A Case Study in Vanishing Viscosity

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Abstract. We consider a special 2×2 viscous hyperbolic system of conservation laws of the form $u_t + A(u)u_x = \varepsilon u_{xx}$, where $A(u) = Df(u)$ is the Jacobian of some flux function f. For initial data with small total variation, we prove that the solutions satisfy a uniform BV bound, independent of ε. Letting $\varepsilon \to 0$, we show that solutions of the viscous system converge to the unique entropy weak solutions of the hyperbolic system $u_t + f(u)_x = 0$. Within the proof, we introduce two new Lyapunov functional which control the interaction of viscous waves of the same family. This provides a first example where uniform BV bounds and convergence of vanishing viscosity solutions are obtained, for a system with a genuinely nonlinear field where shock and rarefaction curves do not coincide.

1 - Introduction

This paper is a contribution toward the understanding of the stability and convergence of vanishing viscosity approximations to hyperbolic systems of conservation laws. Given the $n \times n$ strictly hyperbolic system

$$u_t + f(u)_x = 0. \tag{1.1}$$

a long standing open question is whether the solutions of the viscous approximation

$$u_t + A(u)u_x = \varepsilon u_{xx} \tag{1.2}$$

with $A(u) = Df(u)$ are uniformly stable and converge to entropy weak solutions of (1.1) as $\varepsilon \to 0$.

We recall that, by the results in [4, 5, 9, 10], the entropy weak solutions of the hyperbolic system (1.1) form a uniformly Lipschitz continuous semigroup $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ defined on a closed domain $\mathcal{D} \subset \mathbf{L}^1$ containing all functions with suitably small total variation. A comprehensive description of the recent uniqueness and stability theory can be found in the monograph [3]. In earlier literature, vanishing viscosity limits have been studied, with partial success, by techniques of compensated compactness [6] and singular perturbations [8].

The eventual goal of our research is to prove that, for each $\varepsilon > 0$, the system (1.2) also generates a continuous semigroup S^ε, and that the convergence $S^\varepsilon \to S$ holds as $\varepsilon \to 0$. In particular, if the total variation of a solution is initially small, then it remains small for all times $t \geq 0$, uniformly w.r.t. ε.

The first step of this program was accomplished in [1], where the authors proved that the above result indeed holds for $n \times n$ systems where all characteristic curves are straight lines. For such systems, new oscillations can only be produced by the interaction of viscous waves of distinct families. The result in [1] was indeed obtained by carefully controlling this type of interactions.

In the present paper, we concentrate on interactions of viscous waves of the same family. Namely, we study in detail one particular 2×2 system, which provides the most elementary case where viscous waves of the same family can interact and produce oscillations in another family. We establish uniform BV bounds, and hence the (strong) convergence of vanishing viscosity approximations.

The heart of the matter is the derivation of a priori BV bounds, which we obtain by introducing two new Lyapunov functionals. These are related to the length and to the area swept by a planar curve whose components are the conserved quantity and the flux, for a scalar viscous conservation law.

The present paper thus achieves the second step in our research program, by understanding the interaction of viscous waves of the same family. The third and final step, extending all results to general $n \times n$ viscous hyperbolic systems, is the the subject of current research.

2 - Evolution of gradient components

Our main concern is to provide uniform BV estimates for solutions to the viscous system (1.2). We always assume that the $n \times n$ matrix $A(u)$ is strictly hyperbolic, i.e. it has real distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$ and dual bases of left and right eigenvectors $l_1, \ldots, l_n,\ r_1, \ldots, r_n$, such that

$$l_i(u) \cdot r_j(u) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{2.1}$$

The directional derivative of a function $\phi = \phi(u)$ in the direction of the eigenvector r_i is written

$$r_i \bullet \phi(u) \doteq \lim_{h \to 0} \frac{\phi\big(u + h r_i(u)\big) - \phi(u)}{h},$$

while

$$[r_j, r_k] \doteq r_j \bullet r_k - r_k \bullet r_j$$

denotes the usual Lie bracket. Moreover, we write $u_x^i \doteq l_i \cdot u_x$ for the i-th component of u_x. From (1.2) and (2.1) respectively it follows

$$u_t + \sum_i \lambda_i u_x^i r_i = \varepsilon \sum_i (u_x^i r_i)_x = \varepsilon \sum_i (u_x^i)_x r_i + \varepsilon \sum_{i,j} u_x^i u_x^j (r_j \bullet r_i)\,, \tag{2.2}$$

