m}\left(\frac{e^{\beta\xi}-1}{\beta}\right)\frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle} \\ &+ \left(\frac{v^j_m}{\sigma_m}\right)^2 \frac{D\lambda^j_m r^j_m}{\lambda^j_m}\left(\frac{1+e^{-\beta}}{2\beta^2\big(1-e^{-\beta}\big)}\big(e^{\beta\xi}-1\big)^2 + \frac{1-e^{-\beta}}{\beta\big((1+\beta)e^{-\beta}-1\big)}\left(\xi e^{\beta\xi} - \frac{e^{\beta\xi}-1}{\beta}\right)\right)\frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle^2} \\ &+ \left(\frac{v^j_m}{\sigma_m}\right)^2 \sum_{n\neq m}\langle l^j_n, Dr^j_m r^j_m\rangle\left(\frac{\big(\lambda^j_n-\lambda^j_m\big)\big(1+e^{-\beta}\big)}{2\beta^2\Big(\lambda^j_n\big(1+e^{-\beta}\big)-2\lambda^j_m\Big)}\big(e^{2\beta\xi}-1\big) - \frac{e^{\beta\xi}-1}{\beta^2}\right)\frac{r^j_n}{\langle l_m(u_0), r^j_m\rangle^2} \\ &- \frac{\big(v^j_m\big)^2}{\sigma_m}\sum_{n\neq m}\frac{\langle l^j_n, Dr^j_m r^j_m\rangle\langle l_m(u_0), r^j_n\rangle}{\lambda^j_n\big(1+e^{-\beta}\big)-2\lambda^j_m}\left(\frac{e^{\beta\xi}-1}{\beta}\right)\frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle^3}. \end{aligned}$$

where β is obtained by the linearized dispersion relation

$$\frac{\sigma_m}{\lambda_m(u^j)} = \frac{1-e^{-\beta}}{\beta}. \tag{3.19}$$

If we denote by ϕ the travelling profile of (3.1) *satisfying $\phi(0) = u^j$, $\phi'(0) = -v^j_m/\sigma_m$, $\sigma = \sigma_m$, then*

$$\big\|\phi - u\big\|_{C^0[-1,0]} = \mathcal{O}\big(v^j_m\big)^3. \tag{3.20}$$

Proof. In Appendix A it is proved that (3.18) satisfies (3.1) up to third order in v^j_m, i.e.,

$$-\sigma_m u_\xi + f\big(u(\xi)\big) - f\big(u(\xi-1)\big) = \mathcal{O}\big(v^j_m\big)^3.$$

Moreover we have that $u(0) = u^j$ and

$$\begin{aligned}\langle l_m(u_0), -\sigma_m u'(0)\rangle &= v^j_m - \frac{\big(v^j_m\big)^2}{\big\langle l_m(u_0), r^j_m\big\rangle^2}\left\langle l_m(u_0), \sum_n \frac{\langle l^j_n, Dr^j_m r^j_m\rangle}{\lambda^j_n(1+e^{-\beta}) - \lambda^j_m}\left(r^j_n - \frac{\langle l_m(u_0), r^j_n\rangle}{\langle l_m(u_0), r^j_m\rangle} r^j_m\right)\right\rangle \\ &= v^j_m.\end{aligned}$$

Note that for $\sigma_m = \lambda_m(u^j)$, i.e., $\beta = 0$, (3.18) reduces to

$$\begin{aligned}u(\xi) = u^j &- \frac{v^j_m}{\lambda^j_m}\xi \frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle} + \left(\frac{v^j_m}{\lambda^j_m}\right)^2 \left(-\frac{1}{6\lambda^j_m} D\lambda^j_m r^j_m \xi^2 + \frac{1}{3\lambda^j_m} D\lambda^j_m r^j_m \xi^3\right)\frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle^2} \\ &+ \left(\frac{v^j_m}{\lambda^j_m}\right)^2 \frac{\langle l^j_n, Dr^j_m r^j_m\rangle}{2\langle l_m(u_0), r^j_m\rangle^2} \sum_{n\neq m}\left(\frac{\lambda^j_m}{(\lambda^j_n - \lambda^j_m)}\left(r^j_n - \frac{\langle l_m(u_0), r^j_n\rangle}{\langle l_m(u_0), r^j_m\rangle} r^j_m\right)\xi + \xi^2 r^j_n\right).\end{aligned}$$

The above equations show that the manifold $C_m : \mathbb{R}^{n+1} \times \mathbb{R}^+ \mapsto C^0([-1,0];\mathbb{R}^n)$, defined by (3.18), is tangent to the null space of the linearized operator, and moreover it satisfies

$$u(0) = u^j, \qquad -\sigma_m\langle l_m(u_0), u'(0)\rangle = v^j_m.$$

By the center manifold theory, we can conclude that C_m approximates $\mathcal{C}_m$ to the second order, i.e., the conclusion (3.20) holds. □

Remark 3.5. As noted before, the center manifold in a neighborhood of an equilibrium $u \equiv u_0$, $\sigma = \lambda_m(u_0)$ contains all the equilibria $u \equiv \bar{u}$, $\sigma = \lambda_m(\bar{u})$, with $\bar{u}$ close to u_0. As the proof of Proposition 3.4 shows, the expansion (3.18) is actually tangent to the center manifold $\mathcal{C}_m$ in all these equilibria.

Note that the choice of the constants in (3.18) is very delicate: in fact, if we choose other constants, we still get an "up to third order" invariant manifold, but for $\beta \to 0$, or equivalently $\sigma_m \to \lambda_m(u^j)$, this manifold will blowup, i.e., we are not approximating the center manifold. This is the same situation as in Example 3.3. Note moreover that, when $D\lambda_m r_m = 0$ and $\langle l_m(u_0), r_n(u^j)\rangle = 0$, (3.18) reduces to

$$\begin{aligned}u(\xi) = u^j &- \frac{v^j_m}{\sigma_m}\left(\frac{e^{\beta\xi}-1}{\beta}\right)\frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle} \\ &+ \left(\frac{v^j_m}{\sigma_m}\right)^2 \sum_{n\neq m}\frac{\langle l^j_n, Dr^j_m r^j_m\rangle}{\langle l_m(u_0), r^j_m\rangle^2}\left(\frac{\big(\lambda^j_n - \lambda^j_m\big)\big(1+e^{-\beta}\big)}{2\beta^2\Big(\lambda^j_n\big(1+e^{-\beta}\big) - 2\lambda^j_m\Big)}\big(e^{2\beta\xi}-1\big) - \frac{e^{\beta\xi}-1}{\beta^2}\right) r^j_n,\end{aligned}$$

which coincides in the case of Example 3.3 with (3.17), because

$$\langle l_2, Dr_1 r_1\rangle = \frac{1}{\lambda_2 - \lambda_1}, \qquad l_1(u_0) = l_1(u) = (1,0), \qquad r_1(u) = \begin{pmatrix} 1 \\ u/(\lambda_2 - \lambda_1)\end{pmatrix}.$$

A consequence of the above proposition is the following corollary:

Corollary 3.6. *Consider the functions $\tilde{r}_m$, $\tilde{p}_m$ of the variables (u^j, v^j_m, σ^j_m), defined by (3.6) for $|u^j - u_0| \leq 3\delta_1$, $|v^j_m| \leq 3\delta_1$, $|\sigma_m - \lambda_m(u_0)| \leq 3\delta_1$. Then the following expansions hold:*

$$\begin{aligned}\text{(3.21)} \quad \tilde{r}_m(u^j, v^j_m, \sigma_m) &= \frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle} \\ &- v^j_m\left(\frac{1}{\langle l_m(u_0), r^j_m\rangle^2}\sum_{n\neq m}\frac{\langle l^j_m, Dr^j_m r^j_m\rangle}{\lambda^j_n\big(1+e^{-\beta^j_m}\big) - 2\lambda^j_m}\left(r^j_n - \frac{\langle l_m(u_0), r^j_n\rangle}{\langle l_m(u_0), r^j_m\rangle} r^j_m\right)\right) + \mathcal{O}(1)\big(v^j_m\big)^2.\end{aligned}$$

(3.22)

$$\tilde p_m(u^j, v^j_m, \sigma_m) = e^{-\beta^j_m} + \frac{v^j_m}{\lambda^j_m}\frac{D\lambda^j_m r^j_m}{\lambda^j_m}\left(\frac{1+e^{-\beta^j_m}}{1-e^{-\beta^j_m}}e^{-\beta^j_m} + \frac{\beta^j_m}{(1+\beta^j_m)e^{-\beta^j_m}-1}e^{-\beta^j_m}\right)\frac{1}{\langle l_m(u_0), r^j_m\rangle}$$
$$+ \frac{v^j_m}{\lambda^j_m}\sum_{n\neq m}\langle l^j_n, Dr^j_m r^j_m\rangle \frac{(\lambda^j_n-\lambda^j_m)(1+e^{-\beta^j_m})}{\lambda^j_m(1+e^{-\beta^j_m})-2\lambda^j_m}e^{-\beta^j_m}\frac{\langle l_m(u_0), r^j_n\rangle}{\langle l_m(u_0), r^j_m\rangle^2} + \mathcal{O}(1)(v^j_m)^2,$$

where β^j_m *is given by*

$$\frac{\sigma_m}{\lambda(u^j_m)} = \frac{1-e^{-\beta^j_m}}{\beta^j_m}. \tag{3.23}$$

Remark 3.7. Note that (3.21) implies

$$\tilde r_m(u^j, 0, \sigma_m) = r_m(u^j), \tag{3.24}$$

and

$$\tilde r_{m,v}(u^j, 0, \sigma_m) = -\sum_{n\neq m}\langle l^j_n, Dr^j_m r^j_m\rangle\left(\frac{1}{\lambda^j_n(1+e^{-\beta^j_m})-\lambda^j_m}\right)\left(r^j_n - \langle l_m(u_0), r^j_m\rangle r^j_n\right). \tag{3.25}$$

Moreover by (3.22)

$$\tilde p_m(u^j, 0, \sigma_m) = e^{-\beta^j_m}. \tag{3.26}$$

Proof. From (3.18) we get

$$-\sigma_m\frac{d}{d\xi}\phi(\xi) = v^j_m e^{\beta^j_m\xi}\frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle}$$
$$- \frac{(v^j_m)^2}{\sigma_m}\frac{D\lambda^j_m r^j_m}{\lambda^j_m}\left(\frac{1+e^{-\beta^j_m}}{\beta^j_m(1-e^{-\beta^j_m})}(e^{\beta^j_m\xi}-1)e^{\beta^j_m\xi} + \frac{1-e^{-\beta^j_m}}{(1+\beta^j_m)e^{-\beta^j_m}-1}\xi e^{\beta^j_m\xi}\right)\frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle^2}$$
$$- \frac{(v^j_m)^2}{\sigma_m}\sum_{n\neq m}\langle l^j_n, Dr^j_m r^j_m\rangle\left(\frac{(\lambda^j_n-\lambda^j_m)(1+e^{-\beta^j_m})}{\beta^j_m(\lambda^j_n(1+e^{-\beta^j_m})-2\lambda^j_m)}e^{2\beta^j_m\xi} - \frac{e^{\beta^j_m\xi}}{\beta^j_m}\right)\frac{r^j_n}{\langle l_m(u_0), r^j_m\rangle^2}$$
$$+ (v^j_m)^2\sum_{n\neq m}\frac{\langle l^j_n, Dr^j_m r^j_m\rangle}{\lambda^j_n(1+e^{-\beta^j_m})-2\lambda^j_m}\frac{\langle l_m(u_0), r^j_n\rangle}{\langle l_m(u_0), r^j_m\rangle}e^{\beta^j_m\xi}\frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle^2} + \mathcal{O}(v^j_m)^3,$$

so that we obtain the expansion

$$v^j = -\sigma\left.\frac{d\phi}{d\xi}\right|_{\xi=0}$$
$$= v^j_m\frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle} - \frac{(v^j_m)^2}{\langle l_m(u_0), r^j_m\rangle^2}\sum_{n\neq m}\frac{\langle l^j_n, Dr^j_m r^j_m\rangle}{\lambda^j_n(1+e^{-\beta^j})-2\lambda^j_m}\left(r^j_n - \frac{\langle l_m(u_0), r^j_n\rangle}{\langle l_m(u_0), r^j_m\rangle}r^j_m\right) + \mathcal{O}(1)(v^j_m)^3,$$

from which (3.21) follows. Evaluating now $d\phi/d\xi$ at $\xi=-1$, we get

$$v^{j-1} = v^j_m e^{-\beta^j_m}\frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle} - \frac{(v^j_m)^2}{\sigma_m}\frac{D\lambda^j_m r^j_m}{\lambda^j_m}\left(-\frac{1+e^{-\beta^j_m}}{\beta^j_m} - \frac{1-e^{-\beta^j_m}}{(1+\beta^j_m)e^{-\beta^j_m}-1}\right)\frac{e^{-\beta^j_m}r^j_m}{\langle l_m(u_0), r^j_m\rangle^2}$$
$$+ \frac{(v^j_m)^2}{\sigma_m}\sum_{n\neq m}\langle l^j_n, Dr^j_m r^j_m\rangle\frac{\lambda^j_n(1+e^{-\beta^j_m})-\lambda^j_m(2+e^{-\beta^j_m})}{\lambda^j_n(1+e^{-\beta^j_m})-2\lambda^j_m}\frac{1-e^{-\beta^j_m}}{\beta^j_m}e^{-\beta^j_m}\frac{r^j_n}{\langle l_m(u_0), r^j_m\rangle^2}$$
$$+ (v^j_m)^2\sum_{n\neq m}\frac{\langle l^j_n, Dr^j_m r^j_m\rangle}{\lambda^j_n(1+e^{-\beta^j_m})-2\lambda^j_m}\frac{\langle l_m(u_0), r^j_n\rangle}{\langle l_m(u_0), r^j_m\rangle}e^{-\beta^j_m}\frac{r^j_m}{\langle l_m(u_0), r^j_m\rangle^2} + \mathcal{O}(v^j_m)^3,$$

so it follows that

$$v_m^{j-1} = v_m^j e^{-\beta_m^j} + \frac{(v_m^j)^2}{\lambda_m^j} \frac{D\lambda_m^j r_m^j}{\lambda_m^j} \left(\frac{1+e^{-\beta_m^j}}{1-e^{-\beta_m^j}} e^{-\beta_m^j} + \frac{\beta_m^j}{(1+\beta_m^j)e^{-\beta_m^j} - 1} e^{-\beta_m^j} \right) \frac{1}{\langle l_m(u_0), r_m^j \rangle}$$
$$+ \frac{(v_m^j)^2}{\lambda_m^j} \sum_{n \neq m} \langle l_n^j, D r_m^j r_m^j \rangle \frac{(\lambda_n^j - \lambda_m^j)(1+e^{-\beta_m^j})}{\lambda_m^j (1+e^{-\beta_m^j}) - 2\lambda_m^j} e^{-\beta_m^j} \frac{\langle l_m(u_0), r_n^j \rangle}{\langle l_m(u_0), r_m^j \rangle^2} + \mathcal{O}(\epsilon^3).$$

This concludes the proof. □

Remark 3.8. Note that, even if the speed σ is constant, in general β_m^j is not. This is a consequence of the fact that the dispersion relation depends on u^j through the eigenvalue λ_m.

Example 3.9. We now consider an explicit example where we can define a travelling profile manifold in a genuinely nonlinear case. We consider the flux function f given by

$$f(u) = e^u,$$

so that the corresponding semidiscrete scheme is

$$u_t^j + e^{u^j} - e^{u^{j-1}} = 0. \tag{3.27}$$

We now perform a change of variable analogous to the Cole-Hopf transformation: if u^j is a solution to (3.27) with bounded total variation and satisfying $u^{-\infty} = 0$, define the sequence z^j by

$$u^j = \log z^{j-1} - \log z^j \quad \Longrightarrow \quad z^j = \exp\left(- \sum_{k=-\infty}^{j} u^k \right). \tag{3.28}$$

Adding up from $-\infty$ to j, equation (3.27) yields

$$0 = \left(\sum_{k=-\infty}^{j} u^k \right)_t + e^{u^j} - 1 = \frac{1}{e^{-\sum_{-\infty}^{j} u^k}} \left(-z_t^j + e^{-\sum_{-\infty}^{j-1} u^k} - e^{-\sum_{-\infty}^{j} u^k} \right),$$

so that we conclude that the variable z^j satisfies the linear equation

$$z_t^j + z^j - z^{j-1} = 0. \tag{3.29}$$

Conversely it is easy to verify that if z^j is a strictly positive solution of (3.29), then u^j defined by (3.28) is a solution to (3.27). In fact

$$u_t^j = \frac{z_t^{j-1}}{z^{j-1}} - \frac{z_t^j}{z^j} = \frac{z^{j-2} - z^{j-1}}{z^{j-1}} - \frac{z^{j-1} - z^j}{z^j} = \frac{z^{j-2}}{z^{j-1}} - \frac{z^{j-1}}{z^j} = e^{u^{j-1}} - e^{u^j}.$$

By means of the transformation (3.28), we can obtain all the travelling profiles of (3.27). In fact, a travelling profile for z is

$$z(\xi) = a + b e^{\beta \xi},$$

so that the invariant manifold of travelling profiles of (3.27) is

$$\phi(\xi) = \alpha + \log\left(\frac{a + b e^{\beta(\xi-1)}}{a + b e^{\beta\xi}} \right) = \alpha + \log\left(\frac{\epsilon + e^{\beta(\xi-1)}}{\epsilon + e^{\beta\xi}} \right), \qquad \alpha, \beta, \epsilon \in \mathbb{R}. \tag{3.30}$$

The quantity α is equal to $\phi(-\infty)$, and the speed is given by

$$\sigma = e^{\alpha} \frac{e^{\beta} - 1}{\beta}, \tag{3.31}$$

which is the Rankine-Hugoniot condition if $\epsilon > 0$. Otherwise we are considering an unbounded profile, corresponding to a rarefaction in the hyperbolic case.

We can now check in this special case the decomposition given by Proposition 3.4. We need to rewrite the travelling profiles given by (3.30) using the coordinates $\phi(0) = u^j$, $\phi'(0) = -v^j/\sigma$ and σ.

Imposing the restriction that $\phi(0) = u^j$, we obtain

$$\phi(\xi) = u^j + \log\left(\frac{\epsilon + 1}{\epsilon + e^{\beta}} \right) - \log\left(\frac{\epsilon + e^{-\beta\xi}}{\epsilon + e^{-\beta(\xi-1)}} \right),$$

for $\epsilon \in \mathbb{R}$. Moreover the speed is given by

$$\sigma = e^{u^j} \frac{\epsilon + 1}{\epsilon + e^\beta} \frac{e^\beta - 1}{\beta},$$

and

$$\phi'(0) = -\frac{v^j}{\sigma} = \frac{\epsilon\beta\big(e^{-\beta} - 1\big)}{\big(1+\epsilon\big)\big(1 + \epsilon e^{-\beta}\big)}.$$

We thus keep u^j, σ fixed and we let $v^j \to 0$, so that the size of the jump tends to 0. Equivalently we can let $\epsilon \to 0$.

The speed relation implies that β is a function of ϵ. In fact we have that

$$\beta = \beta^j + \epsilon \frac{\partial \beta}{\partial \epsilon}\bigg|_{\epsilon=0} + \mathcal{O}(1)\epsilon^2,$$

where

$$\frac{\sigma}{e^{u^j}} = \frac{1 - e^{-\beta^j}}{\beta^j}, \qquad \frac{\partial \beta}{\partial \epsilon}\bigg|_{\epsilon=0} = \frac{\beta^j\big(1 - e^{-\beta^j}\big)\big(e^{-\beta^j} - 1\big)}{\big(1+\beta^j\big)e^{-\beta^j} - 1}.$$

We thus obtain the following expansion for $\phi(\xi)$:

$$\begin{aligned}
\phi(\xi) &= u^j + \log\big(1+\epsilon\big) - \log\big(1 + \epsilon e^{\beta\xi}\big) + \log\big(1 + \epsilon e^{\beta(\xi-1)}\big) - \log\big(1 + \epsilon e^{-\beta}\big) \\
&= u^j + \epsilon\big(e^{-\beta} - 1\big)\big(e^{\beta\xi} - 1\big) - \frac{\epsilon^2}{2}\big(e^{-2\beta} - 1\big)\big(e^{2\beta\xi} - 1\big) + \mathcal{O}(1)\epsilon^3 \\
&= u^j + \phi'(0)\frac{e^{\beta\xi} - 1}{\beta}\big(1 + \epsilon\big(1 + e^{-\beta}\big) + \mathcal{O}(1)\epsilon^2\big) - \phi'(0)\frac{\epsilon}{2\beta}\big(1 + e^{-\beta}\big)\big(e^{2\beta\xi} - 1\big) + \mathcal{O}(1)\epsilon^3 \\
&= u^j + \phi'(0)\frac{e^{\beta\xi} - 1}{\beta} + \frac{\phi'(0)^2}{2}\frac{1 + e^{-\beta}}{\beta^2\big(1 - e^{-\beta}\big)}\big(e^{\beta\xi} - 1\big)^2 + \mathcal{O}(1)\epsilon^3 \\
&= u^j - \frac{v^j}{\sigma}\frac{e^{\beta^j\xi} - 1}{\beta^j} \\
&\quad + \left(\frac{v^j}{\sigma}\right)^2 \left[\frac{1 + e^{-\beta^j}}{2(\beta^j)^2\big(1 - e^{-\beta^j}\big)}\big(e^{\beta^j\xi} - 1\big)^2 + \frac{1 - e^{-\beta^j}}{\beta^j\big((1+\beta^j)e^{-\beta^j} - 1\big)}\left(\xi e^{\beta^j\xi} - \frac{e^{\beta^j\xi} - 1}{\beta}\right)\right] \\
&\quad + \mathcal{O}(1)\big(v^j_m\big)^3.
\end{aligned}$$

This computation shows that the two second order terms have different meanings: the first one is due to the Taylor expansion of the profile, while the other gives the correction of the local dispersion relation (3.19) to the terms β, when passing from $\phi(0) = u^j$ to $\phi(\xi)$.

4. Wave decomposition and the Glimm interaction functional for the scalar case

Following the same approach as in the vanishing viscosity approximation [4], the next step towards a proof of BV estimates is to use the center manifold of travelling profiles to decompose a solution u^j of (2.1). Differently from the hyperbolic or parabolic case, however, we cannot expect to identify a travelling profile only looking at the local information at a point j.

In the scalar parabolic case, for example, given a solution of

$$u_t + \lambda(u)u_x = u_{xx},$$

a corresponding tangent travelling wave profile is found by solving the ODE

$$-\sigma u_x + \lambda(u)u_x - u_{xx} = 0, \qquad \begin{cases} \phi(x_0) = u(x_0) \\ \phi'(x_0) = u_x(x_0) \end{cases} \qquad \sigma = \lambda\big(u(0)\big) - \frac{u_{xx}(0)}{u_x(0)}.$$

In the hyperbolic case the Rankine-Hugoniot condition plays the same role, given the value u^- and the jump in the m-th direction $u^+_m - u^-_m$ at $x = j$.

In the semidiscrete upwind scheme, the only local information is u^j, because knowing for example v^j is equivalent to knowing u^{j-1}. Roughly speaking, since in the scalar case a travelling profile is identified by three data (the position, the "jump" and the speed), we expect that we will need to know u^j, u^{j-1} and u^{j-2} to identify the travelling profile in $x = 0$.

Consider the semidiscrete scheme for a scalar equation,

$$u^j_t + f\big(u^j\big) - f\big(u^{j-1}\big) = 0, \qquad u^j \in \mathbb{R}. \tag{4.1}$$

The aim of this section is first to show how we can decompose the solution u^j to (4.1) in travelling profiles ϕ^j, identified by the values u^j, u^{j-1}, u^{j-2}. When this first result is achieved, a natural question is if there is a Glimm-type functional $Q(u)$ of the form

$$Q(u) = \sum_{j<k} \Big[\text{strenght of the wave in } j\Big] \cdot \Big[\text{strenght of the wave in } k\Big] \cdot \Big[\text{difference in speeds}\Big].$$

It is clear that in the general case, i.e., when $u^j \in \mathbb{R}^N$, this functional is very important because we will need it to bound the interaction among travelling profiles of the same family, see [16].

The first step is to find a way to identify the travelling wave at j. While u^j is given and $v^j = f(u^{j-1}) - f(u^j)$, to identify the speed of the travelling profile we recall that, by (3.5) and Corollary 3.6, if ϕ is the travelling profile with speed σ such that

$$\phi(0) = u^j, \quad -\sigma\phi'(0) = v^j, \quad -\sigma\phi'(-1) = v^{j-1},$$

then we have the relation

$$v^{j-1} = \pi\big(u^j, v^j, \sigma\big) = v^j \tilde{p}\big(u^j, v^j, \sigma\big), \tag{4.2}$$

where $\tilde{p}$ is a smooth function in a neighborhood of $(u_0, 0, f'(u_0))$ such that by (3.22)

$$\tilde{p}\big(u^j, v^j, \sigma\big) = e^{-\beta^j} + \mathcal{O}(1)v^j, \tag{4.3}$$

and β^j is given by the dispersion relation:

$$\frac{\sigma}{f'(u^j)} = \frac{1 - e^{-\beta^j}}{\beta^j}.$$

We define $\alpha^j \doteq v^{j-1}/v^j$, so that we can rewrite (4.3) as

$$\alpha^j = \tilde{p}\big(u^j, v^j, \sigma\big). \tag{4.4}$$

We can compute the derivative of $\tilde{p}$ w.r.t. σ when $v^j = 0$, obtaining

$$\frac{\partial \tilde{p}}{\partial \sigma}\bigg|_{v^j=0} = -e^{-\beta^j} \frac{\partial \beta^j}{\partial \sigma} = -\frac{(\beta^j)^2 e^{-\beta^j}}{f'(u^j)\big((1+\beta^j)e^{-\beta^j} - 1\big)}. \tag{4.5}$$

Note that this derivative is uniformly different from zero in a neighborhood of $\beta^j = 0$, corresponding to $\sigma^j = f'(u^j)$, because we have

$$\lim_{\beta^j \to 0} \frac{(\beta^j)^2}{(1+\beta^j)e^{-\beta^j} - 1} = -2.$$

This implies that $\tilde{p}$ is invertible for σ close to $\lambda(u_0)$, so that we can write the speed σ as a function of u^j, v^j, v^{j-1} if $\alpha^j = v^{j-1}/v^j$ is sufficiently close to 1, i.e., β^j close to 0.

We have proved the following proposition:

Proposition 4.1. *Given the three points u^j, u^{j-1}, u^{j-2}, such that*

$$\big|u^j - u_0\big| \leq 3\delta_1, \quad \big|f(u^j) - f(u^{j-1})\big| \leq 3\delta_1, \quad \left|\frac{f(u^{j-2}) - f(u^{j-1})}{f(u^{j-1}) - f(u^j)} - 1\right| \leq 3\delta_1$$

for δ_1 sufficiently small, then there exists a unique travelling profile ϕ such that

$$\phi(0) = u^j, \qquad -\sigma\phi'(0) = v^j, \qquad -\sigma\phi'(-1) = v^{j-1}.$$

Remark 4.2. Note that, since from (4.1)

$$v^j = f\big(u^{j-1}\big) - f\big(u^j\big) \simeq \lambda(u^j)\big(u^{j-1} - u^j\big),$$

we can state the assumptions of Proposition 4.1 as

$$\big|u^j - u_0\big| \leq 3\delta_1, \quad \big|u^j - u^{j-1}\big| \leq 3\delta_1, \quad \left|\frac{u^{j-2} - u^{j-1}}{u^{j-1} - u^j} - 1\right| \leq 3\delta_1,$$

and its conclusions as

$$\phi(0) = u^j, \qquad \phi(-1) = u^{j-1}, \qquad \phi(-2) = u^{j-2}.$$

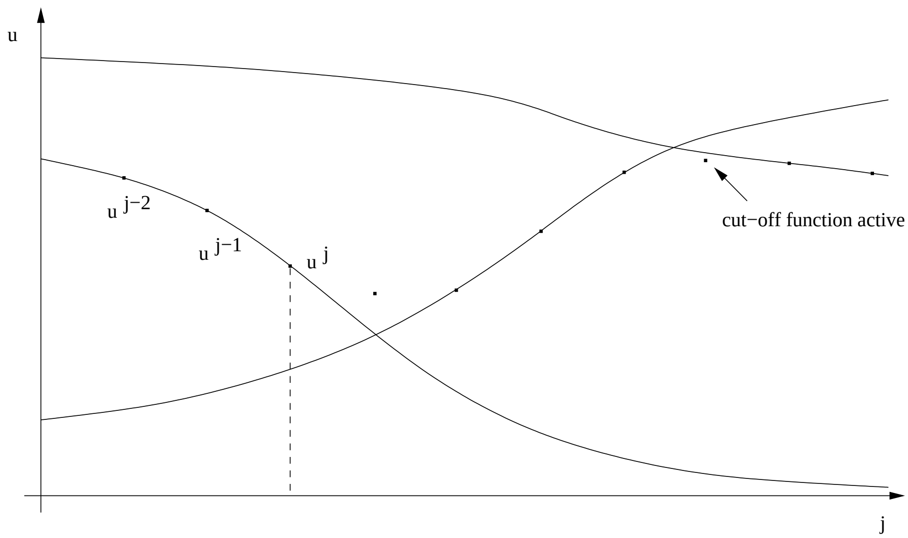

FIGURE 2. Interpolation of the solution u^j with travelling profiles

This means that the travelling profile interpolates the three given points (fig. 2).

Let ψ be the cutoff function,

$$\psi(x) = \begin{cases} 1 & |x-1| \leq \delta_1 \\ \text{smooth connection} & \delta_1 \leq |x-1| \leq 3\delta_1 \\ 0 & |x-1| \geq 3\delta_1 \end{cases} \tag{4.6}$$

and such that $|x\phi(x)| \leq 2\delta_1$, $x \in \mathbb{R}$. A simple consequence of the above proposition is that we can always define a travelling profile in any point j, which fits the points u^j, u^{j-1}, and "tries" to fit also u^{j-2}.

Corollary 4.3. *Given the points u^j, v^j, v^{j-1}, with*

$$\left|u^j - u_0\right| \leq 3\delta_1, \quad \left|f(u^j) - f(u^{j-1})\right| \leq 3\delta_1,$$

there exists a unique travelling profile ϕ such that

$$\phi(0) = u^j, \qquad -\sigma\phi'(0) = v^j,$$

with speed

$$\sigma = \tilde{p}^{-1}\big(u^j, v^j, 1 + \psi(\alpha^j)(\alpha^j - 1)\big),$$

where $\tilde{p}^{-1}$ denotes the inverse function of $\tilde{p}$.

Before considering the case when α^j is far from 1, i.e., the cut-off function (4.6) is active, we want to show how the dispersion relation (3.19) enters in the Glimm type functional. As we will see, we have to weight the speed of the waves through a nonlinear function, which is obtained from the dispersion relation.

As a measure of the local strength of the wave ϕ, a natural choice seems to be $u^j - u^{j-1}$. We will use as a measure of the strength of the wave the time derivative of u^j, i.e., $v^j \doteq u^j_t$. This is due to the fact that v^j is a particular solution of the equation for a perturbation ζ, namely

$$\zeta^j_t + f'\big(u^j\big)\zeta^j - f'\big(u^{j-1}\big)\zeta^{j-1} = 0.$$

This will help in studying the stability of the solution u, see Section 7. Note that for linear equations v^j is proportional to $u^j - u^{j-1}$, so that this choice will not influence the computations of the next example.

