INTERACTION ESTIMATES AND GLIMM FUNCTIONAL FOR GENERAL HYPERBOLIC SYSTEMS

STEFANO BIANCHINI

Abstract. We consider the problem of writing Glimm type interaction estimates for the hyperbolic system

$$u_t + A(u)u_x = 0. \tag{0.1}$$

The aim of these estimates is to prove that there is Glimm-type functional $Q(u)$ such that

$$\text{Tot.Var.}(u) + C_1 Q(u) \text{ is lower semicontinuous w.r.t. } L^1 - \text{norm}, \tag{0.2}$$

with C_1 sufficiently large, and u with small BV norm.

In the first part we analyze the more general case of quasilinear hyperbolic systems. We show that in general this result is not true if the system is not in conservation form: there are Riemann solvers, identified by selecting an entropic conditions on the jumps, which do not satisfy the Glimm interaction estimate (0.2). Next we consider hyperbolic systems in conservation form, i.e. $A(u) = Df(u)$. In this case, there is only one entropic Riemann solver, and we prove that this particular Riemann solver satisfies (0.2) for a particular functional Q, which we construct explicitly. The main novelty here is that we suppose only the Jacobian matrix $Df(u)$ strictly hyperbolic, without any assumption on the number of inflection points of f.

These results are achieved by an analysis of the growth of Tot.Var.(u) when nonlinear waves of (0.1) interact, and the introduction of a Glimm type functional Q, similar but not equivalent to Liu's interaction functional [11].

1. Introduction

In this paper we consider the problem of obtaining Glimm-type estimates for interactions of non linear waves of the quasilinear hyperbolic system

$$u_t + A(u)u_x = 0. \tag{1.1}$$

In [5] the author introduces a general method for constructing solutions to Riemann problems for general strictly hyperbolic systems, i.e. the $n \times n$ system (1.1) with the initial data

$$u(0,x) = \begin{cases} u^- & x \leq 0 \\ u^+ & x > 0 \end{cases} \tag{1.2}$$

The idea is the following. Consider a singular approximation to (1.1), for which there exists a smooth manifold of travelling profiles: for example parabolic approximations, relaxation schemes or semidiscrete schemes. We say that the jump $[w^-, w^+]$ in the solution $u(t)$ to (1.1), (1.2) is admissible for the singular approximation if there exists a travelling profiles ϕ for the approximation considered such that

$$\lim_{x \to \pm\infty} \phi(x) = w^\pm.$$

The speed of the shock $[w^-, w^+]$ is the speed of the corresponding travelling profile ϕ. Roughly speaking, when we choose an approximation to (1.1), we give an admissibility criterium to select the jumps which we consider as "entropic".

In [5] it is shown that there exists only one Riemann solver such that each jump in the solution u to (1.1), (1.2) satisfies the admissibility criterium selected by choosing the singular approximation. The advantages of this construction is that it can be applied to quasilinear strictly hyperbolic systems, without any assumption on the matrix $A(u)$, a part from strictly hyperbolicity. In particular, if $A(u) = Df(u)$, i.e. (1.1) is in conservation form, then in [5] it is shown that there is a unique Riemann solver, independent

Date: September 28, 2002.

1991 *Mathematics Subject Classification.* 35L65.

Key words and phrases. Hyperbolic systems, conservation laws, Riemann problem.

of the approximation. In fact, in this case it is known that Liu's stability condition for shocks [11] is equivalent to the existence of a travelling profile (see [5], [13], [14]). As we said before, we do not assume genuinely nonlinearity, linearly degeneracy or a finite number of inflection points.

Consider now a piecewise constant solution u of (1.1), with jumps in x_α, $\alpha = 1, 2, \dots$ and with sufficiently small total variation. For simplicity, we can think that each jump is an admissible (in the above sense) shock of the i-th family for (1.1), with strength $s_i(x_\alpha)$ and speed σ_α. Suppose now that at time $\bar{t}$ two of these admissible jumps $s_i(x_{\bar\alpha})$, $s_i(x_{\bar\alpha+1})$ of the same family i interact, so that to construct the solution u' for $t > \bar{t}$ one has to solve the new Riemann problem generated at $\bar{t}$. In general, the total variation of the solution u increases, and the first question we consider in this paper is to estimate the growth of Tot.Var.(u).

In [11], it is shown that for systems in conservation form the following estimate holds:

$$\text{Tot.Var.}(u') \leq \text{Tot.Var.}(u) + \mathcal{O}(1)\big|s_i(x_{\bar\alpha})\big|\big|s_j(x_{\bar\alpha+1})\big|\big|\sigma_{\bar\alpha} - \sigma_{\bar\alpha+1}\big|. \tag{1.3}$$

This estimate leads the author to introduce the following Glimm type functional,

$$Q(u) = \sum_{i<j}\sum_{x_\alpha < x_{\alpha'}} \big|s_i(x_\alpha)\big|\big|s_j(x_{\alpha'})\big| + \sum_i \sum_{x_\alpha < x_{\alpha'}} \big|s_i(x_\alpha)\big|\big|s_i(x_{\alpha'})\big| P\big(x_\alpha, x_{\alpha'}\big). \tag{1.4}$$

The weight P_i is computed in the following way: consider all the waves $s_i(x_\beta)$ of the i-th family with $x_{\bar\alpha} \leq x_\beta \leq x_{\bar\alpha'}$. If σ_β is their speed, then the weight is defined as

$$P_i\big(x_\alpha, x_{\alpha'}\big) \doteq \sum_{\beta=\bar\alpha}^{\bar\alpha'-1} \big[\sigma_\beta - \sigma_{\beta+1}\big]^+,$$

where we denote with $[\cdot]^+$ the positive part. By means of this functional, one proves that

$$Q(u') \leq Q(u) - c|s_1||s_2||\sigma_1 - \sigma_2|, \qquad c > 0, \tag{1.5}$$

so that for C_1 sufficiently large one has Tot.Var.$(u) + C_1 Q(u)$ decreasing at each interaction. As a consequence, if we can extend the solution u to all $t \geq 0$ (by means of the Glimm scheme, for example), we have an a priori estimate of Tot.Var.$(u(t))$:

$$\text{Tot.Var.}(u(t)) \leq \text{Tot.Var.}(u(0)) + C_1 Q(u(0)). \tag{1.6}$$

The above estimate is used in [12], where the authors prove the existence of a solution if the flux has a finite number of inflection points, by means of the Glimm scheme (see also [1] for the wavefront tracking scheme).

In this paper we consider the general case: we do not assume that the number of inflection points of the flux f is finite. As a consequence, the estimates we will obtain will not depend on the number of inflection points of the flux f.

As a preliminary result, we show that if the system (1.1) is not in conservation form, then in general the estimate (1.3) does not hold. As a consequence, in general there will be no a functional Q such that (1.6) holds. We prove this result by considering an explicit example.

We thus restrict our analysis to the conservative case $A(u) = Df(u)$. The idea is to rewrite (1.3) as

$$\text{Tot.Var.}(u') \leq \text{Tot.Var.}(u) + \mathcal{O}(1)\left\{\sum_{i<j}\big|s_i(x_{\bar\alpha+1})\big|\big|s_j(x_{\bar\alpha})\big| + \sum_i I_i(\bar\alpha, \bar\alpha+1)\right\}. \tag{1.7}$$

The first quantity in the right hand side corresponds to the standard interaction estimate among waves of different families. We introduce the quantity I_i, which we call the *Amount of Interaction*, which measures how the waves of the same family change when they interact. This quantity becomes the difference in speed when we reduce to the case of the interaction of two Liu admissible jumps. In general, $I_i(\bar\alpha, \bar\alpha+1)$ is related to the strengths and speeds of the waves of the i-th family of the two interacting Riemann problems.

The interaction estimate (1.7) is then used to write a functional $Q(u)$ such that

$$Q(u') - Q(u) \leq -c\left\{\sum_{i<j}\big|s_i(x_{\bar\alpha+1}\big|\big|s_j(x_{\bar\alpha})\big| + \sum_i I_i(\bar\alpha, \bar\alpha+1)\right\}, \tag{1.8}$$

where c is a strictly positive constant. If the jumps of u in x_α are admissible jumps, then the form of the functional is

$$Q(u) = \sum_{i<j} \sum_{x_\alpha < x_{\alpha'}} \big|s_i(x_\alpha)\big|\big|s_j(x_{\alpha'})\big| + \sum_{i} \sum_{x_\alpha < x_{\alpha'}} \big|s_i(x_\alpha)\big|\big|s_i(x_{\alpha'})\big|\big|\sigma_{\alpha,i} - \sigma_{\alpha',i}\big|. \tag{1.9}$$

As a consequence of (1.8), the functional Tot.Var.$(u) + C_1 Q(u)$ is decreasing if the constant C_1 is sufficiently large. This last estimate is obtained in the case of piecewise constant functions. Using the same techniques of [4], one can extend (1.9) to general BV functions, with small total variation. We observe that the form (1.9) is directly deduced from the vanishing viscosity limit, see [7]. Using this functional, one can prove the existence of a weak solution to the hyperbolic system

$$u_t + f(u)_x = 0,$$

without any assumption on the number of inflection points of f. Moreover in these estimates we do not need the decrease of total variation to estimate interactions of waves of the same family but with different sign, as it is done in [12]: the functional Q is sufficient to prove uniform BV bounds for all kinds of interactions.

To enter in the heart of the matter, we recall briefly the results of [5] on the construction of a Riemann solver for quasilinear hyperbolic systems of the form

$$u_t + A(u)u_x = 0, \tag{1.10}$$

i.e. the construction of the solution (in some weak sense) of the above equation with the initial data

$$u(0,x) = \begin{cases} u^- & x < 0 \\ u^+ & x \geq 0 \end{cases} \tag{1.11}$$

The matrix $A(u)$ in (1.10) is assumed to be strictly hyperbolic, and we will denote with $r_i(u)$ ($l_i(u)$) the base of right (left) eigenvalues, normalized by

$$\big|r_i(u)\big| = 1, \qquad \big\langle l_j(u), r_i(u)\big\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.12}$$

The eigenvalues are denoted with $\lambda_i(u)$, and the strict hyperbolicity assumption implies as

$$\lambda_1(u) < \ldots < \lambda_n(u). \tag{1.13}$$

The construction of a solution to the Riemann problem (1.10), (1.11) for conservative systems, is based on the definition of the admissible curves $T^i_s u$, $s \in [-\delta, \delta]$, parameterized by s, the i-th component of $u(s)$, and passing through u for $s = 0$. The index i varies in $1, \ldots, n$ (see [11]). Each point $u(s) = T^i_s u$ of the curve T^i can be connected to u by means of rarefactions or jumps of the i-th family. In the conservative case it is assumed that each jump $[u^-_\alpha, u^+_\alpha]$ satisfies Liu's stability condition: if $S_s u^-_\alpha$, $s \in [0, \bar{s}]$, is the shock curve connecting u^-_α to u^+_α, with $S_0 u^-_\alpha = u^-_\alpha$, $S_{\bar{s}} u^-_\alpha = u^+_\alpha$, and if $\sigma(s, u^-_\alpha)$ denotes the speed of the shock $[u^-_\alpha, S_s u^-_\alpha]$, then

$$\sigma\big(s, u^-_\alpha\big) \geq \sigma(u^+_\alpha, u^-_\alpha), \tag{1.14}$$

for all $s \in [0, \bar{s}]$.

In the non conservative setting, as we say in the introduction, we associate a singular approximation to the quasilinear system (1.10), and we define the jump $[u^-_\alpha, u^+_\alpha]$ *admissible* if it corresponds to a travelling profile ϕ for the singular approximation considered.

Example 1.1. As an example, we consider the quasilinear parabolic-hyperbolic system

$$u_t + A(u)u_x = \epsilon B(u) u_{xx}. \tag{1.15}$$

We thus consider the jump $[u^-_\alpha, u^+_\alpha]$ *admissible* if there is travelling profile for (1.15), i.e. a function ϕ satisfying for some σ

$$-\sigma\phi' + A(u)\phi' = \epsilon B(u)\phi'',$$

and such that its limit points are

$$\lim_{x \to \pm\infty} \phi(x) = u^\pm_\alpha.$$

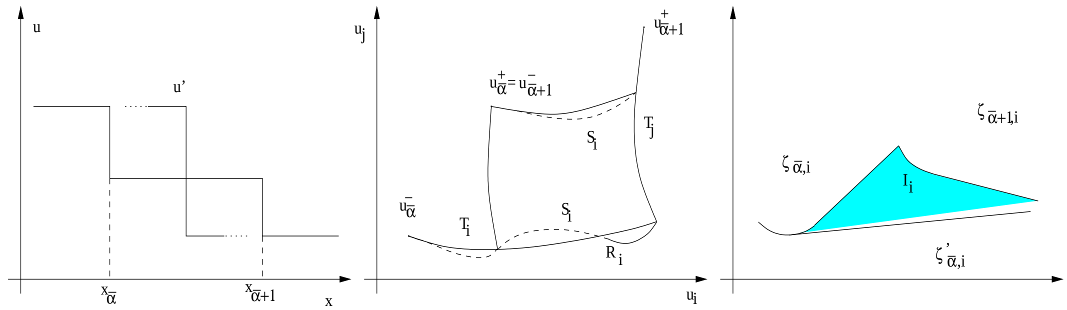

FIGURE 1. The replacement of two consecutive jumps: R_i, S_i are the rarefaction and shock curves, and the colored area is the amount of interaction I_i in (1.20).

The speed we associate to the jump $[u_\alpha^-, u_\alpha^+]$ is σ. Under some general assumptions on the matrix $B(u)$, in [5] it is shown that this condition characterizes a unique Riemann solver. Moreover the speed σ is close to one of the eigenvalues λ_i, so that we can associate the jump $[u_\alpha^-, u_\alpha^+]$ to the i-th family.

The curves T_i are defined in the same way as in the conservative case: each point $u(s) = T_s^i u$ of the curve T^i can be connected to u by means of rarefactions or admissible jumps of the i-th family.

Once the curves $T_s^i u$ are constructed, for fixed u^-, the solution to a Riemann problem is given by inverting the function

$$\mathbb{R}^n \ni s \mapsto T_{s_n}^n \circ \ldots \circ T_{s_1}^1 u^- = w \in \mathbb{R}^n. \tag{1.16}$$

This can be done if the vector $u^- - u^+$ is sufficiently small, because in [5] it is shown that the curves $T_s^i u$ are tangent to $r_i(u^-)$ for $s = 0$. Fixed u^-, u^+, we thus obtain the vector $(s_1, \ldots, s_n)$. This means that each point p_k, with

$$p_0 \doteq u^-, \qquad p_k \doteq T_{s_k}^k p_{k-1}, \qquad p_n \doteq u^+, \tag{1.17}$$

can be connected to p_{k-1} by means of a sequence of rarefactions and admissible jumps of the k-th family. The solution to (1.10) with initial data (1.11) is the self-similar function obtained by piecing together the n solutions to $[p_{k-1}, p_k]$.

Consider now the elementary Riemann problem $[p_{k-1}, p_k]$. By construction, the point p_k can be connected to the point p_{k-1} by means a sequence of rarefactions and admissible jumps, such that the speed of these waves is increasing as we move from p_{k-1} to p_k. A consequence of the smallness assumption is that we can parameterize the curve T^k and the rarefactions and admissible jumps generating the solution to $[p_{k-1}, p_k]$ by means of the k coordinate of the vector u, i.e.

$$s \doteq \big[T_s^k p_{k-1} - p_{k-1}\big]_k,$$

where we denote with $[w]_k$ the k-th component of the vector w. In particular, for the elementary jump $[p_{k-1}, p_k]$, we obtain the speed σ as a function of s with $s \in [0, s_k]$. We define the elementary curve $\zeta_k \in \mathbb{R}^2$ by

$$\zeta_k(0) = 0, \qquad \frac{d}{ds}\zeta_k(s) = \left(\begin{array}{c} 1 \\ \sigma(s) \end{array} \right), \qquad s \in [0, s_k] \quad \big(\text{or } s \in [s_k, 0]\big). \tag{1.18}$$

This curve is the vector analog of the curve obtained in the scalar case by considering the convex (concave) envelope of the flux function f.

Let u be a piecewise constant function, with a countable number of jumps in x_α, $\alpha \in \mathbb{N}$. To each jump $[u(x_\alpha-), u(x_\alpha+)]$ we can associate the elementary curve $\zeta_{\alpha,i}$ using the same procedure described above: if $\sigma_{\alpha,i}(s)$ is the parameterization of the speed of the waves of the i-th family, then

$$\zeta_{\alpha,i}(0) = 0, \qquad \frac{d}{ds}\zeta_{\alpha,i}(s) = \left(\begin{array}{c} 1 \\ \sigma_{\alpha,i}(s) \end{array} \right), \qquad s \in [0, s_{\alpha,i}] \quad \big(\text{or } s \in [s_{\alpha,i}, 0]\big). \tag{1.19}$$

To the function u we can now associate the curves ζ_i by piecing together the curves $\zeta_{\alpha,i}$.