$$u_x = \sum_i u_x^i r_i\,. \tag{2.3}$$

Differentiating (2.2) w.r.t. x and (2.3) w.r.t. t and equating the results we find

$$\begin{aligned} u_{xt} = & \sum_i (u_x^i)_t r_i - \sum_{ij} u_x^i u_x^j \lambda_j (r_j \bullet r_i) \\ & + \varepsilon \sum_{i,j} u_x^i (u_x^j)_x (r_j \bullet r_i) + \varepsilon \sum_{i,j,k} u_x^i u_x^j u_x^k (r_j \bullet r_k) \bullet r_i \,, \end{aligned}$$

$$\begin{aligned} u_{tx} + & \sum_i (\lambda_i u_x^i)_x r_i + \sum_{i,j} \lambda_i u_x^i u_x^j (r_j \bullet r_i) \\ & = \varepsilon \sum_i (u_x^i)_{xx} r_i + \varepsilon \sum_{i,j} (u_x^i)_x u_x^j (r_j \bullet r_i) + \varepsilon \sum_{i,j} (u_x^i)_x u_x^j (r_j \bullet r_i) \\ & \quad + \varepsilon \sum_{i,j} u_x^i (u_x^j)_x (r_j \bullet r_i) + \varepsilon \sum_{i,j,k} u_x^i u_x^j u_x^k \, r_k \bullet (r_j \bullet r_i) \,, \end{aligned}$$

$$\begin{aligned} & \sum_i (u_x^i)_t r_i + \sum_i (\lambda_i u_x^i)_x r_i + \sum_{j \neq k} \lambda_j [r_k, r_j] u_x^j u_x^k \\ & \quad = \varepsilon \left\{ \sum_i (u_x^i)_{xx} r_i + 2 \sum_{i,j} (u_x^i)_x u_x^j (r_j \bullet r_i) + \sum_{i,j,k} u_x^i u_x^j u_x^k [r_k, \ r_j \bullet r_i] \right\} . \end{aligned} \tag{2.4}$$

Let now a smooth initial condition

$$u(0,x) = \bar{u}(x) \tag{2.5}$$

be assigned. The rescaling $s = t/\varepsilon, \ y = x/\varepsilon$ transforms the Cauchy problem (1.2), (2.5) into

$$u_s + A(u) u_y = u_{yy}, \qquad\qquad u(0,y) = \bar{u}^\varepsilon(y) \doteq \bar{u}(\varepsilon y)$$

Observe that, as $\varepsilon \to 0$, the initial data $\bar{u}^\varepsilon$ has constant total variation, all its derivatives approach zero, but its $\mathbf{L}^1$ norm approaches infinity. To study a priori BV bounds on solutions of (1.2), one can equivalently consider the system

$$u_t + A(u) u_x = u_{xx}, \tag{2.6}$$

and derive uniform estimates on the total variation of $u(t,\cdot)$, for initial data which have small BV norm but whose $\mathbf{L}^1$ norm is arbitrarily large.

Taking the inner product of (2.4) with $l_i(u)$ and assuming $\varepsilon = 1$, we obtain

$$\begin{aligned} & (u_x^i)_t + (\lambda_i u_x^i)_x - (u_x^i)_{xx} \\ & \quad = l_i \cdot \left\{ \sum_{j \neq k} \lambda_j [r_j, r_k] u_x^j u_x^k + 2 \sum_{j,k} (r_k \bullet r_j)(u_x^j)_x u_x^k + \sum_{j,k,\ell} [r_\ell, \ r_k \bullet r_j] u_x^j u_x^k u_x^\ell \right\} \\ & \quad = \sum_{j \neq k} G_{ijk}(u) u_x^j u_x^k + \sum_{j,k} H_{ijk}(u) (u_x^j)_x u_x^k + \sum_{j,k,\ell} K_{ijk\ell}(u) u_x^j u_x^k u_x^\ell . \end{aligned} \tag{2.7}$$

Setting $v^i \doteq u^i_x$, we thus need to estimate the $\mathbf{L}^1$ norm of solutions to

$$v^i_t + (\lambda_i v^i)_x - v^i_{xx} = \sum_{j\neq k} G_{ijk} v^j v^k + \sum_{j,k} H_{ijk} v^j_x v^k + \sum_{j,k,\ell} K_{ijk\ell} v^j v^k v^\ell. \tag{2.8}$$

We regard (2.8) as a parabolic system of n scalar equations, coupled through the terms G, H, K defined by (2.7). These coupling terms can be split in two groups:

- *Transversal terms* involving at least two distinct components, such as $v^j v^k$, $v^j_x v^k$, $v^j v^k v^\ell$ with $j \neq k$,
- *Non-transversal terms* involving one single component, such as $v^j v^j_x$, $v^j v^j v^j$.