Since we can give locally the strength and the speed of a travelling profile, we may suspect that the functional Q is

$$Q(u) = \sum_{j<k} \left|v^j\right|\left|v^k\right|\left|\sigma^j - \sigma^k\right|.$$

To see the implications of this choice, we study a linear scalar equation, where the computations can be performed explicitly.

Example 4.4. We consider the linear semidiscrete scheme,

$$u^j_t + \lambda u^j - \lambda u^{j-1} = 0,$$

and we assume for simplicity that v^{j-1}/v^j is close to 1, where as usual we define $v^j = u^j_t$. In the linear case it is easy to prove that the speed of a travelling profile is given by

$$\frac{\sigma^j}{\lambda} = \frac{v^{j-1}/v^j - 1}{\log v^{j-1} - \log v^j} \doteq g\big(v^{n-1}/v^n\big),$$

and the equation satisfied by $v^j \sigma^j$ is

$$\begin{aligned}(4.7)\qquad \big(v^j\sigma^j\big)_t + \lambda\big(v^j\sigma^j\big) - \lambda\big(v^{j-1}\sigma^{j-1}\big) &= \lambda^2 v^{j-1} g'\big(v^{j-1}/v^j\big)\left(\frac{v^{j-2}}{v^{j-1}} - \frac{v^{j-1}}{v^j}\right) \\ &\quad + \lambda^2 v^{j-1}\Big(g\big(v^{j-1}/v^j\big) - g\big(v^{j-2}/v^{j-1}\big)\Big) \\ &= -\lambda^2 v^{j-1} g''(y)\left(\frac{v^{j-2}}{v^{j-1}} - \frac{v^{j-1}}{v^j}\right)^2,\end{aligned}$$

where y is an intermediate point between v^{j-2}/v^{j-1} and v^{j-1}/v^j. For v^{j-1}/v^j, v^{j-2}/v^{j-1} close to 1, i.e., when the speeds are close to the eigenvalue λ, we have $g''(y) \simeq g''(1) = -1/12$. In the following we denote by $e^j(t)$ the right-hand side of the above equation.

We are thus considering the 2×2 system of equations

$$\left\{\begin{array}{rcl} v^j + \lambda v^j - \lambda v^{j-1} &=& 0 \\ \big(v^j\sigma^j\big)_t + \lambda\big(v^j\sigma^j\big) - \lambda\big(v^{j-1}\sigma^{j-1}\big) &=& e^j(t). \end{array}\right.$$

If we compute the time derivative of the functional $Q(t)$ defined as

$$(4.8)\qquad Q(t) \doteq \sum_{j<k}\Big|v^j\big(v^k\sigma^k\big) - v^k\big(v^j\sigma^j\big)\Big|,$$

we obtain

$$\begin{aligned}\frac{dQ}{dt} &= \frac{d}{dt}\sum_{j<k}\Big|v^j\big(v^k\sigma^k\big) - v^k\big(v^j\sigma^j\big)\Big| \\ &= 2\sum_{j<k}\operatorname{sgn}\Big(v^j\big(v^k\sigma^k\big) - v^k\big(v^j\sigma^j\big)\Big)\Big(\lambda\big(v^{j-1} - v^j\big)\big(v^k\sigma^k\big) - \big(v^k\big)\lambda\Big(\big(v^{j-1}\sigma^{j-1}\big) - \big(v^j\sigma^j\big)\Big)\Big) \\ &\quad + 2\sum_{j<k}\operatorname{sgn}\Big(v^j\big(v^k\sigma^k\big) - v^k\big(v^j\sigma^j\big)\Big)v^j e^k \\ &= 2\lambda\sum_k\sum_{j=j_i(k)+1}^{j_{i-1}(k)}\operatorname{sgn}\Big(v^j\big(v^k\sigma^k\big) - v^k\big(v^j\sigma^j\big)\Big)\Big(\lambda\big(v^{j-1} - v^j\big)\big(v^k\sigma^k\big) - \big(v^k\big)\lambda\Big(\big(v^{j-1}\sigma^{j-1}\big) - \big(v^j\sigma^j\big)\Big)\Big) \\ &\quad + 2\sum_{j<k}\operatorname{sgn}\Big(v^j\big(v^k\sigma^k\big) - v^k\big(v^j\sigma^j\big)\Big)v^j e^k \\ &= -4\lambda\sum_k\sum_{i>0}\Big|v^{j_i(k)-1}\big(v^k\sigma^k\big) - v^k\big(v^{j_i(k)-1}\sigma^{j_i(k)-1}\big)\Big| - 2\lambda\sum_k\Big|v^{k-1}\big(v^k\sigma^k\big) - v^k\big(v^{k-1}\sigma^{k-1}\big)\Big| \\ &\quad + 2\sum_{j<k}\operatorname{sgn}\Big(v^j\big(v^k\sigma^k\big) - v^k\big(v^j\sigma^j\big)\Big)v^j e^k \\ &\le -2\lambda\sum_k\Big|v^{k-1}\big(v^k\sigma^k\big) - v^k\big(v^{k-1}\sigma^{k-1}\big)\Big| + 2\sum_{j<k}\operatorname{sgn}\Big(v^j\big(v^k\sigma^k\big) - v^k\big(v^j\sigma^j\big)\Big)v^j e^k,\end{aligned}$$

where the points $j_i(k)$ are obtained by

$$\operatorname{sgn}\Big(v^{j_i}\big(v^k\sigma^k\big) - v^k\big(v^{j_i}\sigma^{j_i}\big)\Big)\cdot\operatorname{sgn}\Big(v^{j_i-1}\big(v^k\sigma^k\big) - v^k\big(v^{j_i-1}\sigma^{j_i-1}\big)\Big) = -1,$$

and they are monotone decreasing with the index i. Without loss of generality we assume that the points $j_i(k)$ are finite; the general case follows by approximation because the solution u^j is in BV.

We see that to show that the above functional is decreasing, we need to estimate e^j. However e^j is not related to the negative part of the derivative of Q, because the latter is of second order w.r.t. the quantity v^j, v^{j-1}, while e^j is only first order. In fact, if we rescale $v^j \mapsto \nu v^j$, then Q becomes $\nu^2 Q$, but $e^j \mapsto \nu e^j$. Thus we cannot expect that dQ/dt controls the source term e^j.

As in the parabolic case [8], we can relate the above functional to the area swept by the curve $\gamma(t) \in \mathbb{R}^2$, obtained by connecting the points

$$P^j \doteq \left(\sum_{k=-\infty}^{j} v^k, \sum_{k=-\infty}^{j} v^k \sigma^k \right) = \left(-\lambda u^j, \sum_{k=-\infty}^{j} v^k \sigma^k \right).$$

Similarly to the parabolic case, we can consider another functional, the Length Functional, i.e., the length of the line γ:

$$L(t) = \sum_j \sqrt{\big(v^j\big) + \big(v^j \sigma^j\big)^2}.$$

Differentiating $L(t)$ w.r.t. t we have

$$\begin{aligned}
\frac{dL}{dt} &= \sum_j \frac{1}{\sqrt{\big(v^j\big) + \big(v^j\sigma^j\big)^2}} \Big(v^j \lambda \big(v^{j-1} - v^j\big) + v^j \sigma^j \lambda \big(v^{j-1}\sigma^{j-1} - v^j \sigma^j \big) \Big) + \sum_j \frac{v^j \sigma^j}{\sqrt{\big(v^j\big) + \big(v^j\sigma^j\big)^2}} e^j \\
&= \lambda \sum_j \Bigg[\frac{1}{\sqrt{\big(v^j\big) + \big(v^j\sigma^j\big)^2}} \Big(v^j v^{j-1} + v^j \sigma^j v^{j-1} \sigma^{j-1} \Big) \\
&\qquad - \frac{1}{\sqrt{\big(v^{j-1}\big) + \big(v^{j-1}\sigma^{j-1}\big)^2}} \Big(\big(v^{j-1}\big)^2 + \big(v^{j-1}\sigma^{j-1}\big)^2 \Big) \Bigg] + \sum_j \frac{v^j\sigma^j}{\sqrt{\big(v^j\big) + \big(v^j \sigma_j\big)^2}} e^j \\
&= \lambda \sum_j \frac{\big|v^{j-1}\big|}{\sqrt{1 + (\sigma^j)^2}} \Big(\big(1 + \sigma^j \sigma^{j-1}\big) - \sqrt{1+(\sigma^j)^2}\sqrt{1 + (\sigma^{j-1})^2} \Big) + \sum_j \frac{\sigma^j e^j}{\sqrt{1 + \big(\sigma^j\big)^2}} \\
&= -\lambda \sum_j \frac{\big|v^{j-1}\big| \big(\sigma^j - \sigma^{j-1}\big)^2}{\sqrt{1+(\sigma^j)^2} \Big(1 + |\sigma^j||\sigma^{j-1}| + \sqrt{1+(\sigma^j)^2}\sqrt{1+(\sigma^{j-1})^2} \Big)} + \sum_j \frac{\sigma^j e^j}{\sqrt{1 + \big(\sigma^j\big)^2}}.
\end{aligned}$$

A simple computation shows that, for σ close to λ, the coefficient in front of the terms in the first sum is

$$\frac{\lambda}{2\big(1+\lambda^2\big)^{3/2}} \big(\sigma'(0)\big)^2 = \frac{\lambda^3}{8\big(1+\lambda^2\big)^{3/2}},$$

so that, in general, we cannot expect decreasing or boundedness of the length using the negative part of the the derivative of $L(t)$, because both coefficients are of $\mathcal{O}(1)$.

Remark 4.5. If $\lambda < 1/\sqrt{2}$, then the coefficient in front of the negative part in dL/dt is greater than the source, for σ close to λ. This is a consequence of the relation between g' and g''.

One can rescale t so that λ is sufficiently small. We will show below that there is a more elegant way to handle these source terms, and actually to make them of higher order w.r.t. $|v^j|$.

A more natural choice, at least in the linear case, is the following: consider the line $\gamma(t)$ obtained by connecting the points

$$P^j = \begin{pmatrix} u^j \\ u^{j-1} \end{pmatrix}. \tag{4.9}$$

Since the system is linear, it follows immediately that

$$P^j_t + \lambda P^j - \lambda P^{j-1} = 0.$$

Using the same techniques as in [8], it can be shown that the above equation implies that γ moves in the direction of curvature.

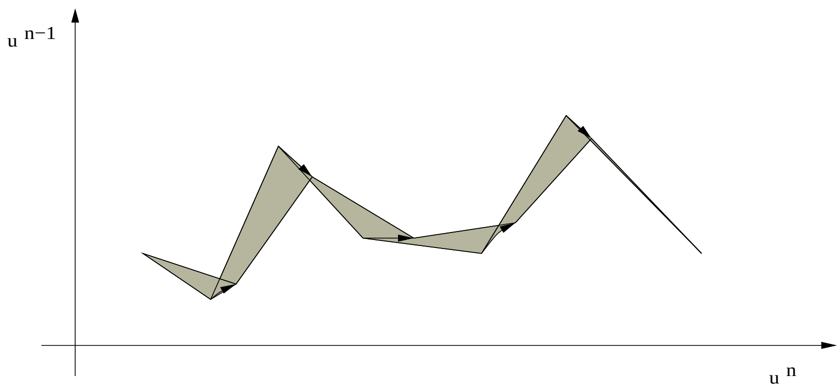

FIGURE 3. Motion by curvature of the curve γ

We can write the following Length and Area Functionals:

$$L(\gamma) = \sum_j \sqrt{\big(u^j - u^{j-1}\big)^2 + \big(u^{j-1} - u^{j-2}\big)^2} = \frac{1}{\lambda} \sum_j \sqrt{\big(v^j\big)^2 + \big(v^{j-1}\big)^2}, \tag{4.10}$$

$$Q(\gamma) = \frac{\lambda}{2} \sum_{j<k} \left| \begin{pmatrix} u^j - u^{j-1} \\ u^{j-1} - u^{j-2} \end{pmatrix} \wedge \begin{pmatrix} u^k - u^{k-1} \\ u^{k-1} - u^{k-2} \end{pmatrix} \right| = \frac{1}{2\lambda} \sum_{j<k} \Big| v^j v^{k-1} - v^k v^{j-1} \Big|. \tag{4.11}$$

With computation similar to the ones performed before and denoting as usual $\alpha^j = v^{j-1}/v^j$, we have

$$\begin{aligned}
\frac{dL}{dt} &= \sum_j \frac{|v^{j-1}|}{\sqrt{1+(\alpha^j)^2}} \Big(\mathrm{sgn}(\alpha^j)\big(1 + \alpha^j \alpha^{j-1}\big) - \sqrt{1+(\alpha^j)^2}\sqrt{1+(\alpha^{j-1})^2} \Big) \\
&\leq -\sum_j \frac{|v^{j-1}|\big(\alpha^j - \alpha^{j-1}\big)^2}{\sqrt{1+(\alpha^j)^2}\Big(1 + \alpha^j\alpha^{j-1} + \sqrt{1+(\alpha^j)^2}\sqrt{1+(\alpha^{j-1})^2}\Big)} \chi\Big\{ j : |\alpha^j - 1|, |\alpha^{j-1} - 1| \leq 5\delta_1 \Big\} \\
&\quad - \frac{1}{2} \sum_j \delta_1^2 |v^{j-1}| \big(1 + |\alpha^{j-1}|\big) \chi\Big\{ |\alpha^j - 1| \geq 5\delta_1, |\alpha^{j-1}| \leq 3\delta_1 \text{ or } |\alpha^j - 1| \leq 3\delta_1, |\alpha^{j-1} - 1| \geq 5\delta_1 \Big\},
\end{aligned}$$

$$\frac{dQ}{dt} \leq -\sum_j \Big| \big(v^{j-1}\big)^2 - v^j v^{j-2} \Big| = -\sum_j \left| v^j v^{j-1} \left(\frac{v^{j-1}}{v^j} - \frac{v^{j-2}}{v^{j-1}} \right) \right|.$$

In the previous equation we have used the fact that

$$\left(\frac{\mathrm{sgn}(x)(1+xy)}{\sqrt{1+x^2}} - \sqrt{1+y^2} \right) \chi\Big\{ |x-1| \geq 5\delta_1, |y-1| \leq 3\delta_1 \text{ or } |x-1| \leq 3\delta_1, |y-1| \geq 5\delta_1 \Big\} \leq -\frac{1}{2}\delta_1^2\big(1 + |y|\big) \tag{4.12}$$

if δ_1 is sufficiently small.

We can interpret the above functional in the following way: instead of using the real speed σ, we use the quantity α, which is related to σ by (see fig. 4)

$$\frac{\sigma}{\lambda} = \frac{\alpha - 1}{\log \alpha}.$$

As we see from the pictures, the "weight" α^j given to the "jump" v^j by the dispersion relation is greater than the real speed σ when $\sigma > 1$.

In the nonlinear case, following the above example, one may look for a variable w^j which is a solution to the equation

$$w^j_t + \lambda^j w^j - \lambda^{j-1} w^{j-1} = 0, \tag{4.13}$$

and such that the local speed σ^j is a function of the ratio w^j/v^j. Thus w^j is a function of u^j, u^{j-1}, u^{j-2}: $w^j = w^j(u^j, u^{j-1}, u^{j-2})$. However it can be shown that such a function does not exists: this is a

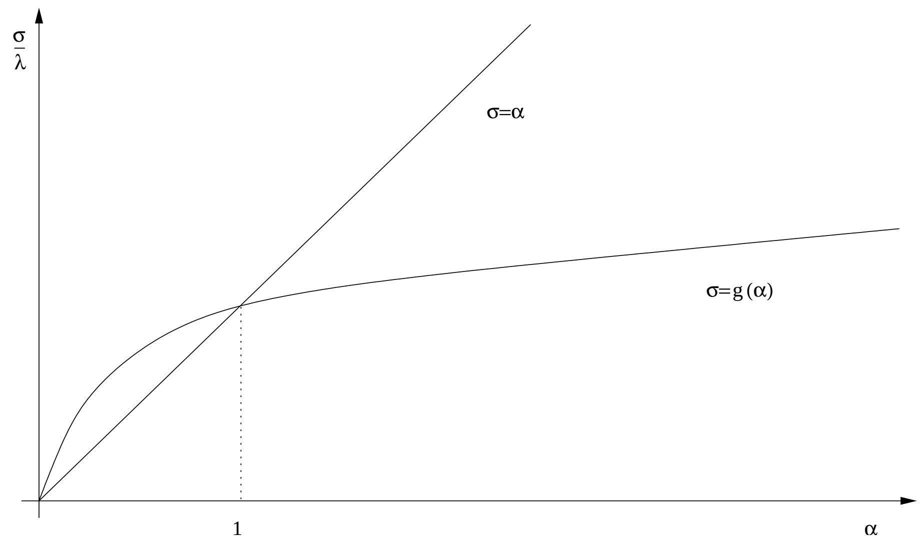

FIGURE 4. Relation between σ and α

consequence of the fact that the derivative of u^{j-2} depends on u^{j-3}, so that w^j should depend also on u^{j-3}, and so on. The proof is in Appendix B.

Remark 4.6. Of course there could be variables $z^j = z(u^j, u^{j-1}, \dots)$ such that z^j and z^{j-1} satisfy the same equation, for example using the Cole-Hopf transformation of Example 3.9.

What is proved in Appendix B is that there are no variables w^j satisfying the same equation of v^j, and such that w^j is a function of u^j, u^{j-1},…,u^{j-k} for some $k > 1$, if (4.1) is nonlinear. Thus for general semidiscrete schemes the only function with these properties is $w^j \equiv v^j$. This is very different from the parabolic case, where such a function is given by (see [8])

$$w = f(u)_x - u_{xx} = u_t.$$

In the following we prove the existence of a function $w^j = w^j(u^j, v^j, \alpha^j)$, such that

$$w^j_t + \lambda^j w^j - \lambda^{j-1} v^{j-1} = e^j(t),$$

with $e^j(t)$ integrable, and precisely

$$\sum_j \int_0^{+\infty} \big|e^j(t)\big| dt = \mathcal{O}(1)\mathcal{V}(u)^2.$$

The main idea is to use, as a measure of the speed of the travelling profile in j, the quantity s^j given by

$$\frac{\sigma^j}{\lambda(u_0)} = \frac{s^j - 1}{\log s^j},$$

i.e., using the dispersion relation at the point

$$u_0 = \lim_{j \to -\infty} u^j.$$

The above choice will generate an error, but locally this error will be of the order of the distance of u^j from u_0 multiplied by v^j, i.e., when summing w.r.t. j, of the order of the total variation squared of u.

Let g be the function

$$g(s) = \frac{s-1}{\log s} \qquad s \geq 0.$$

We can extend it for $s \leq 0$ to an odd function by defining $g(s) = -g(-s)$ if $s \leq 0$, with $g(0) = 0$. This function is of course globally invertible, but not differentiable in 0.

The variable s^j is defined by

$$g(s^j) = \frac{s^j - 1}{\log s^j} = \frac{\sigma^j}{\lambda(u_0)} = \frac{\lambda(u^j)}{\lambda(u_0)} \frac{e^{-\beta^j} - 1}{\log\big(e^{-\beta^j}\big)}, \tag{4.14}$$

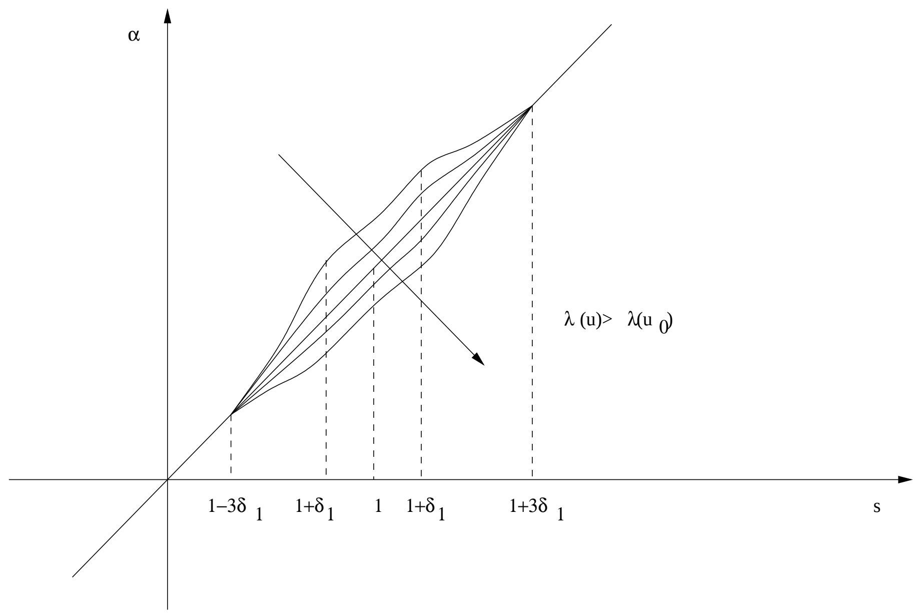

FIGURE 5. Graph of the function $\alpha = \alpha(u, 0, s)$ given by (4.16).

where σ^j is the speed of the travelling profile located in j. Recall that from the center manifold theorem we have

$$\alpha^j = \tilde{p}\big(u^j, v^j, \sigma^j\big),$$

with $\tilde{p}$ invertible in a neighborhood of $\alpha^j = 1$. Using (3.22) and (4.14), for s^j close to 1 we obtain

$$\alpha^j = \tilde{p}\big(u^j, v^j, \lambda(u_0) g(s^j)\big) = g^{-1}\left(\frac{\lambda(u_0)}{\lambda^j} g(s^j)\right) + v^j \tilde{q}\big(u^j, v^j, \lambda(u_0) g(s^j)\big)$$

for some smooth function $\tilde{q}$. Define the cutoff function

$$\psi(x) = \begin{cases} 1 & |x-1| \leq \delta_1 \\ \text{smooth connection} & \delta_1 \leq |x-1| \leq 3\delta_1 \\ 0 & |x-1| \geq 3\delta_1 \end{cases} \tag{4.15}$$

We can extend the right hand side of the above equation to an invertible function, defined on the whole real line $\mathbb{R}$ and with range equal to $\mathbb{R}$: in fact, for fixed u^j, v^j, consider the function

$$\alpha^j = g^{-1}\left(\frac{1}{1 - \psi(s^j)(\lambda(u_0) - \lambda^j)/\lambda(u_0)} g(s^j)\right) + v^j \tilde{q}\Big(u^j, v^j, \lambda(u_0)\big(1 + (g(s^j) - 1)\psi(s^j)\big)\Big) \tag{4.16}$$

It is clear that for u^j sufficiently close to u_0 and v^j small, the function $\alpha^j = \alpha^j(s^j)$ defined in (4.16) is invertible. In fact we have

$$\frac{\partial \alpha^j}{\partial s^j} = 1 + \mathcal{O}(1)\mathcal{V}(u).$$

We can then write

$$\begin{aligned} s^j &= g^{-1}\left(\left(1 + \frac{\lambda^j - \lambda(u_0)}{\lambda(u_0)} \psi(s^j)\right) g(\alpha^j)\right) + v^j \kappa\Big(u^j, v^j, \alpha^j\Big) \\ &= h\big(u^j, v^j, \alpha^j\big) + v^j \kappa\big(u^j, v^j, \alpha^j\big). \end{aligned} \tag{4.17}$$

Here κ is a smooth function, defined for u^j close to u_0, v^j small and depending on α^j only when $\psi(s^j) \neq 0$. In particular, when $|s^j - 1| > 5\delta_1$, we have directly from (4.16)

$$s^j = \alpha^j - v^j \tilde{q}\big(u^j, v^j, \lambda(u_0)\big).$$

In the following, with an abuse of notation, we will write

$$v^{j-1} = v^j \tilde{p}\big(u^j, v^j, s^j\big), \tag{4.18}$$

where $\tilde{p}$ is given by (4.16). A consequence of the definition (4.17) is that $v^j s^j$ is of the order $(|v^j|+|v^{j-1}|)$, and then it is a bounded function. We define this product to be

$$w^j \doteq v^j s^j. \tag{4.19}$$

Note that in the linear case we obtain $w^j = v^{j-1}$, so that by Example 4.4 we do not have any error term e^j.

Assume first that $v^j \neq 0$, so that α^j is well defined and we can take the time derivative of s^j. The equation satisfied by w^j is

(4.20)

$$\begin{aligned}
w^j_t + \lambda^j w^j - \lambda^{j-1} w^{j-1} &= v^j s^j_t + \lambda^{j-1} v^{j-1}\big(s^j - s^{j-1}\big) \\
&= v^j \frac{\partial h^j}{\partial \alpha} \alpha^j_t + \big(v^j\big)^2 \frac{\partial h^j}{\partial u} + v^j \big(\lambda^{j-1} v^{j-1} - \lambda^j v^j\big) \frac{\partial h^j}{\partial v^j} \\
&\quad + \lambda^{j-1} v^{j-1} \Big(h\big(u^j, v^j, \alpha^j\big) - h\big(u^{j-1}, v^{j-1}, \alpha^{j-1}\big) + v^j \kappa^j - v^{j-1} \kappa^{j-1} \Big) \\
&\quad + v^j \Big(\kappa^j_u \big(v^j\big)^2 + \big(\kappa^j + v^j \kappa^j_v\big) \big(\lambda^{j-1} v^{j-1} - \lambda^j v^j\big) + \kappa^j_\alpha v^j \alpha^j_t \Big) \\
&= \lambda^{j-1} v^{j-1} \left(\frac{\partial h}{\partial \alpha}\big(u^j, v^j, \alpha^j\big) \Big(\alpha^{j-1} - \alpha^j\Big) + h\big(u^j, v^j, \alpha^j\big) - h\big(u^j, v^j, \alpha^{j-1}\big) \right) \\
&\quad + P\big(u^j, v^j, v^{j-1}, v^{j-2}\big),
\end{aligned}$$

where P denotes a second order polynomial in v^j, v^{j-1}, v^{j-2}. We have used the following computations:

$$\begin{aligned}
v^j \alpha^j_t = v^j \left(\frac{v^{j-1}}{v^j} \right)_t &= v^j \frac{\lambda^{j-2} v^{j-2} - \lambda^{j-1} v^{j-1}}{v^j} - v^{j-1} \frac{\lambda^{j-1} v^{j-1} - \lambda^j v^j}{v^j} \\
&= \big(\lambda^{j-2} - \lambda^{j-1}\big) v^{j-2} + \lambda^{j-1} v^{j-1} \big(\alpha^{j-1} - \alpha^j\big) - \big(\lambda^{j-1} - \lambda^j\big) v^{j-1} \\
&= \lambda^{j-1} v^{j-1} \big(\alpha^{j-1} - \alpha^j\big) + \mathcal{O}(1) v^{j-1} v^{j-2} + \mathcal{O}(1) v^{j-1} v^j,
\end{aligned}$$

$$h\big(u^{j-1}, v^{j-1}, \alpha^{j-1}\big) - h\big(u^j, v^j, \alpha^{j-1}\big) = \mathcal{O}(1)\big(u^{j-1} - u^j\big) + \mathcal{O}(1)\big(v^{j-1} - v^j\big) = \mathcal{O}(1) v^j + \mathcal{O}(1) v^{j-1}.$$

Since $\partial h/\partial u$, $\partial h/\partial v$, $\partial \kappa/\partial \alpha$ are different from 0 only when α^j is close to 1, the polynomial P has smooth coefficients.

We first study the terms which are of first order w.r.t. v^j, v^{j-1} or v^{j-2}. These terms are the most difficult to bound, because they will correspond to the Length functional, which is of first order too. As we observed in Example 4.4, the constant in front of them should be very small. We will show now that with the choice (4.19) this constant is of the order of the total variation.

The first order term in the right-hand side can be computed as

$$\lambda^{j-1} v^{j-1} \left(\frac{\partial h}{\partial \alpha}\big(u^j, v^j, \alpha^j\big) \Big(\alpha^{j-1} - \alpha^j\Big) + h\big(u^j, v^j, \alpha^j\big) - h\big(u^j, v^j, \alpha^{j-1}\big) \right) = \tag{4.21}$$

$$\begin{cases}
\mathcal{O}(1)\mathcal{V}(u) v^{j-1} \Big(\alpha^j - \alpha^{j-1}\Big)^2 & \big|s^j - 1\big|, \big|s^{j-1} - 1\big| \leq 5\delta_1 \\
\mathcal{O}(1)\mathcal{V}(u) v^{j-2} + \mathcal{O}(1)\mathcal{V}(u) v^{j-1} & \big|s^j - 1\big| \geq 5\delta_1, \big|s^{j-1} - 1\big| \leq 3\delta_1 \text{ or} \\
& \quad \big|s^{j-1} - 1\big| \geq 5\delta_1, \big|s^{j-1} - 1\big| \leq 3\delta_1 \\
0 & \text{otherwise}
\end{cases}$$

In fact with easy computation or directly from figure 5 we have

$$\frac{\partial h}{\partial \alpha}\big(u^j, v^j, \alpha^j\big) \Big(\alpha^{j-1} - \alpha^j\Big) + h\big(u^j, v^j, \alpha^j\big) - h\big(u^j, v^j, \alpha^{j-1}\big) =$$

$$\begin{cases}
\partial^2 h(u^j, x)/\partial \alpha^2 \Big(\alpha^j - \alpha^{j-1}\Big)^2 & \big|s^j - 1\big|, \big|s^{j-1} - 1\big| \leq 5\delta_1 \\
\big(\partial h(u^j, \alpha^j)/\partial \alpha - \partial h(u^j, x)/\partial \alpha\big) \big(\alpha^{j-1} - \alpha^j\big) & \big|s^{j-1} - 1\big| \geq 5\delta_1, \big|s^j - 1\big| \leq 3\delta_1 \\
h(u^j, \alpha^{j-1}) - \alpha^{j-1} & \big|s^{j-1} - 1\big| \leq 3\delta_1, \big|s^j - 1\big| \geq 5\delta_1 \\
0 & \text{otherwise}
\end{cases}$$

where x denotes a point between α^j, α^{j-1}. It is easy to show that for all α

$$\frac{\partial h}{\partial \alpha}(u^j, v^j, \alpha) - 1 = \mathcal{O}(1)\big(u - u_0\big) + \mathcal{O}(1)v^j, \qquad \frac{\partial^2 h}{\partial \alpha^2}(u^j, v^j, \alpha) = \mathcal{O}(1)\big(u - u_0\big) + \mathcal{O}(1)v^j,$$

so that, using the regularity estimates (2.22), (4.21) follows.