Now, replace two consecutive jumps

$$[u(x_{\bar\alpha}-), u(x_{\bar\alpha}+)],\ [u(x_{\bar\alpha+1}-), u(x_{\bar\alpha+1}+)] \quad \text{with the jump} \quad [u(x_{\bar\alpha}-), u(x_{\bar\alpha+1}+)].$$

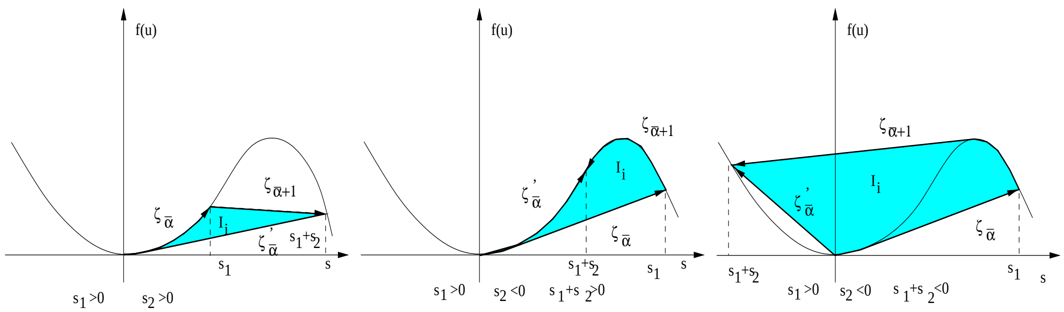

FIGURE 2. The curves ζ and the Amount of Interaction I in the scalar case.

Denote with u' the new function obtained.

In terms of the curves ζ, this means that, for each family i, we are substituting in ζ_i the two arcs $\zeta_{\bar\alpha,i} \cup \zeta_{\bar\alpha+1,i}$ with a new arc $\zeta'_{\bar\alpha,i}$. If $s_{\bar\alpha,i} > 0$, the amount of interaction $I_i(\bar\alpha, \bar\alpha+1)$ is defined as follows:

(1) if $s_{\bar\alpha,i} \cdot s_{\bar\alpha+1,i} \geq 0$, then I_i is the area of the region limited by $\zeta_{\bar\alpha,i} \cup \zeta_{\bar\alpha+1,i}$ and its lower convex envelope,

$$I_i(\bar\alpha, \bar\alpha+1) \doteq \text{Area}\Big(\text{convex envelope}\big(\zeta_{\bar\alpha,i} \cup \zeta_{\bar\alpha+1,i}\big)\Big) - \text{Area}\Big(\text{convex envelope}\big(\zeta_{\bar\alpha,i}\big)\Big) - \text{Area}\Big(\text{convex envelope}\big(\zeta_{\bar\alpha+1,i}\big)\Big); \tag{1.20}$$

(2) if $s_{\bar\alpha,i} \cdot s_{\bar\alpha+1,i} < 0$ and $\zeta_{\bar\alpha,i}$ does not intersect $\zeta_{\bar\alpha+1,i}$, then I_i is the area of the convex envelope of $\zeta_{\bar\alpha,i} \cup \zeta_{\bar\alpha+1,i}$ minus the area of the convex envelops of $\zeta'_{\bar\alpha,i}$,

$$I_i(\bar\alpha, \bar\alpha+1) \doteq \text{Area}\Big(\text{convex envelope}\big(\zeta_{\bar\alpha,i} \cup \zeta_{\bar\alpha+1,i}\big)\Big) - \text{Area}\Big(\text{convex envelope}\big(\zeta'_{\bar\alpha,i}\big)\Big). \tag{1.21}$$

If $\zeta_{\bar\alpha,i}$ intersects $\zeta_{\bar\alpha+1,i}$, we define

(1.22)

$$I_i(\bar\alpha, \bar\alpha+1) \doteq \frac{1}{4}\int_0^{|s_{\bar\alpha,i}|}\int_0^{|s_{\bar\alpha,i}|}\Big|\sigma_{\bar\alpha,i}(s) - \sigma_{\bar\alpha,i}(s')\Big|dsds' + \frac{1}{4}\int_0^{|s_{\bar\alpha+1,i}|}\int_0^{|s_{\bar\alpha+1,i}|}\Big|\sigma_{\bar\alpha+1,i}(s) - \sigma_{\bar\alpha+1,i}(s')\Big|dsds' + \frac{1}{2}\int_0^{|s_{\bar\alpha,i}|}\int_0^{|s_{\bar\alpha+1,i}|}\Big|\sigma_{\bar\alpha,i}(s) - \sigma_{\bar\alpha+1,i}(s')\Big|dsds' - \frac{1}{4}\int_0^{|s'_{\bar\alpha,i}|}\int_0^{|s'_{\bar\alpha,i}|}\Big|\sigma'_{\bar\alpha,i}(s) - \sigma'_{\bar\alpha,i}(s')\Big|dsds'.$$

If $s_{\bar\alpha,i} < 0$, we just consider (1.19) with a minus in front of the derivative, to reduce to the above cases.

Note that (1.20), (1.21) are a generalization of (1.3): in fact, in case of two admissible jumps of the i-th family, the above formulas reduce to (1.3). We observe that (1.22) can be though as an extension of formula (1.21), because of the equivalence (see [9])

$$\text{Area}\Big(\text{conv}\big(\zeta_{\alpha,i}\big)\Big) = \frac{1}{4}\int_0^{|s_{\alpha,i}|}\int_0^{|s_{\alpha,i}|}\Big|\sigma_{\alpha,i}(s) - \sigma_{\alpha,i}(s')\Big|dsds'. \tag{1.23}$$

It is interesting to write the Amount of Interaction I_i in the case of a scalar equation, i.e. $u \in \mathbb{R}$. In fact, in this case the curves ζ coincide with the convex (concave) envelope if $s > 0$ ($s < 0$), and it is easy to check that the curve $\zeta_{\bar\alpha} \cup \zeta_{\bar\alpha+1} \cup \zeta'_{\bar\alpha}$ is closed, and the following equivalence holds:

$$I_i(\bar\alpha, \bar\alpha+1) = \text{Area}\big(\zeta_{\bar\alpha} \cup \zeta_{\bar\alpha+1} \cup \zeta'_{\bar\alpha}\big). \tag{1.24}$$

The above definition is thus the extension of the notion of Amount of Interaction given in [9] for scalar equations.

Using the curves ζ_i, we can write explicitly the functional $Q(u)$: if $s_{\alpha,i}$ denotes the strength of the waves of the i-th family in the Riemann problem at x_α, we define

$$Q(u) \doteq \sum_{\alpha<\alpha'}\sum_{i<j}\big|s_{\alpha,j}\big|\big|s_{\alpha',i}\big| + \frac{1}{4}\sum_{\alpha,\alpha'}\sum_i \int_0^{|s_{\alpha,i}|}\int_0^{|s_{\alpha',i}|}\Big|\sigma_{\alpha,i}(\tau) - \sigma_{\alpha',i}(\tau')\Big|d\tau d\tau'. \tag{1.25}$$

It is possible to interpret the second part of this functional, i.e. the part corresponding to the approaching waves of the same family, as an area. In fact, it can be shown that the quantity

$$\frac{1}{4}\sum_{\alpha,\alpha'}\sum_i \int_0^{|s_{\alpha,i}|}\int_0^{|s_{\alpha',i}|}\Big|\sigma_{\alpha,i}(\tau)-\sigma_{\alpha',i}(\tau')\Big|d\tau d\tau'$$

controls the the maximal area that can be swept by ζ_i under motion in the direction of curvature: the motion in the direction of curvature corresponds to the interaction of nonlinear waves, see section 4 (or [9] for the scalar case).

The paper is organized as follows.

In section 2, we recall the basic ideas in the construction of the solution to the Riemann problem for quasilinear hyperbolic systems, and we consider in particular an important case, the vanishing viscosity limit with identity viscosity matrix,

$$u_t + A(u)u_x - \epsilon u_{xx} = 0. \tag{1.26}$$

Next we give an example in which the solution to the Riemann problem for non conservative systems does not satisfy the interaction estimate (1.5), even if it is obtained as limit of a singular approximation. A consequence of this result is that in general there are no a priori estimates on the growth of Tot.Var.(u).

In section 3, we consider the special case of systems in conservation form. Since in this case the Riemann solver does not depend on the particular approximation considered, we focus on the estimates that can be obtained by means of the vanishing viscosity with identity viscosity matrix (1.26). This particular singular limit, in fact, has stronger properties, due to the fact that, when $A(u)$ is constant, we can diagonalize simultaneously the viscosity part and the matrix A.

Since the proof of existence of the Riemann solver is constructive, we use the techniques described in section 2 to prove the estimate (1.7). To simplify computations, we give an equivalent definition of amount of interaction I_i, which is strictly related to the approximation used to construct the Riemann solver.

In section 4, we introduce the functional Q and prove the estimate (1.8) for piecewise constant functions. The ideas of the proof are based on the results of [9], i.e. of the area swept by a curve moving in the direction of curvature. As a remark, we describe how to generalize the results of this paper to a general BV function u, in such a way that $Q(u)$ and Tot.Var.$(u) + C_1 Q(u)$ are lower semicontinuous w.r.t. the L^1 convergence. This result is the extension of a theorem of [4] to general hyperbolic systems.

2. Preliminaries

We recall here the approach of [5].

Consider two smooth functions $\tilde{r}_i$, $\tilde{\lambda}_i$, defined on the $n+2$ variables $(u, v_i, \sigma_i) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$. The assumptions on $\tilde{r}_i$, $\tilde{\lambda}_i$ are that, in a fixed point u^0,

$$\tilde{r}_i\big(u^0, 0, \lambda_i^0\big) = r_i^0, \qquad \tilde{\lambda}_i\big(u^0, 0, \lambda_i^0\big) = \lambda_i^0, \qquad \frac{\partial}{\partial\sigma}\tilde{\lambda}_i(u^0, 0, \lambda_i^0) = 0, \tag{2.1}$$

for some vector r_i^0 and scalar λ_i^0. We assume that the vectors $\{r_i^0\}$, $i = 1, \dots, n$, generate a base in $\mathbb{R}^n$, and that the constants λ_i^0 satisfy a strictly inequality:

$$\lambda_1^0 < \ldots < \lambda_n^0. \tag{2.2}$$

While one can construct the solution to the Riemann Problem (1.10), (1.11) only assuming (2.1), it is clear that the functions $\tilde{r}_i$, $\tilde{\lambda}_i$ should be related to the hyperbolic system (1.10) by

$$r_i^0 = r_i(u^0), \qquad \lambda_i^0 = \lambda_i(u^0). \tag{2.3}$$

In this case the assumptions (2.2) and that span$\{r_i^0\} = \mathbb{R}^n$ are naturally satisfied. The vectors $\tilde{r}_i$ can be normalized by means of the dual base l_i^0 to r_i^0, assuming that

$$\big|r_i^0\big| = 1, \qquad \big\langle l_j^0, \tilde{r}_i(u, v_i, \sigma_i)\big\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2.4}$$

We give now an example of a general method to obtain the functions $\tilde{r}_i$, $\tilde{\lambda}_i$, such that (2.3) holds. As we said in the introduction, the idea is to consider any singular approximation to (1.10) such that its travelling profiles belong to a center manifold, and consider the dynamics on this manifold to obtain the two functions $\tilde{r}_i$, $\tilde{\lambda}_i$. For more general examples, see [5].

Example 2.1. We construct the vectors $\tilde{r}_i$ and the scalar $\tilde{\lambda}_i$ by means of the center manifold applied to the ODE [5]

$$-\sigma_i u_x + A(u)u_x - u_{xx} = 0. \tag{2.5}$$

This is the equation for travelling profiles with speed σ of the parabolic approximation

$$u_t + A(u)u_x - u_{xx} = 0.$$

Written in first order form, (2.5) becomes

$$\left\{\begin{array}{rcl} u_x &=& v \\ v_x &=& \big(A(u) - \sigma I\big)v \\ \sigma_x &=& 0 \end{array}\right. \tag{2.6}$$

and by means of the Center Manifold Theorem one obtains a vector function $\tilde{r}_i(u, v_i, \sigma)$, normalized by (2.4), such that

$$v = v_i \tilde{r}_i(u, v_i, \sigma). \tag{2.7}$$

In the above equations v_i is the i-th component of v in the base $\{r_i(u^0)\}$. The equation (2.7) is thus the representation of the center manifold for (2.6), parameterized by (u, v_i, σ).

From the tangency of the center manifold to the null space of (2.6), linearized in $(u^0, 0, \lambda_i(u^0))$, it follows that

$$\tilde{r}_i\big(u^0, 0, \lambda_i(u^0)\big) = r_i(u^0), \tag{2.8}$$

where $r_i(u)$ is the right eigenvector associated to $A(u)$. By writing the reduced ODE (2.6) on the center manifold, one obtains

$$\left\{\begin{array}{rcl} u_x &=& v_i \tilde{r}_i(u, v_i, \sigma) \\ v_{i,x} &=& \big(\tilde{\lambda}_i(u, v_i, \sigma) - \sigma\big)v_i \\ \sigma_x &=& 0 \end{array}\right. \tag{2.9}$$

where

$$\tilde{\lambda}_i(u, v_i, \sigma_i) = \big\langle l_i^0, A(u)\tilde{r}_i(u, v_i, \sigma_i)\big\rangle. \tag{2.10}$$

In general, given the functions $\tilde{r}_i$, $\tilde{\lambda}_i$ and fixed any s, $\bar{u}$ with $|\bar{u} - u^0|$, s sufficiently small, we can construct n curves $T^i_s \bar{u}$, $i = 1, \dots, n$ and $\bar{u} \in \mathbb{R}^n$, with values in $\mathbb{R}^n$ by solving for $0 \le \tau \le s$ the integral system

$$\left\{\begin{array}{rcl} u(\tau) &=& \bar{u} + \displaystyle\int_0^\tau \tilde{r}_i\big(u(\xi), v_i(\xi), \sigma_i(\xi)\big)d\xi \\ v_i(\tau) &=& \tilde{f}_i(\tau) - \mathrm{conv}_{[0,s]}\tilde{f}_i(\tau) \\ \sigma_i(\tau) &=& \dfrac{d}{d\tau}\mathrm{conv}_{[0,s]}\tilde{f}_i(\tau) \end{array}\right. \tag{2.11}$$

where

$$\tilde{f}_i(\tau) = \int_0^\tau \tilde{\lambda}_i\big(u(\xi), v_i(\xi), \sigma_i(\xi)\big)d\xi. \tag{2.12}$$

If $s < 0$, then the concave envelope of $\tilde{f}_i$ is considered. The admissible curve $T^i_s \bar{u}$ is defined as the value at $\tau = s$ of u, solution to (2.11) in $[0, s]$. Note that, by (2.4), this implies that T^i is parameterized by the i-th component of u. The envelopes of the functions $\tilde{f}$ are considered so that the speed σ_i is increasing. By the implicit relation $\sigma(s) = x/t$, we can define a self-similar solution $u(t/x)$: this function is the solution to the Riemann problem $[\bar{u}, T^i_s \bar{u}]$, in which only waves of the i-th family are present. In the following, as a measure of Tot.Var.(u), we consider the sum

$$\sum_\alpha \sum_i \big|s_{\alpha.i}\big| \simeq \mathrm{Tot.Var.}(u). \tag{2.13}$$

In [5] it is shown that (2.11) is a contraction in the set $\Gamma_i(s,\bar u)$ of Lipschitz continuous curves with values in $\mathbb{R}^{n+2}$:

$$\Gamma_i(s,\bar u) = \Big\{ \gamma : [0,s] \mapsto \mathbb{R}^{n+2}, \gamma(\tau) = \big(u(\tau), v_i(\tau), \sigma_i(\tau)\big) \Big\}, \tag{2.14}$$

such that

$$u(0)=\bar u,\ u_i(\tau) = \bar u_i + \tau,\ \big|u(\tau)-\bar u\big| = \tau, \quad \big|v_i(0)\big| = 0,\ \big|v_i(\tau)\big| \leq \delta_1, \quad \big|\sigma_i(\tau) - \lambda_i^0\big| \leq 2C_0\delta_1 \leq 1, \tag{2.15}$$

for some small δ_1 sufficiently small. The distance among curves in $\Gamma_i(s,\bar u)$ is measured by the norm

$$\|\gamma\|_\Gamma = \|u\|_{L^\infty} + \|v_i\|_{L^\infty} + \delta_1 \|\sigma_i\|_{L^\infty}. \tag{2.16}$$

If the system is in conservation form and the flux f has a finite number of inflection points, i.e. the derivative of $\lambda_i(u)$ in the direction of the eigenvalue vanishes only in a finite number of hypersurfaces transversal to $r_i(u)$, the curve T^i constructed in [11] are a subcase of the curve constructed by means of (2.11), independent on the singular approximation (if in conservation form).

As we said in the introduction, an important estimate on the Riemann solver for nonconvex hyperbolic system obtained in [11] is the following: if s, s' are two interacting shock of the i-th family, then the strength of the new waves generated by their interaction is of the order of the product of their strength times the difference in speed $\sigma - \sigma'$. Let $s_{1,i}$, $s_{2,i}$ be the strength of the two jumps of the i-th family,

$$u_1 = T^i_{s_{1,i}} u_0, \qquad u_2 = T^i_{s_{2,i}} u_1, \tag{2.17}$$

and let $s = (s_1,\dots,s_n)$ be the n-dimensional vector obtained by solving the Riemann problem $[u_0,u_2]$, i.e.

$$u_2 = T^n_{s_n} \circ \ldots \circ T^1_{s_1} u_0, \tag{2.18}$$

The estimates of (1.3) thus can be written as

$$\sum_j \big|s_j - (s_{1,j}+s_{2,j})\big| = \big|s_i - (s_{1,i}+s_{2,i})\big| + \sum_{j\neq i} |s_j| \leq \mathcal{O}(1)|s_{1,i}||s_{2,i}|\big|\sigma-\sigma'\big|. \tag{2.19}$$

This estimate is important to bound the total variation of the solution u to (1.10), because its increment is controlled by the decrease of a Glimm-type interaction functional, see [11], [12]. We will discuss it in details later on, when we introduce a Glimm type functional Q.