In [1] we performed a careful study of transversal terms: if these are the only terms present in the equations, we showed that their total contribution is of quadratic order. Our present goal is to study the contribution of a non-transversal term. For this purpose, we focus our attention on a simple 2×2 system where $H_{211} \equiv 1$ and all other terms G_{ijk}, H_{ijk}, K_{ijk} vanish identically.

3 - A special system

The system we want to study is

$$\begin{cases} (u_1)_t + (u_1^2/2)_x = (u_1)_{xx}, \\ (u_2)_t + \big(u_1^3/3 - u_1^2/2 + u_2\big)_x = (u_2)_{xx}. \end{cases} \tag{3.1}$$

This corresponds to (2.6), where the matrix A is defined by

$$A(u) \doteq \begin{pmatrix} u_1 & 0 \\ u_1^2 - u_1 & 1 \end{pmatrix}. \tag{3.2}$$

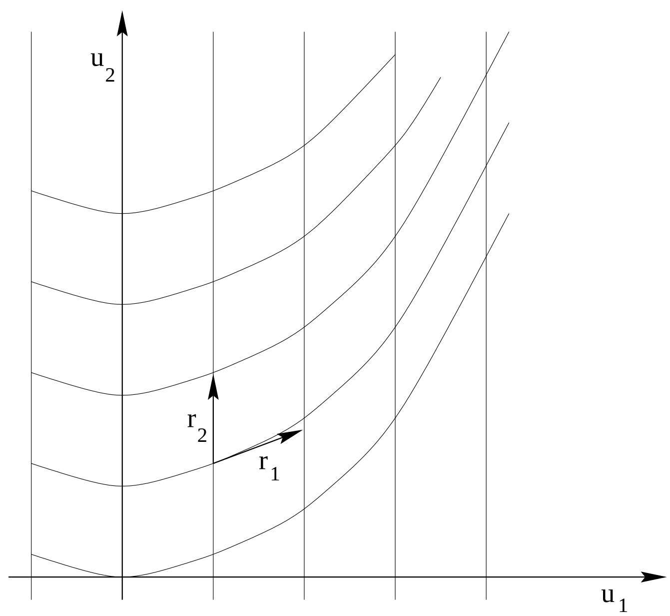

figure 1

As $u = (u_1, u_2)$ varies in a neighborhood of the origin in $\mathbb{R}^2$, the matrix $A(u)$ is strictly hyperbolic. Its eigenvalues are

$$\lambda_1(u) = u_1, \qquad\qquad \lambda_2(u) = 1.$$

Dual bases of right and left eigenvectors (fig. 1) are computed as

$$r_1 = \begin{pmatrix} 1 \\ u_1 \end{pmatrix}, \qquad\qquad r_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$l_1 = (1, 0), \qquad\qquad l_2 = (-u_1, 1).$$

Observe that

$$r_2 \bullet r_2 \;=\; r_1 \bullet r_2 \;=\; r_2 \bullet r_1 \;=\; [r_1, r_2] \;=\; 0,$$

$$r_1 \bullet r_1 \;=\; r_2,$$

$$[r_j,\ r_k \bullet r_\ell] \;=\; 0 \qquad\qquad \text{for all } \ j, k, \ell \in \{1, 2\}\,.$$

In this case, the coefficients in (2.8) become

$$G_{ijk}(u) = 0, \qquad K_{ijk\ell} = 0$$

for all i, j, k, ℓ, while

$$H_{211}(u) = 1, \qquad H_{ijk}(u) = 0 \quad \text{if } (i, j, k) \neq (2, 1, 1).$$

This motivates our interest in the particular system (3.1). Our main result provides the uniform BV bounds for solutions of the viscous system (2.6).