Using (4.16) we rewrite (4.20) as

$$\begin{aligned} w^j_t + \lambda^j w^j - \lambda^{j-1} w^{j-1} &= \mathcal{O}(1)\mathcal{V}(u) v^{j-1}\big(s^j - s^{j-1}\big)^2 \chi\Big\{\big|s^j - 1\big|, \big|s^{j-1} - 1\big| \leq 5\delta_1\Big\} \\ &\quad + \mathcal{O}(1)\mathcal{V}(u)\big(|v^{j-1}| + |v^{j-2}|\big)\chi\Big\{\big|s^j - 1\big| \geq 5\delta_1 \big|s^{j-1} - 1\big| \leq 3\delta_1 \text{ or viceversa}\Big\} \\ &\quad + P\big(u^j, v^j, v^{j-1}, v^{j-2}\big). \end{aligned} \tag{4.22}$$

For $v^j = 0$, we obtain $w^j = v^{j-1}$, so that with direct computations we have

$$\begin{aligned} w^j_t + \lambda^j w^j - \lambda^{j-1} w^{j-1} &= \big(\lambda^j - \lambda^{j-1}\big) v^{j-1} + \lambda^{j-2} v^{j-2} - \lambda^{j-1} v^{j-1}\Big(h\big(u^{j-1}, v^{j-1}, \alpha^{j-1}\big) - v^{j-1}\kappa^{j-1}\Big) \\ &= \mathcal{O}(1)\mathcal{V}(u)\big(|v^{j-1}| + |v^{j-2}|\big)\chi\Big\{\big|a^j - 1\big| \geq 5\delta_1 \big|a^{j-1} - 1\big| \leq 3\delta_1\Big\} \\ &\quad + P\big(u^j, v^j, v^{j-1}, v^{j-2}\big). \end{aligned}$$

Thus (4.22) is valid for all $j \in \mathbb{Z}$.

We now consider the form of the term P, which is by construction a second order polynomial in v^j v^{j-1} and v^{j-2}. A simple analysis shows that P cannot contain the term $(v^{j-2})^2$, because v^{j-2} appears only in the time derivative of v^{j-1}. Thus the form of P is

$$\begin{aligned} P\big(u^j, v^j, v^{j-1}, v^{j-2}\big) &= a_{11}(u^j, v)\big(v^j\big)^2 + a_{12}(u^j, v) v^j v^{j-1} + a_{13}(u^j, v) v^j v^{j-2} \\ &\quad + a_{22}(u^j, v)\big(v^{j-1}\big)^2 + a_{23}(u^j, v) v^{j-1} v^{j-2}, \end{aligned}$$

where the coefficients $a(u^j, v)$ are smooth functions depending on u^j, v^j, v^{j-1} and v^{j-2}.

For an exact travelling profile with speed $\sigma = \lambda(u_0) g(s)$ independent of j, we know that $w^j = v^j s$, so that w^j solves the same equation of v^j, i.e., the source terms vanish, which implies that $P(v^j, \hat{v}^{j-1}, \hat{v}^{j-2}) = 0$. Thus we conclude that

$$\begin{aligned} P\big(v, v^{j-1}, v^{j-2}\big) &= P\big(v, v^{j-1}, v^{j-2}\big) - P\big(v, \hat{v}^{j-1}, \hat{v}^{j-2}\big) \\ &= \Big(a_{12} v^j + a_{22}\big(v^{j-1} + \hat{v}^{j-1}\big) + a_{23}\hat{v}^{j-2}\Big)\big(v^{j-1} - \hat{v}^{j-1}\big) \\ &\quad + \big(a_{13} v^j + a_{23} v^{j-1}\big)\big(v^{j-2} - \hat{v}^{j-2}\big) \\ &\quad + \big(a_{11} - \hat{a}_{11}\big)\big(\hat{v}^j\big)^2 + \big(a_{12} - \hat{a}_{12}\big)\hat{v}^j \hat{v}^{j-1} + \big(a_{13} - \hat{a}_{13}\big)\hat{v}^j \hat{v}^{j-2} \\ &\quad + \big(a_{22} - \hat{a}_{22}\big)\big(\hat{v}^{j-1}\big)^2 + \big(a_{23} - \hat{a}_{23}\big)\hat{v}^{j-1}\hat{v}^{j-2} \\ &= \mathcal{O}(1) v^j\big(v^{j-1} - \hat{v}^{j-1}\big) + \mathcal{O}(1) v^{j-1}\big(v^{j-1} - \hat{v}^{j-1}\big) \\ &\quad + \mathcal{O}(1) v^j\big(v^{j-2} - \hat{v}^{j-2}\big) + \mathcal{O}(1) v^{j-1}\big(v^{j-2} - \hat{v}^{j-2}\big). \end{aligned} \tag{4.23}$$

Define $\check{v}^{j-2}$ as the point of the travelling profile passing through u^{j-1}, v^{j-1} with speed σ^j: using the function $\tilde{p}$ introduced in (3.5) of Section 3, we can write

$$\hat{v}^{j-2} = \hat{v}^{j-1}\tilde{p}\big(u^{j-1}, \hat{v}^{j-1}, s^j\big), \qquad \check{v}^{j-2} = v^{j-1}\tilde{p}\big(u^{j-1}, v^{j-1}, s^j\big). \tag{4.24}$$

Note that this point is different from $\hat{v}^{j-2}$ if $|s^j - 1| \geq \delta_1$: we have the estimate

$$\big|\hat{v}^{j-2} - \check{v}^{j-2}\big| = \big|\pi\big(u^{j-1}, \hat{v}^{j-1}, s^j\big) - \pi\big(u^{j-1}, v^{j-1}, s^j\big)\big| = \mathcal{O}(1)\big|v^{j-1} - \hat{v}^{j-1}\big|. \tag{4.25}$$

Using (4.16) we can then write

$$\begin{aligned} v^j\big(v^{j-2} - \hat{v}^{j-2}\big) &= v^j\big(v^{j-2} - \check{v}^{j-2}\big) + v^j\big(\check{v}^{j-2} - \hat{v}^{j-2}\big) \\ &= v^j v^{j-1}\Big(\tilde{p}\big(u^{j-1}, v^{j-1}, s^{j-1}\big) - \tilde{p}\big(u^{j-1}, v^{j-1}, s^j\big)\Big) + v^j\big(\check{v}^{j-2} - \hat{v}^{j-2}\big) \\ &= \mathcal{O}(1) v^j v^{j-1}\big(s^j - s^{j-1}\big) + \mathcal{O}(1) v^j\big(v^{j-1} - \hat{v}^{j-1}\big), \end{aligned}$$

$$\begin{aligned}
v^{j-1}\big(v^{j-2}-\hat v^{j-2}\big) &= v^{j-1}\big(v^{j-2}-\check v^{j-2}\big)+v^{j-1}\big(\check v^{j-2}-\hat v^{j-2}\big)\\
&= \frac{v^{j-1}}{v^j}v^jv^{j-1}\Big(\tilde p\big(u^{j-1},v^{j-1},s^{j-1}\big)-\tilde p\big(u^{j-1},v^{j-1},s^j\big)\Big)\chi\Big\{\big|s^j-1\big|\le 5\delta_1\Big\}\\
&\quad + v^{j-1}\big(v^{j-2}-\check v^{j-2}\big)\chi\Big\{\big|s^j-1\big|\ge 5\delta_1,\big|s^{j-1}-1\big|\le 3\delta_1\Big\}\\
&\quad + v^{j-1}\big(v^{j-2}-\check v^{j-2}\big)\chi\Big\{\big|s^j-1\big|\ge 5\delta_1,\big|s^{j-1}-1\big|\ge 3\delta_1\Big\}\\
&\quad + v^j\big(\check v^{j-2}-\hat v^{j-2}\big)\\
&= \mathcal O(1)v^jv^{j-1}\big(s^j-s^{j-1}\big)+\mathcal O(1)v^{j-1}\big(v^{j-1}-\hat v^{j-1}\big)\\
&\quad + v^{j-1}\big(v^{j-2}-\check v^{j-2}\big)\chi\Big\{\big|s^j-1\big|\ge 5\delta_1,\big|s^{j-1}-1\big|\le 3\delta_1\Big\}\\
&\quad + v^{j-1}\big(v^{j-2}-\check v^{j-2}\big)\chi\Big\{\big|s^j-1\big|\ge 5\delta_1,\big|s^{j-1}-1\big|\ge 3\delta_1\Big\}.
\end{aligned}$$

Using now the trivial inequality $2ab\le a^2+b^2$, P can be estimated as

$$\begin{aligned}
\Big|P\big(v^j,v^{j-1},v^{j-2}\big)\Big| &\le \mathcal O(1)\Big(\big(v^j\big)^2+\big(v^{j-1}\big)^2\Big)\chi\Big\{\big|s^j-1\big|\ge\delta_1\Big\}\\
&\quad+\mathcal O(1)\Big(\big(v^{j-1}\big)^2+\big(v^{j-2}\big)^2\Big)\chi\Big\{\big|s^{j-1}-1\big|\ge\delta_1\Big\}\\
&\quad+\mathcal O(1)\Big|v^jw^{j-1}-w^jv^{j-1}\Big|\\
&\quad+\mathcal O(1)\big|v^{j-1}\big|\Big(\big|v^{j-2}\big|+\big|v^{j-1}\big|\Big)\chi\Big\{\big|s^j-1\big|\ge 5\delta_1,\big|s^{j-1}-1\big|\le 3\delta_1\Big\}.
\end{aligned}\tag{4.26}$$

We finally show that the equation satisfied by w^j is

$$w^j_t+\lambda^jw^j-\lambda^{j-1}w^{j-1}=e^j(t),\tag{4.27}$$

where, using the estimate $|v|=\mathcal O(1)\mathcal V^2(u)$ of Corollary 2.2 and assuming the constant C_0 sufficiently big, the error term $e^j(t)$ is bounded by

$$\begin{aligned}
\big|e^j(t)\big| &\le \mathcal O(1)\mathcal V(u)\big|v^{j-1}\big|\Big(s^j-s^{j-1}\Big)^2\chi\Big\{\big|s^j-1\big|,\big|s^{j-1}-1\big|\le 5\delta_1\Big\}\\
&\quad+\mathcal O(1)\mathcal V(u)\Big(\big|v^{j-1}\big|+\big|v^{j-2}\big|\Big)\chi\Big\{\big|s^j-1\big|\ge 5\delta_1,\big|s^{j-1}-1\big|\le 3\delta_1\text{ or viceversa}\Big\}\\
&\quad+\mathcal O(1)\Big|w^jv^{j-1}-v^jw^{j-1}\Big|+\mathcal O(1)\Big(\big|v^j\big|^2+\big|v^{j-1}\big|^2\Big)\chi\Big\{\big|s^j-1\big|\ge\delta_1\Big\}\\
&\quad+\mathcal O(1)\Big(\big|v^{j-1}\big|^2+\big|v^{j-2}\big|^2\Big)\chi\Big\{\big|s^{j-1}-1\big|\ge\delta_1\Big\}\\
&\doteq C_0\Big\{\mathcal V(u)I^j_1(t)+I^j_2(t)+I^j_3(t)+I^{j-1}_3(t)\Big\}.
\end{aligned}\tag{4.28}$$

Remark 4.7. We can classify the various terms in $e^j(t)$ in the following categories:

(1) terms due to the approximate "dispersion" relation (4.14), i.e., due to the fact that we are using the dispersion relation in u_0 and not in u^j:

$$\begin{aligned}
I^j_1(t) =&\mathcal V(u)\big|v^{j-1}\big|\Big(s^j-s^{j-1}\Big)^2\chi\Big\{\big|s^j-1\big|,\big|s^{j-1}-1\big|\le 5\delta_1\Big\}\\
&+\mathcal V(u)\Big(\big|v^{j-1}\big|+\big|v^{j-2}\big|\Big)\chi\Big\{\big|s^j-1\big|\ge 5\delta_1,\big|s^{j-1}-1\big|\le 3\delta_1\text{ or viceversa}\Big\};
\end{aligned}$$

(2) terms due to the interactions among travelling profiles, i.e., due to the fact that s^j is not constant w.r.t. j:

$$I^j_2(t)=\Big|v^jw^{j-1}-w^jv^{j-1}\Big|=\Big|v^jv^{j-1}\big(s^{j-1}-s^j\big)\Big|;$$

(3) terms due to the cutoff function (4.15), i.e., due to the fact that there are no travelling profiles with a speed corresponding to s^j:

$$I^j_3(t)=\Big(\big|v^j\big|^2+\big|v^{j-1}\big|^2\Big)\chi\Big\{\big|s^j-1\big|\ge\delta_1\Big\}.$$

As we have already observed, the first order terms are the most difficult to bound, because they correspond to a first order functional, the Length Functional. As we will see later, the second order error terms can be computed using second order functionals. When computing the derivative of these functionals, these terms appear multiplied by a small constant, of the order of the total variation of u.

Remark 4.8. A simple explanation of the presence of the error terms can be obtained by looking at the parabolic case. Consider the scalar equation

$$u_t + f(u)_x - u_{xx} = 0.$$

In [8] it is proved that, defining the auxiliary function

$$w = f(u) - u_x,$$

and using $y = u(t,x)$ as the independent variable, we obtain the equation

$$w_t = \big(f(y) - w\big)^2 w_{yy},$$

so that the functional

$$Q(t) = \frac{1}{2} \iint_{y_1 < y_2} \big|w_y(y_1) - w_y(y_2)\big| dy_1 dy_2$$

is decreasing. If instead of u we consider the nonlinear function $z(w)$, we obtain the equation

$$z_t - \big(f(y) - w\big)^2 z_{yy} = -\big(f(y) - w\big)^2 z'' w_y^2.$$

Going back to the original coordinates, we obtain

$$z(w)_t + f'(u) z(w)_x - z(w)_{xx} = -u_x z''(w) \big(w/u_x\big)_x^2.$$

In the semidiscrete linear case, we have analogous quantities given by $u_x = v^j$, $w = -u_{xx} = v^{j-1} - v^j$, so that we can rewrite the last equation as

$$z_t^j + \lambda^j z^j - \lambda^{j-1} z^{j-1} = -z'' v^j \big(v^{j-2}/v^{j-1} - v^{j-1}/v^j\big)^2.$$

Thus, the error term is due to the fact that we use an approximate dispersion relation to compute speed, i.e., the dispersion relation evaluated at the point u_0, see (4.14).

Since the dispersion relation is not constant, it is easy to show that if we choose $w^j = v^{j-1}$, the source term $e^j(t)$ is only a second order polynomial in v^j, v^{j-1}, v^{j-2}. However this polynomial does not vanish on travelling profiles, because $s^j = \alpha^j$ is a function of u^j and is not constant. This implies that $w^j = v^j s^j$ does not satisfy the same equation of v^j.

We introduce the Energy Functional. Let θ be the function

$$\theta(x) = \begin{cases} 0 & |x-1| \leq 4\delta_1/5 \\ \text{smooth connection} & 4\delta_1/5 \leq |x-1| \leq \delta_1 \\ 1 & |x-1| \geq \delta_1 \end{cases} \tag{4.29}$$

We multiply

$$v_t^j + \lambda\big(u^j\big) v^j - \lambda\big(u^{j-1}\big) v^{j-1} = 0,$$

by $v^j \theta^j = v^j \theta(s^j)$, so that we obtain the equation

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\big(v^j\big)^2 \theta^j\Big) + \lambda^j \big(v^j\big)^2 \theta^j - \lambda^{j-1} v^j v^{j-1} \theta^j &= \frac{1}{2}\big(v^j\big)^2 \theta'\big(s^j\big) s_t^j \\ &= \frac{\lambda^{j-1}}{2} \theta'(s^j) \big(w^j v^{j-1} - w^{j-1} v^j\big) + \frac{\lambda^{j-1}}{2} \theta'(s^j) v^j e^j(t). \end{aligned} \tag{4.30}$$

The sum w.r.t. j of (4.30) yields the equation

$$\begin{aligned} \sum_j \Big(\lambda^j \big(v^j\big)^2 \theta^j - \lambda^{j-1} v^j v^{j-1} \theta^j\Big) &= \frac{1}{2} \sum_j \Big(\lambda^j \big(v^j\big)^2 \theta^j - 2\lambda^{j-1} v^j v^{j-1} \theta^j + \lambda^{j-1} \big(v^{j-1}\big)^2 \theta^{j-1}\Big) \\ &= -\frac{1}{2}\frac{d}{dt} \sum_j \big(v^j\big)^2 \theta^j + \sum_j \frac{\lambda^{j-1}}{2} \theta' \big(w^j v^{j-1} - w^{j-1} v^j\big) + \sum_j \frac{\lambda^{j-1}}{2} \theta' v^j e^j. \end{aligned} \tag{4.31}$$

We write the left-hand side of (4.31) as

$$\begin{aligned}\frac{1}{2}\Big(\lambda^j\big(v^j\big)^2\theta^j-2\lambda^{j-1}v^jv^{j-1}\theta^j+\lambda^{j-1}\big(v^{j-1}\big)^2\theta^{j-1}\Big)\\ =\frac{\lambda^{j-1}}{2}\big(v^j-v^{j-1}\big)^2\theta^j+\frac{1}{2}\big(\lambda^j-\lambda^{j-1}\big)\big(v^j\big)^2\theta^j+\frac{\lambda^{j-1}}{2}\big(v^{j-1}\big)^2\big(\theta^{j-1}-\theta^j\big).\end{aligned}\tag{4.32}$$

As a consequence of (4.16), we have the series of inequalities

$$\left|\frac{v^{j-1}}{v^j}-1\right|\theta^j\geq\Big(\big|s^j-1\big|-\mathcal{O}(1)\mathcal{V}(u)\Big)\theta^j\geq\left(\frac{4}{5}\delta_1-\mathcal{O}(1)\mathcal{V}(u)\right)\theta^j\geq\frac{1}{2}\delta_1\theta^j,$$

if the total variation of u is sufficiently small, and thus we obtain

$$\frac{1}{4}\lambda^{j-1}\big(v^j-v^{j-1}\big)^2\theta^j\geq\frac{1}{8}\lambda^{j-1}\delta_1\big(v^j\big)^2\theta^j\geq\frac{1}{2}\big|\lambda^j-\lambda^{j-1}\big|\big(v^j\big)^2\theta^j.$$

The last inequality follows from the estimate (2.22).

Note that the term $\lambda^{j-1}(v^{j-1})^2(\theta^j-\theta^{j-1})$ in (4.32) is equal to 0 when $|s^j-1|$, $|s^{j-1}-1|$ are greater than δ_1. We can then write

$$\begin{aligned}\frac{\lambda^{j-1}}{2}\big(v^{j-1}\big)^2\big|\theta^{j-1}-\theta^j\big|&=\frac{\lambda^{j-1}}{2}\big(v^{j-1}\big)^2\big|\theta^{j-1}-\theta^j\big|\chi\Big\{\big|s^j-1\big|\leq 5\delta_1\Big\}\\&\quad+\frac{\lambda^{j-1}}{2}\big(v^{j-1}\big)^2\big|\theta^{j-1}-1\big|\chi\Big\{\big|s^j-1\big|\geq 5\delta_1,\big|s^{j-1}-1\big|\leq 3\delta_1\Big\}\\&\leq\frac{1+10\delta_1}{2}\big\|\theta'\big\|_{L^\infty}\lambda^{j-1}\Big|w^jv^{j-1}-v^jw^{j-1}\Big|\chi\Big\{\big|s^j-1\big|\leq 5\delta_1\Big\}\\&\quad+\frac{1}{2}\lambda^{j-1}\big(v^{j-1}\big)^2\chi\Big\{\big|s^j-1\big|\geq 5\delta_1,\big|s^{j-1}-1\big|\leq 3\delta_1\Big\}.\end{aligned}$$

where we used the inequality $|\theta(x)-\theta(y)|\leq\|\theta'\|_{L^\infty}|x-y|$ and, by assuming $\mathcal{V}(u)$ sufficiently small,

$$\left|\frac{v^{j-1}}{v^j}-1\right|\leq 10\delta_1\quad\text{if}\quad\big|s^j-1\big|\leq 5\delta_1.$$

We finally can estimate (4.32) as

$$\begin{aligned}\frac{1}{2}\Big(\lambda^j\big(v^j\big)^2\theta^j-2\lambda^{j-1}v^jv^{j-1}\theta^j+\lambda^{j-1}\big(v^{j-1}\big)^2\theta^{j-1}\Big)\geq\frac{\lambda^{j-1}}{4}\big(v^j-v^{j-1}\big)^2\theta^j\\-\frac{1+10\delta_1}{2}\big\|\theta'\big\|_{L^\infty}\lambda^{j-1}\Big|w^jv^{j-1}-v^jw^{j-1}\Big|-\frac{1}{2}\lambda^{j-1}\big(v^{j-1}\big)^2\chi\Big\{\big|s^j-1\big|\geq 5\delta_1,\big|s^{j-1}-1\big|\leq 3\delta_1\Big\}.\end{aligned}$$

Observe that

$$\big(v^j-v^{j-1}\big)^2\chi\Big\{\big|v^{j-1}/v^j-1\big|\geq\delta_1/2\Big\}\geq\frac{\delta_1^2}{4}\Big(\big|v^j\big|^2+\big|v^{j-1}\big|^2\Big)\chi\Big\{\big|v^{j-1}/v^j-1\big|\geq\delta_1/2\Big\},$$

so that we obtain finally

$$\begin{aligned}\frac{\lambda^{j-1}}{16}\delta_1^2\Big(\big(v^j\big)^2+\big(v^{j-1}\big)^2\Big)\theta^j\leq&-\frac{1}{2}\frac{d}{dt}\sum_j\big(v^j\big)^2\theta^j+\frac{1+5\delta_1}{2}\big\|\theta'\big\|_{L^\infty}\lambda^{j-1}\Big|w^jv^{j-1}-v^jw^{j-1}\Big|\\&+\frac{1}{2}\lambda^{j-1}\big(v^{j-1}\big)^2\chi\Big\{\big|s^j-1\big|\geq 5\delta_1,\big|s^{j-1}-1\big|\leq 3\delta_1\Big\}\\&+\sum_j\frac{\lambda^{j-1}}{2}\big\|\theta'\big\|_{L^\infty}\Big|w^jv^{j-1}-w^{j-1}v^j\Big|+\sum_j\frac{\lambda^{j-1}}{2}\big\|\theta'\big\|_{L^\infty}\big|v^je^j\big|.\end{aligned}$$

Recalling (4.28), the above equation becomes thus

$$
\begin{aligned}
(4.33)\qquad \big(1-\mathcal{O}(1)\mathcal{V}(u)^2\big)\sum_j \Big(\big|v^{j-1}\big|^2+\big|v^j\big|^2\Big)\theta^j \leq{}& -\mathcal{O}(1)\frac{d}{dt}\sum_j \big(v^j\big)^2\theta^j+\mathcal{O}(1)\sum_j\Big|w^jv^{j-1}-v^jw^{j-1}\Big| \\
&+\mathcal{O}(1)\mathcal{V}(u)^2\sum_j\big|v^{j-1}\big|\big(s^j-s^{j-1}\big)^2\chi\Big\{\big|s^j-1\big|,\big|s^{j-1}-1\big|\leq 5\delta_1\Big\} \\
&+\mathcal{O}(1)\mathcal{V}(u)^2\sum_j\Big(\big|v^{j-1}\big|+\big|v^{j-2}\big|\Big)\chi\Big\{\big|s^j-1\big|\geq 5\delta_1,\big|s^{j-1}-1\big|\leq 3\delta_1\Big\} \\
&+\mathcal{O}(1)\mathcal{V}(u)^2\sum_j\Big(\big|v^{j-1}\big|+\big|v^{j-2}\big|\Big)\chi\Big\{\big|s^j-1\big|\leq 3\delta_1,\big|s^{j-1}-1\big|\geq 5\delta_1\Big\},
\end{aligned}
$$

where we have used Corollary 2.2. The same corollary yields the estimate

$$
(4.34)\qquad \sum_j\big(v^j\big)^2\theta^j\leq\sum_j\big(v^j\big)^2\leq\|v\|_{\ell^\infty}\sum_j\big|v^j\big|\leq\mathcal{O}(1)\mathcal{V}^3(u),
$$

and finally

$$
\begin{aligned}
(4.35)\qquad \sum_j\int_0^t I_3^j(t)\leq{}& \sum_j\int_0^t\Big(\big|v^{j-1}\big|^2+\big|v^j\big|^2\Big)\theta^j\,dt \\
\leq{}& \mathcal{O}(1)\sum_j\big|v^j(0)\big|^2+\mathcal{O}(1)\sum_j\int_0^t\Big|w^jv^{j-1}-v^jw^{j-1}\Big|dt \\
&+\mathcal{O}(1)\mathcal{V}(u)^2\sum_j\int_0^t\big|v^{j-1}\big|\big(s^j-s^{j-1}\big)^2\chi\Big\{\big|s^j-1\big|,\big|s^{j-1}-1\big|\leq 5\delta_1\Big\}dt \\
&+\mathcal{O}(1)\mathcal{V}(u)^2\sum_j\int_0^t\Big(\big|v^{j-1}\big|+\big|v^{j-2}\big|\Big)\chi\Big\{\big|s^j-1\big|\geq 5\delta_1,\big|s^{j-1}-1\big|\leq 3\delta_1\Big\}dt \\
&+\mathcal{O}(1)\mathcal{V}(u)^2\sum_j\int_0^t\Big(\big|v^{j-1}\big|+\big|v^{j-2}\big|\Big)\chi\Big\{\big|s^j-1\big|\leq 3\delta_1,\big|s^{j-1}-1\big|\geq 5\delta_1\Big\}dt \\
\leq{}& C_0\mathcal{V}(u)^3+C_0\sum_j\int_0^t\Big\{\mathcal{V}(u)^2I_1^j(t)+I_2^j(t)\Big\}dt,
\end{aligned}
$$

where as usual C_0 denotes a big constant.

We now compute the derivatives of the Length and Area functionals for the line γ obtained by connecting the points

$$
P^j(t)=\left(\sum_{-\infty}^j v^k(t),\sum_{-\infty}^j w^k(t)\right)=\left(-f(u^j),\sum_{-\infty}^j w^k(t)\right).
$$

Following Example 4.4, define the two functionals

$$
(4.36)\qquad L(t)\doteq\frac{1}{\underline{\lambda}}\sum_j\sqrt{\big(v^j\big)^2+\big(w^j\big)^2},\qquad Q(t)\doteq\frac{1}{2\underline{\lambda}}\sum_{j<k}\Big|w^jv^k-w^kv^j\Big|.
$$

Note that by regularity estimates (2.21) we have

$$
\begin{aligned}
(4.37)\qquad L(t)&=\mathcal{O}(1)\mathcal{V}(u), \\
Q(t)&=\frac{1}{2\underline{\lambda}}\sum_{j<k}\Big|\big(w^j-v^j\big)v^k-\big(w^k-v^k\big)v^j\Big| \\
&\leq\frac{1}{\underline{\lambda}}\big\|w-v\big\|_{\ell^1}\big\|v\big\|_{\ell^1}=\mathcal{O}(1)\mathcal{V}^3(u),
\end{aligned}
$$

because $s^j - \alpha^j = \mathcal{O}(1)\mathcal{V}(u)$. With computation similar to the one given in Example 4.4, one gets

(4.38)
$$\begin{aligned}
\frac{dL}{dt} \leq & -\frac{1}{\underline{\lambda}}\sum_j \lambda^{j-1}\frac{\left|v^{j-1}\right|\left(s^j - s^{j-1}\right)^2}{\sqrt{1+(s^j)^2}\left(1+s^j s^{j-1}+\sqrt{1+(s^j)^2}\sqrt{1+(s^{j-1})^2}\right)}\chi\Big\{j : \left|s^j-1\right|, \left|s^{j-1}-1\right| \leq 5\delta_1\Big\} \\
& -\frac{1}{\underline{\lambda}}\sum_j \lambda^{j-1}\delta_1\left|v^{j-1}\right|\left(1+\left|s^{j-1}\right|\right)\chi\Big\{\left|s^j-1\right| \geq 5\delta_1, \left|s^{j-1}-1\right| \leq 3\delta_1 \text{ or viceversa}\Big\} \\
& +\frac{1}{\underline{\lambda}}\sum_j \left\langle \frac{(v^j,w^j)}{\left|(v^j,w^j)\right|}, \begin{pmatrix} 0 \\ e^j(t) \end{pmatrix} \right\rangle \\
\leq & -\Big(1-\mathcal{O}(1)\mathcal{V}(u)\Big)\sum_j I_1^j(t) + \mathcal{O}(1)\sum_j I_2^j(t) + \mathcal{O}(1)\sum_j I_j^3(t) \\
\leq & -\frac{1}{2}\sum_j I_1^j(t) + \mathcal{O}(1)\sum_j I_2^j(t) + \mathcal{O}(1)\sum_j I_j^3(t),
\end{aligned}$$

where we used (4.12).

In a similar way, we obtain

$$\begin{aligned}
\frac{dQ}{dt} \leq & -\frac{1}{\underline{\lambda}}\sum_j \lambda^{j-1}\left|v^j w^{j-1} - v^{j-1}w^j\right| + \mathcal{O}(1)\mathcal{V}(u)\sum_j e^j(t) \\
\leq & -\frac{1}{2}\sum_j I_2^j(t) + \mathcal{O}(1)\mathcal{V}(u)^2\sum_j I_1^j(t) + \mathcal{O}(1)\mathcal{V}(u)\sum_j I_j^3(t).
\end{aligned} \tag{4.39}$$

Integrating in $[0,t]$ (4.38), (4.39), we thus obtain the system

$$\begin{cases}
\sum_j \int_0^t I_1^j(s)ds \leq C_0\left(\mathcal{V}(u) + \sum_j \int_0^t I_2^j(s)ds + \sum_j \int_0^t I_3^j(s)ds\right) \\
\sum_j \int_0^t I_2^j(s)ds \leq C_0\left(\mathcal{V}^3(u) + \mathcal{V}(u)^2\sum_j \int_0^t I_1^j(s)ds + \mathcal{V}(u)\sum_j \int_0^t I_3^j(s)ds\right) \\
\sum_j \int_0^t I_3^j(t)dt \leq C_0\left(\mathcal{V}^3(u) + \mathcal{V}(u)^2\sum_j \int_0^t I_1^j(s)ds + \sum_j \int_0^t I_2^j(s)ds\right)
\end{cases} \tag{4.40}$$

It is now easy to verify that, if the total variation of u is sufficiently small, system (4.40) has a bounded solution such that

$$\sum_j \int_0^t I_1^j(t) \leq 2C_0\mathcal{V}(u), \quad \sum_j \int_0^t I_2^j(t) \leq 4C_0^2\mathcal{V}^3(u), \quad \sum_j \int_0^t I_3^j(t) \leq 7C_0^2\mathcal{V}(u)^3, \tag{4.41}$$

so we conclude that

$$\sum_j \int_0^{+\infty}\left|e^j(t)\right|dt \leq C_0\mathcal{V}^2(u). \tag{4.42}$$

Remark 4.9. Note that a similar result can be obtained if we assume that the characteristic speed $\lambda(u)$ is strictly less than $1/\sqrt{2}$. However, this method does not require any rescaling, and we have a smaller source term, because in the other case it can be proved only that

$$\sum_j \int_0^{+\infty}\left|e^j(t)\right|dt \leq \mathcal{O}(1)\mathcal{V}(u).$$

As we will show later, in the vector case we have to add to e^j another term, due to interaction among waves of different families. It is interesting to note here that this term will have the same order of magnitude as $e^j(t)$.

5. Decomposition in terms of travelling profiles

In this section we will decompose the vector $u^j_t \in \mathbb{R}^N$ as a sum of N vectors $-\sigma^j_m \phi^j_{m,x}$, where ϕ^j_m is a travelling profile for (2.1) with speed σ^j_m:

$$u^j_t = \sum_m -\sigma_m \phi^j_{m,x}(t) = \sum_m v^j_m \tilde{r}_m\big(u, v_m, \sigma_m\big). \tag{5.1}$$

Note that from

$$v^j + f(u^j) - f(u^{j-1}) = 0,$$

we have also

$$f\big(u^{j-1}\big) - f\big(u^j\big) = \sum_m f\big(\phi^{j-1}_m\big) - f\big(\phi^j_m\big).$$

As suggested in the previous section, the idea is to find the speeds σ_i by trying to fit also

$$f\big(u^{j-2}\big) - f\big(u^{j-1}\big) = \sum_m f\big(\phi^{j-2}_m\big) - f\big(\phi^{j-1}_m\big).$$

As Proposition 4.1 of Section 4, this can be achieved only when the two jumps $\phi^j_m - \phi^{j-1}_m$, $\phi^{j-1}_m - \phi^{j-2}_m$ are of the same order.