We now show that, even if we can define a Riemann solver by means of $\tilde r_i$, $\tilde\lambda_i$ for systems not in conservation form, in general this non conservative Riemann solver does not satisfy the interaction estimate (2.19).

Example 2.2. Consider the following triangular system not in conservation form:

$$\left\{ \begin{array}{ccc} u_t + (u-1)^2 u_x & = & u_{xx} \\ w_t + \big((u-1)^2(1+u) - u\big) w_x + u u_x & = & (1+u) w_{xx} \end{array} \right. \tag{2.20}$$

It is clear that the viscosity matrix

$$B(u) = \left[\begin{array}{cc} 1 & 0 \\ 0 & (1+u) \end{array} \right]$$

is uniformly positive definite in a neighborhood of the line $u=1$. In the same point the first eigenvector, normalized by assuming its first component equal to 1, is given by

$$r_1(1) = \left[\begin{array}{c} 1 \\ 1 \end{array} \right]. \tag{2.21}$$

We compute the admissible shock $u \in [1-\epsilon, 1]$, with speed $\sigma = \epsilon^2/3$. We obtain

$$\left\{ \begin{array}{ccc} \big(-\sigma + (u-1)^2\big) u_x & = & u_{xx} \\ \big(-\sigma + (u-1)^2(1+u) - u\big) w_x + u u_x & = & (1+u) w_{xx}, \end{array} \right. \tag{2.22}$$

and one can check that the unique bounded solution is given by

$$w(x) = w^- + \frac{u(x) - u^-}{1-\sigma}, \tag{2.23}$$

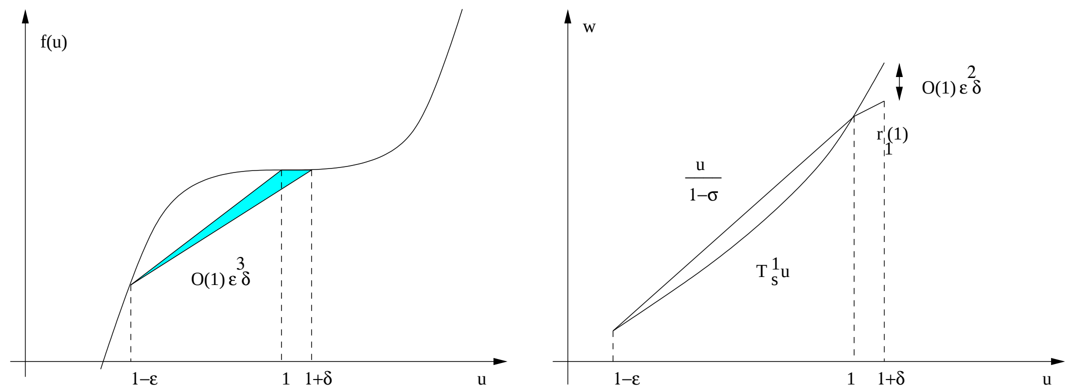

FIGURE 3. Example 2.2.

where $u(x)$ is a solution to the first equation of system (2.22). In particular, the Rankine-Hugoniot condition is given by

$$w^+ - w^- = \frac{u^+ - 1 + \epsilon}{1 - \sigma(u^+, 1-\epsilon)}, \qquad \sigma\big(u^+, 1-\epsilon\big) = \frac{1}{3}\Big((u^+ - 1)^2 + \epsilon(u^+ - 1) + \epsilon^2\Big),$$

and its tangent vector for $u^+ = 1$ is

$$\begin{bmatrix} 1 \\ 1/(1-\epsilon^2/3)^2 \end{bmatrix}.$$

Thus we have that the interaction with a small rarefaction of size $\delta > 0$ generates a new wave in w of size

$$\left(\frac{1}{(1-\epsilon^2/3)^2} - 1\right)\delta = \mathcal{O}(1)\epsilon^2\delta,$$

while the product of the strength times the difference in speed is of the order of $\epsilon^3\delta$, i.e. the area of the triangle in fig. 3. This prove that in general the interaction estimate (2.19) is not valid.

The above example shows that in general, for Riemann solver obtained by singular approximations to quasilinear hyperbolic systems not in conservation form, one cannot expect the existence of a functional Q which controls the growth of the total variation of u. In fact, for nonconvex scalar conservation laws the Glimm functional Q decreases at the interaction considered of an amount of the order of the area.

In the next section we will discuss the special case of systems in conservation form.

3. The conservative case

We consider here the special case of hyperbolic systems in conservation form,

$$u_t + f(u)_x = 0. \tag{3.1}$$

In this case, the Riemann solver does not depend on the singular approximation: in fact, roughly speaking, the shock curves are uniquely defined by the Rankine-Hugoniot condition, and then Liu's stability condition identifies the unique admissible jumps (see [5] for the proof). Since the Riemann solver does not depend on the particular choice of $\tilde{r}_i$, $\tilde{\lambda}_i$ if the system is in conservation form, then in this case we choose a special approximation, i.e. the identity viscosity matrix:

$$u_t + f(u)_x = \epsilon u_{xx}. \tag{3.2}$$

In [7] it is shown that the vectors $\tilde{r}_i$ satisfies the additional property

$$\tilde{r}_{i,\sigma}(u, v_i, \sigma) = \mathcal{O}(1)v_i. \tag{3.3}$$

This property, which can be directly deduced by substituting (2.7) into (2.6), is a consequence of the fact that it is possible to diagonalize simultaneously the viscosity matrix and the Jacobian matrix $A(u) \doteq Df(u)$. The same computations can be used for quasilinear hyperbolic systems, whose Riemann solver is obtained by means of the singular approximation

$$u_t + A(u)u_x = \epsilon u_{xx}.$$

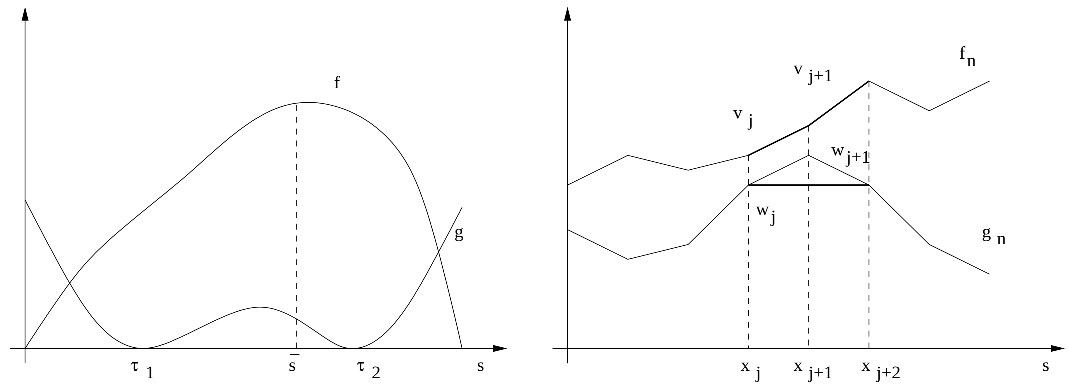

FIGURE 4. The two cases of Lemma 3.1.

These Riemann solvers are of course a subclass of the general case.

We assume thus that (3.3) holds and as in example 2.1 we set

$$\tilde{\lambda}_i(u, v_i, \sigma_i) = \big\langle l_i(u^0), A(u)\tilde{r}_i(u, v_i, \sigma_i)\big\rangle. \tag{3.4}$$

Note that in this case, by the normalization $\langle l_i(u^0), \tilde{r}_i\rangle = 1$, we have that

$$\big\langle l_i(u^0), \partial r_i\big\rangle = 0,$$

where $\partial\tilde{r}_i$ denotes any partial derivative of $\tilde{r}_i$, so that we obtain

$$\tilde{\lambda}_i(u^0, v_i, \sigma_i) = \lambda_i(u^0), \qquad \frac{\partial\tilde{\lambda}_i}{\partial v_i}(u^0, v_i, \sigma_i) = 0, \qquad \frac{\partial\tilde{\lambda}_i}{\partial \sigma_i}(u^0, v_i, \sigma_i) = 0.$$

These identities imply that

$$\left|\frac{\partial}{\partial v_i}\tilde{\lambda}_i(u, v_i, \sigma_i)\right| \leq C_0|u - u^0|, \qquad \left|\frac{\partial}{\partial \sigma_i}\tilde{\lambda}_i(u, v_i, \sigma_i)\right| \leq C_0|v_i||u - u^0|, \tag{3.5}$$

where here and in the following C_0 will denotes a possibly large constant. Note that in the last estimate of (3.5) we have used the assumption (3.3).

We will denote the convex envelope of a continuous function f in the interval $[a, b]$ as

$$\text{conv}_{[a,b]}f(x) = \inf\Big\{\theta f(y) + (1-\theta)f(z),\ x = \theta y + (1-\theta)z;\ y, z \in [a, b],\ \theta \in [0, 1]\Big\}. \tag{3.6}$$

We will write only conv$f$ if there are not ambiguity in what is assumed to be $[a, b]$.

We begin with an easy lemma on convex envelopes.

Lemma 3.1. *Let f, g be C^1 functions on the interval $[0, s]$. Then we have the estimates*

$$\big\|f - \text{conv}_{[0,s]}f - (g - \text{conv}_{[0,s]}g)\big\|_{L^\infty} \leq \frac{1}{2}\big\|df - dg\big\|_{L^1}, \tag{3.7}$$

$$\big\|d\big(\text{conv}_{[0,s]}f\big) - d\big(\text{conv}_{[0,s]}g\big)\big\|_{L^1} \leq \big\|df - dg\big\|_{L^1}. \tag{3.8}$$

These estimates are clearly sharp.

Proof. Since the right hand side of (3.7) is a C^0 function, there is a maximum in some point $\bar{s} \in [0, s]$. We can assume that this maximum is positive and such that

$$f(\bar{s}) - \text{conv}f(\bar{s}) > g(\bar{s}) - \text{conv}g(\bar{s}).$$

This means that there is a subinterval $[s_1, s_2]$, containing $\bar{s}$, such that $f > \text{conv}f$ in (s_1, s_2) and $f(s_1) - \text{conv}f(s_1) = f(s_2) - \text{conv}f(s_2) = 0$. Thus, since the convex envelope of g in $[s_1, s_2]$ is above the convex envelope of g in $[0, s]$, by restricting to $[s_1, s_2]$, we have only to prove (3.7) in the case $f > \text{conv}f$ in $(0, s)$, and moreover we can assume that $\text{conv}f \equiv 0$ in $[0, s]$.

We have two cases. If the maximum is assumed in a point $\bar{s}$ in which $g(\bar{s}) = \mathrm{conv}g(\bar{s})$, then we can assume that $g(\bar{s}) = 0$, $dg(0) = 0$, since adding a linear function to g does not affect (3.7). This implies that $g(0)$ and $g(s)$ are bigger than 0. Then (3.7) follows from the estimate

$$\begin{aligned} f(\bar{s}) \leq& \min\Big\{ f(\bar{s}) - \big(g(\bar{s}) - g(0)\big), f(\bar{s}) - \big(g(\bar{s}) - g(s)\big)\Big\} \\ \leq& \min\left\{ \int_0^{\bar{s}} \big|df(\xi) - dg(\xi)\big| d\xi, \int_{\bar{s}}^s \big|df(\xi) - dg(\xi)\big| d\xi \right\} \leq \frac{1}{2} \int_0^s \big|df(\xi) - dg(\xi)\big| d\xi. \end{aligned}$$

The other case is when the maximum is assume in a point in which $g > \mathrm{conv}g$. Let $[\tau_1, \tau_2]$ be the interval in which the last condition holds. By means of a linear transformation, we can assume that $g(\tau_1) = g(\tau_2) = 0$. This implies again that $g(0)$ and $g(s)$ are bigger than 0, so that with the same estimates as above

$$\big\|f - \big(g - \mathrm{conv}g\big)\big\|_{L^\infty} \leq \big|f(\bar{s}) - g(\bar{s})\big| \leq \min\left\{ \int_0^{\bar{s}} \big|df(\xi) - dg(\xi)\big| d\xi, \int_{\bar{s}}^s \big|df(\xi) - dg(\xi)\big| d\xi \right\}.$$

This conclude the proof of (3.7).

We now show that inequality (3.8) is trivial for piecewise linear function. Consider in fact piecewise functions f_n, g_n defined on the grid $x_i = i/n$, and define $v_i = f_n(x_{i+1}) - f_n(x_i)$, and similarly $w_i = g_n(x_{i+1}) - g_n(x_i)$. We replace two adjacent j, $j+1$ segments with a their convex envelope, and denote with f_n', g_n' the new functions. We have to consider two cases. We can always assume that for at least one function the replacement does not coincide with the original function, let us say $g_n > g_n'$.

If $f_n > f_n'$, then we obtain

$$\big\|df_n' - dg_n'\big\|_{L^1} - \big\|df_n - dg_n\big\|_{L^1} = 2\left|\frac{v_j + v_{j+1}}{2} - \frac{w_j + w_{j+1}}{2}\right| - \big|v_j - w_j\big| - \big|v_{j+1} - w_{j+1}\big| \leq 0.$$

Thus the sum $\sum_i |v_i - w_i|$ is decreasing.

Assume now that $f_n = f_n'$. In this case, the only interesting possibility is when $w_j > v_j$, $w_{j+1} < v_j$. In fact otherwise there is no change in sign and thus

$$\big\|df_n' - dg_n'\big\|_{L^1} = \big\|df_n - dg_n\big\|_{L^1}.$$

We have that

$$\big\|df_n' - dg_n'\big\|_{L^1} - \big\|df_n - dg_n\big\|_{L^1} = \left|v_j - \frac{w_j + w_{j+1}}{2}\right| + \left|v_{j+1} - \frac{w_j + w_{j+1}}{2}\right| - \big|v_j - w_j\big| - \big|v_{j+1} - w_{j+1}\big|.$$

If $v_j + w_{j+1} < 2v_{j+1}$, then the right hand side is $w_{j+1} - w_j \leq 0$. Otherwise, i.e. when $w_j + w_{j+1}$ is above or below $2v_j$, $2v_{j+1}$, then we obtain $2(v_j - w_j) \leq 0$.

Thus in any case a single replacement makes the right hand side of (3.8) smaller for piecewise linear functions. Since the convex envelope is obtained by a sequence of replacements, (3.8) follows for piecewise linear functions. The general case can be obtained by approximation in the C^1-norm. □

Due to the special assumption (3.3) and its consequences (3.5), we can prove that the transformation $\mathcal{T}^i$ in $\Gamma_i(s, \bar{u})$ generated by the system

$$\left\{\begin{array}{rcl} u(\tau) &=& \bar{u} + \displaystyle\int_0^\tau \tilde{r}_i\big(u(\xi), v_i(\xi), \sigma_i(\xi)\big) d\xi \\ v_i(\tau) &=& \tilde{f}_i(\tau;\gamma) - \mathrm{conv}_{[0,s]}\tilde{f}_i(\tau;\gamma) \\ \sigma_i(\tau) &=& \displaystyle\frac{d}{d\tau}\mathrm{conv}_{[0,s]}\tilde{f}_i(\tau;\gamma) \end{array}\right. \tag{3.9}$$

with

$$\tilde{f}_i(\tau;\gamma) = \int_0^\tau \tilde{\lambda}_i\big(u(\xi), v_i(\xi), \sigma_i(\xi)\big) d\xi, \tag{3.10}$$

is a contraction in $\Gamma_i(s, \bar{u})$ for a new distance.

Proposition 3.2. *Define the distance $D(\cdot,\cdot)$ in $\Gamma_i(s, \bar{u})$ by*

$$D(\gamma, \gamma') = \delta_1 \|u - u'\|_{L^\infty} + \|v_i - v_i'\|_{L^1} + \|v_i\sigma_i - v_i'\sigma_i'\|_{L^1}, \tag{3.11}$$

where

$$\gamma = \big(u, v_i, \sigma_i\big) \in \Gamma_i(s, \bar{u}), \qquad \gamma' = \big(u', v_i', \sigma_i'\big) \in \Gamma_i(s, \bar{u}),$$

and $\delta_1 << 1$. *Then, if* $\mathcal{T}^i$ *is the transformation in* $\Gamma_i(s,\bar u)$ *defined by* (3.9)*, the following holds:*

$$D\big(\mathcal{T}^i(s)\gamma, \mathcal{T}^i(s)\gamma'\big) \leq \frac{1}{2} D\big(\gamma,\gamma'\big). \tag{3.12}$$

Note that the choice of $v_i\sigma_i$ instead of σ_i is suggested from the assumption (3.3) on $\tilde r_i$ and its consequences (3.5).