Theorem 1. *Consider the system (2.6), with the matrix $A(u)$ defined at (3.2). Then there exist constants $\delta_1 > \delta_0 > 0$ and L such that the following holds. For every initial data $\bar u \in \mathbf{L}^1$ with Tot.Var.$\{\bar u\} \leq \delta_0$, the Cauchy problem (2.5)-(2.6) has a unique solution, defined for all times $t \geq 0$, such that*

$$\text{Tot.Var.}\big\{u(t, \cdot)\big\} \leq \delta_1 \qquad \text{for all } t \geq 0. \tag{3.3}$$

The next result provides the convergence of vanishing viscosity approximations.

Theorem 2. *Consider the system (1.2), with the matrix $A(u)$ defined at (3.2). Then there exists $\delta_0 > 0$ such that, for every initial data $\bar u \in \mathbf{L}^1$ with Tot.Var.$\{\bar u\} \leq \delta_0$, the corresponding solution $u^\varepsilon = u^\varepsilon(t, x)$ of (1.2) converges in $\mathbf{L}^1_{\text{loc}}$ to the unique entropy solution of (3.1), as $\varepsilon \to 0$.*

The key step in the proof of the above theorems is the derivation of a priori BV bounds on the solutions of (2.6). Introducing the Riemann coordinates

$$z_1 \doteq u_1, \qquad\qquad z_2 \doteq u_2 - \frac{u_1^2}{2}\,,$$

the system (3.1) can be rewritten as

$$\begin{cases} z_{1,t} + (z_1^2/2)_x - z_{1,xx} = 0, \\ \qquad z_{2,t} + z_{2,x} - z_{2,xx} = (z_{1,x})^2. \end{cases} \tag{3.4}$$

Since z_1 satisfies a scalar viscous Burgers' equation, its total variation cannot increase in time. We thus need to provide bounds on the total variation of z_2. The main ideas toward this estimate are outlined in the next two sections. Here we make one preliminary observation. If the total variation of the initial data $\bar u = (\bar u_1, \bar u_2) \in \mathbf{L}^1$ is sufficienlty small, then the solution of (3.1) certainly exists within the time interval $t \in [0,1]$. Moreover, by parabolic regularization, the norms of u and all of its derivatives will be small at time $t = 1$. In particular, for $\alpha = 0, 1, \ldots, 4$ we can assume

$$\sup_x \left|\frac{\partial^\alpha u}{\partial x^\alpha}(1,x)\right| \leq 1, \qquad\qquad \int \left|\frac{\partial^\alpha u}{\partial x^\alpha}(1,x)\right| \, dx \leq 1.$$

By shifting the origin of time, it is thus not restrictive to assume that the first Riemann component z_1 in (3.4) satisfies

$$\left\|\frac{\partial^\alpha z_1}{\partial x^\alpha}(t,\cdot)\right\|_{\mathbf{L}^\infty} \leq 1, \qquad\qquad \left\|\frac{\partial^\alpha z_1}{\partial x^\alpha}(t,\cdot)\right\|_{\mathbf{L}^1} \leq 1, \tag{3.5}$$

for $\alpha = 0, 1, \ldots, 4$ and all $t \geq 0$.

4 - BV estimates for the linear non-homogeneous heat equation

Aim of this section is to derive an estimate on the total variation of the solution to the linear heat equation, in the presence of a source and with unit drift:

$$U_t + U_x - U_{xx} = V(t,x) \qquad\qquad U(0,x) = 0. \tag{4.1}$$

We only consider the Cauchy problem with zero initial data, since the general case follows by linearity. We are mainly interested in the case where V is not integrable in the t-x plane, but it admits a front tracing representation, as specified below.

Definition. We say that a scalar function $V = V(t,x)$ defined on $\mathbb{R}_+ \times \mathbb{R}$ admits a *front tracing representation* if there exists some velocity function $\eta = \eta(t,x)$ such that the following integral is bounded:

$$E \doteq \int_0^\infty \int_{-\infty}^\infty \Big\{ \big|V_t + (\eta V)_x\big| + |V| \cdot |\eta_t + \eta\eta_x| \Big\} \, dxdt < \infty \tag{4.2}$$

Interpreting V as the density of particles and η as their velocity, the above quantity can be interpreted as

$$\begin{aligned} E \;&= [\text{total amount of particles created or destroyed}] \\ &\qquad + [\text{mass}] \times [\text{change in speed}] \end{aligned}$$