An important property of the decomposition (5.1) is that the speed cannot be given looking at the solution locally: to identify the N travelling profiles in u^j_t, we need actually $N + 2N$ conditions, i.e., the vectors u^j, u^j_t and u^{j-1}_t. Thus, in some sense, the speed of one wave will depend on the value of $u^{j-1}_{m,t}$, which, on the other hand, depends on the solution at the point u^{j-2}_m. As we will see later on, a consequence of this fact in that the evolution of the travelling profile depends on all the previous travelling profiles in $k = j, j-1, \dots$. It is not surprising then that we will have non-local interaction terms, but a travelling profile located in j will interact with all the waves from $-\infty$ to j. However, this interaction is exponentially decreasing as $|j-k| \to \infty$, i.e., when the distance of the profiles increases.

From Section 3, we know that for a travelling profile ϕ of the m-th family, with $\phi(0) = u$ and $\langle l_m(u_0), \phi'(0)\rangle = -v_m/\sigma$ and with speed σ, the quantities $-\sigma\phi'(0)$ and $-\sigma\phi'(-1)$ are given by

$$-\sigma\phi'(0) \doteq v = v_m \tilde{r}_m\big(u, v_m, \sigma\big), \qquad -\sigma\phi'(-1) \doteq v^{(-1)} = v^{(-1)}_m \tilde{p}_m\big(\phi(-1), v^{j-1}_m, \sigma\big).$$

Moreover, by equation (3.22) of Corollary 3.6, we have the relation

$$\frac{v^{(-1)}_m}{v^j_m} = \tilde{p}_m\big(u, v_m, \sigma\big) = e^{-\beta_m} + \big(v_m\big)^2 \tilde{q}_m\big(u^j, v^j_m, \sigma^j_m\big), \tag{5.2}$$

where β_m is given by the local dispersion relation

$$\frac{\sigma}{\lambda_m(u)} = \frac{1 - e^{-\beta_m}}{\beta_m}.$$

From (5.2), if v_m is sufficiently small and $v^{(-1)}_m / v_m$ is sufficiently close to 1, then the function $\tilde{p}_m$ in (5.2) is invertible, so that we can obtain β_m and thus the speed σ as a function of u, v_m and $\alpha^j_m \doteq v^{(-1)}_m / v_m$. Following (4.4), we will write

$$\sigma = \tilde{p}^{-1}_m\big(u, v_m, v^{(-1)}_m / v_m\big) = \tilde{p}^{-1}_m\big(u, v_m, \alpha_m\big). \tag{5.3}$$

Recalling Corollary 4.3, we prove the following lemma:

Lemma 5.1. *Let u^j, v^j, $j \in \mathbb{Z}$ be given, with*

$$\big|u^j - u_0\big| \leq 3\delta_1, \qquad \big|v^j\big| \leq 3\delta_1,$$

where δ_1 is sufficiently small. Then there exists a unique decomposition of v^j such that, for all $j \in \mathbb{Z}$,

$$v^j = \sum_m v^j_m \tilde{r}_m\big(u^j, v^j_m, \sigma^j_m\big), \tag{5.4}$$

where σ^j_m is given by

$$\sigma^j_m = \tilde{p}^{-1}_m\Big(u^j, v^j_m, 1 + \psi(v^{j-1}_m / v^j_m)\big(v^{j-1}_m / v^j_m - 1\big)\Big), \tag{5.5}$$

ψ being the cutoff function defined in (4.6).

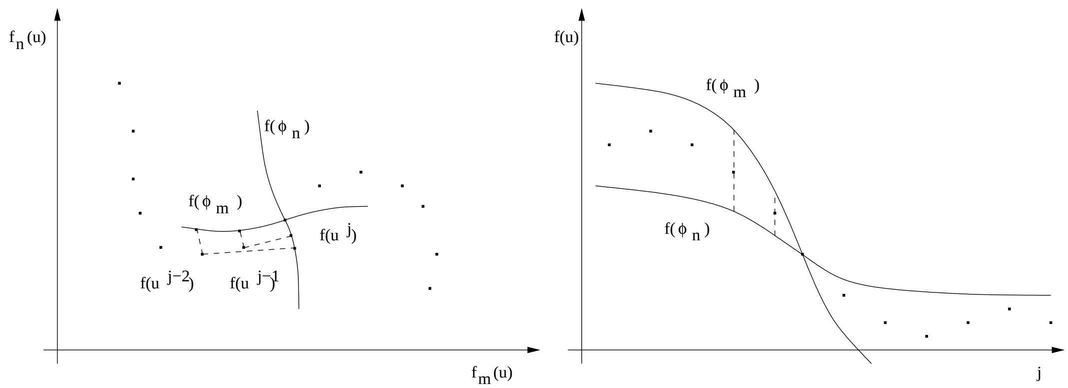

FIGURE 6. Decomposition of v^j in the vectors case.

Remark 5.2. With the above decomposition, when $v^j = u^j_t = f(u^{j-1}) - f(u^j)$, we are trying to fit the vector u^j_t, with the n scalar components v^j_m, using a nonlinear map given by the functions $\tilde{r}_m$ and $\tilde{p}^{-1}_m$.

Assume first that for j, $j-1$, $m = 1, \dots, N$ the ratio $\alpha^j_m = v^{j-1}_m / v^j_m$ is sufficiently close to 1 and that $\sigma^j_m = \sigma^{j-1}_m$, and let ϕ_m be the travelling profile defined by the quantities $\phi(0) = u^j$, $\langle l_m(u_0), -\sigma^j_m \phi'(0) \rangle = v^j_m$ and σ^j_m given by (5.5). Then from Lemma 5.1 we obtain

$$\sum_m -\sigma_m \phi_m(-1) = \sum_m v^{j-1}_m \tilde{r}_m\big(u^{j-1}, v^{j-1}_m, \sigma_m\big) = v^{j-1}, \qquad v^{j-1}_m = v^j_m \tilde{p}\big(u^j, v^j_m, \sigma^j_m\big), \tag{5.6}$$

or equivalently that

$$\sum_m f\big(\phi_m(-2)\big) - f\big(\phi_m(0)\big) = f\big(u^{j-2}\big) - f\big(u^j\big).$$

Note that the identity

$$\sum_m f\big(\phi_m(-1)\big) - f\big(\phi_m(0)\big) = f\big(u^{j-1}\big) - f\big(u^j\big)$$

is always verified, see fig. 6.

On the other hand, when one α_m is not close to 1, so that v^{j-1}_m cannot be computed using $\tilde{p}_m$, or $\sigma^j_m \neq \sigma^{j-1}_m$, then (5.6) is not verified. In these regions we are just fitting $f(u^j)$ and $f(u^{j-1})$.

Note moreover that we decompose the sequence u^j as a whole. Of course we can consider only the points u^j, u^{j-1}, u^{j-2}, and decompose the vectors v^j, v^{j-1} as

$$\begin{cases} v^j &= \sum_m v^j_m \tilde{r}_m\big(u^j, v^j_m, \sigma^j_m\big) \\ v^{j-1} &= \sum_m v^{j,-1}_m \tilde{r}_m\big(u^{j,-1}, v^{j,-1}_m, \sigma^j_m\big) \end{cases} \tag{5.7}$$

where we are computing σ^j_m using again (5.5) and we define $u^{j,-1}$ by

$$v^j_m + f(u^j) - f(u^{j,-1}) = 0.$$

However in this case the decomposition of v^{j-1} will have two different scalar components: the components $v^{j,-1}_m$ computed using (5.7) at the point j and the components v^{j-1}_m using (5.7) at the point $j-1$. In general these components are different, so that the analysis of the source terms is more complicated.

We now give the proof of Lemma 5.1.

Proof. Fixed u^j, we consider the map ω defined in (5.4) as a map from $\ell^\infty(\mathbb{Z}, \mathbb{R}^N)$ to $\ell^\infty(\mathbb{Z}, \mathbb{R}^N)$:

$$\omega\Big(\big\{v^j_m\big\}\Big) = \left\{ \sum_m v^j_m \tilde{r}_m\big(u^j, v^j_m, \sigma^j_m\big) \right\},$$

with σ given by (5.5). With easy computations we have that

(5.8)
$$\begin{aligned}\Big\|D\omega\Big(\{v^j_m\}\Big)-I\Big\|_{\ell^\infty} &= \max_j\Big|\sum_m\Big\{\tilde r^j_m - r_m(u_0) + v^j_m\tilde r^j_{m,v^j_m} + v^j_m\tilde r^j_{m,\sigma}\left(-\frac{\partial\sigma^j_m}{\partial v^j_m}\frac{v^{j-1}_m}{(v^j_m)^2}+\frac{\partial\sigma^j_m}{\partial v^{j-1}_m}\frac{1}{v^j_m}\right)\Big\}\Big| \\ &\leq \mathcal{O}(1)\Big(\|v^j\|_{\ell^\infty}+\|u^j-u_0\|_{\ell^\infty}\Big)\leq\frac{1}{2},\end{aligned}$$

which together with $\omega(0)=0$, implies that $D\omega$ is very close to identity, hence invertible, if δ_1 is sufficiently small. Moreover we have

$$\|\omega^{-1}\|_{L(\ell^\infty,\ell^\infty)}\leq 2, \tag{5.9}$$

in a neighborhood of $v^j=0$. A similar computation proves that

$$\Big\|\omega\Big(\{v^j_m\}\Big)-\Big(\{v^j_m\}\Big)\Big\|_{\ell^1}\leq\mathcal{O}(1)\delta_1\sum_{m,j}|v^j_m|\leq\frac{1}{2}\|v^j\|_{\ell^1}, \tag{5.10}$$

and that, for v^j sufficiently small,

$$\|\omega^{-1}\|_{L(\ell^1,\ell^1)}\leq 2. \tag{5.11}$$

□

From Corollary 2.2, the assumption (2.24) and estimates (5.9), (5.11) we obtain

Corollary 5.3. *Assume now that $u^j(t)$ is a solution to (2.1) and that $v^j=f(u^{j-1})-f(u^j)$. Assume moreover that in $[0,T]$ the sequence $u^j(t)$ satisfies*

$$\sum_j|v^j(0)|\leq\frac{\delta_0}{2},\qquad\sum_j|v^j(t)|=\sum_j\big|f(u^j(t))-f(u^{j-1}(t))\big|\leq 4N\delta_0.$$

Then the following estimates hold:

$$\|v^j_m(0)\|_{\ell^1}\leq\delta_0,\qquad\|v^j_m(t)\|_{\ell^\infty}\leq\sum_j\big|v^j_m(t)-v^{j-1}_m(t)\big|\leq\mathcal{O}(1)\delta_0^2, \tag{5.12}$$

for $t\in[0,T]$.

We can now compute the equation satisfied by the components v^j_m. Differentiating

$$u^j_t+f(u^j)-f(u^{j-1})=0$$

w.r.t. time and substituting (5.4) we obtain

$$\begin{aligned}&\sum_m v^j_{m,t}\left(\tilde r^j_m+v^j_m\tilde r^j_{m,v}+v^j_m\frac{\partial\tilde p^{-1,j}_m}{\partial v^j_m}\tilde r^j_{m,\sigma}\right)+\sum_m v^{j-1}_{m,t}v^j_m\frac{\partial\tilde p^{-1,j}_m}{\partial v^j_m}\tilde r^j_{m,\sigma} \\ &\qquad+\sum_{m,n}v^j_mv^j_n\big(D\tilde p^{-1,j}_m\tilde r^j_n\big)\tilde r^j_{m,\sigma}+\sum_{m,n}v^j_mv^j_nD\tilde r^j_m\tilde r^j_n+\sum_m v^j_mA^j\tilde r^j_m-\sum_m v^{j-1}_mA^{j-1}\tilde r^{j-1}_m=0,\end{aligned}$$

which can be rewritten as

(5.13)
$$
\begin{aligned}
&\sum_m \Big(v^j_{m,t} + \tilde\lambda^j_m v^j_m - \tilde\lambda^{j-1}_m v^{j-1}_m \Big) \bigg(\tilde r^j_m + v^j_m \tilde r^j_{m,v} + v^j_m \frac{\partial \tilde p^{-1,j}_m}{\partial v^j_m} \tilde r^j_{m,\sigma} \bigg) \\
&\qquad + \sum_m v^j_m \Big(v^{j-1}_{m,t} + \tilde\lambda^{j-1}_m v^{j-1}_m - \tilde\lambda^{j-2}_m v^{j-2}_m \Big) \frac{\partial \tilde p^{-1,j}_m}{\partial v^{j-1}_m} \tilde r^j_{m,\sigma} \\
&\quad = \sum_m \bigg[v^{j-1}_m \Big(A^{j-1} \tilde r^{j-1}_m - \tilde\lambda^{j-1}_m \big(\tilde r^j_m + v^j_m \tilde r^j_{m,v} \big) \Big) - v^j_m \Big(A^j \tilde r^j_m - \tilde\lambda^j_m \big(\tilde r^j_m + v^j_m \tilde r^j_{m,v} \big) \Big) - \big(v^j_m\big)^2 D\tilde r^j_m \tilde r^j_m \bigg] \\
&\qquad + \sum_m v^j_m \bigg[\big(\tilde\lambda^j_m v^j_m - \tilde\lambda^{j-1}_m v^{j-1}_m \big) \frac{\partial \tilde p^{-1,j}_m}{\partial v^j_m} + \big(\tilde\lambda^{j-1}_m v^{j-1}_m - \tilde\lambda^{j-2}_m v^{j-2}_m \big) \frac{\partial \tilde p^{-1,j}_m}{\partial v^{j-1}_m} - v^j_m D\tilde p^{-1,j}_m \tilde r^j_m \bigg] \tilde r^j_{m,\sigma} \\
&\qquad - \sum_{m\neq n} v^j_m v^j_n \Big[D\tilde r^j_m \tilde r^j_n + \big(D\tilde p^{-1,j}_m \tilde r^j_n \big) \tilde r^j_{m,\sigma} \Big] \\
&\quad = \eta^j(t).
\end{aligned}
$$

When the solution consists of exactly one travelling profile, the speed is constant and moreover there is only one component different form 0, let us say v^j_m. Moreover by the definition of $\tilde\lambda^j_m$ we know that v^j_m satisfies the scalar equation

$$
v^j_{m,t} + \tilde\lambda^j_m v^j_m - \tilde\lambda^{j-1}_m v^{j-1}_m = 0,
$$

because $-\sigma v^j_{m,x} = v^j_{m,t}$.

As in Section 4, in the following we denote by $\hat{\cdot}$ the quantities defined by the travelling profiles in j. For example $\hat v^{j-1}_m$ is the time derivative $\phi_{m,t}(-1)$ of the travelling profile ϕ defined by $\phi(0) = u^j$, $\langle l_m(u_0), -\sigma\phi'(0)\rangle = v^j_m$ and speed σ^j_m. We will also denote by $\check v^{j-2}_m$ the quantity related to the travelling profile in u^{j-1}, v^{j-1}_m but with speed σ^j_m. Note that as in (4.25) of Section 4 we have

$$
\check v^{j-2}_m - \hat v^{j-2}_m = \mathcal{O}(1)\big(v^{j-1}_m - \hat v^{j-1}_m \big) + \mathcal{O}(1) \sum_n \big| v^j_n \big|. \tag{5.14}
$$

Taking the derivative w.r.t. time of (5.3) we get a relation among the partial derivatives of $\tilde p^{-1}_m$:

$$
0 = v^j_m D p^{-1,j}_m \tilde r^j_m + \big(\widehat{\tilde\lambda^{j-1}_{m-1}} \hat v^{j-1}_m - \tilde\lambda^j_m v^j_m \big) \frac{\partial \tilde p^{-1,j}_m}{\partial v^j_m} + \big(\widehat{\tilde\lambda^{j-2}_m} \hat v^{j-2}_m - \hat\lambda^{j-1}_m \hat v^{j-1}_m \big) \frac{\partial \tilde p^{-1,j}_m}{\partial v^{j-1}_m}. \tag{5.15}
$$

We recall also the identity (3.12), namely

$$
\hat v^{j-1}_m \Big(\hat A^{j-1} \widehat{\tilde r^{j-1}_m} - \hat\lambda^{j-1}_m \big(\tilde r^j_m + v^j_m \tilde r^j_{m,v} \big) \Big) = v^j_m \Big(A^j \tilde r^j_m - \tilde\lambda^j_m \big(\tilde r^j_m + v^j_m \tilde r^j_{m,v} \big) \Big) + \big(v^j_m \big)^2 D\tilde r^j_m \tilde r^j_m,
$$

where $\hat A^{j-1}$, $\widehat{\tilde r^{j-1}_m}$, $\widehat{\tilde\lambda^{j-1}_m}$ are computed at the point $\hat u^{j-1}_m$ given by

$$
v^j_m \tilde r^j_m + f\big(u^j\big) - f\big(\hat u^{j-1}_m\big) = 0 \qquad \Longrightarrow \qquad \hat u^{j-1}_m = f^{-1}\big(v^j_m \tilde r^j_m + f(u) \big). \tag{5.16}
$$

Observe that the right-hand side of (5.13) is a second order polynomial in v^j_m, v^{j-1}_m and v^{j-2}_m. This follows from the estimates

$$
\tilde r_m(u, v^j_m, \sigma) = \mathcal{O}(1) v^j_m, \qquad \tilde\lambda_m(u, v^j_m, \sigma) = \mathcal{O}(1) v^j_m, \tag{5.17}
$$

the last one being a consequence of the first.

We will collect the terms in the right hand side of (5.13) as the sum of three types of terms:

(1) terms due to the fact that there are more waves in j and $j-1$:

$$
v^j_m v^j_n, \qquad v^j_m v^{j-1}_n, \qquad v^{j-1}_m v^{j-1}_n \qquad m \neq n.
$$

Note that these terms will represent the interaction among waves of different families. Following [5], we will refer to these terms as *transversal terms*.

(2) terms arising because we do not have an exact travelling profile of the m-th family, i.e., the m-th speed associated with the solution at the point j is different from the speed associated at the

point $j-1$. Observing that v^j_m, v^{j-1}_m and v^{j-2}_m are of the same order when the cutoff function (4.6) is different from 0, we can write this term as

$$v^j_m v^{j-1}_m \big(\sigma^j_m - \sigma^{j-1}_m\big).$$

These terms correspond to the interaction of waves of the same family. We will call them *non transversal terms*.

(3) terms due to the cut-off function in (5.5). These terms are second order terms and will contain the factor

$$v^{j-1}_m - \hat v^{j-1}_m.$$

These will be the *cutoff (or energy) terms*.

Using (3.12), the first three terms of the source η^j can be rewritten as

$$\begin{aligned}
(5.18)\quad & \sum_m \bigg[v^{j-1}_m \Big(A^{j-1}\tilde r^{j-1}_m - \tilde\lambda^{j-1}_m \big(\tilde r^j_m + v^j_m \tilde r^j_{m,v}\big)\Big) - v^j_m \Big(A^j \tilde r^j_m - \tilde\lambda^j_m \big(\tilde r^j_m + v^j_m \tilde r^j_{m,v}\big)\Big) - \big(v^j_m\big)^2 D\tilde r^j_m \tilde r^j_m \bigg] \\
&= \sum_m v^{j-1}_m \Big(A^{j-1}\tilde r^{j-1}_m - \tilde\lambda^{j-1}_m \big(\tilde r^j_m + v^j_m \tilde r^j_{m,v}\big)\Big) - \sum_m \hat v^{j-1}_m \Big(\hat A^{j-1}\widehat{\tilde r^{j-1}_m} - \widehat{\tilde\lambda^{j-1}_m} \big(\tilde r^j_m + v^j_m \tilde r^j_{m,v}\big)\Big) \\
&= \sum_m \big(v^{j-1}_m - \hat v^{j-1}_m\big) \Big(\hat A^{j-1}\tilde r^{j-1}_m - \tilde\lambda^{j-1}_m \big(\tilde r^j_m + v^j_m \tilde r^j_{m,v}\big)\Big) + \sum_m \hat v^{j-1}_m \hat A^{j-1} \big(\tilde r^{j-1}_m - \widehat{\tilde r^{j-1}_m}\big) \\
&\quad + \sum_m \hat v^{j-1}_m \big(\tilde\lambda^{j-1}_m - \widehat{\tilde\lambda^{j-1}_m}\big)\big(\tilde r^j_m + v^j_m \tilde r^j_{m,v}\big) + \mathcal O(1) \sum_{m\neq n} \big|v^{j-1}_m\big|\big|v^j_n\big|,
\end{aligned}$$

where the transversal terms arise because

$$A\big(u^{j-1}\big) - A\big(\hat u^{j-1}\big) = A\left(f^{-1}\left(\sum_m v^j_m \tilde r^j_m + f(u) \right)\right) - A\Big(f^{-1}\big(v^j_m \tilde r^j_m + f(u)\big)\Big) = \mathcal O(1) \sum_{n\neq m} \big|v^j_n\big|.$$

Using the estimates

$$\hat A^{j-1}\tilde r^{j-1}_m - \tilde\lambda^{j-1}_m \big(\tilde r^j_m + v^j_m \tilde r^j_{m,v}\big) = \mathcal O(1) v^j_m + \mathcal O(1) v^{j-1}_m,$$

$$\begin{aligned}
\tilde r^{j-1}_m - \widehat{\tilde r^{j-1}_m} &= \Big(\tilde r\big(u^{j-1}, v^{j-1}_m, \sigma^{j-1}_m\big) - \tilde r\big(u^{j-1}, v^{j-1}_m, \sigma^j_m\big)\Big) + \Big(\tilde r\big(u^{j-1}, v^{j-1}_m, \sigma^j_m\big) - \tilde r\big(\hat u^{j-1}, \hat v^{j-1}_m, \sigma^j_m\big)\Big) \\
&= \mathcal O(1) v^{j-1}_m \big(\sigma^j_m - \sigma^{j-1}_m\big) + \mathcal O(1)\big(v^{j-1}_m - \hat v^{j-1}_m\big) + \mathcal O(1) \sum_{n\neq m} \big|v^j_n\big|,
\end{aligned}$$

$$\tilde\lambda^{j-1}_m - \widehat{\tilde\lambda^{j-1}_m} = \mathcal O(1) v^{j-1}_m \big(\sigma^j_m - \sigma^{j-1}_m\big) + \mathcal O(1)\big(v^{j-1}_m - \hat v^{j-1}_m\big) + \mathcal O(1) \sum_{n\neq m} \big|v^j_n\big|,$$

the above term can be bounded by

$$\begin{aligned}
(5.19)\qquad & \mathcal O(1)\big(|v^j_m| + |v^{j-1}_m|\big)\big|v^{j-1}_m - \hat v^{j-1}_m\big| + \mathcal O(1) v^j_m v^{j-1}_m \big(\sigma^j_m - \sigma^{j-1}_m\big) \\
&\quad + \mathcal O(1) \sum_{m\neq n} \big|v^{j-1}_m\big|\big|v^j_n\big| + \mathcal O(1) \sum_{m\neq n} \big|v^j_m\big|\big|v^j_n\big|.
\end{aligned}$$

Using the center manifold expansion (3.18), it is possible to obtain a more precise estimate of the energy terms. This is shown in Appendix C.

Finally, we can use (5.15) to estimate the term which is in front of $\tilde r^j_{m,\sigma}$, i.e., the second summation in $\eta^j(t)$ in (5.13). Note that this term is basically due to the fact that the speed σ^j_m changes in time due to interactions: these interactions occur either with waves of different families, or with wave of the same family. The latter is the case when the speed σ^j_m is different from σ^{j-1}_m.

With some computations and recalling the definition of $\check{v}^{j-2}$, one obtains

$$
\begin{aligned}
& v_m^j \left[\big(\tilde{\lambda}_m^j v_m^j - \tilde{\lambda}_m^{j-1} v_m^{j-1}\big) \frac{\partial \tilde{p}_m^{-1,j}}{\partial v_m^j} + \big(\tilde{\lambda}_m^{j-1} v_m^{j-1} - \tilde{\lambda}_m^{j-2} v_m^{j-2}\big) \frac{\partial \tilde{p}_m^{-1,j}}{\partial v_m^{j-1}} + v_m^j D\tilde{p}_m^{-1,j} \tilde{r}_m^j \right] \\
&= v_m^j \left[\big(\widehat{\tilde{\lambda}_m^{j-1}} \hat{v}_m^{j-1} - \tilde{\lambda}_m^{j-1} v_m^{j-1}\big) \frac{\partial \tilde{p}_m^{-1,j}}{\partial v_m^j} + \frac{\partial \tilde{p}_m^{-1,j}}{\partial v_m^{j-1}} \Big(\big(\tilde{\lambda}_m^{j-1} v_m^{j-1} - \widehat{\tilde{\lambda}_m^{j-1}} \hat{v}_m^{j-1}\big) + \big(\widehat{\tilde{\lambda}_m^{j-2}} \hat{v}_m^{j-2} - \tilde{\lambda}_m^{j-2} v_m^{j-2}\big) \Big) \right] \\
&= v_m^j \left(\frac{\partial \tilde{p}_m^{-1,j}}{\partial v_m^j} - \frac{\partial \tilde{p}_m^{-1,j}}{\partial v_m^{j-1}} \right) \Big(\big(\widehat{\tilde{\lambda}_m^{j-1}} - \tilde{\lambda}_m^{j-1}\big) v_m^{j-1} + \widehat{\tilde{\lambda}_m^{j-1}} \big(\hat{v}_m^{j-1} - v_m^{j-1}\big) \Big) \\
&\quad + v_m^j \frac{\partial \tilde{p}_m^{-1,j}}{\partial v_m^{j-1}} \Big(\big(\widehat{\tilde{\lambda}_m^{j-2}} \hat{v}_m^{j-2} - \check{\lambda}_m^{j-2} \check{v}_m^{j-2}\big) + \big(\check{\lambda}_m^{j-2} \check{v}_m^{j-2} - \tilde{\lambda}_m^{j-2} v_m^{j-2}\big) \Big) \\
&= \mathcal{O}(1) \big(v_m^{j-1} - \hat{v}_m^{j-1}\big) + \mathcal{O}(1) v_m^j v_m^{j-1} \big(\sigma_m^j - \sigma_m^{j-1}\big) + \mathcal{O}(1) \psi' \frac{v_m^{j-1}}{v_m^j} \big(v_m^{j-2} - \check{v}_m^{j-2}\big) \\
&\quad + \mathcal{O}(1) \psi' \frac{v_m^{j-1}}{v_m^j} v_m^{j-2} \big(\sigma_m^{j-2} - \sigma_m^{j-1}\big) + \mathcal{O}(1) \sum_{n \neq m} \Big(\big|v_n^j\big| + \big|v_n^{j-1}\big| \Big),
\end{aligned}
$$

because $\partial p_m^{-1,j}/\partial v_m^j$, $\partial p_m^{-1,j}/\partial v_m^{j-1}$ are of order $1/v_m^j$, and we have used the estimates

$$
\widehat{\tilde{\lambda}_m^{j-1}} \hat{v}_m^{j-1} - \tilde{\lambda}_m^{j-1} v_m^{j-1} = \mathcal{O}(1) \big(v_m^{j-1} - \hat{v}_m^{j-1}\big) + \mathcal{O}(1) v_m^j v_m^{j-1} \big(\sigma_m^{j-1} - \sigma_m^j\big) + \mathcal{O}(1) \sum_{n \neq m} \big|v_n^j\big|,
$$

$$
\widehat{\tilde{\lambda}_m^{j-2}} \hat{v}_m^{j-2} - \check{\lambda}_m^{j-2} \check{v}_m^{j-2} = \mathcal{O}(1) \big(v_m^{j-1} - \hat{v}_m^{j-1}\big) + \mathcal{O}(1) \sum_{n \neq m} \Big(\big|v_n^j\big| + \big|v_n^{j-1}\big| \Big),
$$

$$
\begin{aligned}
\check{\lambda}_m^{j-2} \check{v}^{j-2} - \tilde{\lambda}_m^{j-2} v_m^{j-2} &= \tilde{\lambda}_m \big(\check{u}^{j-2}, v_m^{j-1} \tilde{p}(u^{j-1}, v_m^{j-1}, \sigma_m), \sigma_m\big) v_m^{j-1} \tilde{p}(u^{j-1}, v_m^{j-1}, \sigma_m) \\
&\quad - \tilde{\lambda}_m \big(u^{j-2}, v_m^{j-2}, \sigma_m^{j-2}\big) v_m^{j-2} \\
&= \mathcal{O}(1) v_m^{j-1} \big(\sigma_m^{j-1} - \sigma_m^j\big) + \mathcal{O}(1) \big(v_m^{j-2} - \check{v}_m^{j-2}\big) \\
&\quad + \mathcal{O}(1) v_m^{j-1} v_m^{j-2} \big(\sigma_m^{j-2} - \sigma_m^{j-1}\big) + \sum_{n \neq m} \big|v_n^{j-1}\big|.
\end{aligned}
$$

Note that in the last one we have used $\tilde{\lambda}_m(u^j, 0, \sigma_m^j) = \lambda_m(u)$.