Proof. In [5] it is shown that (2.11) maps uniformly Lipschitz continuous curves into Lipschitz continuous curves with a uniform bound on the Lipschitz constant. The following computations show that (2.11) is a contraction w.r.t. the distance (3.11), for u sufficiently close to u^0:

$$\begin{aligned}\big\|u-u'\big\|_{L^\infty} &= \bigg|\int_0^s \Big(\tilde r_i\big(u,v_i,\sigma_i\big) - \tilde r_i\big(u',v_i',\sigma_i'\big)\Big)d\tau\bigg| \\ &\leq \int_0^s \bigg| D\tilde r_i\big(u-u'\big) + \tilde r_{i,v}\big(v_i-v_i'\big) + \frac{\tilde r_{i,\sigma}}{\max\big\{|v_i(\tau)|,|v_i'(\tau)|\big\}}\max\big\{|v_i(\tau)|,|v_i'(\tau)|\big\}\big(\sigma_i-\sigma_i'\big)\bigg| d\tau \\ &\leq C_0 s\Big\|u-u'\Big\|_{L^\infty} + C_0\Big\|v_i-v_i'\Big\|_{L^1} + C_0\Big\|v_i\sigma_i - v_i'\sigma_i'\Big\|_{L^1},\end{aligned}$$

$$\begin{aligned}\big\|v_i-v_i'\big\|_{L^1} &\leq s\big\|v_i(\tau)-v_i'(\tau)\big\|_{L^\infty} \leq \frac{s}{2}\int_0^s \Big|\tilde\lambda_i\big(u,v_i,\sigma_i\big) - \tilde\lambda_i\big(u',v_i',\sigma_i'\big)\Big| d\zeta \\ &\leq \frac{s}{2}\int_0^s \bigg| D\tilde\lambda_i\big(u-u'\big) + \tilde\lambda_{i,v}\big(v_i-v_i'\big) + \frac{\tilde\lambda_{i,\sigma}}{\max\big\{|v_i(\tau)|,|v_i'(\tau)|\big\}}\max\big\{|v_i(\tau)|,|v_i'(\tau)|\big\}\big(\sigma_i-\sigma_i'\big)\bigg| d\tau \\ &\leq C_0 s^2\Big\|u-u'\Big\|_{L^\infty} + C_0 s\|u-u^0\|_{L^\infty}\Big\|v_i-v_i'\Big\|_{L^1} + C_0 s\|u-u^0\|_{L^\infty}\Big\|v_i\sigma_i-v_i'\sigma_i'\Big\|_{L^1},\end{aligned}$$

$$\begin{aligned}\Big\|\sigma_i-\sigma_i'\Big\|_{L^1} &\leq \Big\|\tilde\lambda_i\big(u,v_i,\sigma_i\big) - \tilde\lambda_i\big(u',v_i',\sigma_i'\big)\Big\|_{L^1} \\ &\leq C_0 s\Big\|u-u'\Big\|_{L^\infty} + C_0\|u-u^0\|_{L^\infty}\Big\|v_i-v_i'\Big\|_{L^1} + C_0\|u-u^0\|_{L^\infty}\Big\|v_i\sigma_i-v_i'\sigma_i'\Big\|_{L^1}.\end{aligned}$$

We have used the fact that

$$\|v_i-v_i'\|_{L^1} + \big\|\max\{|v_i|,|v_i'|\}(\sigma_i-\sigma_i')\big\|_{L^1} \leq \mathcal{O}(1)\Big(\|v_i-v_i'\|_{L^1} + \|v_i\sigma_i - v_i'\sigma_i'\|_{L^1}\Big),$$

since σ_i, σ_i' are uniformly bounded by (2.15). It follows that

$$\begin{aligned}D\big(\mathcal{T}^i(s)\gamma,\mathcal{T}^i(s)\gamma'\big) &\leq C_0 s(\delta_1+2s)\|u-u'\|_{L^\infty} \\ &\quad + C_0\Big(\delta_1 + (1+s)\|u-u^0\|_{L^\infty}\Big)\Big(\|v_i-v_i'\|_{L^1} + \|v_i\sigma_i - v_i'\sigma_i'\|_{L^1}\Big) \leq \frac{1}{2}D\big(\gamma,\gamma'\big).\end{aligned}$$

This shows that (3.9) is a contraction if δ_1 is sufficiently small and u close to u^0. □

It follows from the fact that $\mathcal{T}^i$ is a strict contraction that

$$\lim_{n\to+\infty} D\Big(\big(\mathcal{T}^i\big)^n\gamma,\gamma\Big) \leq 2D\big(\mathcal{T}^i\gamma,\gamma\big). \tag{3.13}$$

A second consequence of the above proposition is the following:

Corollary 3.3. *Denote with* $\gamma(s;\bar u)$ *the solution to system* (3.9) *with initial data* $\bar u$ *and length* s*. If* $0<s'<s$*, then*

$$D\Big(\mathcal{T}^i_{s'}\gamma(s;\bar u)\big|_{[0,s']}, \gamma(s;\bar u)\big|_{[0,s']}\Big) \leq \mathcal{O}(1)|s'||s-s'|. \tag{3.14}$$

Proof. The proof follows directly from the definition of the map $\mathcal{T}^i$ and the estimate (3.13). □

In [5], [7] (or using the Corollary above) it is shown that the curve $T^i_s u$ is Lipschitz continuous w.r.t. s, $\bar u$ (see also the proof of Lemma 3.8 below, for Lipschitz dependence upon $\bar u$). These properties implies that the composed map

$$\mathcal{R} : (s_1,\dots,s_n) \mapsto T^n_{s_n}\circ T^{n-1}_{s_{n-1}}\circ\cdots\circ T^1_{s_1}\bar u, \tag{3.15}$$

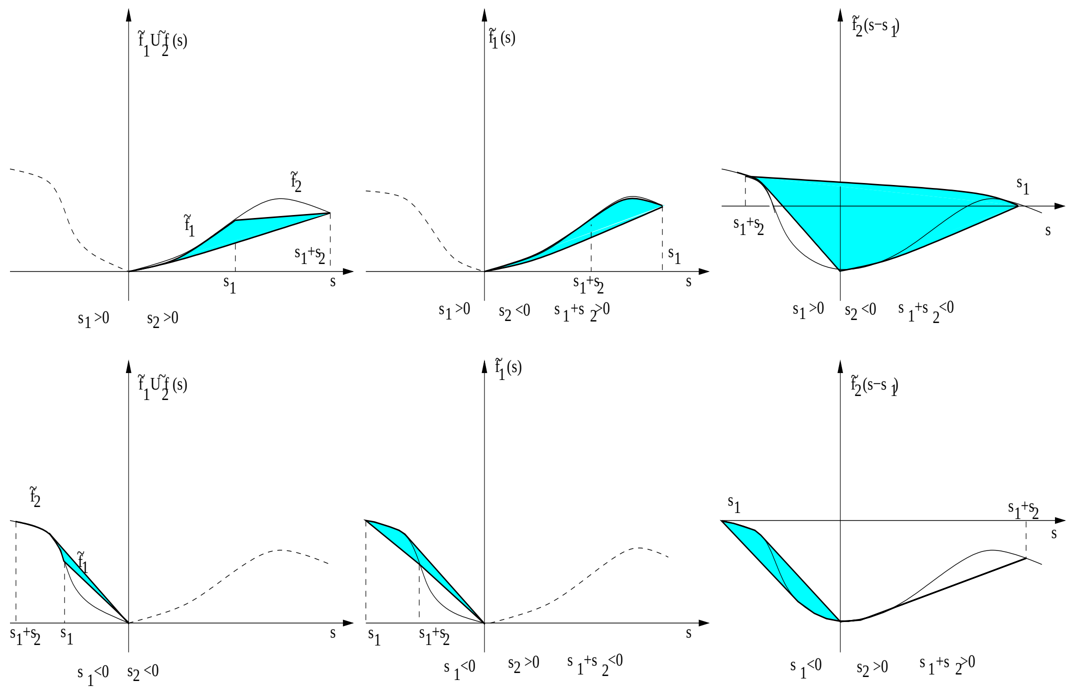

FIGURE 5. The amount of interaction J_i is represented as the colored area.

is Lipschitz continuous w.r.t. u, $s = (s_1, \dots, s_n)$, for $|u - \bar{u}|$, s close to 0. Moreover, it is invertible with Lipschitz inverse function, because one can check that for $s = 0$ the derivative of $\mathcal{R}$ exists and

$$D_s\big(\mathcal{R}_s u\big)\Big|_{s=0} = \Big[r_i(u)\Big].$$

The map $\mathcal{R}$ defines the unique entropic Riemann solver for hyperbolic systems in conservation form.

Remark 3.4. We note that

$$\begin{aligned} \big|u(s) - u'(s)\big| &\leq \big|\bar{u} - \bar{u}'\big| + C_0\Big\{s\big\|u - u'\big\|_\infty + \big\|v - v'\big\|_1 + \big\|v_i\sigma_i - v_i'\sigma_i'\big\|_1\Big\} \\ &= \big|\bar{u} - \bar{u}'\big| + \mathcal{O}(1)D(\gamma, \gamma'), \end{aligned} \tag{3.16}$$

so that we can estimate the distance of the points on $T^i_s u$ by means of the distance of the initial point and the distance of the two curves γ. In fact, one can always think that associated to the map $\mathcal{R}$ there are the reduced scalar fluxes $\tilde{f}_i$, $i = 1, \dots, n$. The convex envelope of these fluxes determines how the Riemann problem $[u, T^i_s u]$ is solved for all $i = 1, \dots, n$.

We observe moreover that the map (3.9) depends essentially on $n+1$ parameter: the starting point $\bar{u}$ and the length parameter s.

In the following we will denote with $(u(\tau; s, \bar{u}), v_i(\tau; s, \bar{u}), \sigma_i(\tau; s, \bar{u}))$ the components of $\gamma \in \Gamma(s, \bar{u})$, solution to (3.9), evaluated at τ.

We now give an equivalent definition of the *Amount of Interaction* I_i, which is strictly related to the singular approximation (3.2), i.e. to the function $\tilde{f}_i$.

Definition 3.5. Consider the point u_0 and $u_1 = T^i_{s_1} u_0$, $u_1' = T^i_{s_2} u_0$, for some $i \in \{1, \dots, n\}$. Assume s_1 positive. Let $\tilde{f}_1$ be the scalar reduced flux function for (3.9) with initial point u_0 in $[0, s_1]$, $\tilde{f}_2$ the reduced flux function for (3.9) in $[0, s_2]$ if $s_2 \geq 0$ or in $[s_2, 0]$ if $s_2 < 0$ and with initial point u_1.

We define *Amount of Interaction* J_i for the Riemann problem $[u_0, u_2]$ the following quantity:

(1) if $s_2 \geq 0$,

$$J_i \doteq \int_0^{s_1} \Big| \mathrm{conv}_{[0,s_1]} \tilde{f}_1(\xi) - \mathrm{conv}_{[0,s_1+s_2]} \big(\tilde{f}_1 \cup \tilde{f}_2\big)(\xi) \Big| d\xi + \int_{s_1}^{s_1+s_2} \Big| \mathrm{conv}_{[0,s_2]} \big(\tilde{f}_i(s_1) + \tilde{f}_2(\xi - s_1)\big) - \mathrm{conv}_{[0,s_1+s_2]} \big(\tilde{f}_1 \cup \tilde{f}_2\big)(\xi) \Big| d\xi, \tag{3.17}$$

where $\tilde{f}_1 \cup \tilde{f}_2$ is the function defined in $[0, s_1+s_2]$ as

$$\tilde{f}_1 \cup \tilde{f}_2(s) \doteq \begin{cases} \tilde{f}_1(s) & s \in [0, s_1] \\ \tilde{f}_1(s_1) + \tilde{f}_2(s - s_1) & s \in [s_1, s_2] \end{cases} \tag{3.18}$$

(2) if $-s_1 \leq s_2 < 0$,

$$J_i \doteq \int_0^{s_1+s_2} \Big| \mathrm{conv}_{[0,s_1]} \tilde{f}_1(\xi) - \mathrm{conv}_{[0,s_1+s_2]} \tilde{f}_1(\xi) \Big| d\xi + \int_{s_1+s_2}^{s_1} \Big| \mathrm{conv}_{[0,s_1]} \tilde{f}_1(\xi) - \mathrm{conc}_{[s_1+s_2,s_1]} \tilde{f}_1(\xi) \Big| d\xi; \tag{3.19}$$

(3) if $s_2 < -s_1$,

$$J_i \doteq \int_{s_1+s_2}^{0} \Big| \mathrm{conc}_{[s_2,0]} \tilde{f}_2(\xi) - \mathrm{conc}_{[s_2,-s_1]} \tilde{f}_2(\xi) \Big| d\xi + \int_0^{s_1} \Big| \mathrm{conc}_{[s_2,0]} \tilde{f}_2(\xi) - \mathrm{conv}_{[-s_1,0]} \tilde{f}_2(\xi) \Big| d\xi. \tag{3.20}$$

If $s_1 < 0$, substitute concave with convex in definition 3.5 (fig. 5).

Remark 3.6. In all cases, we can represent the quantity J_i as an area, fig. 5. This is the extension to the vector case of the interaction estimates for a scalar equation

$$u_t + f(u)_x = 0,$$

obtained in [9]. Here however the flux function $\tilde{f}_i$ does not depend only on u, but on the $n+2$ variables (u, v_i, σ_i), i.e. on the line γ.

We could have written, in case 2) of the above definition,

$$\mathrm{conc}_{[s_1+s_2,s_1]} \tilde{f}_2(\xi - s_1) \quad \text{instead of} \quad \mathrm{conc}_{[s_1+s_2,s_1]} \tilde{f}_1(\xi),$$

and in case 3),

$$\mathrm{conv}_{[0,s_1]} \tilde{f}_1(\xi) \quad \text{instead of} \quad \mathrm{conv}_{[0,s_1]} \tilde{f}_2(\xi - s_1).$$

In fact, it is possible to prove, by using the same procedure of the next lemmas, that, if K_i is the amount of interaction obtained with the substitutions above, then

$$K_i = \big(1 + \mathcal{O}(1)|s_1||s_2|\big) J_i,$$

so that the two quantities are equivalent.

In the same way one can prove the equivalence of definition 3.5 with the definition given in the introduction. Clearly (3.17) is equal to (1.20). Using the contraction property (3.12) and following the same ideas of the next lemmas, one can show that

$$I_i = \big(1 + \mathcal{O}(1)|s_1||s_2|\big) J_i,$$

in the general case. Note that the quantity J_i is defined for the particular singular approximation (3.2). If one consider for example the semidiscrete approximation, then the functions $\tilde{f}_i$ are slightly different, so that in definition 3.5 we will obtain a different value of J_i. Since all these J_i's are equivalent to I_i, it is not important which one we choose to measure the amount of interaction.

We now prove the first result of this note.

Theorem 3.7. *Consider* 3 *points* u_0, u_1, u_2, *and let* s_1, s_2, s *be such that*

$$u_1 = \mathcal{R}(s_1, u_0), \quad u_2 = \mathcal{R}(s_2, u_1), \quad u_2 = \mathcal{R}(s, u_0). \tag{3.21}$$

The following estimate holds

$$\big|s - (s_1 + s_2)\big| \leq \mathcal{O}(1) \left\{ \sum_{i<j} |s_{1,j}||s_{2,i}| + \sum_i J_i \right\}, \tag{3.22}$$

where with J_i *we denote the amount of interaction of the* i*-th family, computed by means of definition 3.5.*

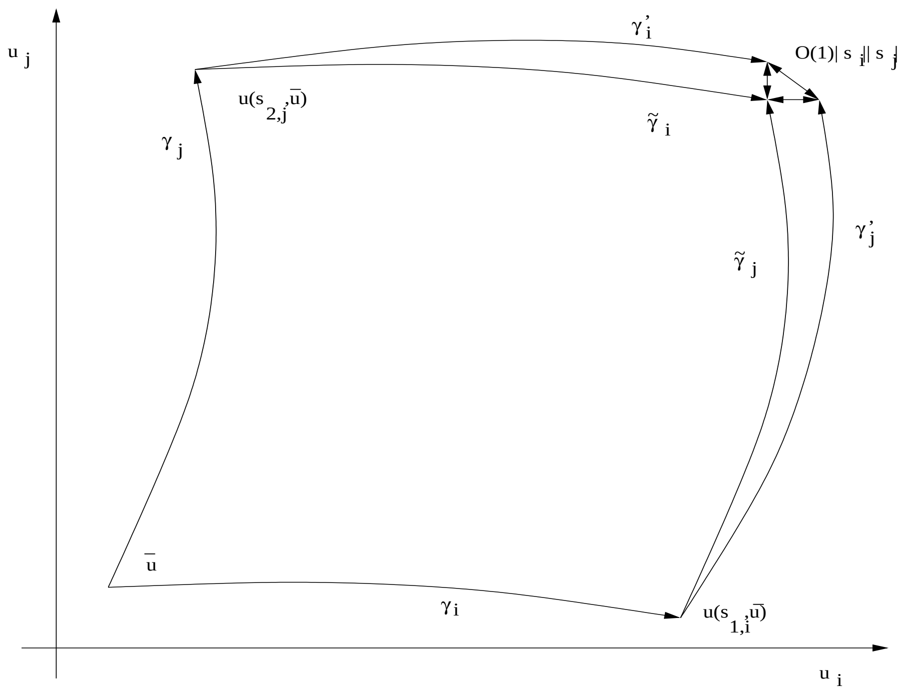

FIGURE 6. Transversal terms estimate of Lemma 3.8.

To prove this theorem, we consider first special cases, i.e. when $[u_0, u_1]$ and $[u_1, u_2]$ are connected by the admissible curve of only one family:

$$u_1 = T^i_{s_{1,i}} u_0, \qquad u_2 = T^j_{s_{2,j}} u_1, \tag{3.23}$$

with $i, j \in \{1, \dots, n\}$. Next we show that the general case follows from these particular results.