Lemma 1. *Assume that the source term V in (4.1) admits a front tracing representation, with*

$$\eta(t,x) \leq \eta^* < 1 \qquad \textit{for all} \quad t, x \tag{4.3}$$

and moreover $\|V(0,\cdot)\|_{\mathbf{L}^1} < \infty$. Then the total variation of the solution U of (4.1) remains uniformly bounded. Indeed, for every $t \geq 0$ there holds

$$\textit{Tot.Var.}\big\{U(t,\cdot)\big\} \leq \frac{2\|V(0,\cdot)\|_{\mathbf{L}^1}}{1-\eta^*} + \frac{2E}{1-\eta^*} + \frac{E}{(1-\eta^*)^2}. \tag{4.4}$$

Proof. Call $\delta(P)$ the Dirac measure consisting of a unit mass at the point P. Consider first the equation

$$U_t + U_x - U_{xx} = \delta(\eta t)\,, \tag{4.5}$$

assuming $\eta < 1$. A solution of (4.5) can be found in the form of a travelling wave:

$$U(t,x) = \phi(\eta, x - \eta t).$$

Substituting into (4.5) one finds that the function $\phi = \phi(\eta,\xi)$ must satisfy

$$-\eta\phi_\xi + \phi_\xi - \phi_{\xi\xi} = \delta(0), \tag{4.6}$$

hence

$$\phi(\eta,\xi) = \begin{cases} (1-\eta)^{-1} & \text{if} \quad \xi \geq 0, \\ (1-\eta)^{-1}e^{(1-\eta)\xi} & \text{if} \quad \xi < 0. \end{cases} \tag{4.7}$$

Observe that

$$\phi_\eta(\eta,\xi) = \begin{cases} (1-\eta)^{-2} & \text{if} \quad \xi \geq 0, \\ \big[(1-\eta)^{-2} - \xi(1-\eta)^{-1}\big]e^{(1-\eta)\xi} & \text{if} \quad \xi < 0. \end{cases} \tag{4.8}$$

Therefore

$$\text{Tot.Var.}\big\{\phi(\eta,\cdot)\big\} = (1-\eta)^{-1}, \qquad \text{Tot.Var.}\big\{\phi_\eta(\eta,\cdot)\big\} = (1-\eta)^{-2}. \tag{4.9}$$

Next, we try to construct a solution of (4.1) in the form

$$U(t,x) = \int V(t,y)\phi\big(\eta(t,y),\, x-y\big)\,dy - v(t,x). \tag{4.10}$$

By successive differentiations we obtain

$$\begin{aligned} U_t &= \int (V_t\phi + V\phi_\eta\eta_t)\,dy - v_t, \\ U_x &= \int V\phi_\xi\,dy - v_x, \\ U_{xx} &= \int V\phi_{\xi\xi}\,dy - v_{xx}. \end{aligned}$$

Here and in the sequel, it is understood that

$$V = V(t,y), \qquad \eta = \eta(t,y), \qquad \phi = \phi\big(\eta(t,y),\, x-y\big).$$

Substituting (4.10) into (4.1) yields

$$\int \big(V_t\phi + V\phi_\eta\eta_t + V\phi_\xi - V\phi_{\xi\xi}\big)\,dy - v_t - v_x + v_{xx} = V. \tag{4.11}$$

Performing an integration by parts and using the property (4.6) of the kernel function ϕ, we obtain the useful identity

$$\begin{aligned} \int \Big(-(\eta V)_y\phi - V\phi_\eta\eta\eta_y + V\phi_\xi - V\phi_{\xi\xi}\Big)dy &= \int \Big(\eta V(\phi_\eta\eta_y - \phi_\xi) - V\phi_\eta\eta\eta_y + V\phi_\xi - V\phi_{\xi\xi}\Big)dy \\ &= \int V\big[(1-\eta)\phi_\xi - \phi_{\xi\xi}\big]\,dy \\ &= V(t,x). \end{aligned} \tag{4.12}$$