We can rewrite the term $v_m^{j-2} - \check{v}_m^{j-2}$ as

$$
\begin{aligned}
v_m^{j-2} - \check{v}_m^{j-2} &= \Big(v_m^{j-2} - v_m^{j-1} \tilde{p}\big(u^{j-1}, v_m^{j-1}, \sigma_m^{j-1}\big) \Big) + v_m^{j-1} \Big(\tilde{p}\big(u^{j-1}, v_m^{j-1}, \sigma_m^{j-1}\big) - \tilde{p}\big(u^{j-1}, v_m^{j-1}, \sigma_m^j\big) \Big) \\
&= \big(v_m^{j-2} - \hat{v}_m^{j-2}\big) + \mathcal{O}(1) v_m^{j-1} \big(\sigma_m^{j-1} - \sigma_m^j\big),
\end{aligned}
$$

where with a slight abuse of notation we have written $\hat{v}_m^{j-2} = v_m^{j-1} \tilde{p}(u^{j-1}, v_m^{j-1}, \sigma_m^{j-1})$. Since we have the estimate

$$
\tilde{r}_{m,\sigma}^j = \mathcal{O}(1) v_m^j,
$$

we can write finally

$$
\begin{aligned}
\eta^j(t) &= \mathcal{O}(1) \sum_m \Big(\big|v_m^j\big| + \big|v_m^{j-1}\big| \Big) \big|v_m^{j-1} - \hat{v}_m^{j-1}\big| + \mathcal{O}(1) \sum_m \big|v_m^{j-1}\big| \big|v_m^{j-2} - \hat{v}_m^{j-2}\big| \\
&\quad + \mathcal{O}(1) \sum_m \Big| v_m^j v_m^{j-1} \big(\sigma_m^{j-1} - \sigma_m^j\big) \Big| + \mathcal{O}(1) \sum_m \Big| v_m^{j-1} v_m^{j-2} \big(\sigma_m^{j-2} - \sigma_m^{j-1}\big) \Big| \\
&\quad + \mathcal{O}(1) \sum_{m \neq n} |v_m^j| \big(|v_n^j| + |v_n^{j-1}|\big).
\end{aligned} \tag{5.20}
$$

Using Lemma 5.1, we can write

$$
v_{m,t}^j + \tilde{\lambda}_m^j v_m^j - \tilde{\lambda}_m^{j-1} v_m^{j-1} = \omega_m^j(t), \tag{5.21}
$$

where, by (5.9) and Corollary 5.3,

$$\big|\omega_m^j(t)\big| \leq \mathcal{O}(1) \max_j \bigg\{ \sum_n \Big(\big|v_n^j\big| + \big|v_n^{j-1}\big| \Big) \big|v_n^{j-1} - \hat{v}_n^{j-1}\big| + \sum_n \Big| v_n^j v_n^{j-1} \big(\sigma_n^{j-1} - \sigma_n^j\big) \Big| + \sum_{n \neq p} |v_n^j| \big(|v_p^j| + |v_p^{j-1}|\big) \bigg\} \leq \mathcal{O}(1)\delta_0^4 \tag{5.22}$$

for $t \in [0,T]$. This implies that the RFDE (5.21) with source term $\omega_m^j(t)$ is well defined. Moreover, using (5.11), we have

$$\sum_j \big|\omega^j(t)\big| \leq \mathcal{O}(1) \sum_{n,j} \Big(\big|v_n^j\big| + \big|v_n^{j-1}\big| \Big) \big|v_n^{j-1} - \hat{v}_n^{j-1}\big| + \mathcal{O}(1) \sum_{n,j} \big| v_n^j v_n^{j-1} \big(\sigma_n^{j-1} - \sigma_n^j\big) \big| + \mathcal{O}(1) \sum_{n \neq p, j} |v_n^j| \big(|v_p^j| + |v_p^{j-1}|\big) \leq \mathcal{O}(1)\delta_0^3. \tag{5.23}$$

As a consequence, to prove that the ℓ^1 norm of v_m^j is uniformly bounded is equivalent to proving that the quantities

$$\sum_{m \neq n} \int_0^{+\infty} |v_m^j| \big(|v_n^j| + |v_n^{j-1}|\big) dt, \qquad \sum_m \int_0^{+\infty} \Big| v_m^j v_m^{j-1} \big(\sigma_m^{j-1} - \sigma_m^j\big) \Big| dt,$$

$$\sum_m \int_0^{+\infty} \big(|v_m^j| + |v_m^{j-1}|\big) \big|v_m^{j-1} - \hat{v}_m^{j-1}\big| dt,$$

are bounded. We now give the idea of the proof.

We assume $\tilde{t} = 0$, and that the estimates of Corollary 2.2 hold. Suppose that in the time interval $[0,T]$ the ℓ^1 norm of the components v^i is less than $2\delta_0$: this is true if t is small because of (5.23). Then, if $2\delta_0$ is sufficiently small, $u^j(t)$ takes values in a neighborhood of u_0 and is of bounded total variation. Using standard techniques, we can extend the solution u to the interval $[T, T+\delta t]$, where δt depends only on δ_0. Let T be the first time such that

$$\sum_j \int_0^T \big|\omega_m^j(t)\big| dx dt = \hat{C}\big(2\delta_0\big)^2, \tag{5.24}$$

where $\hat{C}$ is a big constant.

We conclude that

$$\sum_j \big|v_m^j(t)\big| \leq 2\delta_0$$

if $\delta_0 \leq 1/(4C_0)$. This implies that the solution u remains in K_1 and satisfies

$$\mathcal{V}(u) \leq \sum_{j,m} \big|v_m^j\big| \big|\tilde{r}_m^j\big| \leq 4N\delta_0,$$

and all the a priori estimates are verified.

In the following section we will prove that the condition $\|v^j\|_{L^1} \leq 2\delta_0$ implies that

$$\sum_j \int_0^T \big|\phi^i(t,x)\big| dx dt < \hat{C}\big(2\delta_0\big)^2,$$

contradicting our assumption. Thus the source term will always have ℓ^1-norm in less than $4C_0\delta_0^2$ in $\mathbb{R}^+ \times \mathbb{Z}$, and the solution can be prolonged up to $+\infty$ with uniformly bounded total variation.

6. Estimates of the source terms

As we saw in the previous section, to prove the BV estimate we must prove that the terms

$$\Big|v_m^j v_m^{j-1}\big(\sigma_m^{j-1}-\sigma_m^j\big)\Big|, \qquad \Big(|v_m^j|+|v_m^{j-1}|\Big)\Big|v_m^{j-1}-\hat v_m^{j-1}\Big|, \qquad |v_m^j|\Big(|v_n^j|+|v_n^{j-1}|\Big), \quad m\neq n,$$

are integrable in $\mathbb{R}^+\times\mathbb{Z}$ and their integrals are of the order of $\mathcal{V}(u)^2$. We recall that by definition

$$\hat v_m^j = v_m^j \tilde p\big(u^j, v_m^j, \sigma_m^j\big).$$

We will now study separately the different terms.

6.1. **Transversal terms.** We first prove the following Lemma [3].

Lemma 6.1. *Consider the following* 2×2 *semidiscrete system*

$$\left\{\begin{array}{lcl} z_{1,t}^j+\lambda_1^j(t)z_1^j-\lambda_1^{j-1}(t)z_1^{j-1} &=& 0\\ z_{2,t}^j+\lambda_2^j(t)z_2^j-\lambda_2^{j-1}(t)z_2^{j-1} &=& 0\end{array}\right. \tag{6.1}$$

and assume that

$$0<\lambda_1^j(t)\le L<L+c\le\lambda_2^j(t)<+\infty \quad \forall j\in Z, \tag{6.2}$$

where c is defined in (2.12). *Then the following estimate holds:*

$$\sum_j\int_0^{+\infty}\big|z_1^j(t)\big|\big|z_2^j(t)\big|dt\le\frac{1}{c}\bigg(\sum_j\big|z_1^j(0)\big|\bigg)\bigg(\sum_j\big|z_2^j(0)\big|\bigg). \tag{6.3}$$

Proof. Note that the maximum principle for each equation of system (6.1) implies that

$$\big|z_m^j(t)\big|_t+\lambda_m^j(t)\big|z_m^j(t)\big|-\lambda_m^{j-1}(t)\big|z_m^{j-1}(t)\big|\le 0, \qquad m=1,2.$$

Consider now the functional

$$Q(t)=Q\big(z_1(t),z_2(t)\big)\doteq\sum_{j,k}P(j-k)\big|z_1^j(t)\big|\big|z_2^k(t)\big|, \tag{6.4}$$

where the weight function is defined by

$$P(j)=\begin{cases}1/c\cdot\big(1+c/L\big)^j & j<0\\ 1/c & j\ge 0\end{cases} \tag{6.5}$$

With some computations we have

$$\begin{aligned}\frac{dQ}{dt} &= \sum_{j,k}P(j-k)\Big(\big|z_1^j\big|_t\big|z_2^k\big|+\big|z_1^j\big|\big|z_2^k\big|_t\Big)\\ &\le \sum_{j,k}P(j-k)\Big[\Big(-\lambda_1^j\big|z_1^j\big|+\lambda_1^{j-1}\big|z_1^{j-1}\big|\Big)\big|z_2^k\big|+\big|z_1^j\big|\Big(-\lambda_2^k\big|z_2^k\big|+\lambda_2^{k-1}\big|z_2^{k-1}\big|\Big)\Big]\\ &= \sum_{j,k}\Big(\lambda_1^jP(j-k+1)-\big(\lambda_1^j+\lambda_2^k\big)P(j-k)+\lambda_2^kP(j-k-1)\Big)\big|z_1^j\big|\big|z_2^k\big|\\ &= \frac{1}{c}\sum_{j-k\le-1}\left(1+\frac{c}{L}\right)^{j-k-1}\frac{c}{L}\left(\lambda_1^j\frac{c}{L}-\lambda_2^k+\lambda_1^j\right)\big|z_1^j\big|\big|z_2^k\big|-\sum_j\frac{\lambda_2^j}{c+L}\big|z_1^j\big|\big|z_2^j\big|\\ &\le -\sum_j\big|z_1^j\big|\big|z_2^j\big|.\end{aligned}$$

Integrating in t we obtain (6.3). □

It is now very easy prove that the terms $v_m^jv_n^j$, $v_m^jv_m^{j-1}$ are bounded. In fact, using the same functional Q and (5.24), one obtains

$$\frac{d}{dt}\sum_{j,k}P(j-k)\big|v_m^j\big|\big|v_n^k\big|\le-\sum_j\big|v_m^j\big|\big|v_n^j\big|+\frac{1}{c}2\delta_0\sum_j\Big(\big|\omega_m^j\big|+\big|\omega_n^{j-1}\big|\Big),$$

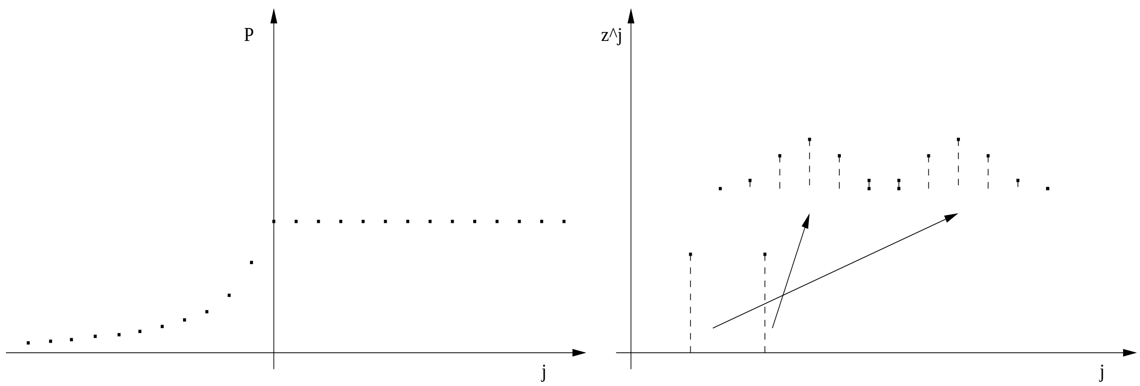

FIGURE 7. Weight function P and crossing of solutions.

$$\frac{d}{dt}\sum_{j,k} P(j-k)\big|v_m^j\big|\big|v_n^{k-1}\big| \leq -\sum_j \big|v_m^j\big|\big|v_n^{j-1}\big| + \frac{1}{c}2\delta_0 \sum_j \Big(\big|\omega_m^j\big| + \big|\omega_n^{j-2}\big|\Big),$$

where P is given by (6.5) and $L = \max \lambda_m$, $m < n$, so that

$$\sum_j \int_0^t \big|v_m^j\big|\Big(\big|v_n^j\big| + \big|v_n^{j-1}\big|\Big)dt \leq \frac{2}{c}\Big(\delta_0 + \hat{C}\big(2\delta_0\big)^2\Big)^2 \leq \frac{2}{c}\big(2\delta_0\big)^2. \tag{6.6}$$

Remark 6.2. Note that Q and P are the semidiscrete version of the hyperbolic interaction potential for different families, introduced by Glimm [16], and of the parabolic transversal functional [5].

As in the parabolic case, we can interpret the weight function P in the following way (see fig. 7). Assume that $z_1^j(0) = \delta_{j\bar{j}}$, $z_1^k(0) = \delta_{k\bar{k}}$. Then the functional Q becomes

$$Q(t=0) = P\big(\bar{j} - \bar{k}\big) = \begin{cases} 1/c \cdot \big(1 + c/L\big)^{\bar{j}-\bar{k}} & \bar{j} < \bar{k} \\ 1/c & \bar{j} \geq \bar{k} \end{cases}$$

If $\bar{j} \geq \bar{k}$, i.e., the slower solution z_1 starts in front of the faster z_2, the functional Q is constant, which means that the two solutions will cross each other, no matter how apart they start at $t = 0$.

Conversely, if $\bar{j} < \bar{k}$, then the numerical diffusion will make the solution interact, but only at an exponentially decaying rate. Note that due to dispersion, the decay rate depends on the speed through the constant L. Observe finally that the right hand side of (6.3) can be interpreted as the expected number of crossings of two particles whose probability distribution is given by $z_1^j(t)$, $z_2^j(t)$ [14].

A method based on Fourier transform to compute the weight function P is explained in [3].

6.2. **Non transversal terms.** In this section we repeat the computation of the scalar case studied in Section 4 to bound the non transversal terms

$$\Big(|v_m^j| + |v_m^{j-1}|\Big)\big|v_m^{j-1} - \hat{v}_m^{j-1}\big|, \qquad \Big|v_m^j v_m^{j-1}\big(\sigma_m^{j-1} - \sigma_m^j\big)\Big|.$$

We recall that from the results of Section 5, we can define the variable s_m^j by

(6.7)
$$\frac{v_m^{j-1}}{v_m^j} = g^{-1}\left(\frac{g(s_m^j)}{1 + \psi(s_m^j)(\lambda_m(u) - \lambda_m(u_0))/\lambda_m(u_0)}\right) + v_m^j \tilde{q}_m\Big(u^j, v_m^j, \lambda_m(u_0)\big(1 + (g(s_m^j) - 1)\psi(s_m^j)\big)\Big),$$

where $\tilde{q}_m$ is defined in (5.2). The function ψ is the standard cut-off function, defined in (4.15). Note that by assuming δ_0 sufficiently small, we have

$$\big|s_m^j - 1\big| \leq 3\delta_1 \qquad \text{if} \qquad \left|\frac{\sigma_m^j}{\lambda_m(u_0)} - 1\right| \leq 2\delta_1.$$

Moreover we can assume that $|s_m^j - 1| \geq 3\delta_1/5$ if $|v_m^{j-1}/v_m^j - 1| \geq 4\delta_1/5$.

For $m = 1, \dots, N$, we define the variable w_m^j as

$$w_m^j \doteq v_m^j s_m^j = v_m^j h_m\big(u^j, v_m^j, \alpha_m^j\big) + \big(v_m^j\big)^2 \kappa_m\big(u^j, v_m^j, \alpha_m^j\big), \tag{6.8}$$

where h_m is given by

$$h_m(u^j, v^j_m, \alpha^j_m) = g^{-1}\left(\left(1 + \frac{\lambda^j_m - \lambda_m(u_0)}{\lambda_m(u_0)}\psi(s^j_m)\right) g(\alpha^j_m)\right), \tag{6.9}$$

and $\tilde{\kappa}_m$ is a smooth function.

Repeating the computations of Section 4 or using equation (D.12) of appendix D, we find that the equation satisfied by w^j_m is

(6.10)

$$\begin{aligned}
w^j_{m,t} &+ \tilde{\lambda}^j_m w^j_m - \tilde{\lambda}^{j-1}_m w^{j-1}_m = \mathcal{O}(1)\mathcal{V}(u) v^{j-1}_m \big(s^j_m - s^{j-1}_m\big)^2 \chi\Big\{\big|s^j_m - 1\big|, \big|s^{j-1}_m - 1\big| \le 5\delta_1\Big\} \\
&+ \mathcal{O}(1)\mathcal{V}(u)\big(|v^{j-1}_m| + |v^{j-2}_m|\big)\chi\Big\{\big|s^j_m - 1\big| \ge 5\delta_1, \big|s^{j-1}_m - 1\big| \le 3\delta_1 \text{ or viceversa}\Big\} \\
&+ \mathcal{O}(1)\big|v^j_m w^{j-1}_m - v^{j-1}_m w^j_m\big| + \mathcal{O}(1)\big|v^{j-1}_m w^{j-2}_m - v^{j-2}_m w^{j-1}_m\big| \\
&+ \mathcal{O}(1)\Big(\big(v^j_m\big)^2 + \big(v^{j-1}_m\big)^2\Big)\chi\Big\{\big|s^j_m - 1\big| \ge 3\delta_1/5\Big\} + \mathcal{O}(1)\Big(\big(v^{j-1}_m\big)^2 + \big(v^{j-2}_m\big)^2\Big)\chi\Big\{\big|s^{j-1}_m - 1\big| \ge 3\delta_1/5\Big\} \\
&+ \mathcal{O}(1)\sum_{n \neq m} \big|v^j_m\big|\big|v^{j-1}_n\big| + \mathcal{O}(1)\omega^j_m + \mathcal{O}(1)\omega^{j-1}_m \\
&= \mathcal{O}(1)\left\{\mathcal{V}(u)\mathcal{I}^j_{1,m}(t) + \mathcal{I}^j_{2,m}(t) + \mathcal{I}^j_{3,m}(t) + \mathcal{I}^{j-1}_{3,m}(t) + \sum_{n \neq m}\big|v^j_m\big|\big|v^j_n\big| + \omega^j_m + \omega^{j-1}_m\right\} \\
&= e^j_m(t).
\end{aligned}$$

In the above equation we have used the notation of Remark 4.7.

We now can rewrite the non transversal source terms as

$$\Big(\big|v^j_m\big| + \big|v^{j-1}_m\big|\Big)\big|v^{j-1}_m - \hat{v}^{j-1}_m\big| \le \mathcal{O}(1)\Big(\big|v^j_m\big|^2 + \big|v^{j-1}_m\big|^2\Big)\chi\Big\{\big|s^{j-1}_m - 1\big| \ge 3\delta_1/5\Big\}, \tag{6.11}$$

$$\begin{aligned}
\Big|v^j_m v^{j-1}_m\big(\sigma^{j-1}_m - \sigma^j_m\big)\Big| \le{}& \mathcal{O}(1)\Big|v^j_m v^{j-1}_m\big(s^j_m - s^{j-1}_m\big)\Big| \\
&+ \mathcal{O}(1)\Big(\big|v^j_m\big| + \big|v^{j-1}_m\big|\Big)\big|v^{j-1}_m - \hat{v}^{j-1}_m\big| + \mathcal{O}(1)\sum_{n \neq m}\big|v^j_m\big|\big|v^j_n\big|.
\end{aligned} \tag{6.12}$$

We introduce the three non transversal functionals which are needed to bound the source terms in w^j_m, and in ω^j_m.

6.3. **Energy Functional.** Let ψ be the cutoff function defined as

$$\theta(x) = \begin{cases} 0 & |x-1| \le \delta_1/5 \\ \text{smooth connection} & \delta_1/5 \le |x-1| \le 2\delta_1/5 \\ 1 & |x-1| \ge 2\delta_1/5 \end{cases} \tag{6.13}$$

Multiplying (5.21) by $v^j_m \theta^j_m = v^j_m \theta(s^j_m)$ and taking the sum w.r.t. j we obtain the equation

(6.14)

$$\begin{aligned}
\sum_j \Big(\tilde{\lambda}^j_m\big(v^j_m\big)^2\theta^j_m - \tilde{\lambda}^{j-1}_m v^j_m v^{j-1}_m \theta^j_m\Big) &= \frac{1}{2}\sum_j \Big(\tilde{\lambda}^j_m\big(v^j_m\big)^2\theta^j_m - 2\tilde{\lambda}^{j-1}_m v^j_m v^{j-1}_m\theta^j_m + \tilde{\lambda}^{j-1}_m\big(v^{j-1}_m\big)^2\theta^{j-1}_m\Big) \\
= -\frac{1}{2}\frac{d}{dt}\sum_j \big(v^j_m\big)^2\theta^j_m &+ \sum_j \frac{\tilde{\lambda}^{j-1}_m}{2}\theta'\big(w^j_m v^{j-1}_m - w^{j-1}_m v^j_m\big) + \sum_j \frac{\lambda^{j-1}}{2}\theta' v^j_m e^j_m + \sum_j \theta^j_m v^j_m \omega^j_m.
\end{aligned}$$

With the same computation of the scalar case we can write the left hand side as

$$\begin{aligned}
\frac{1}{2}\Big(\tilde{\lambda}^j_m\big(v^j_m\big)^2\theta^j_m - 2\tilde{\lambda}^{j-1}_m v^j_m v^{j-1}_m\theta^j_m + \tilde{\lambda}^{j-1}_m\big(v^{j-1}_m\big)^2\theta^{j-1}_m\Big) \ge \frac{\tilde{\lambda}^j_m}{100}\delta_1^2\Big(\big(v^j_m\big)^2 + \big(v^{j-1}_m\big)^2\Big)\theta^j_m \\
+ \mathcal{O}(1)\Big|w^j v^{j-1} - v^j w^{j-1}\Big| + \mathcal{O}(1)\big(v^{j-1}\big)^2\chi\Big\{\big|s^j - 1\big| \ge 5\delta_1, \big|s^{j-1} - 1\big| \le 3\delta_1\Big\},
\end{aligned}$$

so that we obtain

$$
\begin{aligned}
\delta_1^2 \sum_j \mathcal{I}^j_{3,m}(t) \le \delta_1^2 \sum_j \Big(\big|v^{j-1}_m\big|^2 + \big|v^j_m\big|^2 \Big) \theta^j &\le -\mathcal{O}(1) \frac{d}{dt} \sum_j \big(v^j_m\big)^2 \theta^j_m + \mathcal{O}(1) \sum_j \mathcal{I}^j_{2,m} \\
&+ \mathcal{O}(1) \mathcal{V}(u)^2 \sum_j \mathcal{I}^j_{1,m} + \mathcal{O}(1) \mathcal{V}(u)^2 \sum_{j,n\neq m} \big|v^j_m\big| \big|v^{j-1}_n\big| \\
&+ \mathcal{O}(1) \mathcal{V}(u)^2 \sum_j \big|\omega^j_m\big| + \mathcal{O}(1) \mathcal{V}(u)^2 \sum_j \big|\omega^{j-1}_m\big|.
\end{aligned} \tag{6.15}
$$

6.4. **Length functional.** Consider the functional

$$
L_m(t) \doteq \sum_j \sqrt{\big(v^j_m\big)^2 + \big(w^j_m\big)^2}. \tag{6.16}
$$

We have

$$
\begin{aligned}
\sum_j \mathcal{I}^j_{1,m}(t) \le & -\mathcal{O}(1) \frac{dL_m}{dt} + \mathcal{O}(1) \sum_j \mathcal{I}^j_{2,m} + \mathcal{O}(1) \sum_j \mathcal{I}^j_{3,m} \\
& + \mathcal{O}(1) \sum_{j,n\neq m} \big|v^j_m\big| \big|v^{j-1}_n\big| + \mathcal{O}(1) \sum_j \big|\omega^j_m\big| + \mathcal{O}(1) \sum_j \big|\omega^{j-1}_m\big|.
\end{aligned} \tag{6.17}
$$

6.5. **Area functional.** Consider the functional

$$
Q_m \doteq \frac{1}{2} \sum_{j<k} \big| v^j_m w^k_m - v^k_m w^j_m \big|. \tag{6.18}
$$

We obtain that

$$
\begin{aligned}
\sum_j \mathcal{I}^j_{2,m}(t) \le & -\mathcal{O}(1) \frac{dQ_m}{dt} + \mathcal{O}(1) \mathcal{V}^2(u) \sum_j \mathcal{I}^j_{1,m} + \mathcal{O}(1) \mathcal{V}(u) \sum_j \mathcal{I}^j_{3,m} \\
& + \mathcal{O}(1) \mathcal{V}(u) \sum_{j,n\neq m} \big|v^j_m\big| \big|v^{j-1}_n\big| + \mathcal{O}(1) \mathcal{V}(u) \sum_j \big|\omega^j_m\big| \\
& + \mathcal{O}(1) \mathcal{V}(u) \sum_j \big|\omega^{j-1}_m\big|.
\end{aligned} \tag{6.19}
$$

Using (6.15), (6.17), (6.19) and the assumption (5.24), we obtain that if C_0 is sufficiently large

$$
\left\{
\begin{array}{lll}
\displaystyle\sum_j \int_0^T \mathcal{I}^j_1(t) dt & \le & C_0 \left(\delta_0 + \displaystyle\sum_j \int_0^T \mathcal{I}^j_2(t) dt + \sum_j \int_0^T \mathcal{I}^j_3(t) dt \right) \\
\displaystyle\sum_j \int_0^T \mathcal{I}^j_2(t) dt & \le & C_0 \left(\delta_0^3 + \delta_0^2 \displaystyle\sum_j \int_0^T \mathcal{I}^j_1(t) dt + \delta_0 \sum_j \int_0^T \mathcal{I}^j_3(t) dt \right) \\
\displaystyle\sum_j \int_0^T \mathcal{I}^j_3(t) dt & \le & C_0 \left(\delta_0^3(u) + \delta_0^2 \displaystyle\sum_j \int_0^T \mathcal{I}^j_1(t) dt + \sum_j \int_0^T \mathcal{I}^j_2(t) dt \right)
\end{array}
\right.
$$

so that if δ_0 is sufficiently small,

$$
\sum_j \int_0^T \mathcal{I}^j_{1,m}(s) ds = 2C_0 \delta_0, \quad \sum_j \int_0^t \mathcal{I}^j_{2,m}(s) ds = 4C_0^2 \delta_0^3, \quad \sum_j \int_0^t \mathcal{I}^j_{3,m}(s) ds = 7C_0^2 \delta_0^3. \tag{6.20}
$$

In particular, using (6.11), (6.12), we have

$$
\sum_j \int_0^T \big|e^j(t)\big| dt, \sum_j \int_0^T \big|\omega^j_m(t)\big| dt < \hat{C} \big(2\delta_0\big)^2 \tag{6.21}
$$

if $\hat{C}$ is sufficiently big. The above inequality proves that $\|v^j_m\|_{\ell^1}$ is always below $2\delta_0$. Note that we obtain an estimate on the source term for w^j_m, which is of the order of δ_0^2 too.

This concludes the proof of the BV bounds for the solution u^j.

7. Stability estimates

Consider the equation for a perturbation ζ^j of the solution u^j,

$$\zeta_t^j + A(u^j)\zeta^j - A(u^{j-1})\zeta^{j-1} = 0. \tag{7.1}$$

Without any loss of generality, we assume that $\|\zeta(0)\|_{\ell^1} = \delta_0/4$. Thus, by the results of Section 2, at time $\tilde{t}$ we have

$$\big\|\zeta^j(\tilde{t})\big\|_{\ell^1} \leq \frac{\delta_0}{2}, \quad \big\|\zeta^j(\tilde{t})\big\|_{\ell^\infty} \leq \sum_j \big|\zeta^j(\tilde{t}) - \zeta^{j-1}(\tilde{t})\big| \leq \mathcal{O}(1)\delta_0^2. \tag{7.2}$$

As before we set $\tilde{t} = 0$ and we assume that estimates (7.2) hold. We decompose ζ^j as

$$\zeta^j = \sum_m \zeta_m^j \tilde{r}_m\big(u^j, v_m^j, \sigma_m^j\big), \tag{7.3}$$

so that by Lemma 5.1 we have the estimates

$$\big\|\zeta_m^j(0)\big\|_{\ell^1} \leq \delta_0, \qquad \big\|\zeta_m^j(0)\big\|_{\ell^\infty} \leq \sum_j \big|\zeta_m^j(0) - \zeta_m^{j-1}(0)\big| \leq \mathcal{O}(1)\delta_0^2. \tag{7.4}$$

Remark 7.1. We note that, differently from the vanishing viscosity case, here the natural decomposition of the perturbation h using the generalized eigenvectors of v gives integral terms, at least for a fixed time T. The main difference is that in the vanishing viscosity case we cannot control the space derivatives of the vectors $\tilde{r}_m$, while here the spatial discretization removes this problem.