The basic ingredient in the proof of the next lemmas is the following: we construct a line $\tilde{\gamma}$ for which the left hand side of (3.22) is 0. Next we show that the following holds:

$$D\big(\mathcal{T}^i\tilde{\gamma}, \tilde{\gamma}\big) \simeq \text{ left hand side of (3.22)},$$

so that, if γ is the right solution, by the strict contraction one obtains

$$D\big(\gamma, \tilde{\gamma}\big) \simeq \text{ left hand side of (3.22)}.$$

As first simple case, we consider $i \neq j$ in (3.23), i.e. the interaction of waves of different families.

Lemma 3.8. *For any $i \neq j$ and $\bar{u}$ close to u^0, we have*

$$\Big| T^i_{s_{1,i}} \circ T^j_{s_{2,j}} \bar{u} - T^j_{s_{2,j}} \circ T^i_{s_{1,i}} \bar{u} \Big| \leq \mathcal{O}(1) \big| s_{1,i} \big| \big| s_{2,j} \big|. \tag{3.24}$$

Proof. We first prove the following estimate. Let γ be the solution to (2.11) starting in $\bar{u}$ with length s, and let γ' be the solution starting in $\bar{u}'$ with the same length s. Define

$$\tilde{\gamma} = \Big(\bar{u}' + \big(u(\tau; s, \bar{u}) - \bar{u}\big), v_i(\tau; s, \bar{u}), \sigma_i(\tau; s, \bar{u})\Big),$$

i.e. the translated of the curve γ in $\bar{u}'$. With a direct computation and using the strict contraction property, one has that

$$D(\gamma', \tilde{\gamma}) \leq \mathcal{O}(1) D\big(\mathcal{T}\tilde{\gamma}, \tilde{\gamma}\big) = \mathcal{O}(1)|s|\big|\bar{u}' - \bar{u}\big|. \tag{3.25}$$

To prove the lemma, we now consider the following curves, fig. 6:

- the curve $\gamma_i(s_{1,i}, \bar{u})$ of the i-th family starting in $\bar{u}$ with length $s_{1,i}$, and the curve $\gamma_j(s_{2,j}, \bar{u})$ of the j-th family starting in $\bar{u}$ with length $s_{2,j}$;

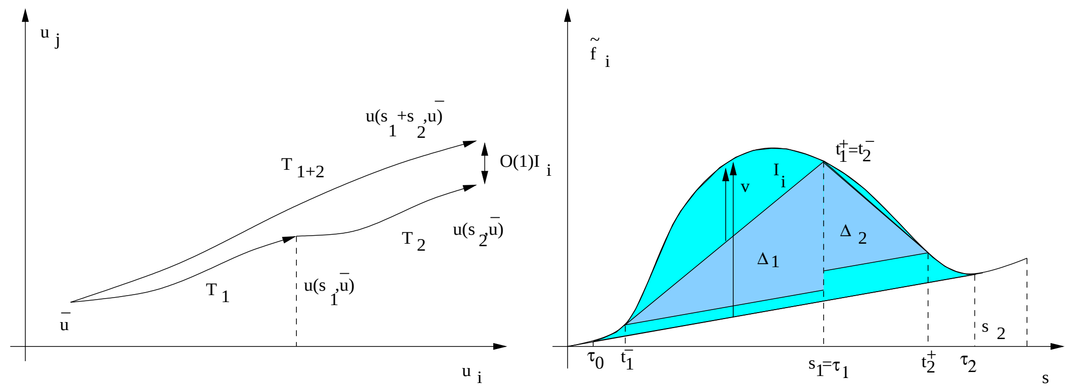

FIGURE 7. The case $s_1 \cdot s_2 > 0$.

- the curve $\gamma'_i(s_{1,i}, T^j_{s_{2,j}}\bar u)$ of the i-th family with length $s_{1,i}$ starting in $T^j_{s_{2,j}}\bar u$, the end point of γ_j, and similarly the curve $\gamma'_j(s_{2,j}, T^i_{s_{1,i}}\bar u)$ of the j-th family with length $s_{2,j}$ starting in $T^i_{s_{1,i}}\bar u$, the end point of γ_i;
- the curve
$$\tilde\gamma_i(\tau) \doteq \Big(T^j_{s_{2,j}}\bar u + \big(u(\tau; s_{1,j}, \bar u) - \bar u\big), v_i(\tau; s, \bar u), \sigma_i(\tau; s, \bar u)\Big),$$
which is the translated curve γ_i in $T^j_{s_{2,j}}\bar u$, and the curve
$$\tilde\gamma_j(\tau) \doteq \Big(T^i_{s_{1,i}}\bar u + \big(u(\tau; s, \bar u) - \bar u\big), v_j(\tau; s, \bar u), \sigma_j(\tau; s, \bar u)\Big),$$
i.e. the translated curve γ_j in $T^i_{s_{1,i}}\bar u$.

Using (3.25) it follows that
$$D(\gamma'_j, \tilde\gamma_j) = \mathcal{O}(1)|s_i||s_j|, \qquad D(\gamma'_i, \tilde\gamma_i) = \mathcal{O}(1)|s_i||s_j|.$$
Since the two endpoints $u(\tilde\gamma_i)$, $u(\tilde\gamma_j)$ of $\tilde\gamma_i$ and $\tilde\gamma_j$ coincide, we have with easy computations
$$\begin{aligned}\Big|T^i_{s_{1,i}} \circ T^j_{s_{2,j}}\bar u - T^j_{s_{2,j}} \circ T^i_{s_{1,i}}\bar u\Big| &= \Big|T^i_{s_{1,i}} \circ T^j_{s_{2,j}}\bar u - u(\tilde\gamma_i) + u(\tilde\gamma_j) - T^j_{s_{2,j}} \circ T^i_{s_{1,i}}\bar u\Big| \\ &\le \Big|T^i_{s_{1,i}} \circ T^j_{s_{2,j}}\bar u - u(\tilde\gamma_i)\Big| + \Big|u(\tilde\gamma_j) - T^j_{s_{2,j}} \circ T^i_{s_{1,i}}\bar u\Big| \\ &\le D\Big(\tilde\gamma_i, \gamma'_i\big(s_{1,i}, T^j_{s_{2,j}}\bar u\big)\Big) + D\Big(\tilde\gamma_j, \gamma'_j\big(s_{2,j}, T^i_{s_{1,i}}\bar u\big)\Big) \\ &= \mathcal{O}(1)\big|s_{1,i}\big|\big|s_{2,j}\big|.\end{aligned}$$
This concludes the proof of the lemma. □

Next we study the case $i = j$. Assuming $s_{1,i} > 0$ for definiteness, we have to consider 3 cases. As we said in definition 3.5, if $s_{1,i} < 0$, one should exchange convex envelopes with concave ones, and viceversa.

For simplicity, in the following we shall write s_1, s_2 instead of $s_{1,i}$, $s_{2,j}$.

Lemma 3.9. *For $s_1, s_2 > 0$, the following holds*
$$\begin{aligned}(3.26)\qquad \Big|T^i_{s_1+s_2}\bar u - T^i_{s_2} \circ T^i_{s_1}\bar u\Big| &\le \mathcal{O}(1)\int_0^{s_1}\Big|\mathrm{conv}_{[0,s_1]}\tilde f_1(\xi) - \mathrm{conv}_{[0,s_1+s_2]}\big(\tilde f_1 \cup \tilde f_2\big)(\xi)\Big|d\xi \\ &\quad + \mathcal{O}(1)\int_{s_1}^{s_1+s_2}\Big|\mathrm{conv}_{[s_1,s_2]}\tilde f_2(\xi - s_1) - \mathrm{conv}_{[0,s_1+s_2]}\big(\tilde f_1 \cup \tilde f_2\big)(\xi)\Big|d\xi,\end{aligned}$$
i.e. the distance of the two end points of $T^i_{s_1+s_2}\bar u$ and $T^i_{s_2} \circ T^i_{s_1}\bar u$ is controlled by the amount of interaction J_i of definition 3.5.

Proof. In the following we assume that $\tilde f_2$, $\gamma_2(s_2, T^i_{s_1}\bar u)$ are defined on the interval $[s_1, s_1 + s_2]$. Let us write
$$\gamma_1(\tau; s_1, \bar u) = \Big(u(\tau), v_i(\tau), \sigma_i(\tau)\Big) \qquad \tau \in [0, s_1],$$

for the solution of (3.9) starting in $\bar{u}$ with length s_1, and

$$\gamma_2\big(\tau; s_2, T^i s_1 \bar{u}\big) = \Big(u'(\tau), v_i'(\tau), \sigma_i'(\tau)\Big) \qquad \tau \in [s_1, s_1+s_2],$$

for the solution to (3.9) starting in $T^i_{s_1}\bar{u}$. Denote moreover with $v_i''(\tau)$, $\sigma_i''(\tau)$, $\tau \in [0, s_1+s_2]$ the functions

$$v_i''(\tau) = \tilde{f}_1 \cup \tilde{f}_2(\tau) - \mathrm{conv}_{[0,s_1+s_2]}\tilde{f}_1 \cup \tilde{f}_2(\tau), \qquad \sigma_i''(\tau) = \frac{d}{d\tau}\mathrm{conv}_{[0,s_1+s_2]}\tilde{f}_1 \cup \tilde{f}_2(\tau).$$

We recall that $\tilde{f}_1 \cup \tilde{f}_2$ is defined in (3.18). By a direct computation, we have the estimate

$$\begin{aligned}(3.27)\quad & D\big(\mathcal{T}_i(\gamma_1\cup\gamma_2), \gamma_1\cup\gamma_2\big) \\ &\le \big\|v_i - v_i''\big\|_{L^1(0,s_1)} + \big\|v_i' - v_i''\big\|_{L^1(s_1,s_1+s_2)} + \big\|v_i\sigma_i - v_i''\sigma_i''\big\|_{L^1(0,s_1)} + \big\|v_i'\sigma_i' - v_i''\sigma_i''\big\|_{L^1(s_1,s_1+s_2)} \\ &\le \Big\|\big(\mathrm{conv}_{[0,s_1]}\tilde{f}_1\big) \cup \big(\mathrm{conv}_{[s_1,s_1+s_2]}\tilde{f}_2\big) - \mathrm{conv}_{[0,s_1+s_2]}\tilde{f}_1 \cup \tilde{f}_2\Big\|_{L^1(0,s_1+s_2)} \\ &\quad + \Big\|\big(|v_1|+|v_2|\big)\Big(d\big((\mathrm{conv}_{[0,s_1]}\tilde{f}_1) \cup (\mathrm{conv}_{[s_1,s_1+s_2]}\tilde{f}_2)\big) - d\mathrm{conv}_{[0,s_1+s_2]}\big(\tilde{f}_1 \cup \tilde{f}_2\big)\Big)\Big\|_{L^1(0,s_1+s_2)}.\end{aligned}$$

Let τ_0, τ_2 be the two points where the convex envelope of $\tilde{f}_1 \cup \tilde{f}_2$ in $[0, s_1+s_2]$ meets the convex envelopes of $\tilde{f}_1$ and $\tilde{f}_2$ in $[0,s_1]$, $[s_1, s_1+s_2]$, respectively, and let $\tau_1 = s_1$ be the last point of the interval of definition of $\tilde{f}_1$. Of course, the integrals in the right hand side of (3.27) can be restricted to the interval $[\tau_0, \tau_2]$. Let (t_α^-, t_α^+), $\alpha \in \mathbb{N}$, be the segments in which $\mathrm{conv}_{[0,s_1]}\tilde{f}_1 \le \tilde{f}_1$ and $d\mathrm{conv}_{[0,s_1]}\tilde{f}_1/d\tau$ is constant. Note that, since f_1 is $C^{1,1}$, because it is the integral of a Lipschitz function, we can estimate v_i in (t_α^-, t_α^+) as

$$v_i = \mathcal{O}(1)(t_\alpha^+ - t_\alpha^-).$$

The same is valid for v_i', i.e. if $(t_{\alpha'}^-, t_{\alpha'}^+)$ denotes the intervals where $\mathrm{conv}_{[s_1,s_1+s_2]}\tilde{f}_2 \le \tilde{f}_2$ and the derivative of $\mathrm{conv}_{[s_1,s_1+s_2]}\tilde{f}_2$ is constant, then

$$v_i' = \mathcal{O}(1)(t_{\alpha'}^+ - t_{\alpha'}^-),$$

in $(t_{\alpha'}^-, t_{\alpha'}^+)$. After some computations one obtains

$$\begin{aligned}&\Big\|\big(|v_1|+|v_2|\big)\Big(d\big((\mathrm{conv}_{[0,s_1]}\tilde{f}_1) \cup (\mathrm{conv}_{[s_1,s_1+s_2]}\tilde{f}_2)\big) - d\mathrm{conv}_{[0,s_1+s_2]}\big(\tilde{f}_1 \cup \tilde{f}_2\big)\Big)\Big\|_{L^1(0,s_1+s_2)} \\ &= \sum_\alpha \int_{t_\alpha^-}^{t_\alpha^+} \Big|v_i\big(d\mathrm{conv}_{[0,s_1]}\tilde{f}_1 - d\mathrm{conv}_{[0,s_1+s_2]}(\tilde{f}_1\cup\tilde{f}_2)\big)\Big| d\xi \\ &\quad + \sum_{\alpha'} \int_{t_{\alpha'}^-}^{t_{\alpha'}^+} \Big|v_i'\big(d\mathrm{conv}_{[s_1,s_1+s_2]}\tilde{f}_2 - d\mathrm{conv}_{[0,s_1+s_2]}(\tilde{f}_1\cup\tilde{f}_2)\big)\Big| d\xi \\ &\le \mathcal{O}(1)\sum_\alpha \big(t_\alpha^+ - t_\alpha^-\big)\big(t_\alpha^+ - t_\alpha^-\big)\big(\sigma_\alpha - \sigma\big) + \mathcal{O}(1)\sum_{\alpha'} \big(t_{\alpha'}^+ - t_{\alpha'}^-\big)\big(t_{\alpha'}^+ - t_{\alpha'}^-\big)\big(\sigma_{\alpha'} - \sigma\big)\end{aligned}$$

where

$$\sigma_\alpha = \frac{\tilde{f}_1(t_\alpha^+) - \tilde{f}_1(t_\alpha^-)}{t_\alpha^+ - t_\alpha^-}, \qquad \sigma_{\alpha'} = \frac{\tilde{f}_2(t_{\alpha'}^+) - \tilde{f}_2(t_{\alpha'}^-)}{t_\alpha^+ - t_\alpha^-}, \qquad \sigma = \frac{\tilde{f}_1 \cup \tilde{f}_2(\tau_2) - \tilde{f}_1(\tau_0)}{\tau_2 - \tau_0}.$$

It is clear that the last sum is bounded by twice J_i, because it is twice the area of the small triangles

$$\begin{aligned}\Delta_\alpha &= \Big(\big(t_\alpha^-, \tilde{f}_1(t_\alpha^-)\big), \big(t_\alpha^+, \tilde{f}_1(t_\alpha^-) + \sigma(t_\alpha^+ - t_\alpha^-)\big), \big(t_\alpha^+, \tilde{f}_1(t_\alpha^+)\big)\Big), \\ \Delta_{\alpha'} &= \Big(\big(t_{\alpha'}^-, \tilde{f}_2(t_{\alpha'}^-)\big), \big(t_{\alpha'}^-, \tilde{f}_2(t_{\alpha'}^+) + \sigma(t_{\alpha'}^- - t_{\alpha'}^+)\big), \big(t_{\alpha'}^+, \tilde{f}_2(t_{\alpha'}^+)\big)\Big).\end{aligned}$$

These triangles are certainly contained in the area enclosed by $\tilde{f}_1 \cup \tilde{f}_2$ and $\mathrm{conv}_{[0,s_1+s_2]}\tilde{f}_1 \cup \tilde{f}_2$. Thus we obtain finally that

$$D\big(\mathcal{T}_i(\gamma_1\cup\gamma_2), \gamma_1\cup\gamma_2\big) \le \mathcal{O}(1)J_i,$$

and, by the strict contraction property, (3.26) follows. □

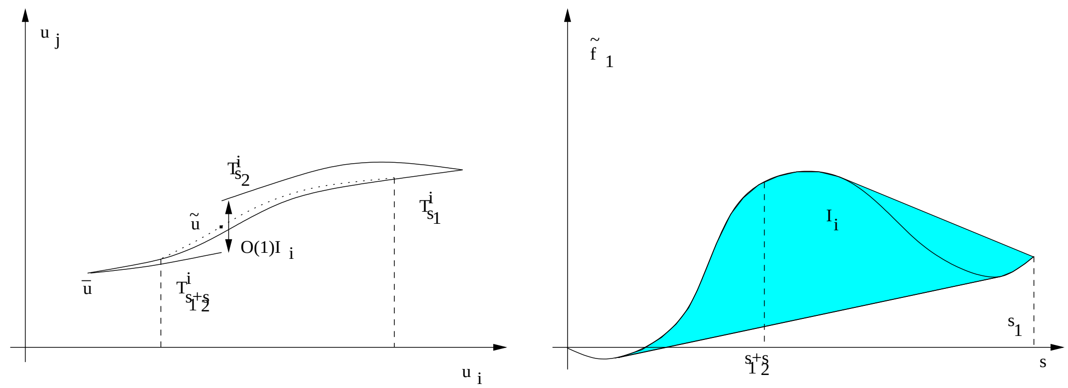

FIGURE 8. The case $-s_1 \le s_2 < 0$. Note that the point $\tilde u$ is on a travelling profile, not on $T^i_s \bar u$.