By (4.11) and (4.12), if U in (4.10) provides a solution to the Cauchy problem (4.1), then v must satisfy

$$v_t + v_x - v_{xx} = \int \Big[\big(V_t + (\eta V)_y \big) \phi + V(\eta_t + \eta\eta_y)\phi_\eta \Big] \, dy, \tag{4.13}$$

$$v(0,x) = \int V(0,y)\phi\big(\eta(0,y),\, x-y\big)\, dy. \tag{4.14}$$

Recalling (4.3) and (4.9), from (4.13)-(4.14) we obtain

$$\begin{aligned} \text{Tot.Var.}\{v(t,\cdot)\} \leq & \left(\big\|V(0,\cdot)\big\|_{\mathbf{L}^1} + \iint \big|V_t + (\eta V)_y\big| \, dydt \right) \cdot \text{Tot.Var.}\{\phi(\eta^*,\cdot)\} \\ & + \left(\iint |V| \, \big|\eta_t + \eta\eta_y\big| \, dydt \right) \cdot \text{Tot.Var.}\{\phi_\eta(\eta^*,\cdot)\} \\ \leq & \frac{\big\|V(0,\cdot)\big\|_{\mathbf{L}^1}}{1-\eta^*} + \frac{E}{1-\eta^*} + \frac{E}{(1-\eta^*)^2}. \end{aligned} \tag{4.15}$$

Furthermore, by (4.2) it follows that the quantity V approximately satisfies a conservation law with flux ηV. Hence

$$\int \big|V(t,y)\big| \, dy \leq \big\|V(0,\cdot)\big\|_{\mathbf{L}^1} + E. \tag{4.16}$$

From (4.10), using (4.16), (4.9) and (4.15) we recover (4.4).

5 - Front-tracing representations for the viscous Burgers' equation

Let $u = u(t,x)$ be a solution of the scalar viscous conservation law

$$u_t + f(u)_x = u_{xx}. \tag{5.1}$$

Toward the study of (3.1) it suffices to take $f(u) = u^2/2$, but at this stage we shall consider any smooth function f. In view of Lemma 1, in order to show that the total variation of $z_2(t,\cdot)$ in (3.5) remains bounded, we need to prove that, for any solution u of (5.1) with suitably small total variation, the quantity $V(t,x) \doteq u_x^2(t,x)$ admits a front tracing representation, with a speed $\eta = \eta(t,x) < 1$.

More generally, given any BV solution $u = u(t,x)$ of (5.1), assuming that

$$\Big| f'\big(u(t,x)\big) \Big| \leq \eta^* \qquad \text{for all } \ t,x, \tag{5.2}$$

we shall prove that the quantity $V = u_x^2$ admits a front tracing representation, determined by the velocity function

$$\eta = \pi_{[-\eta^*,\ \eta^*]} \left(-\frac{u_t}{u_x} \right) = \pi_{[-\eta^*,\ \eta^*]} \left(f'(u) - \frac{u_{xx}}{u_x} \right). \tag{5.3}$$

Here π is the projection on the interval $[-\eta^*,\ \eta^*]$, i.e.

$$\pi_{[-\eta^*,\ \eta^*]}(\lambda) \doteq \begin{cases} \lambda & \text{if} \quad \lambda \in [-\eta^*,\ \eta^*], \\ -\eta^* & \text{if} \quad \lambda < -\eta^*, \\ \eta^* & \text{if} \quad \lambda > \eta^*. \end{cases}$$