Substituting in (7.1) we obtain

$$\begin{aligned} &\sum_m \zeta_{m,t}^j \tilde{r}_m^j + \sum_{m,n} \zeta_m^j v_n^j \big(D\tilde{r}_m^j \tilde{r}_n^j + \tilde{r}_{m,\sigma}^j D\tilde{p}^{-1,j}\tilde{r}_n^j\big) + \sum_m \zeta_m^j v_{m,t}^j \left(\tilde{r}_{m,v}^j + \tilde{r}_{m,\sigma}^j \frac{\partial p_m^{-1,j}}{\partial v_m^j}\right) \\ &\quad + \sum_m \zeta_m^j v_{m,t}^{j-1} \tilde{r}_{m,\sigma}^j \frac{\partial p_m^{-1,j}}{\partial v_m^{j-1}} + \sum_m \zeta_m^j A(u^j)\tilde{r}_m^j - \sum_m \zeta_m^{j-1} A(u^{j-1})\tilde{r}_m^{j-1} = 0. \end{aligned}$$

Using (3.12) and (5.15), after some computations one gets

$$\begin{aligned} &\sum_m \big(\zeta_{m,t}^j + \tilde{\lambda}_m^j \zeta_m^j - \tilde{\lambda}_m^{j-1}\zeta_m^{j-1}\big)\tilde{r}_m^j \\ &= \sum_m \zeta_m^j \tilde{r}_{m,\sigma}^j \left[\left(\frac{\partial p_m^{-1,j}}{\partial v_m^j} - \frac{\partial p_m^{-1,j}}{\partial v_m^{j-1}}\right)\big(\widehat{\tilde{\lambda}_m^{j-1}}\hat{v}_m^{j-1} - \tilde{\lambda}_m^{j-1}v_m^{j-1}\big) + \frac{\partial p_m^{-1,j}}{\partial v_m^{j-1}}\big(\widehat{\tilde{\lambda}_m^{j-2}}\hat{v}_m^{j-2} - \tilde{\lambda}_m^{j-2}v_m^{j-2}\big)\right] \\ &\quad + \sum_m \zeta_m^{j-1}\Big(A^{j-1}\tilde{r}_m^{j-1} - \tilde{\lambda}^{j-1}\big(\tilde{r}_m^j + v_m^j \tilde{r}_{m,v}^j\big)\Big) - \sum_m \zeta_m^j \tilde{p}_m\Big(\hat{A}^{j-1}\widehat{\tilde{r}_m^{j-1}} - \widehat{\tilde{\lambda}_m^{j-1}}\big(\tilde{r}_m^j + v_m^j \tilde{r}_{m,v}^j\big)\Big) \\ &\quad + \sum_m \tilde{\lambda}_m^{j-1}\big(\zeta_m^{j-1}v_m^j - \zeta_m^j v_m^{j-1}\big)\tilde{r}_{m,v}^j + \mathcal{O}(1)\omega_m^j(t) + \mathcal{O}(1)\omega_m^{j-1}(t). \end{aligned} \tag{7.5}$$

As in Section 5, we get the following estimates:

$$\begin{aligned} &\zeta_m^j \tilde{r}_{m,\sigma}^j \left[\left(\frac{\partial p_m^{-1,j}}{\partial v_m^j} - \frac{\partial p_m^{-1,j}}{\partial v_m^{j-1}}\right)\big(\widehat{\tilde{\lambda}_m^{j-1}}\hat{v}_m^{j-1} - \tilde{\lambda}_m^{j-1}v_m^{j-1}\big) + \frac{\partial p_m^{-1,j}}{\partial v_m^{j-1}}\big(\widehat{\tilde{\lambda}_m^{j-2}}\hat{v}_m^{j-2} - \tilde{\lambda}_m^{j-2}v_m^{j-2}\big)\right] \\ &\quad = \mathcal{O}(1)\zeta_m^j\big(v_m^{j-1} - \hat{v}_m^{j-1}\big) + \mathcal{O}(1)\zeta_m^j\big(v_m^{j-2} - \hat{v}_m^{j-2}\big) + \mathcal{O}(1)\zeta_m^j v_m^j v_m^{j-1}\big(\sigma_m^j - \sigma_m^{j-1}\big) \\ &\qquad + \mathcal{O}(1)\zeta_m^j v_m^{j-1}v_m^{j-2}\big(\sigma_m^{j-1} - \sigma_m^{j-2}\big) + \mathcal{O}(1)\sum_{n\neq m}\big|\zeta_m^j\big|\Big(\big|v_n^j\big| + \big|v_n^{j-1}\big|\Big) \\ &\quad = \mathcal{O}(1)\zeta_m^j\big(v_m^{j-1} - \hat{v}_m^{j-1}\big) + \mathcal{O}(1)\zeta_m^j\big(v_m^{j-2} - \check{v}_m^{j-2}\big) + \mathcal{O}(1)\zeta_m^j v_m^j v_m^{j-1}\big(s_m^j - s_m^{j-1}\big) \\ &\qquad + \mathcal{O}(1)\zeta_m^j v_m^{j-1}v_m^{j-2}\big(s_m^{j-1} - s_m^{j-2}\big) + \mathcal{O}(1)\sum_{n\neq m}\big|\zeta_m^j\big|\Big(\big|v_n^j\big| + \big|v_n^{j-1}\big|\Big), \end{aligned}$$

$$\begin{aligned}
&\zeta^{j-1}_m\Big(A^{j-1}\tilde r^{j-1}_m - \tilde\lambda^{j-1}\big(\tilde r^j_m + v^m_j \tilde r^j_{m,v}\big)\Big) - \zeta^j_m \tilde p^j_m\Big(\hat A^{j-1}\widehat{\tilde r^{j-1}_m} - \widehat{\tilde\lambda^{j-1}_m}\big(\tilde r^j_m + v^m_j \tilde r^j_{m,v}\big)\Big) \\
&\quad = \zeta^{j-1}_m\bigg[\Big(A^{j-1}\tilde r^{j-1}_m - \tilde\lambda^{j-1}\big(\tilde r^j_m + v^m_j\tilde r^j_{m,v}\big)\Big) - \Big(\hat A^{j-1}\widehat{\tilde r^{j-1}_m} - \widehat{\tilde\lambda^{j-1}_m}\big(\tilde r^j_m + v^m_j\tilde r^j_{m,v}\big)\Big)\bigg] \\
&\qquad + \Big[\big(\zeta^{j-1}_m v^j_m - \zeta^j_m v^{j-1}_m\big) + \zeta^j_m\big(v^{j-1}_m - \hat v^{j-1}_m\big)\Big]\frac{1}{v^j_m}\Big(\hat A^{j-1}\widehat{\tilde r^{j-1}_m} - \widehat{\tilde\lambda^{j-1}_m}\big(\tilde r^j_m + v^m_j\tilde r^j_{m,v}\big)\Big) \\
&\quad = \mathcal O(1)\sum_{n\neq m}\zeta^{j-1}_m v^j_n + \mathcal O(1)\zeta^j_m\big(v^{j-1}_m - \hat v^{j-1}_m\big) + \mathcal O(1)\zeta^{j-1}_m\big(v^{j-1}_m - \hat v^{j-1}_m\big) \\
&\qquad + \mathcal O(1)\zeta^j_m v^{j-1}_m\big(\sigma^j_m - \sigma^{j-1}_m\big) + \mathcal O(1)\big(\zeta^j_m v^{j-1}_m - \zeta^{j-1}_m v^j_m\big) \\
&\quad = \mathcal O(1)\sum_{n\neq m}\zeta^{j-1}_m v^j_n + \mathcal O(1)\zeta^j_m v^{j-1}_m\big(s^j_m - s^{j-1}_m\big)\chi\Big\{\big|s^j_m - 1\big| \le 3\delta_1\Big\} + \mathcal O(1)\zeta^j_m v^{j-1}_m\chi\Big\{\big|s^j_m-1\big| > 3\delta_1\Big\} \\
&\qquad + \mathcal O(1)\zeta^j_m\big(v^{j-1}_m - \hat v^{j-1}_m\big) + \mathcal O(1)\zeta^{j-1}_m\big(v^{j-1}_m - \hat v^{j-1}_m\big) + \mathcal O(1)\big(\zeta^j_m v^{j-1}_m - \zeta^{j-1}_m v^j_m\big).
\end{aligned}$$

We thus obtain the following equation for the component ζ^j_m:

$$\begin{aligned}
\zeta^j_{m,t} + \tilde\lambda^j_m\zeta^j_m - \tilde\lambda^{j-1}_m\zeta^{j-1}_m ={}& \mathcal O(1)\sum_{n\neq p}\Big(\big|\zeta^j_n\big| + \big|\zeta^{j-1}_n\big|\Big)\Big(\big|v^j_p\big| + \big|v^{j-1}_p\big|\Big) + \mathcal O(1)\sum_n\big|\zeta^j_n v^{j-1}_n - \zeta^{j-1}_n v^j_n\big| \\
&+ \mathcal O(1)\sum_n\Big|\zeta^j_n v^{j-1}_n\big(s^j_n - s^{j-1}_n\big)\Big|\chi\Big\{\big|s^j_n-1\big|\le 3\delta_1\Big\} + \mathcal O(1)\sum_n\big|\zeta^j_n\big|\big|v^{j-1}_n\big|\chi\Big\{\big|s^j_n - 1\big| > 3\delta_1\Big\} \\
&+ \mathcal O(1)\sum_n\Big|\zeta^j_n\big(v^{j-1}_n - \hat v^{j-1}_n\big)\Big| + \mathcal O(1)\sum_n\Big|\zeta^{j-1}_n\big(v^{j-1}_n - \hat v^{j-1}_n\big)\Big| + \mathcal O(1)\sum_n\Big|\zeta^j_n\big(v^{j-2}_n - \check v^{j-2}_n\big)\Big| \\
&+ \mathcal O(1)\sum_n\Big|\zeta^j_n\big(v^{j-1}_n w^{j-2}_n - w^{j-1}_n v^{j-2}_n\big)\Big| + \mathcal O(1)\sum_n\Big|\zeta^j_n\big(v^j_n w^{j-1}_n - w^j_n v^{j-1}_n\big)\Big| \\
&+ \mathcal O(1)\sum_n\big|\omega^j_n(t)\big| + \mathcal O(1)\sum_n\big|\omega^{j-1}_n(t)\big|.
\end{aligned} \tag{7.6}$$

Further computations allow us to write

$$\begin{aligned}
\zeta^j_m v^{j-1}_m\big(s^j_m - s^{j-1}_m\big)\chi\Big\{\big|s^j_m - 1\big|\le 3\delta_1\Big\} ={}& \big(\zeta^{j-1}_m w^j_m - \zeta^j_m w^{j-1}_m\big)\chi\Big\{\big|s^j_m-1\big|\le 3\delta_1\Big\} \\
&+ s^j_m\big(\zeta^j_m v^{j-1}_m - \zeta^{j-1}_m v^j_m\big)\chi\Big\{\big|s^j_m - 1\big| \le 3\delta_1\Big\} \\
={}& \mathcal O(1)\big(\zeta^{j-1}_m w^j_m - \zeta^j_m w^{j-1}_m\big) + \mathcal O(1)\big(\zeta^j_m v^{j-1}_m - \zeta^{j-1}_m v^j_m\big),
\end{aligned}$$

$$\begin{aligned}
\big|\zeta^j_m v^{j-1}_m\big|\chi\Big\{\big|s^j_m - 1\big| > 3\delta_1\Big\} ={}& \big|\zeta^j_m v^{j-1}_m\big|\chi\Big\{\big|s^j_m - 1\big| > 3\delta_1, \big|\zeta^{j-1}_m/\zeta^j_m - 1\big| \le 2\delta_1\Big\} \\
&+ \big|\zeta^j_m v^{j-1}_m\big|\chi\Big\{\big|s^j_m - 1\big| > 3\delta_1, \big|\zeta^{j-1}_m/\zeta^j_m - 1\big| > 2\delta_1\Big\},
\end{aligned}$$

$$\begin{aligned}
&\zeta^j_m\big(v^{j-2}_m - \check v^{j-2}_m\big) = \zeta^j_m\big(v^{j-2}_m - v^{j-1}_m\tilde p_m(u^{j-1}, v^{j-1}_m, \sigma^{j-1}_m)\big) + \zeta^j_m\big(v^{j-1}_m\tilde p_m(u^{j-1}, v^{j-1}_m, \sigma^{j-1}_m) - \check v^{j-1}_m\big) \\
&\quad = \frac{\zeta^j_m}{\zeta^{j-1}_m}\zeta^{j-1}_m\big(v^{j-2}_m - v^{j-1}_m\tilde p_m(u^{j-1}, v^{j-1}_m, \sigma^{j-1}_m)\big)\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m - 1\big| \le 4\delta_1/5\Big\} \\
&\qquad + \zeta^j_m\big(v^{j-2}_m - v^{j-1}_m\tilde p_m(u^{j-1}, v^{j-1}_m, \sigma^{j-1}_m)\big)\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m - 1\big| > 4\delta_1/5\Big\} + \mathcal O(1)\zeta^j_m v^{j-1}_m\big(\sigma^{j-1}_m - \sigma^j_m\big) \\
&\quad = \frac{\zeta^j_m}{\zeta^{j-1}_m}\zeta^{j-1}_m\big(v^{j-2}_m - v^{j-1}_m\tilde p_m(u^{j-1}, v^{j-1}_m, \sigma^{j-1}_m)\big)\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m - 1\big|, \big|\zeta^{j-2}_m/\zeta^{j-1}_m - 1\big| \le 4\delta_1/5\Big\} \\
&\qquad + \frac{\zeta^j_m}{\zeta^{j-1}_m}\zeta^{j-1}_m\big(v^{j-2}_m - v^{j-1}_m\tilde p_m(u^{j-1}, v^{j-1}_m, \sigma^{j-1}_m)\big)\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m - 1\big| \le 4\delta_1/5, \big|\zeta^{j-2}_m/\zeta^{j-1}_m - 1\big| > 4\delta_1/5\Big\} \\
&\qquad + \zeta^j_m\big(v^{j-2}_m - v^{j-1}_m\tilde p_m(u^{j-1}, v^{j-1}_m, \sigma^{j-1}_m)\big)\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m - 1\big| > 4\delta_1/5\Big\} \\
&\qquad + \mathcal O(1)\zeta^j_m v^{j-1}_m\big(s^j_m - s^{j-1}_m\big)\chi\Big\{\big|s^j_m - 1\big| \le 3\delta_1\Big\} + \mathcal O(1)\zeta^j_m v^{j-1}_m\chi\Big\{\big|s^j_m - 1\big| > 3\delta_1\Big\} \\
&\qquad + \mathcal O(1)\sum_n\big|\zeta^j_m\big|\big|v^j_n\big| + \mathcal O(1)\zeta^j_m\big(v^{j-1}_m - \hat v^{j-1}_m\big),
\end{aligned}$$

$$\begin{aligned}\zeta_m^j\big(v_m^{j-1}-\hat v_m^{j-1}\big) &= \zeta_m^j\big(v_m^{j-1}-\hat v_m^{j-1}\big)\chi\Big\{\big|\zeta_m^{j-1}/\zeta_m^j-1\big|\le 4\delta_1/5\Big\}\\ &\quad+\zeta_m^j\big(v_m^{j-1}-\hat v_m^{j-1}\big)\chi\Big\{\big|\zeta_m^{j-1}/\zeta_m^j-1\big|\ge 4\delta_1/5\Big\},\end{aligned}$$

$$\begin{aligned}\zeta_m^{j-1}\big(v_m^{j-1}-\hat v_m^{j-1}\big) &= \mathcal{O}(1)\zeta_m^j\big(v_m^{j-1}-\hat v_m^{j-1}\big)\chi\Big\{\big|\zeta_m^{j-1}/\zeta_m^j-1\big|\le 4\delta_1/5\Big\}\\ &\quad+\zeta_m^{j-1}\big(v_m^{j-1}-\hat v_m^{j-1}\big)\chi\Big\{\big|\zeta_m^{j-1}/\zeta_m^j-1\big|\ge 4\delta_1/5\Big\},\end{aligned}$$

Note that

$$\begin{aligned}&\zeta_m^j\big(v_m^{j-1}-\hat v_m^{j-1}\big)\chi\Big\{\big|\zeta_m^{j-1}/\zeta_m^j-1\big|\le 4\delta_1/5\Big\}\\ &\qquad= \big(\zeta_m^j v_m^{j-1}-\zeta_m^{j-1}v_m^j\big)\chi\Big\{\big|s_m^j-1\big|\ge\delta_1,\big|\zeta_m^{j-1}/\zeta_m^j-1\big|\le 4\delta_1/5\Big\}\\ &\qquad\quad+\mathcal{O}(1)\zeta_m^{j-1}v_m^j\chi\Big\{\big|s_m^j-1\big|\ge\delta_1,\big|\zeta_m^{j-1}/\zeta_m^j-1\big|\le 4\delta_1/5\Big\},\end{aligned}$$

and, if δ_0 is sufficiently small so that $|s^j-1|>\delta_1$ implies $|v_m^{j-1}/v_m^j-1|>9\delta_1/10$, that

$$\begin{aligned}&\big|\zeta_m^{j-1}v_m^j-\zeta_m^j v_m^{j-1}\big|\chi\Big\{\big|s_m^j-1\big|\ge\delta_1,\big|\zeta_m^{j-1}/\zeta_m^j-1\big|\le 4\delta_1/5\Big\}\\ &\qquad\ge \big|\zeta_m^{j-1}v_m^j-\zeta_m^j v_m^{j-1}\big|\chi\Big\{\big|v_m^{j-1}/v_m^j-1\big|\ge 9\delta_1/10,\big|\zeta_m^{j-1}/\zeta_m^j-1\big|\le 4\delta_1/5\Big\}\\ &\qquad=\mathcal{O}(1)\frac{1}{\delta_1}\Big(\big|\zeta_m^{j-1}v_m^j\big|+\big|v_m^{j-1}\zeta_m^j\big|\Big),\end{aligned}\tag{7.7}$$

$$\big|\zeta_m^{j-1}v_m^j-\zeta_m^j v_m^{j-1}\big|\chi\Big\{\big|s_m^j-1\big|\ge 3\delta_1,\big|\zeta_m^{j-1}/\zeta_m^j-1\big|\le 2\delta_1\Big\}\ge\mathcal{O}(1)\frac{1}{\delta_1}\Big(\big|\zeta_m^{j-1}v_m^j\big|+\big|v_m^{j-1}\zeta_m^j\big|\Big).$$

Using the trivial estimate $2ab\le a^2+b^2$, we can thus rewrite (7.6) as

(7.8)

$$\begin{aligned}&\zeta_{m,t}^j+\tilde\lambda_m^j\zeta_m^j-\tilde\lambda_m^{j-1}\zeta_m^{j-1}=\mathcal{O}(1)\sum_{n\ne p}\Big(\big|\zeta_n^j\big|+\big|\zeta_m^{j-1}\big|\Big)\Big(\big|v_p^j\big|+\big|v_p^{j-1}\big|\Big)\\ &\quad+\mathcal{O}(1)\sum_n\big|\zeta_n^j v_n^{j-1}-\zeta_n^{j-1}v_n^j\big|+\mathcal{O}(1)\sum_n\big|\zeta_n^{j-1}v_n^{j-2}-\zeta_n^{j-2}v_n^{j-1}\big|+\mathcal{O}(1)\sum_n\Big|\zeta_n^j w_n^{j-1}-\zeta_n^{j-1}w_n^j\Big|\\ &\quad+\mathcal{O}(1)\sum_n\Big|\zeta_n^j\big(v_n^j w_n^{j-1}-v_n^{j-1}w_n^j\big)\Big|+\mathcal{O}(1)\sum_n\Big|\zeta_n^j\big(v_n^{j-1}w_n^{j-2}-v_n^{j-2}w_n^{j-1}\big)\Big|\\ &\quad+\mathcal{O}(1)\sum_n\big|\omega_n^j\big|+\mathcal{O}(1)\sum_n\Big(\big|\zeta_n^j\big|^2+\big|\zeta_n^{j-1}\big|^2\Big)\chi\Big\{\big|\zeta_n^{j-1}/\zeta_n^j-1\big|\ge 4\delta_1/5\Big\}\\ &\quad+\mathcal{O}(1)\sum_n\big|\omega_n^{j-1}\big|+\mathcal{O}(1)\sum_n\Big(\big|\zeta_n^{j-1}\big|^2+\big|\zeta_n^{j-2}\big|^2\Big)\chi\Big\{\big|\zeta_n^{j-2}/\zeta_n^{j-1}-1\big|\ge 4\delta_1/5\Big\}\\ &\quad+\mathcal{O}(1)\sum_n\Big(\big|v_n^j\big|^2+\big|v_n^{j-1}\big|^2\Big)\chi\Big\{\big|v_n^{j-1}/v_n^j-1\big|\ge 4\delta_1/5\Big\}\\ &\quad+\mathcal{O}(1)\sum_n\Big(\big|v_n^{j-1}\big|^2+\big|v_n^{j-2}\big|^2\Big)\chi\Big\{\big|v_n^{j-2}/v_n^{j-1}-1\big|\ge 4\delta_1/5\Big\}\\ &=\mu_m^j(t).\end{aligned}$$

As in the previous section, we assume that

$$\sum_j\int_0^\tau\big|\mu_m^j(t)\big|dt<\hat C\big(2\delta_0\big)^2,\tag{7.9}$$

for $\tau\in[0,T]$ and let T be the first time that equality holds. The above equation implies that

$$\sum_j\big|\zeta_m^j(t)\big|\le 2\delta_0$$

if $\delta_0 \le 1/4\hat{C}$. We will prove that as a consequence of $|\zeta^j_m| \le 2\delta_0$ we obtain that

$$\sum_j \int_0^T \big|\mu^j_m(t)\big| dt < \hat{C}\big(2\delta_0\big)^2, \tag{7.10}$$

which yields a contradiction. Thus (7.9) holds for all $\tau \ge 0$. As a consequence one gets $\|\zeta^j_m(t)\|_{\ell^1} \le 2\delta_0 = 8\|\zeta^j_m(0)\|_{\ell^1}$, which implies that for any perturbation ζ

$$\big\|\zeta(t)\big\|_{\ell^1} \le L\big\|\zeta(0)\big\|_{\ell^1} \tag{7.11}$$

for some constant L. A standard homotopy argument yields

$$\big\|u(t) - u'(t)\big\|_{\ell^1} \le L\big\|u(0) - u'(0)\big\|_{\ell^1}$$

for any solutions u, u'. This proves that the semigroup generated by the semidiscrete upwind scheme (2.1) is Lipschitz continuous w.r.t. the ℓ^1-norm.

In the remainder of this section we establish a priori bounds on all the source terms in (7.8), thus proving (7.10).

7.1. **Transversal terms.** Using the results of Section 6.1, we obtain

$$\sum_j \int_0^T \Big(\big|\zeta^j_m\big| + \big|\zeta^{j-1}_n\big|\Big)\Big(\big|v^j_n\big| + \big|v^{j-1}_n\big|\Big) dt \le \mathcal{O}(1)\delta_0^2. \tag{7.12}$$

7.2. **Non transversal terms.** We now introduce a variable which contains informations about the speed of the perturbation. As in Section 4 and in Section 6.2, we define

$$\varsigma^j_m \doteq h_m\big(u^j, \varsigma^j_m, \zeta^{j-1}_m/\zeta^j_m\big) + v^j_m \kappa_m\big(u^j, v^j_m, \zeta^{j-1}_m/\zeta^j_m\big), \tag{7.13}$$

where h_m is defined in (6.9) and g is the function $g(s) = (s-1)/\log s$. The function ψ is the standard cut-off function, defined in (4.15). Define

$$\iota^j_m \doteq \zeta^j_m \varsigma^j_m = \zeta^j_m h_m\big(u^j, v^j_m, \zeta^{j-1}_m/\zeta^j_m\big) + \zeta^j_m v^j_m \kappa_m\big(u^j, v^j_m, \zeta^{j-1}_m/\zeta^j_m\big). \tag{7.14}$$

Using the computation in appendix D, we obtain

(7.15)

$$\begin{aligned}
\iota^j_{m,t} &+ \tilde{\lambda}^j_m \iota^j_m - \tilde{\lambda}^{j-1}_m \iota^{j-1}_m = \mathcal{O}(1)\mathcal{V}(u)\zeta^{j-1}_m\big(\varsigma^j_m - \varsigma^{j-1}_m\big)^2 \chi\Big\{\big|\varsigma^j_m - 1\big|, \big|\varsigma^{j-1}_m - 1\big| \le 5\delta_1\Big\} \\
&+ \mathcal{O}(1)\mathcal{V}(u)\big(|\zeta^{j-1}_m| + |\zeta^{j-2}_m|\big)\chi\Big\{\big|\varsigma^j_m - 1\big| \ge 5\delta_1 \big|\varsigma^{j-1}_m - 1\big| \le 3\delta_1 \text{ or viceversa}\Big\} \\
&+ \mathcal{O}(1)\big|\zeta^j_m v^{j-1}_m - \zeta^{j-1}_m v^j_m\big| + \mathcal{O}(1)\big|\zeta^j_m w^{j-1}_m - \zeta^{j-1}_m w^j_m\big| + \mathcal{O}(1)\big|\zeta^{j-1}_m v^{j-2}_m - \zeta^{j-2}_m v^{j-1}_m\big| \\
&+ \mathcal{O}(1)\Big(\big(v^j_m\big)^2 + \big(v^{j-1}_m\big)^2\Big)\chi\left\{\big|s^j_m - 1\big| \ge \frac{3\delta_1}{5}\right\} + \mathcal{O}(1)\Big(\big(v^{j-1}_m\big)^2 + \big(v^{j-2}_m\big)^2\Big)\chi\left\{\big|s^{j-1}_m - 1\big| \ge \frac{3\delta_1}{5}\right\} \\
&+ \mathcal{O}(1)\Big(\big(\zeta^j_m\big)^2 + \big(\zeta^{j-1}_m\big)^2\Big)\chi\left\{\big|\varsigma^j_m - 1\big| \ge \frac{3\delta_1}{5}\right\} + \mathcal{O}(1)\Big(\big(\zeta^{j-1}_m\big)^2 + \big(\zeta^{j-2}_m\big)^2\Big)\chi\left\{\big|\varsigma^{j-1}_m - 1\big| \ge \frac{3\delta_1}{5}\right\} \\
&+ \mathcal{O}(1)\sum_n \big|v^j_n\big|\big|\zeta^{j-1}_n\big| + \mathcal{O}(1)\mu^j_m(t) + \mathcal{O}(1)\mu^{j-1}_m(t) + \mathcal{O}(1)\omega^j_m(t) \\
= C_0 &\left\{\mathcal{V}(\zeta)\mathcal{J}^j_{1,m} + \mathcal{J}^j_{2,m} + \mathcal{J}^j_{3,m} + \mathcal{J}^{j-1}_{3,m} + \sum_n \big|v^j_n\big|\big|\zeta^{j-1}_n\big| + \mu^j_m(t) + \mu^{j-1}_m(t) + \omega^j_m(t)\right\} \\
&+ C_0\Big\{\big|\zeta^j_m v^{j-1}_m - \zeta^{j-1}_m v^j_m\big| + \big|\zeta^j_m w^{j-1}_m - \zeta^{j-1}_m w^j_m\big| + \big|\zeta^{j-1}_m v^{j-2}_m - \zeta^{j-2}_m v^{j-1}_m\big|\Big\} = \tilde{e}^j_m(t).
\end{aligned}$$

We have used the notation of Remark 4.7. Note that, if the total variation of u^j is sufficiently small, we can write the energy terms in (7.8) as

$$\Big(\big|\zeta^j_m\big| + \big|\zeta^{j-1}_m\big|\Big)\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m - 1\big| \ge 4\delta_1/5\Big\} \le \Big(\big|\zeta^j_m\big| + \big|\zeta^{j-1}_m\big|\Big)\chi\Big\{\big|\varsigma^j_m - 1\big| \ge 3\delta_1/5\Big\}. \tag{7.16}$$

We now introduce the non transversal functionals for the perturbation h. Note that, by means of the functional

$$Q_{v\zeta} = \sum_{j<k} \Big| v_m^j h_m^{k-1} - v_m^{j-1} h_m^k \Big|, \quad Q_{w\zeta} = \sum_{j<k} \Big| w_m^j h_m^{k-1} - w_m^{j-1} h_m^k \Big|, \tag{7.17}$$

we have immediately the estimates

$$\sum_j \int_0^T \big| v_m^j \zeta_m^{j-1} - \zeta_m^j v_m^{j-1} \big| dt, \sum_j \int_0^T \big| w_m^j \zeta_m^{j-1} - \zeta_m^j w_m^{j-1} \big| dt \leq \mathcal{O}(1)\delta_0^3. \tag{7.18}$$

7.3. **Energy estimates.** Multiplying (7.8) by $\zeta^j \hat{\theta}^j = \zeta^j \theta(\varsigma^j)$ and repeating the computations of Section 6.3, we obtain

$$\begin{aligned} \delta_1^2 \sum_j \mathcal{J}_{3,m}^j(t) \leq \delta_1^2 \sum_j \Big(\big|\zeta_m^{j-1}\big|^2 + \big|\zeta_m^j\big|^2 \Big) \hat{\theta}^j &\leq -\mathcal{O}(1) \frac{d}{dt} \sum_j \big(\zeta_m^j\big)^2 \theta_m^j + \mathcal{O}(1) \sum_j \mathcal{J}_{2,m}^j \\ &+ \mathcal{O}(1)\mathcal{V}(\zeta)^2 \sum_j \mathcal{J}_{1,m}^j + \mathcal{O}(1)\mathcal{V}(\zeta)^2 \sum_{j,n\neq m} \big|v_m^j\big|\big|\zeta_n^{j-1}\big| \\ &+ \mathcal{O}(1)\mathcal{V}(\zeta)^2 \sum_j \big|\omega_m^j\big| + \mathcal{O}(1)\mathcal{V}(\zeta)^2 \sum_j \big|\omega_m^{j-1}\big| \\ &+ \mathcal{O}(1)\mathcal{V}(\zeta)^2 \sum_j \big|\mu_m^j\big| + \mathcal{O}(1)\mathcal{V}(\zeta)^2 \sum_j \big|\mu_m^{j-1}\big|. \end{aligned} \tag{7.19}$$

7.4. **Length functional.** Consider the functional

$$\mathcal{L}_m(t) \doteq \sqrt{\big(\zeta_m^j(t)\big)^2 + \big(\iota_m^j(t)\big)^2}. \tag{7.20}$$

With the computations of Section 6.4 we obtain

$$\begin{aligned} \sum_j \mathcal{J}_{1,m}^j(t) \leq & -\mathcal{O}(1) \frac{d\mathcal{L}_m}{dt} + \mathcal{O}(1) \sum_j \mathcal{J}_{2,m}^j + \mathcal{O}(1) \sum_j \mathcal{J}_{3,m}^j \\ & + \mathcal{O}(1) \sum_{j,n\neq m} \big|\zeta_m^j\big|\big|v_n^{j-1}\big| + \mathcal{O}(1) \sum_j \big|\omega_m^j\big| + \mathcal{O}(1) \sum_j \big|\omega_m^{j-1}\big| \\ & + \mathcal{O}(1) \sum_j \big|\mu_m^j\big| + \mathcal{O}(1) \sum_j \big|\mu_m^{j-1}\big|. \end{aligned} \tag{7.21}$$

7.5. **Area functional.** Consider the functional

$$\mathcal{Q}_m(t) \doteq \frac{1}{2} \sum_{j<k} \big| \zeta_m^j \iota_m^k - \zeta_m^k \iota_m^j \big|. \tag{7.22}$$

With the computations of Section 6.5 we obtain

$$\begin{aligned} \sum_j \mathcal{J}_{2,m}^j(t) \leq & -\mathcal{O}(1) \frac{d\mathcal{Q}_m}{dt} + \mathcal{O}(1)\mathcal{V}^2(\zeta) \sum_j \mathcal{J}_{1,m}^j + \mathcal{O}(1)\mathcal{V}(\zeta) \sum_j \mathcal{J}_{3,m}^j \\ & + \mathcal{O}(1)\mathcal{V}(\zeta) \sum_{j,n\neq m} \big|\zeta_m^j\big|\big|v_n^{j-1}\big| + \mathcal{O}(1)\mathcal{V}(u) \sum_j \big|\omega_m^j\big| \\ & + \mathcal{O}(1)\mathcal{V}(u) \sum_j \big|\omega_m^{j-1}\big| + +\mathcal{O}(1) \sum_j \big|\mu_m^j\big| + \mathcal{O}(1) \sum_j \big|\mu_m^{j-1}\big|. \end{aligned} \tag{7.23}$$

Using (6.15), (6.17) and (6.19), we obtain

$$\sum_j \int_0^T \mathcal{I}_{1,m}^j(s) ds = \mathcal{O}(1)\delta_0, \quad \sum_j \int_0^t \mathcal{I}_{2,m}^j(s) ds = \mathcal{O}(1)\delta_0^3, \quad \sum_j \int_0^t \mathcal{I}_{3,m}^j(s) ds = \mathcal{O}(1)\delta_0^3, \tag{7.24}$$

so that, by means of (6.20), (6.21), (7.16), we get

$$\sum_j \int_0^T \big|\mu_m^j(s)\big| ds < 4\hat{C}\delta_0^2. \tag{7.25}$$

This proves the stability estimate (7.11), and concludes the proof of Theorem 1.1.

8. The hyperbolic limit of the semidiscrete scheme

In this section we prove that the limit as $\epsilon \to 0$ of (1.4) exists and it coincides with the trajectory of a Riemann Semigroup. The trajectories of this semigroup can be uniquely characterized as Viscosity Solutions.

By means of the rescaling $t \mapsto \epsilon t$, $x \mapsto \epsilon x$, and using the conclusions of Theorem 1.1, we have obtained the following results. The solution $u^{j,\epsilon}$ to the semidiscrete equation

$$u^j_t + \frac{1}{\epsilon}\big(f(u^j) - f(u^{j-1})\big) = 0 \tag{8.1}$$

satisfies the uniform bound on BV norm:

$$\sum_j \big|u^{j,\epsilon}(t) - u^{j-1,\epsilon}(t)\big| \leq \frac{1}{L_1}\mathcal{V}\big(u^{j,\epsilon}(t)\big) \leq \frac{4N\delta_0}{L_1} \tag{8.2}$$

if $\mathcal{V}(\bar{u}^j) \leq \delta_0/4$. Moreover, from Section 7, it follows that

$$\big\|u^{j,\epsilon}(t) - z^{j,\epsilon}(t)\big\|_{\ell^1} \leq L\big\|u^{j,\epsilon}(0) - v^{j,\epsilon}(0)\big\|_{\ell^1}. \tag{8.3}$$

We define the function u^ϵ by

$$u^\epsilon(t,x) = u^{j,\epsilon}(t) \qquad (j-1)\epsilon < x \leq j\epsilon. \tag{8.4}$$

where u^j is the solution of (8.1) with initial data

$$u^{j,\epsilon}(0) = \bar{u}(j\epsilon), \qquad \bar{u} \in BV,\ u \text{ right continuous}, \ \lim_{x \to -\infty} \bar{u}(x) = u_0 \in K_0. \tag{8.5}$$

Since we have

$$\big\|u^{j,\epsilon}\big\|_{\ell^1} = \big\|u^\epsilon\big\|_{L^1}, \quad \sum_j \big|u^{j,\epsilon}(t) - u^{j-1,\epsilon}(t)\big| = \mathcal{V}\big(u^\epsilon\big),$$

up to a subsequence we can show that the limit as $\epsilon \to 0$ of u^ϵ exists and it defines a Lipschitz continuous semigroup $\mathcal{S}$ on a domain of functions with sufficiently small BV norm.