Lemma 3.10. *If $s_2 < 0 < s_1$ and $s_1 + s_2 \ge 0$, then*

$$\big|T^i_{s_1+s_2}\bar u - T^i_{s_1}\circ T^i_{s_2}\bar u\big| \le \mathcal{O}(1)\Big\|\mathrm{conv}_{[0,s_1]}\tilde f_1 - \mathrm{conv}_{[0,s_1+s_2]}\tilde f_1\Big\|_{L^1(0,s_1+s_2)} + \mathcal{O}(1)\Big\|\mathrm{conv}_{[0,s_1]}\tilde f_1 - \mathrm{conc}_{[s_1+s_2,s_1]}\tilde f_1\Big\|_{L^1(s_1+s_2,s_1)}. \tag{3.28}$$

Proof. The proof is similar to the previous one.

Define $s_1 + s_2 = \tau_1$. Let $\tilde u = u(\tau_1; s_1, \bar u)$ be the value of the solution to (2.11) starting in $\bar u$ with length s_1, evaluated in τ_1. We can write first

$$\Big|T^i_{\tau_1}\bar u - T^i_{s_1}\circ T^i_{s_2}\bar u\Big| \le \big|T^i_{\tau_1}\bar u - \tilde u\big| + \big|\tilde u - T^i_{s_1}\circ T^i_{s_2}\bar u\big|. \tag{3.29}$$

One can use the same procedure of Lemma 3.9 to estimate the first part of the left hand side of (3.29) as

$$\big|T^i_{\tau_1}u - \bar u\big| \le \mathcal{O}(1)\Big\|\mathrm{conv}_{[0,s_1]}\tilde f_1 - \mathrm{conv}_{[0,s_1+s_2]}\tilde f_1\Big\|_{L^1(0,s_1+s_2)}. \tag{3.30}$$

Regarding the second part of the left hand side of (3.29), consider the curve $\tilde\gamma$ starting in $T^i_{s_1}u$ with length s_2 defined as

$$\tilde\gamma(\tau) \doteq \gamma(s_1-\tau; s_1, \bar u) = \Big(u(s_1-\tau), v_i(s_1-\tau), \sigma_i(s_1-\tau)\Big), \qquad \tau \in [s_2, 0].$$

We will use $\tilde\gamma$ to estimate the distance between $\tilde u$ and $T^i_{s_2}\circ T^i_{s_1}\bar u$.

A direct computation using the integral transformation (3.9) considered in $[s_2, 0]$ gives

$$D\big(\mathcal{T}_i\tilde\gamma, \tilde\gamma\big) \le \Big\|\mathrm{conv}_{[0,s_1]}\tilde f_1 - \mathrm{conc}_{[\tau_1,s_1]}\tilde f_1\Big\|_{L^1(\tau_1,s_1)} + \Big\|\big(\tilde f_1 - \mathrm{conv}_{[0,s_1]}\tilde f_1\big)d\mathrm{conv}_{[0,s_1]}\tilde f_1 - \big(\tilde f_1 - \mathrm{conc}_{[\tau_1,s_1]}\tilde f_1\big)d\mathrm{conc}_{[\tau_1,s_1]}\tilde f_1\Big\|_{L^1(\tau_1,s_1)}. \tag{3.31}$$

Since $\tilde f_1 - \mathrm{conv}_{[0,s_1]}\tilde f_1$ and $\tilde f_1 - \mathrm{conc}_{[\tau_1,s_1]}\tilde f_1$ have different signs and the speeds σ_i are uniformly bounded, it follows that

$$\Big\|\big(\tilde f_1 - \mathrm{conv}_{[0,s_1]}\tilde f_1\big)d\mathrm{conv}_{[0,s_1]}\tilde f_1 - \big(\tilde f_1 - \mathrm{conc}_{[\tau_1,s_1]}\tilde f_1\big)d\mathrm{conc}_{[\tau_1,s_1]}\tilde f_1\Big\|_{L^1(\tau_1,s_1)} \le \mathcal{O}(1)\big\|\mathrm{conv}_{[0,s_1]}\tilde f_1 - \mathrm{conc}_{[\tau_1,s_1]}\tilde f_1\big\|_{L^1(\tau_1,s_1)}. \tag{3.32}$$

Using (3.30), (3.31), (3.32) and the strict contraction property (3.13), we obtain (3.28). □

Remark 3.11. Using the same estimates of the above proof, one can substitute

$$\mathrm{conc}_{[s_1+s_2,s_1]}\tilde f_1 \qquad \text{with} \qquad \mathrm{conc}_{[s_1+s_2,s_1]}\tilde f_2$$

in (3.28). In fact, sing the same techniques of the previous lemmas, we can show that

$$\big\|\mathrm{conv}_{[\tau_1,s_1]}\tilde f_1 - \mathrm{conc}_{[\tau_1,s_1]}\tilde f_1\big\|_{L^1(\tau_1,s_1)} \simeq \big\|\mathrm{conv}_{[\tau_1,s_1]}\tilde f_1 - \mathrm{conc}_{[\tau_1,s_1]}\tilde f_2\big\|_{L^1(\tau_1,s_1)}. \tag{3.33}$$

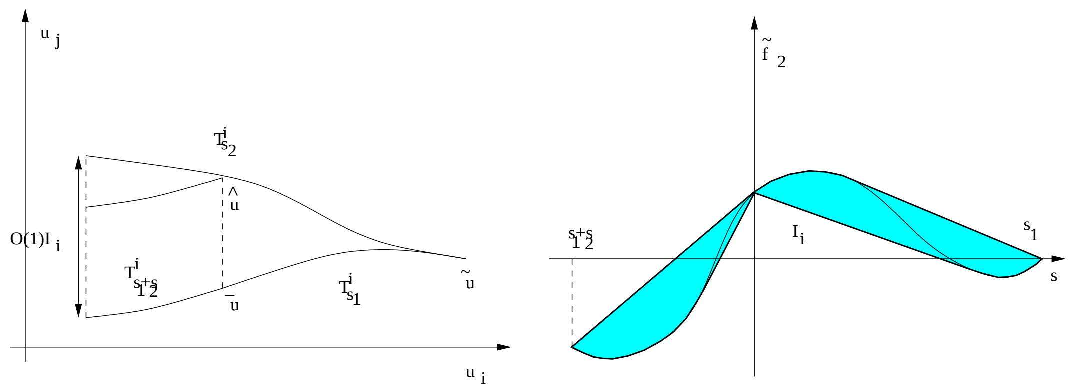

FIGURE 9. The case $s_1 \cdot s_2 < 0$, with $s_1 + s_2 < 0$.

This shows again that J_i is equivalent to I_i, because we can replace $d\mathrm{conv}_{[\tau_1,s_1]}\tilde f_1/d\tau$ with the speeds of the i-th waves in the Riemann problem $[u_0, u_2]$.

Lemma 3.12. *We have for $s_2 < -s_1 < 0$,*

$$\left|T^i_{s_1+s_2}\bar u - T^i_{s_1} \circ T^i_{s_2}\bar u\right| \le \mathcal{O}(1)\left\|\mathrm{conc}_{[s_2+s_1,0]}\tilde f_2 - \mathrm{conc}_{[s_2+s_1,s_1]}\tilde f_2\right\|_{L^1(s_2+s_1,0)} + \mathcal{O}(1)\left\|\mathrm{conv}_{[0,s_1]}\tilde f_2 - \mathrm{conc}_{[s_1+s_2,s_1]}\tilde f_2\right\|_{L^1(0,s_1)}. \tag{3.34}$$

Remark 3.13. The same remark holds here, i.e. we could have written (3.34) as

$$\left|T^i_{s_1+s_2}\bar u - T^i_{s_1} \circ T^i_{s_2}\bar u\right| \le \mathcal{O}(1)\left\|\mathrm{conc}_{[s_2+s_1,0]}\tilde f_2 - \mathrm{conc}_{[s_2+s_1,s_1]}\tilde f_2\right\|_{L^1(s_2+s_1,0)} + \mathcal{O}(1)\left\|\mathrm{conv}_{[0,s_1]}\tilde f_1 - \mathrm{conc}_{[s_1+s_2,s_1]}\tilde f_2\right\|_{L^1(0,s_1)},$$

or

$$\left|T^i_{s_1+s_2}\bar u - T^i_{s_1} \circ T^i_{s_2}\bar u\right| \le \mathcal{O}(1) I_i,$$

where I_i is defined in the introduction.

Proof. Let $\tilde u$ be the point $T^i_{s_1}\bar u$, and denote with $\hat u$ the point

$$\hat u \doteq u\big(-s_1; s_2, \tilde u\big). \tag{3.35}$$

By Lemma 3.10 and remark 3.11, it follows that

$$\big|\hat u - \bar u\big| \le \mathcal{O}(1)\left\|\mathrm{conv}_{[0,s_1]}\tilde f_2 - \mathrm{conc}_{[s_1+s_2,s_1]}\tilde f_2\right\|_{L^1(0,s_1)}. \tag{3.36}$$

and

$$\big|T^i_{s_2}\tilde u - T^i_{s_1+s_2}\hat u\big| \le \mathcal{O}(1)\left\|\mathrm{conc}_{[s_1+s_2,0]}\tilde f_2 - \mathrm{conc}_{[s_1+s_2,s_1]}\tilde f_2\right\|_{L^1(s_1+s_2,0)}. \tag{3.37}$$

Moreover, by the Lipschitz dependence of $T^i_s u$ on the initial point u, we have that

$$\big|T^i_{s_2+s_1}\hat u - T^i_{s_2+s_1}\bar u\big| \le \mathcal{O}(1)\left\|\mathrm{conv}_{[0,s_1]}\tilde f_2 - \mathrm{conc}_{[s_1+s_2,s_1]}\tilde f_2\right\|_{L^1(0,s_1)}. \tag{3.38}$$

Using (3.36), (3.37) and (3.38), we conclude that

$$\begin{aligned}\big|T^i_{s_1+s_2}\bar u - T^i_{s_1}\circ T^i_{s_2}\bar u\big| &\le \big|T^i_{s_2}\tilde u - T^i_{s_1+s_2}\hat u\big| + \big|T^i_{s_2+s_1}\hat u - T^i_{s_2+s_1}\bar u\big| \\ &\le \mathcal{O}(1)\left\|\mathrm{conc}_{[s_2+s_1,0]}\tilde f_2 - \mathrm{conc}_{[s_2,s_1]}\tilde f_2\right\|_{L^1(s_2+s_2,0)} \\ &\quad + \mathcal{O}(1)\left\|\mathrm{conv}_{[0,s_1]}\tilde f_1 - \mathrm{conc}_{[0,s_1]}\tilde f_1\right\|_{L^1(0,s_1)}.\end{aligned}$$

This concludes the proof. □

Now we prove Theorem 3.7.

Proof. Since the map is Lipschitz continuous, it is sufficient to prove that
(3.39)
$$T^n_{s_{1,n}+s_{2,n}} \circ \ldots \circ T^1_{s_{1,1}+s_{2,1}} u_0 - \Big(T^n_{s_{2,n}} \circ \ldots \circ T^1_{s_{2,1}} \Big) \circ \Big(T^n_{s_{1,n}} \circ \ldots \circ T^1_{s_{1,1}} \Big) u_0 \leq \mathcal{O}(1) \left\{ \sum_{i<j} |s_{1,j}||s_{2,i}| + \sum_i J_i \right\},$$
i.e. the final distance of the two points
$$\tilde u \doteq T^n_{s_{1,n}+s_{2,n}} \circ \ldots \circ T^1_{s_{1,1}+s_{2,1}} u_0 \quad \text{and} \quad u_2 = \Big(T^n_{s_{2,n}} \circ \ldots \circ T^1_{s_{2,1}} \Big) \circ \Big(T^n_{s_{1,n}} \circ \ldots \circ T^1_{s_{1,1}} \Big) u_0,$$
is of the order of the right hand side of (3.22).

With a finite number of applications of Lemma 3.8, we obtain the estimate
$$\Big| \big(T^n_{s_{2,n}} \circ T^n_{s_{1,n}} \big) \circ \ldots \circ \big(T^1_{s_{2,1}} \circ T^1_{s_{1,1}} \big) u_0 - u_2 \Big| \leq \mathcal{O}(1) \sum_{i<j} \big|s_{1,j}\big| \big|s_{2,i}\big|. \tag{3.40}$$
We are thus left with proving that
$$T^n_{s_{1,n}+s_{2,n}} \circ \ldots \circ T^1_{s_{1,1}+s_{2,1}} u_0 - \big(T^n_{s_{2,n}} \circ T^n_{s_{1,n}} \big) \circ \ldots \circ \big(T^1_{s_{2,1}} \circ T^1_{s_{1,1}} \big) u_0 = \mathcal{O}(1) \left\{ \sum_{i<j} \big|s_{1,j}\big| \big|s_{2,i}\big| + \sum_i J_i \right\},$$
where J_i is computed by means of definition 3.5. We show the first step, i.e. the estimate
$$T^1_{s_{1,1}+s_{2,1}} u_0 - \big(T^1_{s_{2,1}} \circ T^1_{s_{1,1}} \big) u_0 = \mathcal{O}(1) \left\{ \sum_{i<j} \big|s_{1,j}\big| \big|s_{2,i}\big| + J_1 \right\}. \tag{3.41}$$
From Lemma 3.9, Lemma 3.10 and Lemma 3.12, it follows that
$$T^1_{s_{1,1}+s_{2,1}} u_0 - \big(T^1_{s_{2,1}} \circ T^1_{s_{1,1}} \big) u_0 = J'_i,$$
where J'_i is the amount of interaction for the Riemann problems $[u_0, T^1_{s_{1,1}} u_0]$, $[T^1_{s_{1,1}} u_0, T^1_{s_{2,1}} \circ T^1_{s_{1,1}} u_0]$. If $\tilde f'_{2,i}$ denotes the scalar flux function for the Riemann problem $[T^1_{s_{1,1}} u_0, T^1_{s_{2,1}} \circ T^1_{s_{1,1}} u_0]$, and $\tilde f_{2,i}$ the flux function for the original Riemann problem $[u_1, T^1_{s_{2,1}} u_1]$, then by means of (3.25) we have the estimate
$$\big\| \tilde f'_{2,1} - \tilde f_{2,1} \big\|_{L^\infty} = \mathcal{O}(1) \sum_{i<j} \big|s_{1,j}\big| \big|s_{2,i}\big|,$$
so that it follows
$$\big| J'_i - J_i \big| \leq \mathcal{O}(1) \sum_{i<j} \big|s_{1,j}\big| \big|s_{2,i}\big|.$$
It follows that (3.41) is correct.

For the general case, one has
$$T^k_{s_{1,1}+s_{2,1}} \circ \Big(T^{k-1}_{s_{1,k}+s_{2,k}} \circ \ldots \circ T^1_{s_{1,1}+s_{2,1}} \Big) u_0 - \big(T^k_{s_{2,k}} \circ T^k_{s_{1,k}} \big) \circ \Big(T^{k-1}_{s_{1,k}+s_{2,k}} \circ \ldots \circ T^1_{s_{1,1}+s_{2,1}} \Big) u_0 = J'_i,$$
where J'_k satisfies
$$\big| J'_k - J_k \big| \leq \mathcal{O}(1) \left\{ \sum_{i<j} \big|s_{1,j}\big| \big|s_{2,i}\big| + \sum_{j<k} J_j \right\}.$$
This shows that
$$\begin{aligned} T^k_{s_{1,1}+s_{2,1}} \circ \Big(T^{k-1}_{s_{1,k}+s_{2,k}} \circ \ldots \circ T^1_{s_{1,1}+s_{2,1}} \Big) u_0 - \big(T^k_{s_{2,k}} \circ T^k_{s_{1,k}} \big) \circ \Big(T^{k-1}_{s_{1,k}+s_{2,k}} \circ \ldots \circ T^1_{s_{1,1}+s_{2,1}} \Big) u_0 \\ = \mathcal{O}(1) \left\{ \sum_{i<j} \big|s_{1,j}\big| \big|s_{2,i}\big| + \sum_{j\leq k} J_j \right\}, \end{aligned}$$
and from the above equation in the case $k = n$ and (3.40), the proof of Theorem 3.7 follows. □

We observe that, with a slight change in the proof of the above lemmas, we have actually proved the following stronger result. Associate to each Riemann problem $[u_0, u_1]$, n curves $\gamma_i(u_0, u_1)$ as follows: if

$$u_1 = T^n_{s_n} \circ \ldots \circ T^1_{s_1} u_0,$$

and the points p_i, $i = 0, \ldots, n$, are defined by

$$p_0 \doteq u_0, \qquad p_i \doteq T^i_{s_i} p_{i-1}, \qquad p_n \doteq u_1, \tag{3.42}$$

then each curve $\gamma_i(u_0, u_1)$ is the solution to (3.9), starting in p_{i-1} and with length s_i,

$$\gamma_i(u_0, u_1) \doteq \gamma_i(s_i, p_{i-1}). \tag{3.43}$$

Note that for each i we solve (3.9) with a different generalized eigenvector $\tilde{r}_i$, one for each family.