Note that in principle different subsequences could converge to different semigroups. To prove uniqueness of the limit, we use the definition of *Viscosity Solution*.

As a first step we prove that the limiting semigroup $\mathcal{S}$ has a finite speed of propagation. In fact, directly from (7.1) we have

$$\big|\zeta^j\big|_t = \big\langle \zeta^j/|\zeta^j|, A(u^{j-1})\zeta^{j-1}\big\rangle - \big\langle \zeta^j/|\zeta^j|, A(u^j)\zeta^j\big\rangle \leq \big\|A\big\|_{L^\infty}\Big(\big|\zeta^j\big| + \big|\zeta^{j-1}\big|\Big). \tag{8.6}$$

If at $t = 0$ we have $|\zeta^j(0)| \leq e^{-j}$, then we obtain

$$\sigma e^{\sigma t - x} = \big\|A\big\|_{L^\infty}(1+e)e^{\sigma t - x} \quad \text{if} \quad \sigma = \bar{\sigma} = \big\|A\big\|_{L^\infty}(1+e),$$

so that $|h^j(t)| \leq e^{\bar{\sigma}t - x}$. When rescaling we conclude that

$$\big|\zeta^{j,\epsilon}\big| \leq \exp\left(-\frac{\epsilon j - \bar{\sigma}t}{\epsilon}\right) \quad \text{if} \quad \big|\zeta^{j,\epsilon}(0)\big| \leq e^{-j},$$

so that the function defined as

$$\zeta^\epsilon(t,x) = \zeta^{j,\epsilon}(t) \quad \text{if} \quad (j-1)\epsilon < x \leq j\epsilon \tag{8.7}$$

satisfies the bound

$$\big|\zeta^\epsilon(t,x)\big| \leq \exp\left(-\frac{x - \bar{\sigma}t}{\epsilon}\right) \quad \text{if} \quad \big|\zeta^\epsilon(0,x)\big| \leq e^{-x/\epsilon}.$$

Consider now two solutions $u^{j,\epsilon}$, $z^{j,\epsilon}$. A simple homotopy argument shows that

$$\big|u^\epsilon(t,x) - z^\epsilon(t,x)\big| \leq \big\|u(0) - z(0)\big\|_{L^\infty}\exp\left(-\frac{x - \bar{\sigma}t}{\epsilon}\right) \quad \text{if} \quad \big|u(0,x) - z(0,x)\big| = 0, \quad x > 0.$$

Passing to the limit we obtain

$$\big|u^\epsilon(t,x) - z^\epsilon(t,x)\big| = 0, \quad x \geq \bar{\sigma}t \quad \text{if} \quad \big|u(0,x) - z(0,x)\big| = 0, \quad x > 0. \tag{8.8}$$

This proves that the speed of propagation of a perturbation is less than $\bar{\sigma}$. Note that trivially this speed is greater or equal to 0, because the Green kernel (2.13) has support in $j \geq 0$. This concludes the proof of finite speed propagation.

A particular solution of the limiting semigroup is the trajectory of $\mathcal{S}$ with initial data

$$\bar{u}(0,x) = \begin{cases} u^- & x \leq 0 \\ u^+ & x > 0 \end{cases}$$

with u^-, u^+ in K_0, $|u^- - u^+|$ small. For this special initial data there is a general technique to determine which is the limiting solution by studying the evolution equation on the center manifold (3.11). In [2] it is shown that this limit does not depend on the approximating sequence, and coincides with the Riemann Solver constructed by the vanishing viscosity limit. This Riemann Solver can be uniquely determined by requiring that each jump in the solution $\mathcal{S}_t \bar{u}$ satisfies Liu's stability condition [18].

The final step in proving uniqueness is to prove that any limit of a convergent subsequence is a Viscosity Solution to

$$u_t + f(u)_x = 0$$

in the sense of [10] In fact it follows that any trajectory is a trajectory of the semigroup $\mathcal{S}$, i.e., the limit does not depend on the subsequence.

We recall that a *Viscosity Solution* of a quasilinear hyperbolic system

$$u_t + A(u) u_x = 0 \tag{8.9}$$

is defined as follows.

Let $u(t,x)$ be a BV function w.r.t. x. Given a point (τ,ξ), denote by $U^\sharp_{(u;\tau,\xi)}$ the solution to the Riemann problem

$$u(\tau,x) = \begin{cases} \lim\limits_{y \to \xi-} u(\tau,y) & x \leq \xi \\ \lim\limits_{y \to \xi+} u(\tau,y) & x > \xi \end{cases} \tag{8.10}$$

This solution is obtained by the Riemann solver defined in [2], i.e., it is the unique limit of $u^\epsilon(t)$ with the special initial data (8.10).

We denote by $U^\flat_{(u;\tau,\xi)}$ the solution to the linear system

$$u_t + A\big(u(\tau,\xi)\big) u_x = 0, \tag{8.11}$$

with initial data $u(\tau,x)$.

A *Viscosity Solution* to (8.9) is now a function $u(t,x)$ satisfying the integral estimates:

(i) At every point (τ,ξ), for some $\beta' > 0$

$$\lim_{h \to 0+} \frac{1}{h} \int_{\xi - h\beta}^{\xi + h\beta} \Big| u(\tau+h,x) - U^\sharp_{(u;\tau,\xi)}(\tau+h,x) \Big| dx = 0. \tag{8.12}$$

(ii) There are constant C, $\beta \leq \beta'$ such that for every $a < \xi < b$

$$\lim_{h \to 0+} \frac{1}{h} \int_{a + h\beta}^{b - h\beta} \Big| u(\tau+h,x) - U^\flat_{(u;\tau,\xi)}(\tau+h,x) \Big| dx \leq C \mathcal{V}\big(u;]a,b[\big)^2. \tag{8.13}$$

For an account of viscosity solution of hyperbolic systems we refer to [10].

At this point, using the same technique of [4], one can prove the following Lemma:

Lemma 8.1. *Let $\mathcal{S} : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$ be a semigroup of solutions, constructed as the limit of a sequence S^{ϵ_m} of the semidiscrete scheme* (8.1) *and defined on a domain $\mathcal{D} \subset L_1$ of functions with small total variation. Let $u : [0,T] \mapsto \mathcal{D}$ be Lipschitz continuous w.r.t. time, i.e.,*

$$\big\| u(t) - u(s) \big\|_{L^1} \leq L |t-s| \tag{8.14}$$

for some constant L and all $s,t \in [0,T]$. Then

$$u(t) = \mathcal{S}_t u(0) \qquad \text{for all } t \in [0,T] \tag{8.15}$$

if and only if u is a viscosity solution of (8.9).

.

In particular $\mathcal{S}_t u$ is a viscosity solution to (8.9).

What remains to be proved is that the whole family of semidiscrete approximations converges to a unique limit, i.e.,

$$\lim_{\epsilon \to 0+} S_t^\epsilon \bar{u} = \mathcal{S}_t \bar{u}, \tag{8.16}$$

where the limit holds over all real values of ϵ and not only along a particular sequence $\{\epsilon_m\}$. If (8.16) fails, we can find $\bar{v}$, τ and two different sequences $\epsilon_m, \epsilon'_m \to 0$ such that

$$\lim_{m \to \infty} S_\tau^{\epsilon_m} \bar{v} \neq \lim_{m \to \infty} S_\tau^{\epsilon'_m} \bar{v} \,. \tag{8.17}$$

By extracting further subsequences, we can assume that the limits

$$\lim_{m \to \infty} S_t^{\epsilon_m} \bar{u} = \mathcal{S}_t \bar{u} \,, \qquad \lim_{m \to \infty} S_t^{\epsilon'_m} \bar{u} = \mathcal{S}'_t \bar{u}$$

exist in L^1, for all $t \geq 0$ and $\bar{u} \in \mathcal{D}$. By the analysis in Section 13 in [4], both $\mathcal{S}$ and $\mathcal{S}'$ are semigroups of vanishing viscosity solutions. In particular, the necessity part of Lemma 8.1 implies that the map $t \mapsto v(t) \doteq \mathcal{S}_t \bar{v}$ is a viscosity solution of (8.9), while the sufficiency part implies $v(t) = \mathcal{S}'_t v(0)$ for all $t \geq 0$. But this is in contradiction with (8.17), hence (8.16) must hold.

This concludes the proof of uniqueness of the limit.

Appendix A. Computation of the approximated manifold

The travelling profile is a solution to the RFDE

$$-\sigma_m u_\xi + f\big(u(\xi)\big) - f\big(u(\xi-1)\big) = 0.$$

We look for an expansion of the form

$$u(\xi) = u^j + \epsilon a(\xi) + \epsilon^2 b(\xi) + \mathcal{O}(1)\epsilon^3.$$

Substituting this into the equation we obtain

$$-\sigma_m\Big(\epsilon a'(\xi) + \epsilon^2 b'(\xi)\Big) + A\big(u^j\big)\Big(\epsilon\big(a(\xi) - a(\xi-1)\big) + \epsilon^2\big(b(\xi) - b(\xi-1)\big)\Big) + \frac{\epsilon^2}{2} DA\big(u^j\big)\Big(a(\xi)\otimes a(\xi) - a(\xi-1)\otimes a(\xi-1)\Big) + \mathcal{O}(1)\epsilon^3 = 0.$$

At first order in ϵ we get the equation

$$-\sigma_m a'(\xi) + A\big(u^j\big)\Big(a(\xi) - a(\xi-1)\Big) = 0 \qquad \Longrightarrow \qquad a(\xi) = \frac{e^{\beta\xi}-1}{\beta} r_m(u^j), \tag{A.1}$$

where

$$\frac{\sigma_m}{\lambda_m(u^j)} = \frac{1-e^{-\beta}}{\beta}. \tag{A.2}$$

The second order in ϵ gives

$$-\sigma_m b'(\xi) + A\big(u^j\big)\big(b(\xi) - b(\xi-1)\big) + \frac{1}{2} DA\big(u^j\big) r^j_m \otimes r^j_m \left(\frac{\sigma_m}{\beta\lambda_m(u^j)}\Big(e^{2\beta\xi}\big(1+e^{-\beta}\big) - 2e^{\beta\xi}\Big)\right) = 0. \tag{A.3}$$

Using now the relation

$$DA\big(u^j\big) r^j_m \otimes r^j_m + A\big(u^j\big) Dr^j_m r^j_m = \big(D\lambda^j_m r^j_m\big) r^j_m + \lambda^j_m Dr^j_m r^j_m, \tag{A.4}$$

we obtain

$$\Big\langle l^j_n, DA\big(u^j\big) r^j_m \otimes r^j_m \Big\rangle = \begin{cases} D\lambda^j_m r^j_m & n = m \\ \big(\lambda^j_m - \lambda^j_n\big)\langle l^j_n, Dr^j_m r^j_m\rangle & n \neq m \end{cases} \tag{A.5}$$

Projecting (A.3) on $l_m(u^j)$ we obtain

$$-\sigma_m b'_m(\xi) + \lambda^j_m\big(b_m(\xi) - b_m(\xi-1)\big) = -\frac{\sigma_m}{2\lambda^j_m\beta}\big(D\lambda^j_m \bar r_m\big)\Big(e^{2\beta\xi}\big(1+e^{-\beta}\big) - 2e^{\beta\xi}\Big),$$

so that a solution has the form

$$\begin{aligned} b_m(\xi) = {} & \frac{1}{2\lambda^j_m}\big(D\lambda^j_m r^j_m\big)\frac{\sigma_m}{\lambda^j_m}\frac{1+e^{-\beta}}{\beta\big(1-e^{-\beta}\big)^2}\big(e^{\beta\xi}-1\big)^2 \\ & + \frac{1}{\lambda^j_m}\big(D\lambda^j_m r^j_m\big)\frac{\sigma_m}{\lambda^j_m}\frac{1}{\big(1+\beta\big)e^{-\beta}-1}\Big(\xi e^{\beta\xi} - \big(e^{\beta\xi}-1\big)/\beta\Big). \\ & - \sigma_m \sum_{n\neq m} \frac{\big\langle l_n(u^j), Dr_m(u^j) r_m(u^j)\big\rangle}{\lambda_n(u^j)(1+e^{-\beta}) - 2\lambda^j_m}\frac{\big\langle l_m(u_0), r^j_n\big\rangle}{\big\langle l_m(u_0), r^j_m\big\rangle}\left(\frac{e^{\beta\xi}-1}{\beta}\right). \end{aligned} \tag{A.6}$$

Projecting on l^j_n, $n \neq m$, we have

$$-\sigma_m b'_n + \lambda^j_n\big(b_n(\xi) - b_n(\xi-1)\big) = \frac{\sigma_m}{2\lambda^j_m\beta}\big(\lambda^j_n - \lambda^j_m\big)\langle l^j_n, Dr^j_m r^j_m\rangle\Big(e^{2\beta\xi}\big(1+e^{-\beta}\big) - 2e^{\beta\xi}\Big),$$

which admits the solution

$$\begin{aligned} b_n(\xi) = {} & \langle l^j_n, Dr^j_m r^j_m\rangle\frac{\sigma_m}{\lambda_m}\big(\lambda^j_n - \lambda^j_m\big)\frac{1+e^{-\beta}}{2\beta\Big(\lambda^j_n\big(1-e^{-2\beta}\big) - 2\lambda_m\big(1-e^{-\beta}\big)\Big)}\big(e^{2\beta\xi}-1\big) \\ & - \langle l^j_n, Dr^j_m r^j_m\rangle\frac{\sigma_m}{\lambda_m}\frac{1}{\beta\big(1-e^{-\beta}\big)}\big(e^{\beta\xi}-1\big). \end{aligned} \tag{A.7}$$

Finally, using (A.1), (A.2), (A.6) and (A.7) we obtain

(A.8)

$$
\begin{aligned}
b(\xi) = \frac{D\lambda^j_m r^j_m}{\lambda^j_m} & \left(\frac{1+e^{-\beta}}{2\beta^2\big(1-e^{-\beta}\big)} \big(e^{\beta\xi}-1\big)^2 + \frac{1-e^{-\beta}}{\beta\big((1+\beta)e^{-\beta}-1\big)} \Big(\xi e^{\beta\xi} - \big(e^{\beta\xi}-1\big)/\beta \Big) \right) r_m(u^j) \\
& + \sum_{n \neq m} \big\langle l_n(u^j), Dr_m(u^j) r_m(u^j) \big\rangle \left(\frac{\big(\lambda_n(u^j)-\lambda_m(u^j)\big)\big(1+e^{-\beta}\big)}{2\beta^2\Big(\lambda_n(u^j)\big(1+e^{-\beta}\big) - 2\lambda_m(u^j)\Big)} \big(e^{2\beta\xi}-1\big) - \frac{e^{\beta\xi}-1}{\beta^2} \right) r_n(u^j) \\
& - \sigma_m \sum_{n\neq m} \frac{\big\langle l_n(u^j), Dr_m(u^j) r_m(u^j) \big\rangle \big\langle l_m(u_0), r^j_n \big\rangle}{\lambda_n(u^j)\big(1+e^{-\beta}\big) - 2\lambda^j_m} \left(\frac{e^{\beta\xi}-1}{\beta} \right) \frac{r_m(u^j)}{\big\langle l_m(u_0), r_m(u^j)\big\rangle}.
\end{aligned}
$$

Choosing now

$$
\epsilon = - \frac{v^j_m}{\sigma_m \big\langle l_m(u_0), r_m(u^j)\big\rangle},
$$

we obtain (3.18).

Appendix B. Non existence of a good variable w^j depending on the speed σ^j and satisfying (4.13)

Let $w^j = w(u^j, u^{j-1}, u^{j-2})$ satisfy

$$
w^j_t + \lambda(u^j) w^j - \lambda^{j-1} w^{j-1} = 0.
$$

This implies that

$$
\begin{aligned}
\partial_1 w^j \big(f(u^{j-1}) - f(u^j)\big) + \partial_2 w^j \big(f(u^{j-2}) - f(u^{j-1})\big) & \\
+ \partial_3 w^j \big(f(u^{j-3}) - f(u^{j-2})\big) &= \lambda^{j-1} w^{j-1} - \lambda^j w^j.
\end{aligned}
$$

for all u^j, u^{j-1}, u^{j-2}, u^{j-3}. In particular choosing $u^j = u^{j-1} = u^{j-2} = x$, $u^{j-3} = y$, we obtain

$$
\partial_3 w(x,x,x)\big(f(y) - f(x)\big) = \lambda(x) w(x,x,y) - \lambda(x) w(x,x,x),
$$

so that we have

$$
w(x,x,y) = w(x,x,x) + \partial_3 w(x,x,x) \frac{f(y)-f(x)}{\lambda(x)}. \tag{B.1}
$$

In a similar way, if $u^j = u^{j-1} = x$, $u^{j-2} = y$, $u^{j-3} = z$, we conclude that

$$
\partial_2 w(x,x,y)\big(f(y)-f(x)\big) + \partial_3 w(x,x,y)\big(f(z)-f(y)\big) = \lambda(x) w(x,y,z) - \lambda(x) w(x,x,y),
$$

so that using (B.1) we have

$$
\begin{aligned}
w(x,y,z) &= w(x,x,y) + \partial_2 w(x,x,y) \frac{f(y)-f(x)}{\lambda(x)} + \partial_3 w(x,x,y) \frac{f(z)-f(y)}{\lambda(x)} \\
&= w(x,x,y) + \partial_2 w(x,x,y) \frac{f(y)-f(x)}{\lambda(x)} + \partial_3 w(x,x,x) \frac{\lambda(y)}{\lambda(x)} \frac{f(z)-f(y)}{\lambda(x)}.
\end{aligned} \tag{B.2}
$$

It follows that

$$
\partial_3 w(x,y,z) = \partial_3 w(x,x,x) \frac{\lambda(y)\lambda(z)}{\lambda^2(x)},
$$

and that

$$
\begin{aligned}
\partial_2 w(x,y,z) &= \partial_3 w(x,x,y) + \partial_{23} w(x,x,y) \frac{f(y)-f(x)}{\lambda(x)} + \partial_2 w(x,x,y) \frac{\lambda(y)}{\lambda(x)} \\
&\quad + \partial_3 w(x,x,x) \frac{\lambda'(y)}{\lambda(x)} \frac{f(z)-f(y)}{\lambda(x)} - \partial_3 w(x,x,x) \frac{\lambda^2(y)}{\lambda^2(x)} \\
&= \partial_3 w(x,x,y) + \partial_3 w(x,x,x) \frac{\lambda'(x)}{\lambda(x)} \frac{f(y)-f(x)}{\lambda(x)} + \partial_2 w(x,x,y) \frac{\lambda(y)}{\lambda(x)} \\
&\quad + \partial_3 w(x,x,x) \frac{\lambda'(y)}{\lambda(x)} \frac{f(z)-f(y)}{\lambda(x)} - \partial_3 w(x,x,x) \frac{\lambda^2(y)}{\lambda^2(x)}.
\end{aligned}
$$

If now $x = y$ we obtain

$$\partial_2 w(x,x,z) = \partial_2 w(x,x,x) + \partial_3 w(x,x,x) \frac{\lambda'(x)}{\lambda(x)} \frac{f(z) - f(x)}{\lambda(x)},$$

so that we can write

$$w(x,y,z) = a(x) + c(x)\frac{f(y) - f(x)}{\lambda(x)} + \left(b(x) + c(x)\frac{\lambda'(x)}{\lambda(x)}\frac{f(y) - f(x)}{\lambda(x)} \right) \frac{f(y) - f(x)}{\lambda(x)} + c(x)\frac{\lambda(y)}{\lambda(x)}\frac{f(z) - f(y)}{\lambda(x)}. \tag{B.3}$$

We have defined the quantities $a(x) \doteq w(x,x,x)$, $b(x) \doteq \partial_2 w(x,x,x)$, $c(x) = \partial_3 w(x,x,x)$. Using now $u^j = x$, $u^{j-1} = u^{j-2} = u^{j-3} = y$, we obtain

$$\partial_1 w(x,y,y)\big(f(y) - f(x)\big) = \lambda(y) w(y,y,y) - \lambda(x) w(x,y,y),$$

so that, taking the x derivative for $x = y$ we conclude

$$-\lambda'(x) w(x,x,x) = 0 \quad \Longrightarrow \quad a(x) = 0,$$

since we are assuming that $\lambda' \neq 0$. The two above equations give

$$\frac{\partial_1 w(x,y,y)}{w(x,y,y)} = \frac{\lambda(x)}{f(x) - f(y)},$$

so that $w(x,y,y) = k(f(x) - f(y))$. Using (B.3), we obtain the relation

$$k\big(f(x) - f(y)\big) = \big(c(x) + b(x)\big)\frac{f(y) - f(x)}{\lambda(x)} + c(x)\frac{\lambda'(x)}{\lambda(x)} \left(\frac{f(y) - f(x)}{\lambda(x)}\right)^2.$$

This implies that $c(x) = 0$, so we conclude finally that $w^j = k(f(u^{j-1}) - f(u^j)) = kv^j$.

Note that a basic assumption is that $\lambda' \neq 0$, otherwise there are infinite functions w due to linearity. Note moreover that the same proof holds for functions of the form $w(u^j, \dots, u^{j-k})$.

Appendix C. Computation of the terms due to wrong speed choice

Let $s(v^j_m, v^{j-1}_m)$ be the function

$$\begin{aligned} s\big(v^j_m, v^{j-1}_m\big) &\doteq v^{j-1}_m \Big(\hat{A}^{j-1} \tilde{r}^{j-1}_m - \tilde{\lambda}^{j-1}_m \big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big)\Big) \\ &= v^{j-1}_m \Big(\hat{A}\big(u^{j-1}\big) \tilde{r}_m\big(u^{j-1}, v^{j-1}_m, \sigma^{j-1}_m\big) - \big\langle l_m(u_0), A^{j-1}\tilde{r}^{j-1}_m \big\rangle \big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big)\Big). \end{aligned} \tag{C.1}$$

Note that we can write

$$v^{j-1}_m \Big(\hat{A}^{j-1} \tilde{r}^{j-1}_m - \tilde{\lambda}^{j-1}_m \big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big)\Big) - \hat{v}^{j-1}_m \Big(\hat{A}^{j-1} \hat{r}^{j-1}_m - \hat{\lambda}^{j-1}_m \big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big)\Big) = s\big(v^j_m, v^{j-1}_m\big) - s\big(v^j_m, \hat{v}^{j-1}_m\big),$$

where v^{j-1}_m is a function of v^j_m. We compute the expansion of s near $v^i_m = v^{j-1}_m = 0$.

The first derivatives of s are

$$\begin{aligned} \frac{\partial s}{\partial v^j_m} &= v^{j-1}_m D\hat{A}^{j-1}\tilde{r}^{j-1}_m \otimes \big(\hat{A}^{j-1}\big)^{-1}\big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big) + v^{j-1}_m \hat{A}^{j-1} D\tilde{r}^{j-1}_m \big(\hat{A}^{j-1}\big)^{-1}\big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big) \\ &\quad - v^{j-1}_m \Big\langle l_m(u_0), D\hat{A}^{j-1}\tilde{r}^{j-1}_m \otimes \big(\hat{A}^{j-1}\big)^{-1}\big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big)\Big\rangle \big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big) \\ &\quad - v^{j-1}_m \Big\langle l_m(u_0), \hat{A}^{j-1} D\tilde{r}^{j-1}_m \big(\hat{A}^{j-1}\big)^{-1}\big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big)\Big\rangle \big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big) \\ &\quad - v^{j-1}_m \Big\langle l_m(u_0), \hat{A}^{j-1}\tilde{r}^{j-1}_m \Big\rangle \big(2\tilde{r}^j_{m,v} + v^j_m \tilde{r}^j_{m,vv}\big). \end{aligned}$$

$$\frac{\partial s}{\partial v^{j-1}_m} = \hat{A}^{j-1}\big(\tilde{r}^{j-1}_m + v^{j-1}_m \tilde{r}^{n-1}_{i,v}\big) - \Big\langle l_m(u_0), \hat{A}^{j-1}\big(\tilde{r}^{j-1}_m + v^{j-1}_m \tilde{r}^{j-1}_{m,v}\big)\Big\rangle \big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big),$$

where we used the relation

$$\frac{\partial \hat{u}^{j-1}}{\partial v^{j-1}_m} = \big(\hat{A}^{j-1}\big)^{-1}\big(\tilde{r}^j_m + v^j_m \tilde{r}^j_{m,v}\big).$$

We obtain with easy computation

$$\frac{\partial s}{\partial v^j_m}\bigg|_{v=0} = 0, \qquad \frac{\partial s}{\partial v^{j-1}_m}\bigg|_{v=0} = -A^j r^j_m + \Big\langle l_m(u_0), A^j r^j_m \Big\rangle r^j_m = 0.$$

Computing now the second derivatives for $v^j_m = v^{j-1}_m = 0$ and using the estimates (3.25), one gets

$$\frac{\partial^2 s}{\big(\partial v^j_m\big)^2}\bigg|_{v=0} = 0,$$

$$\begin{aligned}
\frac{\partial^2 s}{\partial v^j_m \partial v^{j-1}_m}\bigg|_{v=0} &= DA^j r^j_m \otimes \big(A^j\big)^{-1} r^j_m + A^j Dr^j_m \big(A^j\big)^{-1} r^j_m \\
&\quad - \Big\langle l_m(u_0), DA^j r^j_m \otimes \big(A^j\big)^{-1} r^j_m + A^j Dr^j_m \big(A^j\big)^{-1} r^j_m \Big\rangle r^j_m - 2\lambda^j_m \, \tilde r^j_{m,v}\big|_{v=0} \\
&= \big(1 + e^{-\beta^j_m}\big) \sum_{j \neq i} \left(\frac{\lambda^j_n}{\lambda^j_n\big(1 + e^{-\beta^j_m}\big) - 2\lambda^j_m} \right) \Big(r^n_j - \big\langle l_m(u_0), r^n_j \big\rangle r^j_m \Big),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 s}{\big(\partial v^{j-1}_m\big)^2}\bigg|_{v=0} &= 2A^j \tilde r^j_{m,v} - 2\big\langle l_m(u_0), A^j \tilde r^j_{m,v} \big\rangle r^j_m \\
&= \; -2 \sum_{j \neq i} \left(\frac{\lambda^j_n}{\lambda^j_n\big(1 + e^{-\beta^j_m}\big) - 2\lambda^m_j} \right) \Big(r^n_j - \big\langle l_m(u_0), r^n_j \big\rangle r^j_m \Big).
\end{aligned}$$

Using the same computations we obtain

$$\begin{aligned}
\frac{\partial^2 s\big(v^j_m, \hat v^{j-1}_m\big)}{\big(\partial v^j_m\big)^2}\bigg|_{v=0} &= \frac{\partial^2 s}{\big(\partial v^j_m\big)^2} + 2 \frac{\partial^2 s}{\partial v^j_m \partial \hat v^{j-1}_m} \frac{\partial \hat v^{j-1}_m}{\partial v^j_m} + \frac{\partial^2 s}{\big(\partial \hat v^{j-1}_m\big)^2} \left(\frac{\partial \hat v^{j-1}_m}{\partial v^{j,m}} \right)^2 + \frac{\partial s}{\partial \hat v^{j-1}_m} \frac{\partial^2 \hat v^{j-1}_m}{\big(\partial v^j_m\big)^2} \\
&= 2e^{-\beta^j_m} \sum_{j \neq i} \left(\frac{\lambda^j_m}{\lambda^j_n\big(1 + e^{-\beta^j_m}\big) - 2\lambda^j_m} \right) \Big(r^n_j - \big\langle l_m(u_0), r^n_j \big\rangle r^j_m \Big),
\end{aligned}$$

so that we can write

(C.2) $s\big(v^j_m, v^{j-1}_m\big) - s\big(v^j_m, \hat v^{j-1}_m\big)$

$$\begin{aligned}
&= \Big(v^j_m - v^{j-1}_m \Big) \Big(e^{-\beta^j_m} v^j_m - v^{j-1}_m \Big) \sum_{j \neq i} \left(\frac{2\lambda^n_i \lambda^n_j}{2\big(\lambda^n_j - \lambda^n_i\big) - \lambda^n_j g(\sigma^n)\sigma^n/\lambda_i} \right) \Big(r^n_j - \big\langle l_m(u_0), r^n_j \big\rangle r^j_m \Big) \\
&\quad + \mathcal{O}(1) \Big(\big|v^j_m\big| + \big|v^{j-1}_m\big| \Big)^2 \Big(\hat v^{j-1}_m - v^{j-1}_m \Big) \\
&= \Big(v^j_m - v^{j-1}_m \Big) \Big(\hat v^{j-1}_m - v^{j-1}_m \Big) \sum_{j \neq i} \left(\frac{2\lambda^n_i \lambda^n_j}{2\big(\lambda^n_j - \lambda^n_i\big) - \lambda^n_j g(\sigma^n)\sigma^n/\lambda_i} \right) \Big(r^n_j - \big\langle l_m(u_0), r^n_j \big\rangle r^j_m \Big) \\
&\quad + \mathcal{O}(1) \Big(\big|v^j_m\big| + \big|v^{j-1}_m\big| \Big)^2 \Big(\hat v^{j-1}_m - v^{j-1}_m \Big).
\end{aligned}$$

Appendix D. Computation of the equation satisfied by ι^i_m

Let ζ^j_m be the solution to the equation

$$\zeta^j_{m,t} + \tilde\lambda(u^j, v^j_m, \sigma^j_m)\zeta^j_m - \tilde\lambda(u^{j-1}, v^{j-1}_m, \sigma^{j-1}_m)\zeta^{j-1}_m = \mu^j_m(t). \tag{D.1}$$

Following the computations of Section 4, define implicitly now the variable ς^j_m by

(D.2)

$$\frac{\zeta^{j-1}_m}{\zeta^j_m} = \beta^j_m = g^{-1}\left(\frac{g(\varsigma^j_m)}{1 + \psi(\varsigma^j_m)\big(\lambda^j_m - \lambda_m(u_0)\big)/\lambda_m(u_0)} \right) + v^j_m \tilde q_m\Big(u^j, v^j_m, \lambda(u_0)\big(1 + (g(\varsigma^j_m) - 1)\psi(\varsigma^j_m)\big) \Big),$$

where $\tilde{q}_m$ is defined in (6.7) and ψ is the cutoff function (4.6). Let ι^j_m be the function

$$\begin{aligned}\iota^j_m &= \zeta^j_m \varsigma^j_m \\ &= \zeta^j_m g^{-1}\left(\left(1+\frac{\lambda^j_m-\lambda_m(u_0)}{\lambda_m(u_0)}\psi(\varsigma^j_m)\right)g(\beta^j_m)\right)+\zeta^j_m v^j_m \tilde{\kappa}_m\big(u^j,v^j,\beta^j_m\big) \\ &= \zeta^j_m h^j_m\big(u^j,v^j,\beta^j_m\big)+\zeta^j_m v^j_m\tilde{\kappa}_m\big(u^j,v^j,\beta^j_m\big).\end{aligned} \tag{D.3}$$

As in the scalar case, one has the estimate $|\iota^j_m| \leq |\zeta^j_m| + |\zeta^{j-1}_m|$.

We now compute the equation satisfied by ι^j_m under the assumption that $\psi > 0$, i.e., $|\varsigma^j_m - 1| \leq 3\delta_1$ and β^j_m is bounded. One can check that the following computations are valid even if $\psi = 0$, i.e., $\iota^j_m = \zeta^{j-1}_m + \zeta^j_m v^j_m \kappa^j_m$. Following Section 4 we have

$$\begin{aligned}\iota^j_{m,t} + \tilde{\lambda}^j_m \iota^j_m - \tilde{\lambda}^{j-1}_m \iota^{j-1}_m &= \zeta^j_m \varsigma^j_t + \tilde{\lambda}^{j-1}_m \zeta^{j-1}_m\big(\varsigma^j_m - \varsigma^{j-1}_m\big) + \mathcal{O}(1)\mu^j_m \\ &= \zeta^j_m \frac{\partial h^j_m}{\partial\beta}\beta^j_t + \zeta^j_m v^j_m \frac{\partial h^j_m}{\partial u} + \zeta^j_m\big(\tilde{\lambda}^{j-1}_m v^{j-1}_m - \tilde{\lambda}^j_m v^j_m\big)\frac{\partial h^j_m}{\partial v^j_m} \\ &\quad + \zeta^j_m\Big(\kappa^j_{m,u}\big(v^j_m\big)^2 + \big(\kappa^j_m + v^j_m\kappa^j_{m,v}\big)\big(\tilde{\lambda}^{j-1}_m v^{j-1}_m - \tilde{\lambda}^j_m v^j_m\big) + v^j_m\kappa^j_{m,\beta}\beta^j_{m,t}\Big) \\ &\quad + \tilde{\lambda}^{j-1}_m\zeta^{j-1}_m\Big(h_m\big(u^j,v^j_m,\beta^j_m\big) - h_m\big(u^{j-1},v^{j-1}_m,\beta^{j-1}_m\big) + v^j_m\kappa^j_m - v^{j-1}_m\kappa^{j-1}_m\Big) \\ &\quad + \mathcal{O}(1)\mu^j_m + \mathcal{O}(1)\omega^j_m \\ &= \tilde{\lambda}^{j-1}_m\zeta^{j-1}_m\left(\frac{\partial h}{\partial\beta}\big(u^j,v^j_m,\beta^j_m\big)\Big(\beta^{j-1}_m - \beta^j_m\Big) + h\big(u^j,v^j_m,\beta^j_m\big) - h\big(u^j,v^j_m,\beta^{j-1}_m\big)\right) \\ &\quad + Q\big(u^j,v^j,v^{j-1},v^{j-1}_m,\zeta^j_m,\zeta^{j-1}_m,\zeta^{j-2}_m\big) + \mathcal{O}(1)\mu^j_m + \mathcal{O}(1)\mu^{j-1}_m \\ &\quad + \mathcal{O}(1)\omega^j_m(t),\end{aligned} \tag{D.4}$$

where Q denotes a second order polynomial in v, ζ. We have used the following computations:

$$\begin{aligned}\zeta^j_m\beta^j_{m,t} = \zeta^j_m\left(\frac{\zeta^{j-1}_m}{\zeta^j_m}\right)_t &= \zeta^j_m\frac{\tilde{\lambda}^{j-2}_m\zeta^{j-2}_m - \tilde{\lambda}^{j-1}_m\zeta^{j-1}_m}{\zeta^j_m} - \zeta^{j-1}_m\frac{\tilde{\lambda}^{j-1}_m\zeta^{j-1}_m - \tilde{\lambda}^j_m\zeta^j_m}{\zeta^j_m} + \mu^{j-1}_m - \frac{\zeta^{j-1}_m}{\zeta^j_m}\mu^j_m \\ &= \big(\tilde{\lambda}^{j-2}_m - \tilde{\lambda}^{j-1}_m\big)\zeta^{j-2}_m + \tilde{\lambda}^{j-1}_m\zeta^{j-1}_m\big(\beta^{j-1}_m - \beta^j_m\big) - \big(\tilde{\lambda}^{j-1}_m - \tilde{\lambda}^j_m\big)\zeta^{j-1}_m \\ &\quad + \mathcal{O}(1)\mu^j_m + \mu^{j-1}_m \qquad \text{(because } \varsigma^j_m \text{ is bounded)} \\ &= \tilde{\lambda}^{j-1}_m\zeta^{j-1}_m\big(\beta^{j-1}_m - \beta^j_m\big) + \mathcal{O}(1)v^{j-2}_m\zeta^{j-2}_m + \mathcal{O}(1)\sum_n\big|v^{j-1}_n\big|\zeta^{j-2}_m \\ &\quad + \mathcal{O}(1)v^{j-1}_m\zeta^{j-1}_m + \mathcal{O}(1)\sum_n\big|v^j_n\big|\zeta^{j-1}_m + \mathcal{O}(1)\mu^j_m + \mu^{j-1}_m,\end{aligned}$$

$$\begin{aligned}h\big(u^{j-1},v^{j-1}_m,\beta^{j-1}_m\big) - h\big(u^j,v^j_m,\beta^{j-1}_m\big) &= \mathcal{O}(1)\big(u^{j-1}-u^j\big) + \mathcal{O}(1)\big(v^{j-1}_m - v^j_m\big) \\ &= \mathcal{O}(1)\sum_n\big|v^j_n\big| + \mathcal{O}(1)v^{j-1}_m.\end{aligned}$$

Since $\partial h/\partial u$, $\partial h/\partial v$, $\partial\kappa/\partial\beta$ are different from 0 only when β^j is close to 1, then the polynomial Q has certainly smooth coefficients. Moreover it is easy to check that Q is linear in ζ.

We first study the terms which are of first order w.r.t. v, ζ. As in Section 4 we have

$$\tilde{\lambda}^{j-1}_m\zeta^{j-1}_m\left(\frac{\partial h}{\partial\alpha}\big(u^j,v^j_m,\beta^j_m\big)\Big(\beta^{j-1}_m - \beta^j_m\Big) + h\big(u^j,v^j_m,\beta^j_m\big) - h\big(u^j,v^j_m,\beta^{j-1}_m\big)\right) = \tag{D.5}$$

$$\begin{cases}\mathcal{O}(1)\mathcal{V}(u)\zeta^{j-1}_m\Big(\beta^j_m - \beta^{j-1}_m\Big)^2 & \big|\varsigma^j_m - 1\big|, \big|\varsigma^{j-1}_m - 1\big| \leq 5\delta_1 \\ \mathcal{O}(1)\mathcal{V}(u)\zeta^{j-2}_m + \mathcal{O}(1)\mathcal{V}(u)\zeta^{j-1}_m & \big|\varsigma^j_m - 1\big| \geq 5\delta_1, \big|\varsigma^{j-1}_m - 1\big| \leq 3\delta_1 \text{ or} \\ & \qquad \big|\varsigma^{j-1}_m - 1\big| \geq 5\delta_1, \big|\varsigma^{j-1}_m - 1\big| \leq 3\delta_1 \\ 0 & \text{otherwise}\end{cases}$$

Using (D.2) we rewrite (D.4) as

$$
\begin{aligned}
\iota^j_{m,t} + \tilde\lambda^j_m \iota^j_m - \tilde\lambda^{j-1}_m \iota^{j-1}_m = {}& \mathcal{O}(1)\mathcal{V}(u)\zeta^{j-1}_m \big(\varsigma^j_m - \varsigma^{j-1}_m\big)^2 \chi\Big\{\big|\varsigma^j_m - 1\big|, \big|\varsigma^{j-1}_m - 1\big| \leq 5\delta_1\Big\} \\
&+ \mathcal{O}(1)\mathcal{V}(u)\big(|\zeta^{j-1}_m| + |\zeta^{j-2}_m|\big)\chi\Big\{\big|\varsigma^j - 1\big| \geq 5\delta_1 \big|\varsigma^{j-1} - 1\big| \leq 3\delta_1 \text{ or viceversa}\Big\} \\
&+ Q\big(u^j, v^j, v^{j-1}, v^{j-2}, \zeta^j_m, \zeta^{j-1}_m, \zeta^{j-2}_m\big) + \mathcal{O}(1)\mu^j_m + \mathcal{O}(1)\mu^{j-1}_m + \mathcal{O}(1)\omega^j_m(t).
\end{aligned} \tag{D.6}
$$

We now consider the form of the term Q, linear in ζ_m:

$$
Q\big(u^j, v, \zeta_m\big) = \sum_{r_1,r_2=0}^{2} c_{r_1 r_2} v^{j-r_1}_m \zeta^{j-r_2}_m + \sum_{n \neq m} \sum_{r_1,r_2=0}^{2} d^n_{r_1 r_2} v^{j-r_1}_n \zeta^{j-r_2}_m, \tag{D.7}
$$

where the coefficients c, d are smooth functions depending on u^j, v and ζ_m.

If ζ is proportional to v and v is an exact travelling profile, then $\omega = \mu = 0$ and the right hand side of (D.6) is equal to zero. Note that this implies also that $\varsigma^j_m = \varsigma^{j-1}_m = s^j_m$, where s^j_m is defined in (6.7). Denote by $\hat\zeta^{j-1}_m$, $\hat\zeta^{j-2}_m$ the quantities

$$
\hat\zeta^{j-1}_m \doteq \zeta^j \tilde p_m\big(u^j, v^j_m, \sigma^j_m\big), \qquad \hat\zeta^{j-2}_m \doteq \hat\zeta^{j-1}_m \tilde p_m\big(\hat u^{j-1}, \hat v^{j-1}_m, \sigma^j_m\big). \tag{D.8}
$$

Observe that, if $\varsigma \leq \delta_1$, from (D.2) it follows that

$$
\zeta^{j-1}_m \doteq \zeta^j \tilde p_m\big(u^j, v^j_m, \lambda_m(u_0) g(\varsigma^j_m)\big), \qquad \zeta^{j-2}_m \doteq \zeta^{j-1}_m \tilde p_m\big(\hat u^{j-1}, \hat v^{j-1}_m, \lambda_m(u_0) g(\varsigma^j_m)\big). \tag{D.9}
$$

Since $Q(u^j, \hat v, \hat\zeta_m) = 0$, we can write

$$
\begin{aligned}
Q\big(u^j, v, \zeta_m\big) &= Q\big(u^j, v, \zeta_m\big) - Q\big(u^j, \hat v, \hat\zeta_m\big) \\
&= \sum_{r_1,r_2=0}^{2} \big(c_{r_1 r_2} - \hat c_{r_1,r_2}\big) \hat v^{j-r_1}_m \hat\zeta^{j-r_2}_m + \sum_{r_1,r_2=0}^{2} c_{r_1 r_2} \big(v^{j-r_1}_m \zeta^{j-r_2}_m - \hat v^{j-r_1}_m \hat\zeta^{j-r_1}_m\big) \\
&\quad + \sum_{n \neq m} \sum_{r_1,r_2=0}^{2} d^n_{r_1 r_2} v^{j-r_1}_n \zeta^{j-r_2}_m \\
&= \mathcal{O}(1) v^j_m \big(\zeta^{j-1}_m - \hat\zeta^{j-1}_m\big) + \mathcal{O}(1)\zeta^j_m \big(v^{j-1}_m - \hat v^{j-1}_m\big) + \mathcal{O}(1)\big(v^{j-1}_m \zeta^{j-1}_m - \hat v^{j-1}_m \hat\zeta^{j-1}_m\big) \\
&\quad + \mathcal{O}(1) v^j_m \big(\zeta^{j-2}_m - \hat\zeta^{j-2}_m\big) + \mathcal{O}(1)\zeta^j_m \big(v^{j-2}_m - \hat v^{j-2}_m\big) + \mathcal{O}(1)\big(v^{j-1}_m \zeta^{j-2}_m - \hat v^{j-1}_m \hat\zeta^{j-2}_m\big) \\
&\quad + \mathcal{O}(1)\big(v^{j-2}_m \zeta^{j-1}_m - \hat v^{j-2}_m \hat\zeta^{j-1}_m\big) + \mathcal{O}(1)\big(v^{j-2}_m \zeta^{j-2}_m - \hat v^{j-2}_m \hat\zeta^{j-2}_m\big) \\
&\quad + \mathcal{O}(1) \sum_{n \neq m} \sum_{r_1,r_2=0}^{2} \big|v^{j-r_1}_n\big| \big|\zeta^{j-r_2}_m\big|.
\end{aligned}
$$

Using similar computation to that in Section 4 and following a similar approach to the vanishing viscosity case [4], we estimate the above terms as

$$
\Big|v^j_m \big(\zeta^{j-1}_m - \hat\zeta^{j-1}_m\big)\Big| \leq \Big|v^j_m \zeta^{j-1}_m - v^{j-1}_m \zeta^j_m\Big| + \Big|\zeta^j_m \big(v^{j-1}_m - \hat v^{j-1}_m\big)\Big|,
$$

$$
\begin{aligned}
\Big|\zeta^j_m \big(v^{j-1}_m - \hat v^{j-1}_m\big)\Big| \leq {}& \mathcal{O}(1)\big(\zeta^j_m\big)^2 \chi\Big\{\big|\zeta^{j-1}_m / \zeta^j_m - 1\big| \geq 4\delta_1/5\Big\} \\
&+ \mathcal{O}(1)\Big(\big(v^j_m\big)^2 + \big(v^{j-1}_m\big)^2\Big)\chi\Big\{\big|v^{j-1}_m / v^j_m - 1\big| \geq 9\delta_1/10\Big\} \\
&+ \mathcal{O}(1)\big|v^j_m \zeta^{j-1}_m - v^{j-1}_m \zeta^j_m\big| / \delta_1,
\end{aligned}
$$

where we used the computation $2ab \leq a^2 + b^2$ and, for $|\zeta^{j-1}/\zeta^j_m - 1| < 4\delta_1/5$, $|v^{j-1}_m / v^j_m - 1| \geq 9\delta_1/10$,

$$
\Big|\zeta^j_m v^{j-1}_m - \zeta^{j-1}_m v^j_m\Big| \geq \mathcal{O}(1)\delta_1 \big|\zeta^j_m v^{j-1}_m\big| + \mathcal{O}(1)\delta_1 \big|\zeta^{j-1}_m v^j_m\big| \geq \mathcal{O}(1)\delta_1 \Big|\zeta^j_m \big(v^{j-1}_m - \hat v^{j-1}_m\big)\Big|, \tag{D.10}
$$

$$
\begin{aligned}
\Big|v_m^{j-1}\zeta_m^{j-1} - \hat{v}_m^{j-1}\hat{\zeta}_m^{j-1}\Big| &\le \Big|\hat{v}_m^{j-1}\big(\zeta_m^{j-1} - \hat{\zeta}_m^{j-1}\big)\Big| + \Big|\big(v_m^{j-1} - \hat{v}_m^{j-1}\big)\zeta_m^{j-1}\Big| \\
&\le \mathcal{O}(1)\Big|v_m^j\big(\zeta_m^{j-1} - \hat{\zeta}_m^{j-1}\big)\Big| \\
&\quad + \Big|\big(v_m^{j-1} - \hat{v}_m^{j-1}\big)\zeta_m^{j-1}\Big|\chi\Big\{\big|\zeta_m^{j-1}/\zeta_m^j - 1\big| \ge 4\delta_1/5\Big\} \\
&\quad + \mathcal{O}(1)\Big|\big(v_m^{j-1} - \hat{v}_m^{j-1}\big)\zeta_m^j\Big|\chi\Big\{\big|\zeta_m^{j-1}/\zeta_m^j - 1\big| \le 4\delta_1/5\Big\} \\
&\le \mathcal{O}(1)\Big(\big(v_m^j\big)^2 + \big(v_m^{j-1}\big)^2\Big)\chi\Big\{\big|v_m^{j-1}/v_m^j - 1\big| \ge 4\delta_1/5\Big\} \\
&\quad + \mathcal{O}(1)\Big(\big(\zeta_m^j\big)^2 + \big(\zeta_m^{j-1}\big)^2\Big)\chi\Big\{\big|\zeta_m^{j-1}/\zeta_m^j - 1\big| \ge 3\delta_1/5\Big\} \\
&\quad + \mathcal{O}(1)\big|v_m^j\zeta_m^{j-1} - v_m^{j-1}\zeta_m^j\big|/\delta_1,
\end{aligned}
$$

$$
\begin{aligned}
\Big|v_m^j\big(\zeta_m^{j-2} - \hat{\zeta}_m^{j-2}\big)\Big| &\le \Big|v_m^j\big(\hat{\zeta}_m^{j-1} - \zeta_m^{j-1}\big)\tilde{p}_m^{j-1}\Big| + \Big|v_m^j\big(\zeta_m^{j-2} - \zeta_m^{j-1}\tilde{p}_m(u^{j-1}, v_m^{j-1}, \sigma_m^{j-1})\big)\Big| \\
&\quad + \Big|v_m^j\zeta_m^{j-1}\big(\tilde{p}_m(u^{j-1}, v_m^{j-1}, \sigma_m^{j-1}) - \tilde{p}_m(\hat{u}^{j-1}, \hat{v}_m^{j-1}, \sigma_m^j)\big)\Big| \\
&\le \mathcal{O}(1)\Big|v_m^j\big(\zeta_m^{j-1} - \hat{\zeta}_m^{j-1}\big)\Big| \\
&\quad + \mathcal{O}(1)\Big|v_m^{j-1}\big(\zeta_m^{j-2} - \zeta_m^{j-1}\tilde{p}_m(u^{j-1}, v_m^{j-1}, \sigma_m^{j-1})\big)\Big|\chi\Big\{\big|v_m^{j-1}/v_m^j - 1\big| \le \delta_1\Big\} \\
&\quad + \mathcal{O}(1)\big|v_m^j\zeta_m^{j-1}\big|\chi\Big\{\big|\zeta_m^{j-2}/\zeta_m^{j-1} - 1\big| \le 4\delta_1/5, \big|v_m^{j-1}/v_m^j - 1\big| \ge \delta_1\Big\} \\
&\quad + \mathcal{O}(1)\big|v_m^j\big|\Big(\big|\zeta_m^{j-2}\big| + \big|\zeta_m^{j-1}\big|\Big)\chi\Big\{\big|\zeta_m^{j-2}/\zeta_m^{j-1} - 1\big| \ge 4\delta_1/5, \big|v_m^{j-1}/v_m^j - 1\big| \ge \delta_1\Big\} \\
&\quad + \mathcal{O}(1)\sum_{n\neq m}\big|v_n^j\big|\big|\zeta_m^{j-1}\big| + \mathcal{O}(1)\big|v_m^j\zeta_m^{j-1}\big(v_m^{j-1} - \hat{v}_m^{j-1}\big)\Big| \\
&\quad + \mathcal{O}(1)\Big|v_m^j\zeta_m^{j-1}\big(\sigma_m^j - \sigma_m^{j-1}\big)\Big|\chi\Big\{\big|\zeta_m^{j-2}/\zeta_m^{j-1} - 1\big| \le 4\delta_1/5, \big|v_m^{j-1}/v_m^j - 1\big| \le \delta_1\Big\} \\
&\le \mathcal{O}(1)\Big|v_m^j\big(\zeta_m^{j-1} - \hat{\zeta}_m^{j-1}\big)\Big| \\
&\quad + \mathcal{O}(1)\Big|v_m^{j-1}\big(\zeta_m^{j-2} - \zeta_m^{j-1}\tilde{p}(u^{j-1}, v^{j-1}, \sigma_m^{j-1})\big)\Big|\chi\Big\{\big|v_m^{j-1}/v_m^j - 1\big| \le \delta_1\Big\} \\
&\quad + \mathcal{O}(1)\Big(\big(v_m^j\big)^2 + \big(v_m^{j-1}\big)^2\Big)\chi\Big\{\big|v_m^{j-1}/v_m^j - 1\big| \ge \delta_1\Big\} \\
&\quad + \mathcal{O}(1)\Big(\big(\zeta_m^{j-1}\big)^2 + \big(\zeta_m^{j-2}\big)^2\Big)\chi\Big\{\big|\zeta_m^{j-2}/\zeta_m^{j-1} - 1\big| \ge 4\delta_1/5\Big\} \\
&\quad + \mathcal{O}(1)\Big|\zeta_m^{j-1}v_m^j - v_m^{j-1}\zeta_m^j\Big| + \mathcal{O}(1)\Big|\zeta_m^{j-1}w_m^j - w_m^{j-2}\zeta_m^j\Big| + \mathcal{O}(1)\sum_{n\neq m}\big|v_n^j\big|\big|\zeta_m^{j-1}\big|,
\end{aligned}
$$

where, for $|v_m^{j-1}/v_m^j - 1| \le \delta_1$, we have used the computation

$$
\begin{aligned}
v_m^j\zeta_m^{j-1}\big(\sigma_m^j - \sigma_m^{j-1}\big) &= \mathcal{O}(1)\big(v_m^j\zeta_m^{j-1} - v_m^{j-1}\zeta_m^j\big) + \mathcal{O}(1)v_m^{j-1}\zeta_m^j\Big(s_m^j - s_m^{j-1}\Big) \\
&= \mathcal{O}(1)\big(v_m^{j-1}\zeta_m^j - \zeta_m^{j-1}v_m^j\big)\big(1 + |s_m^j|\big) + \mathcal{O}(1)\Big(\zeta_m^{j-1}v_m^js_m^j - \zeta_m^jv_m^{j-1}s_m^{j-1}\Big) \\
&= \mathcal{O}(1)\big(v_m^{j-1}\zeta_m^j - \zeta_m^{j-1}v_m^j\big) + \mathcal{O}(1)\big(\zeta_m^{j-1}w_m^j - \zeta_m^jw_m^{j-1}\big),
\end{aligned}
\tag{D.11}
$$

$$\begin{aligned}
\Big|\zeta^j_m\big(v^{j-2}_m-\hat v^{j-2}_m\big)\Big| \le{}& \Big|\zeta^j_m\big(v^{j-2}_m-v^{j-1}_m\tilde p_m(u^{j-1},v^{j-1}_m,\sigma^{j-1}_m)\big)\Big| \\
&+\Big|\zeta^j_m\big(v^{j-1}_m\tilde p(u^{j-1},v^{j-1}_m,\sigma^{j-1}_m)-\hat v^{j-1}_m\tilde p(\hat u^{j-1},\hat v^{j-1}_m,\sigma^j_m)\big)\Big| \\
\le{}& \mathcal{O}(1)\big|\zeta^{j-1}_m\big(v^{j-2}_m-v^{j-1}_m\tilde p_m(u^{j-1},v^{j-1}_m,\sigma^{j-1}_m)\big)\big|\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m-1\big|\le\delta_1\Big\} \\
&+\mathcal{O}(1)\Big(\big(\zeta^j_m\big)^2+\big(\zeta^{j-1}_m\big)^2\Big)\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m-1\big|\ge\delta_1\Big\} \\
&+\mathcal{O}(1)\Big(\big(v^{j-1}_m\big)^2+\big(v^{j-2}_m\big)^2\Big)\chi\Big\{\big|v^{j-2}_m/v^{j-1}_m-1\big|\ge 4\delta_1/5\Big\} \\
&+\mathcal{O}(1)\Big|\zeta^j_m\big(v^{j-1}_m-\hat v^{j-1}_m\big)\Big|+\mathcal{O}(1)\Big|\zeta^j_m v^{j-1}_m-\zeta^{j-1}_m v^j_m\Big| \\
&+\Big|v^j_m\zeta^{j-1}_m\big(\tilde p_m(u^{j-1},v^{j-1}_m,\sigma^{j-1}_m)-\tilde p_m(\hat u^{j-1},\hat v^{j-1}_m,\sigma^j_m)\big)\Big|,
\end{aligned}$$

$$\begin{aligned}
\Big|v^{j-1}_m\zeta^{j-2}_m-\hat v^{j-1}_m\hat\zeta^{j-2}_m\Big| \le{}& \Big|\hat v^{j-1}_m\big(\zeta^{j-2}_m-\hat\zeta^{j-2}_m\big)\Big|+\Big|\zeta^{j-2}_m\big(v^{j-1}_m-\hat v^{j-1}_m\big)\Big| \\
\le{}& \mathcal{O}(1)\Big|v^j_m\big(\zeta^{j-2}_m-\hat\zeta^{j-2}_m\big)\Big| \\
&+\Big|\zeta^{j-2}_m\big(v^{j-1}_m-\hat v^{j-1}_m\big)\Big|\chi\Big\{\big|\zeta^{j-2}_m/\zeta^{j-1}_m-1\big|\ge\delta_1\Big\} \\
&+\mathcal{O}(1)\Big|\zeta^{j-1}_m\big(v^{j-1}_m-\hat v^{j-1}_m\big)\Big|\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m-1\big|\ge\delta_1\Big\} \\
&+\mathcal{O}(1)\Big|\zeta^j_m\big(v^{j-1}_m-\hat v^{j-1}_m\big)\Big|,
\end{aligned}$$

$$\begin{aligned}
\Big|\zeta^{j-1}_m v^{j-2}_m-\hat\zeta^{j-1}_m\hat v^{j-2}_m\Big| \le{}& \Big|\hat v^{j-2}_m\big(\zeta^{j-1}_m-\hat\zeta^{j-1}_m\big)\Big|+\Big|\zeta^{j-1}_m\big(v^{j-2}_m-\hat v^{j-2}_m\big)\Big| \\
\le{}& \mathcal{O}(1)\Big|v^j_m\big(\zeta^{j-1}_m-\hat\zeta^{j-1}_m\big)\Big|\chi\Big\{\big|v^{j-1}_m/v^j_m-1\big|\le\delta_1\Big\} \\
&+\mathcal{O}(1)\big|v^{j-1}_m\zeta^j_m\big|\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m-1\big|\le 4\delta_1/5,\big|v^{j-1}_m/v^j_m-1\big|\ge\delta_1\Big\} \\
&+\mathcal{O}(1)\big|v^{j-1}_m\big|\Big(\big|\zeta^j_m\big|+\big|\zeta^{j-1}_m\big|\Big)\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m-1\big|\ge 4\delta_1/5,\big|v^{j-1}_m/v^j_m-1\big|\ge\delta_1\Big\} \\
&+\Big|\zeta^{j-1}_m\big(v^{j-2}_m-v^{j-1}_m\tilde p(u^{j-1}_m,v^{j-1}_m,\sigma^{j-1}_m)\big)\Big|+\mathcal{O}(1)\sum_{n\neq m}\big|v^j_n\big|\big|\zeta^{j-1}_m\big| \\
&+\mathcal{O}(1)\Big|\zeta^{j-1}_m\big(v^{j-1}_m-\hat v^{j-1}_m\big)\Big|+\mathcal{O}(1)\Big|\zeta^{j-1}_m v^j_m\big(\sigma^j_m-\sigma^{j-1}_m\big)\Big|,
\end{aligned}$$

$$\begin{aligned}
\Big|\zeta^{j-2}_m v^{j-2}_m-\hat\zeta^{j-2}_m\hat v^{j-2}_m\Big| \le{}& \Big|\zeta^{j-2}_m v^{j-2}_m-\zeta^{j-1}_m\tilde p^{j-1}_m v^{j-1}_m\tilde p^{j-1}_m\Big|+\Big|\zeta^{j-1}_m\tilde p^{j-1}_m v^{j-1}_m\tilde p^{j-1}_m-\hat\zeta^{j-2}_m\hat v^{j-2}_m\Big| \\
\le{}& \Big|\zeta^{j-2}_m v^{j-2}_m-\zeta^{j-1}_m\tilde p^{j-1}_m v^{j-1}_m\tilde p^{j-1}_m\Big|+\mathcal{O}(1)\Big|\zeta^{j-1}_m\big(v^{j-1}_m-\hat v^{j-1}_m\big)\Big| \\
&+\mathcal{O}(1)\Big|\zeta^{j-1}_m v^j_m\big((\tilde p^{j-1}_m)^2-\big(\tilde p_m(\hat u^{j-1},\hat v^{j-1}_m,\sigma^j_m)\big)^2\big)\Big|+\mathcal{O}(1)\Big|v^j_m\big(\zeta^{j-1}_m-\hat\zeta^{j-1}_m\big)\Big| \\
\le{}& \Big|\zeta^{j-2}_m v^{j-2}_m-\zeta^{j-1}_m\tilde p^{j-1}_m v^{j-1}_m\tilde p^{j-1}_m\Big|+\mathcal{O}(1)\Big|v^j_m\big(\zeta^{j-1}_m-\hat\zeta^{j-1}_m\big)\Big| \\
&+\mathcal{O}(1)\Big|\zeta^j_m\big(v^{j-1}_m-\hat v^{j-1}_m\big)\Big|+\mathcal{O}(1)\Big|\zeta^{j-1}_m\big(v^{j-1}_m-\hat v^{j-1}_m\big)\Big|\chi\Big\{\big|\zeta^{j-1}_m/\zeta^j_m-1\big|\ge\delta_1\Big\} \\
&+\mathcal{O}(1)\Big|\zeta^{j-1}_m v^j_m\big((\tilde p^{j-1}_m)^2-\big(\tilde p_m(\hat u^{j-1},\hat v^{j-1}_m,\sigma^j_m)\big)^2\big)\Big|.
\end{aligned}$$

We finally obtain the equation for ι_m^j:

(D.12)

$$\begin{aligned}
\iota_{m,t}^j + \tilde{\lambda}_m^j \iota_m^j - \tilde{\lambda}_m^{j-1} \iota_m^{j-1} = {} & \mathcal{O}(1)\mathcal{V}(u)\zeta_m^{j-1}\big(\varsigma_m^j - \varsigma_m^{j-1}\big)^2 \chi\Big\{\big|\varsigma_m^j - 1\big|, \big|\varsigma_m^{j-1} - 1\big| \leq 5\delta_1\Big\} \\
& + \mathcal{O}(1)\mathcal{V}(u)\big(|\zeta_m^{j-1}| + |\zeta_m^{j-2}|\big)\chi\Big\{\big|\varsigma_m^j - 1\big| \geq 5\delta_1 \big|\varsigma_m^{j-1} - 1\big| \leq 3\delta_1 \text{ or viceversa}\Big\} \\
& + \mathcal{O}(1)\big|\zeta_m^j v_m^{j-1} - \zeta_m^{j-1} v_m^j\big| + \mathcal{O}(1)\big|\zeta_m^j w_m^{j-1} - \zeta_m^{j-1} w_m^j\big| + \mathcal{O}(1)\big|\zeta_m^{j-1} v_m^{j-2} - \zeta_m^{j-2} v_m^{j-1}\big| \\
& + \mathcal{O}(1)\Big(\big(v_m^j\big)^2 + \big(v_m^{j-1}\big)^2\Big)\chi\left\{\big|s_m^j - 1\big| \geq \frac{3\delta_1}{5}\right\} + \mathcal{O}(1)\Big(\big(v_m^{j-1}\big)^2 + \big(v_m^{j-2}\big)^2\Big)\chi\left\{\big|s_m^{j-1} - 1\big| \geq \frac{3\delta_1}{5}\right\} \\
& + \mathcal{O}(1)\Big(\big(\zeta_m^j\big)^2 + \big(\zeta_m^{j-1}\big)^2\Big)\chi\left\{\big|\varsigma_m^j - 1\big| \geq \frac{3\delta_1}{5}\right\} + \mathcal{O}(1)\Big(\big(\zeta_m^{j-1}\big)^2 + \big(\zeta_m^{j-2}\big)^2\Big)\chi\left\{\big|\varsigma_m^{j-1} - 1\big| \geq \frac{3\delta_1}{5}\right\} \\
& + \mathcal{O}(1)\sum_n \big|v_n^j\big|\big|\zeta_n^{j-1}\big| + \mathcal{O}(1)\mu_m^j(t) + \mathcal{O}(1)\mu_m^{j-1}(t) + \mathcal{O}(1)\omega_m^j(t).
\end{aligned}$$

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