The distance (3.11) is suitable for curves with the same length s. We now generalize it to compare curves γ_i for different Riemann problems: if γ_i, γ_i' are curves in $\Gamma_i(s_i, u)$, $\Gamma_i(s_i', u')$, parameterized as in (2.15), define in fact the distance P as

$$P(\gamma, \gamma') \doteq D\big(\gamma_i|_{[o,\underline{s}_i]}, \gamma_i'|_{[0,\underline{s}_i]}\big) + \big|s_i - s_i'\big|, \tag{3.44}$$

where

$$\underline{s}_i = \min\big\{s_i, s_i'\big\}. \tag{3.45}$$

Note that when $s_i = s_i'$, P reduces to D.

If s_i, s_i' are greater than 0, we define the curve $\gamma_i \cup \gamma_i'$ as

$$\gamma_i \cup \gamma_i'(\tau) = \begin{cases} \gamma_i(\tau) & 0 \leq \tau \leq s_i \\ \gamma_i'(\tau - s_i) + \gamma_i(s_i) & s_i \leq \tau \leq s_i' \end{cases} \tag{3.46}$$

Following the same ideas of definition 3.5, and assuming for simplicity that $s_i > 0$, we define

$$\hat{\gamma}_i \doteq \begin{cases} \gamma_i \cup \gamma_i' & s_i, s_i' \geq 0 \\ \gamma_i\big|_{[0, s_i + s_i']} & -s_i \leq s_i' < 0 \\ \gamma_i'\big|_{[s_i', s_i + s_i']} & s_i' < -s_i < 0 \end{cases} \tag{3.47}$$

A similar definition can be given if $s_i < 0$.

Observing that one can state the above lemmas by replacing the distance of the final point by the distance of the curves γ associated to them, we can prove the following theorem:

Theorem 3.14. *Fix 3 point u_0, u_1, u_2. Let $\gamma_i(u_0, u_1)$, $\gamma_i(u_1, u_2)$, $i = 1, \ldots, n$ be the curves associated with the Riemann problems $[u_0, u_1]$, $[u_1, u_2]$, respectively, and let $\gamma_i(u_0, u_2)$, $ii = 1, \ldots, n$, be the curves associated with $[u_0, u_2]$. Denote with $\hat{\gamma}_i(u_0, u_1, u_2)$ the curve defined in (3.47) starting from the curves $\gamma_i(u_0, u_1)$ and $\gamma_i(u_1, u_2)$. Then*

$$\sum_i P\Big(\hat{\gamma}_i(u_0, u_1, u_2), \gamma_i(u_0, u_2)\Big) \leq \mathcal{O}(1) \left\{ \sum_{i<j} |s_{1,j}||s_{2,i}| + \sum_i J_i \right\}. \tag{3.48}$$

4. Decreasing functional for piecewise constant functions

Consider a piecewise constant function u, with Tot.Var.(u) sufficiently small, and let x_α, $\alpha \in \mathbb{N}$, be the points of discontinuity. To each jump $[u(x_\alpha-), u(x_\alpha+)]$ we can associate the n curves $\gamma_{\alpha,i}$, $i = 1, \ldots, n$, with length $s_{\alpha,i}$,

$$\gamma_{\alpha,i}(\tau) = \Big(u_{\alpha,i}(\tau), v_{\alpha,i}(\tau), \sigma_{\alpha,i}(\tau)\Big), \qquad \tau \in [0, s_{\alpha,i}],$$

where the vector $s_\alpha = (s_{\alpha,1}, \ldots, s_{\alpha,n})$ is obtained by solving

$$u(x_\alpha+) = T^n_{s_{\alpha,n}} \circ \ldots \circ T^1_{s_{\alpha,1}} u(x_\alpha-).$$

We will also denote with $\tilde{f}_{\alpha,i}$ the i-th reduced flux function for the curve $\gamma_{\alpha,i}$, computed by means of (3.10).

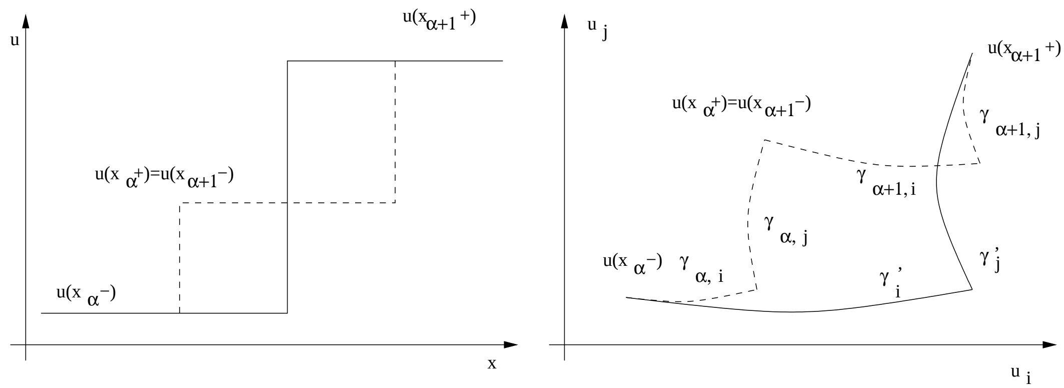

FIGURE 10. The replacement of two consecutive jumps in the function u.

We now replace two adjacent jumps in $x_{\bar\alpha}$, $x_{\bar\alpha+1}$ with the jump $[u(x_{\bar\alpha}-, u(x_{\bar\alpha+1}+)]$. Let u' be the new BV function obtained. Using Theorem 3.7, we have that

$$\text{Tot.Var.}(u') \leq \text{Tot.Var.}(u) + \mathcal{O}(1)\left\{\sum_{i<j} |s_{\bar\alpha+1,i}||s_{\bar\alpha,j}| + \sum_i J_i(\bar\alpha, \bar\alpha+1)\right\}, \tag{4.1}$$

where $J_i(\bar\alpha, \bar\alpha+1)$ is the amount of interaction for the Riemann problem $[u(x_{\bar\alpha}-), u(x_{\bar\alpha+1}+)]$, computed using definition 3.5.

In this section we prove the following proposition:

Proposition 4.1. *Define the functional $Q(u)$ as*

$$Q(u) \doteq \sum_{\alpha<\alpha'}\sum_{i<j} \big|s_{\alpha,j}\big|\big|s_{\alpha',i}\big| + \frac{1}{4}\sum_{\alpha,\alpha'}\sum_i \int_0^{|s_{\alpha,i}|}\int_0^{|s_{\alpha',i}|} \Big|\sigma_{\alpha,i}(\tau) - \sigma_{\alpha',i}(\tau')\Big| d\tau d\tau'. \tag{4.2}$$

Then, if we replace the two adjacent jumps in $x_{\bar\alpha}$, $x_{\bar\alpha+1}$ with the jump $[u(x_{\bar\alpha}-, u(x_{\bar\alpha+1}+)]$, the following holds:

$$Q(u') \leq Q(u) - c\left\{\sum_{i<j} |s_{\bar\alpha+1,i}||s_{\bar\alpha,j}| + \sum_i J_i\big(\bar\alpha, \bar\alpha+1\big)\right\}, \tag{4.3}$$

where c is a strictly positive constant.

A consequence of the above proposition is that, if C_1 is sufficiently large,

$$\text{Tot.Var.}(u') + C_1 Q(u') \leq \text{Tot.Var.}(u) + C_1 Q(u), \tag{4.4}$$

i.e. Q is a Glimm functional for systems in conservation form (3.1), without any assumption on the form of the flux f.

Remark 4.2. If each jump in x_α is composed of n admissible shocks of strength $s_{\alpha,i}$ and with speed $\sigma_{\alpha,i}$ (i.e. $\tilde f_{\alpha,i}$ is convex if $s_{\alpha,i} > 0$ or concave if $s_{\alpha,i} < 0$), then the functional Q becomes

$$Q(u) = \sum_{\alpha<\alpha'}\sum_{i<j} |s_{\alpha,j}||s_{\alpha',i}| + \frac{1}{2}\sum_{\alpha<\alpha'}\sum_i |s_{\alpha,j}||s_{\alpha',i}|\big|\sigma_{\alpha,i} - \sigma_{\alpha',i}\big|. \tag{4.5}$$

Thus, all waves of the same family are approaching, with a weight equal to the difference in speed. A simple computation shows that the second part of the functional, i.e. the part concerning waves of the same family, is of the third order w.r.t. the total variation.

In the original Glimm paper [10], a functional Q was constructed by means of the notion of approaching waves. We recall that two waves of the i-th genuinely nonlinear family are said to be approaching if at least one is a shock, and the functional Q is defined as

$$Q(u) = \sum_{i<j,x<y} |s_j(x)||s_i(y)| + \frac{1}{2}\sum_i \sum_{\text{appr.}} |s_i(x)||s_i(y)|. \tag{4.6}$$

We observe that in this functional, the second part is of second order w.r.t. the total variation. Note moreover that the first part of the functional, i.e. the one measuring the approaching waves of different families, is the same in (4.2).

In [11], the author introduce a similar functional, where the "difference in speed" of two shocks, let us say $s_{\alpha,i}$ and $s_{\alpha',i}$, $\alpha < \alpha'$, is computed by considering all the waves $s_{\beta,i}$ with $\alpha \leq \beta \leq \alpha'$. If $\sigma_{\beta,i}$ is their speed, then the weight is defined as

$$\sum_{\beta=\alpha}^{\alpha'-1} \big[\sigma_{\beta,i} - \sigma_{\beta+1,i}\big]^+,$$

where we denote with $[\cdot]^+$ the positive part. This functional is used in [12], where they prove the existence of a solution is the flux has a finite number of inflection points.

Differently from the above cases, in the form of (4.2) every wave is considered as approaching. In particular even a solution to a Riemann problem has $Q \neq 0$, while the original Glimm functional and Liu's functional are both 0 in this last case.

We finally recall that if each field λ_i has only one inflection point, it is possible to construct a decreasing functional $Q(u)$ in which the part corresponding to the approaching waves of the same family is still of second order w.r.t. the total variation [3]. In this particular case, one can prove also stability of the solution in the L^1-norm for 2×2 systems, see [2].

Before proving (4.3), we study how the value of the functional changes when we perturb the curves.

Lemma 4.3. *Let $\gamma_i(s_i, u)$, $\gamma'_i(s'_i, u')$ be solution to (3.9), with initial data u_0, u'_0 and length s_i, s'_i, respectively. Assume that γ_i is i-th curve of the solution to the Riemann problem in x_α, and replace it (and only this one) by γ'_i. Then the following holds:*

$$\big|Q(\gamma'_i, u) - Q(\gamma_i, u)\big| \leq \mathcal{O}(1) P(\gamma_i, \gamma'_i). \tag{4.7}$$

Note that in general we cannot substitute one curve γ_i without changing the curves γ_α: in fact the curves $\gamma_{\alpha,i}$ depend on the initial data $u(x_\alpha-)$. We imagine here just to change the values of $s_{\alpha,i}$ and $\sigma_{\alpha,i}$ and to keep the other strengths and speed fixed, for $\alpha' \neq \alpha$. With this interpretation, $Q(\gamma'_i, u)$ is not necessarily the value of Q on a BV function u, but is define on the lines γ.

Proof. We have

$$\begin{aligned}
Q(\gamma'_i, u) - Q(\gamma_i, u) = &\sum_{\alpha'<\alpha} \sum_{j>i} |s_{\alpha',j}| \big(|s'_i| - |s_i|\big) + \sum_{\alpha'>\alpha} \sum_{j<i} |s_{\alpha',j}| \big(|s'_i| - |s_i|\big) \\
&+ \frac{1}{2} \sum_{\alpha'} \int_0^{|s_{\alpha',i}|} \int_0^{|\underline{s}_i|} \Big(\big|\sigma_{\alpha',i}(\tau) - \sigma'_i(\tau')\big| - \big|\sigma_{\alpha',i}(\tau) - \sigma_i(\tau')\big| \Big) d\tau d\tau' \\
&+ \mathcal{O}(1) \text{Tot.Var.}(u) \big|s_i - s'_i\big| \\
\leq\ & \mathcal{O}(1) \text{Tot.Var.}(u) |s - s'| + \mathcal{O}(1) \text{Tot.Var.}(u) \big\| d\text{conv} f - d\text{conv} f' \big\|_{L^1} \\
\leq\ & \mathcal{O}(1) \text{Tot.Var.}(u) \Big(\big|s - s'\big| + D(\gamma, \gamma') \Big).
\end{aligned}$$

The values $\underline{s}_i$ is defined in (3.45) and we have used the estimate

$$\big\| d\text{conv} f - \text{conv} f' \big\|_{L^1} \leq \big\| df - df' \big\|_{L^1} \leq \mathcal{O}(1) D(\gamma, \gamma'), \tag{4.8}$$

consequence of (3.5) and the definition of $\tilde{f}_i$, (3.10). □

Remark 4.4. Note that, by means of (4.8), the same result holds if γ_i, γ'_i are obtained by $\gamma_{i,0}$, $\gamma'_{i,0}$ with a single iteration of system (3.9), i.e.

$$\gamma_i = \mathcal{T}_i \gamma_{i,0}, \qquad \gamma'_i = \mathcal{T}_i \gamma'_{i,0}.$$

In this case we have

$$Q(\gamma'_i, u) - Q(\gamma_i, u) = Q\big(\mathcal{T}_i \gamma'_{i,0}, u\big) - Q\big(\mathcal{T}_i \gamma_{i,0}, u\big) = \mathcal{O}(1) P\big(\gamma_{i,0}, \gamma'_{i,0}\big).$$

We first prove special cases of the estimate, namely when the jumps located at $x_{\bar{\alpha}}$, $x_{\bar{\alpha}+1}$ are composed only of waves of one family.

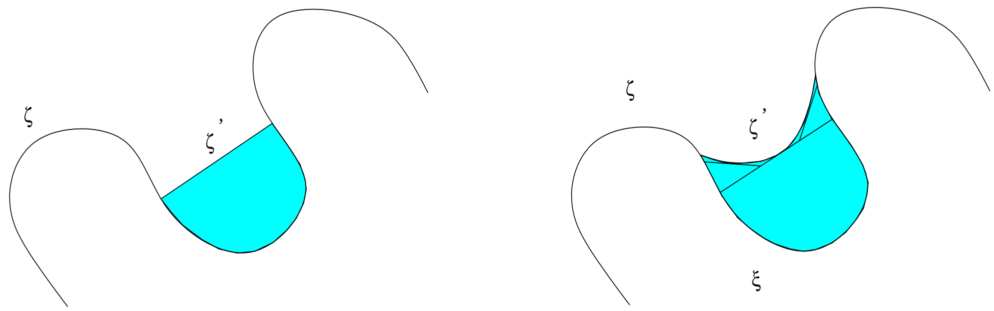

FIGURE 11. Motion in the direction of curvature: the case of a single cut and the general case.

Lemma 4.5. *Assume that* $u(x_{\bar\alpha+1}-) = u(x_{\bar\alpha}+) = T^j_{s_1} u(x_{\bar\alpha}-)$, $u(x_{\bar\alpha+1}+) = T^i_{s_2} u(x_{\bar\alpha+1}-)$, $i < j$. *Let* u' *be the function obtained by joining the jumps in* $x_{\bar\alpha}$, $x_{\bar\alpha+1}$. *Then the following holds:*

$$Q(u') - Q(u) \leq -c|s_1||s_2|. \tag{4.9}$$

Proof. If we substitute $T^i_{s_2} \circ T^j_{s_1} u(x_{\bar\alpha}-)$ with the line $T^j_{s_1} \circ T^i_{s_2} u(x_{\bar\alpha}-)$, then by Lemma 3.8 we obtain

$$D(\gamma'_i, \gamma_{\bar\alpha+1,i}) \leq \mathcal{O}(1)|s_1|s_2|, \qquad D(\gamma'_j, \gamma_{\bar\alpha,j}) \leq \mathcal{O}(1)|s_1|s_2|,$$

where we denote with γ'_i the curves of the new Riemann problem $[u(x_{\bar\alpha}-), u(x_{\bar\alpha+1}+)]$. It is clear that in the functional $Q(u')$ the term $|s_1||s_2|$ disappears, so that

$$Q\Big(T^j_{s_1} \circ T^i_{s_2} u(x_{\bar\alpha}-), u\Big) - Q(u) \leq -|s_1||s_2| + \mathcal{O}(1)\text{Tot.Var.}(u)|s_1||s_2| \leq -c|s_1||s_2|.$$

By Theorem 3.14, the distance P between the curves γ'_i corresponding to the new Riemann problem $[u(x_{\bar\alpha}-), u(x_{\bar\alpha}+)]$ and $T^j_{s_1} \circ T^i_{s_2} u(x_{\bar\alpha}-)$ is of the order of $|s_1||s_2|$. We thus obtain by means of Lemma 4.3 that

$$\begin{aligned} Q(u') - Q(u) &= Q(u') - Q\Big(T^j_{s_1} \circ T^i_{s_2} u(x_{\bar\alpha}-), u\Big) + Q\Big(T^j_{s_1} \circ T^i_{s_2} u(x_{\bar\alpha}-), u\Big) - Q(u) \\ &\leq \; -|s_1||s_2| + \mathcal{O}(1)\text{Tot.Var.}(u)|s_1||s_2| \leq -c|s_1||s_2|. \end{aligned}$$

This concludes the proof. □

Before considering the case when $i = j$, we recall the following results from [9]. If $\zeta : [0, s] \mapsto \mathbb{R}^2$ be an absolutely continuous curve in the plane, the by a *cut* we mean the replacement of the arc $\xi = \{\zeta(s), s \in [s_1, s_2] \subseteq [0, s]\}$ with the line $\theta\zeta(s_1) + (1 - \theta)\zeta(s_2)$.

Theorem 4.6. *Let* $\zeta : [0, s] \mapsto \mathbb{R}^2$ *be an absolutely continuous curve in the plane. If* $\zeta' : [0, s'] \mapsto \mathbb{R}^2$ *is a curve obtained by* ζ *by means of a countable number of cuts, then*

$$Area(\zeta', \zeta) \leq \frac{1}{2} \int_0^s \int_\tau^s \left| \frac{d}{ds}\zeta(\tau) \wedge \frac{d}{ds}\zeta(\tau') \right| d\tau d\tau' - \frac{1}{2} \int_0^s \int_\tau^s \left| \frac{d}{ds}\zeta'(\tau) \wedge \frac{d}{ds}\zeta'(\tau') \right| d\tau d\tau'. \tag{4.10}$$

Here $\cdot \wedge \cdot$ *denotes the external product in* $\mathbb{R}^2$.

We say that ζ' is obtained from ζ by *motion in the direction of curvature*. The quantity $\text{Area}(\zeta', \zeta)$ is the area of the regions with an odd winding number, w.r.t. to the closed curve $\zeta \cup \zeta'$.

Corollary 4.7. *Let* ζ' *be obtained by* ζ *by replacing the arc* $\xi = \{\zeta(s), s \in [s_1, s_2] \subseteq [0, s]\}$ *with a convex curve contained in the convex envelope of* ξ, *with the same endpoints. Then* ζ' *is obtained by motion in the direction of curvature.*

As observed in remark 3.6, the amount of interaction J_i can be represented as an area. To estimate the decrease of the functional Q, the idea is to associate a curve ζ_i to u so that the curve ζ'_i of u' is obtained by motion in the direction of curvature, and the area between ζ_i and ζ'_i is of the order of J_i. Following definition 3.5 and remark 3.6, this curve ζ_i clearly is the union of the elementary curves $\zeta_{\alpha,i}$, for each Riemann problem $[u(x_\alpha-), u(x_\alpha+)]$, defined as

$$\zeta_{\alpha,i} \doteq \begin{pmatrix} s \\ \text{conv}_{[0,s_{i,a}} \tilde f_{\alpha,i}(s) \end{pmatrix}, \qquad s \in [0, s_{\alpha,i}]. \tag{4.11}$$

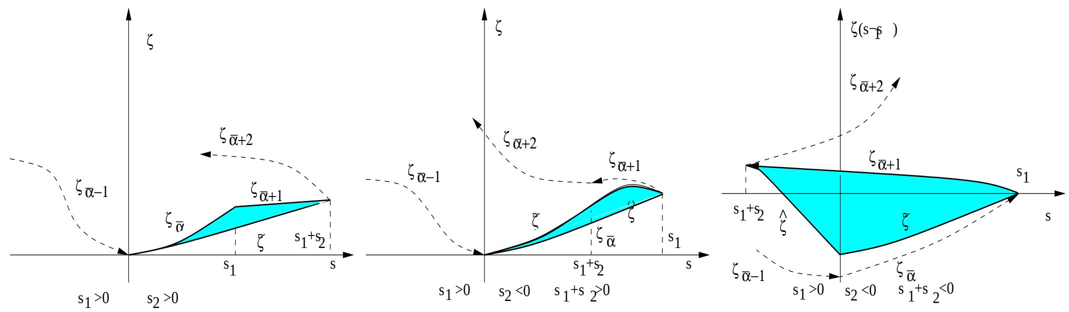

FIGURE 12. The three cases of Lemma 4.8.

Note that with this choice, we have that

$$Q(u) = \sum_{\alpha<\alpha'} \sum_{i<j} |s_{\alpha,j}||s_{\alpha',i}| + \frac{1}{2} \sum_i \int_0^{\sum_\alpha |s_{\alpha,i}|} \int_0^{\sum_\alpha |s_{\alpha,i}|} \left| \frac{d}{ds}\zeta(\tau) \wedge \frac{d}{ds}\zeta(\tau') \right| d\tau d\tau'. \tag{4.12}$$

We can now prove the following lemma:

Lemma 4.8. *If $u(x_{\bar\alpha+1}-) = u(x_{\bar\alpha}+) = T^i_{s_1} u(x_{\bar\alpha}-)$, $u(x_{\bar\alpha+1}+) = T^i_{s_2} u(x_{\bar\alpha+1}-)$, and u' is the function obtained by joining the jumps in $x_{\bar\alpha}$, $x_{\bar\alpha+1}$, then*

$$Q(u') - Q(u) \leq -cJ_i\big(\bar\alpha, \bar\alpha+1\big). \tag{4.13}$$

Proof. We prove only the case $s_1 > 0$, the other cases being completely similar.

Assume $s_2 > 0$, and let $\tilde\gamma_i = (\tilde u, \tilde v_i, \tilde\sigma_i)$ be the line obtained from the first iteration of (3.9) starting with $\gamma_{\alpha,i} \cup \gamma_{\alpha+1,i}$. It is clear that $\tilde\zeta_i$,

$$\tilde\zeta_i \doteq \begin{pmatrix} s \\ \tilde\sigma_i(s) \end{pmatrix}, \qquad s \in [0, s_1+s_2],$$

is convex and contained in the convex envelope of $\zeta_{\bar\alpha,i} \cup \zeta_{\bar\alpha+1,i}$. It follows from Theorem 4.6 and Corollary 4.7 that

$$Q(\tilde\gamma_i, u) - Q(u) \leq -J_i.$$

By Theorem 3.14, the distance between $\tilde\gamma$ and γ' is of the order of $J_i(\bar\alpha, \bar\alpha+1)$, so that we have

$$\begin{aligned} Q(u') - Q(u) &= Q(u') - Q(\tilde\gamma_i, u) + Q(\tilde\gamma_i, u) - Q(u) \\ &\leq \mathcal{O}(1)\text{Tot.Var.}(u) J_i(\bar\alpha, \bar\alpha+1) - J_i(\bar\alpha, \bar\alpha+1) \leq -cJ_i(\bar\alpha, \bar\alpha+1), \end{aligned}$$

if Tot.Var.(u) is sufficiently small.

Assume now $-s_1 \leq s_2 < 0$. Then let $\tilde\gamma_i$ be the line defined by means of the first iteration of (3.9) starting from the line $\gamma_{\bar\alpha,i}$ restricted to $[0, s_1+s_2]$, and let $\hat\gamma_i$ obtained by means of the first iteration to the line $\gamma_{\bar\alpha,i}$ restricted to $[s_1+s_2, s_1]$. We have again that

$$\hat\gamma_i - \gamma_{\bar\alpha+1,i} = \mathcal{O}(1) J_i(\bar\alpha, \bar\alpha+1),$$

and that by curvature motion

$$Q(\tilde\gamma_i, u) - Q(\hat\gamma_i, u) \leq -J_i(\bar\alpha, \bar\alpha+1)$$

where $Q(\hat\gamma_i, u)$ is the functional obtained by substituting $\gamma_{\bar\alpha+1,i}$ with $\hat\gamma_i$. Since $\tilde\zeta_i$ is obtained from $\zeta_{\alpha,i} \cup \hat\zeta_i$ by motion in the direction of curvature, we finally can write

$$\begin{aligned} Q(u') - Q(u) &= Q(u') - Q(\tilde\gamma, u) + Q(\tilde\gamma, u) - Q(\hat\gamma, u) + Q(\hat\gamma, u) - Q(u) \\ &\leq \ -J_i + \mathcal{O}(1)\text{Tot.Var.}(u) J_i \leq -cJ_i. \end{aligned}$$

Finally, for $s_2 < -s_1 \leq 0$, then define $\hat\gamma_i$ as the line obtained by the first iteration of (3.9), starting from $\gamma_{\bar\alpha+1,i}$ in the interval $[-s_2, -s_1]$, with endpoint $u(x_\alpha+)$. Similarly, let $\hat\gamma_i$ be the line obtained by means of the first iteration of (3.9) starting with the curve $\gamma_{\bar\alpha+1,i}$, restricted to $[-s_1, 0]$. Then it follows

$$\hat\gamma_i - \gamma_{\bar\alpha,i} = \mathcal{O}(1) J_i(\bar\alpha, \bar\alpha+1),$$

and, since $\tilde{\gamma}_i$ is obtained by curvature motion from $\tilde{\gamma}_i \cup \gamma_{\bar{\alpha}+1,i}$, by Theorem 4.6 we obtain

$$\begin{aligned} Q(u') - Q(u) &= Q(u') - Q(\tilde{\gamma}_i, u) + Q(\tilde{\gamma}_i, u) - Q(\hat{\gamma}_i, u) + Q(\hat{\gamma}_i, u) - Q(u) \\ &\leq \mathcal{O}(1)\text{Tot.Var.}(u) J_i(\bar{\alpha}, \bar{\alpha}+1) - J_i(\bar{\alpha}, \bar{\alpha}+1) + \mathcal{O}(1)\text{Tot.Var.}(u) J_i(\bar{\alpha}, \bar{\alpha}+1) \\ &\leq -c J_i(\bar{\alpha}, \bar{\alpha}+1). \end{aligned}$$

This concludes the proof in the case $s_1 > 0$. □

Now we prove Proposition 4.1. The proof follows the same line of the proof of Theorem 3.7.

Proof. We first replace $\gamma_{\bar{\alpha},i}$, $\gamma_{\bar{\alpha}+1,i}$, $i = 1, \dots, n$, with the curves $\tilde{\gamma}_i$ obtained by exchanging the waves of different families, i.e. by the curves of the Riemann problem $[u(x_{\bar{\alpha}}-), \tilde{u}^+]$, where

$$\tilde{u}^+ \doteq \big(T^n_{s_{2,n}} \circ T^n_{s_{1,n}}\big) \circ \dots \circ \big(T^1_{s_{2,1}} \circ T^1_{s_{1,1}}\big) u(x_{\bar{\alpha}}-).$$

By means of Lemma 4.3, we obtain that

$$Q(\tilde{\gamma}, u) - Q(u) \leq -c \left\{ \sum_{i<j} |s_{\alpha,j}| |s_{\alpha+1,i}| \right\}.$$

Next, if we replace $\tilde{\gamma}_i$, $i = 1, \dots, n$ with the curve $\hat{\gamma}_i$ of the Riemann problem $[u(x_{\bar{\alpha}}-), \hat{u}^+]$, where

$$\hat{u}^+ \doteq \big(T^n_{s_{2,n}+s_{1,n}} \circ \dots \circ T^1_{s_{2,1}+s_{1,1}}\big) u(x_{\bar{\alpha}}-),$$

by means of Lemma 4.8 and with the same procedure of the proof of Theorem 3.7, we obtain

$$Q(\hat{\gamma}, u) - Q(u) \leq -c \left\{ \sum_{i<j} |s_{\alpha,j}| |s_{\alpha+1,i}| + \sum_i J_i(\bar{\alpha}, \bar{\alpha}+1) \right\}.$$

Since by Theorem 3.14 it follows that

$$Q(u') - Q(\hat{\gamma}, u) = \mathcal{O}(1)\text{Tot.Var.}(u) \left\{ \sum_{i<j} |s_{\alpha,j}| |s_{\alpha+1,i}| + \sum_i J_i(\bar{\alpha}, \bar{\alpha}+1) \right\},$$

the proposition is proved. □

Remark 4.9. If u is a generic BV function, with sufficiently small total variation, we can extend the functional such that it is lower semicontinuous w.r.t. the L^1 norm, and moreover the sum Tot.Var.$(u) + C_1 Q(u)$ is lower semicontinuous. We only describe how to write the functional for general BV functions. We follow the same ideas of [4].

Let u be a BV function, with sufficiently small total variation. Decompose the derivative Du of u is its atomic part μ_a and continuous part μ_c,

$$Du = \mu_a + \mu_c.$$

Denote with x_α, $\alpha \in \mathbb{N}$, the points of discontinuity of u, and let $[u(x_\alpha-), u(x_\alpha+)]$ be the jumps in x_α. It is clear that μ_a has support in the set $\{x_\alpha, \alpha \in \mathbb{N}\}$. By solving the Riemann problem in x_α, one obtain the vector $s_\alpha = (s_{\alpha,1}, \dots, s_{\alpha,n})$. We define the atomic measure $\mu_{i,a}$, whose support is the set $\{x_\alpha, \alpha \in \mathbb{N}\}$, as

$$\mu_{i,a}\big(\{x_\alpha\}\big) = s_{\alpha,i}.$$

Similarly, the continuous measure $\mu_{i,c}$ is defined as follows: for all functions ϕ smooth,

$$\int \phi \, d\mu_c \doteq \int \langle l_i(u), \phi \, d\mu_c \rangle, \tag{4.14}$$

where $l_i(u)$ is the left eigenvector of $Df(u)$. Finally, we define the scalar measure μ_i by

$$\mu_i = \mu_{i,c} + \mu_{i,a}. \tag{4.15}$$

Let s_i be the parameter such that

$$0 \leq s_i \leq |\mu_i|(\mathbb{R}) = |\mu_{c,i}|(\mathbb{R}) + \sum_\alpha |s_{\alpha,i}|. \tag{4.16}$$

The total variation of u can be measured by

$$\text{Tot.Var.}(u) = \sum_i |\mu_i|(\mathbb{R}).$$

Define now the curve $\zeta_i : [0, |\mu_i|(\mathbb{R})] \mapsto \mathbb{R}^2$ as follows: if x_α is an atom for μ_i such that

$$|\mu_i|(x_\alpha-) < s \leq |\mu_i|(x_\alpha-) + |\mu_{i,a}|(x_\alpha) = |\mu_i|(x_\alpha+),$$

then

$$\zeta_i(s) - \zeta_i\big(|\mu_i|(x_\alpha-)\big) \doteq \begin{pmatrix} s - |\mu_i|(x_\alpha-) \\ \text{conv}_{[0,|\mu_{i,a}|(x_\alpha)]} \tilde{f}_{\alpha,i}\big(s - |\mu_i|(x_\alpha-)\big) \end{pmatrix}, \tag{4.17}$$

where $\tilde{f}_{\alpha,i}$ is the reduced flux function corresponding the curve $\gamma_{\alpha,i}$, associated to the Riemann problem in x_α. Otherwise, there exists as x such that

$$|\mu_i|(x) = s,$$

and we define

$$D\zeta_i(s) = \begin{pmatrix} 1 \\ \lambda_i(u(x)) \end{pmatrix} d\mu_{i,c}(x) \tag{4.18}$$

where $\lambda_i(u)$ is the eigenvalue of $Df(u)$.

The functional is written in terms of the lines ζ_i, $i = 1, \dots, n$:

$$Q(u) = \sum_{i<j} \iint_{x<y} |\mu_j|(x)|\mu_i|(y) + \frac{1}{2} \sum_i \int_0^{\bar{s}_i} \int_\tau^{\bar{s}_i} \big|d\zeta_i(\tau) \wedge d\zeta_i(\tau')\big|. \tag{4.19}$$

In particular, if u is piecewise constant, the above functional reduces to (4.2).

To close this remark, note that the second component of the curve ζ_i can be interpreted as the decomposition of the vector u_t, with

$$u_t + f(u)_x = 0.$$

This interpretation implies that for general jumps u_t cannot be though as the derivative of a BV function.

References

[1] F. Ancona and A. Marson. Front tracking solutions for general systems of conservation laws. In preparation.

[2] F. Ancona and A. Marson. Well posedness for general 2×2 conservation laws. Memoir A.M.S., to appear.

[3] F. Ancona and A. Marson. A wave-front tracking alghoritm for $n \times n$ nongenuinely nonlinear conservation laws. *J. Diff. Eq.*, 177:454–493, 2001.

[4] P. Baiti and A. Bressan. Lower semicontinuity of weighter path length in BV. In F. Colombini and N. Lerner, editors, *Geometrical Optics and Related Topics*, pages 31–58. Birkhäuser, 1997.

[5] S. Bianchini. On the Riemann problem for non-conservative hyperbolic systems. Arch. Rational Mech. Anal., to appear.

[6] S. Bianchini and A. Bressan. A case study in vanishing viscosity. *Disc. Cont. Dyn. Sys.*, 7(3):449–476, 2001.

[7] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. preprint SISSA, 2001.

[8] S. Bianchini and A. Bressan. A center manifold technique for tracing viscous travelling waves. *Comm. Pure Appl. Analysis*, 1(2):161–190, 2002.

[9] S. Bianchini and A. Bressan. On a Lyapunov functional relating viscous conservation laws and shortening curves. *Nonlinear Analysis TMA*, 51(4):649–662, 2002.

[10] J. Glimm. Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.*, 18:697–715, 1965.

[11] T.-P. Liu. *Admissible solutions of hyperbolic conservation laws*, volume 240. Memoir A.M.S., 1981.

[12] T.-P. Liu and T. Yang. Weak solutions of general systems of conservation laws. *J. Amer. Math. Soc.*, 2002. to appear.

[13] A. Majda and R. L. Pego. Stable viscosity matrices for systems of conservation laws. *J. Diff. Eq.*, 56:229–262, 1985.

[14] W.-A. Yong and K. Zumbrun. Existence of relaxation shock profiles for hyperbolic conservation laws. *SIAM J. Appl. Math.*, 60(5):1565–1575, 2000.

Istituto per le Applicazioni del Calcolo "M. Picone" - CNR, Viale del Policlinico 137, 00161 Roma (ITALY)

E-mail address: `bianchin@iac.rm.cnr.it`

URL: `http://www.iac.rm.cnr.it/~bianchin/`