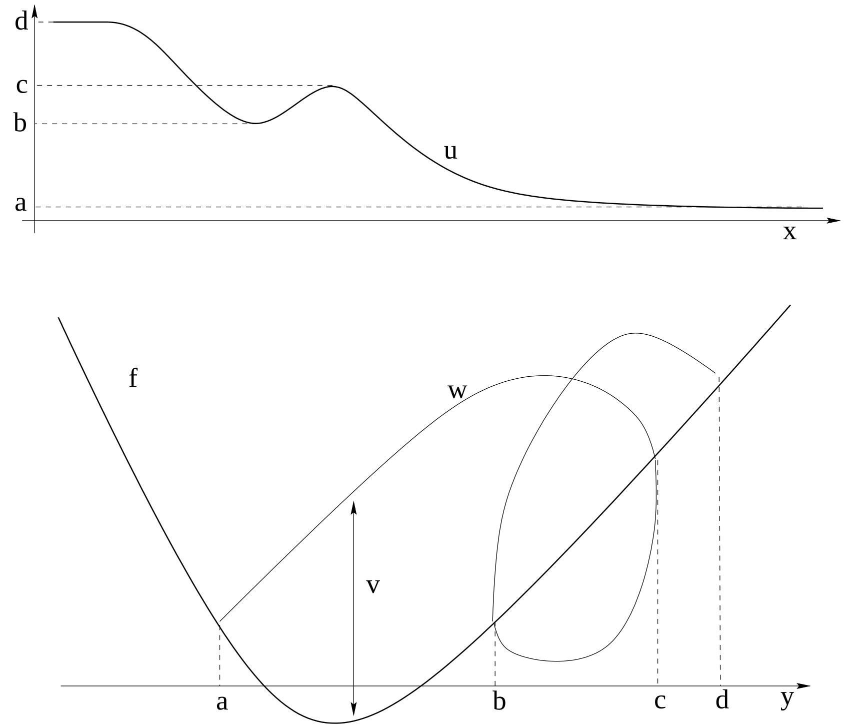

figure 2

Introduce the variables (fig. 2)

$$w \doteq f(u) - u_x, \qquad v \doteq w - f(u) = -u_x, \qquad \begin{cases} \tau \doteq t, \\ y \doteq u(t,x). \end{cases} \tag{5.4}$$

Observe that y and w represent the conserved quantity and the flux, respectively. In regions where u is monotone, we can write w as a function of the independent variables τ, y. Since

$$\tau_t = 1, \qquad \tau_x = 0, \qquad y_t = u_t, \qquad y_x = u_x,$$

for every function φ one has

$$\begin{cases} \varphi_x = \varphi_\tau \tau_x + \varphi_y u_x &= -v\varphi_y, \\ \varphi_t = \varphi_\tau \tau_t + \varphi_y u_t &= \varphi_\tau + v w_y \varphi_y. \end{cases} \tag{5.5}$$

From (5.4)-(5.5) it follows

$$\begin{gathered} w_t + f'(u) w_x = w_{xx}, \\ w_\tau - w_y w_x + f' w_x = \big[w_y (f-w)\big]_x, \\ w_\tau - w_y^2 (f-w) + f' w_y (f-w) = w_{yy}(f-w)^2 + w_y (f-w)_y (f-w). \end{gathered}$$

The flux function $w = w(\tau, y)$ thus satisfies

$$w_\tau = \big(w-f\big)^2 w_{yy}. \tag{5.6}$$

Equivalently,

$$v_\tau = v^2\big(v_{yy} + f''(y)\big). \tag{5.7}$$

Consider the quantity

$$\begin{aligned} E &\doteq \int_0^\infty \int_{-\infty}^{\infty} \Big\{ \big|V_t + (\eta V)_x\big| + |V|\cdot|\eta_t + \eta\eta_x| \Big\}\, dx dt \\ &= \int_0^\infty \int \bigg\{ \Big|V_\tau + v\,\big[V_y w_y - (\eta V)_y\big]\Big| + |V|\cdot\big|\eta_\tau + v\eta_y(w_y - \eta)\big| \bigg\}\cdot \frac{1}{|v|}\,|dy|d\tau. \end{aligned} \tag{5.8}$$

We are mainly interested in the case where $V = v^2$, so that

$$V_\tau = 2vv_\tau = 2v^3 w_{yy}.$$

It is interesting to compute the value of E for various choices of η. In the following, the integrals w.r.t. y range over all branches of the (possibly multivalued) function w.

1. Taking $\eta \equiv \eta^*$ constant, we obtain

$$\begin{aligned} E &= \int_0^\infty \int \big|V_\tau + V_y v(w_y - \eta^*)\big|\cdot\frac{1}{|v|}\,|dy|d\tau \\ &= \int_0^\infty \int 2\Big|v_\tau + vv_y(w_y - \eta^*)\Big|\,|dy|d\tau. \end{aligned} \tag{5.9}$$

2. Taking $\eta \equiv f'(y)$ we obtain

$$\begin{aligned} E &= \int_0^\infty \int \bigg\{ \Big|V_\tau + vV_y w_y - v(f'\cdot V)_y\Big| \;+\; |V|\,\big|vf''v_y\big| \bigg\}\cdot\frac{1}{|v|}\,|dy|d\tau. \\ &= \int_0^\infty \int \bigg\{ \Big|2v^2 w_{yy} + 2vv_y w_y - v^2 f'' - 2vv_y f'\Big| \;+\; \big|v^2 v_y f''\big| \bigg\}\,|dy|d\tau. \\ &= \int_0^\infty \int \bigg\{ \Big|2v^2 w_{yy} + 2vv_y^2 - v^2 f''\Big| \;+\; \big|v^2 v_y f''\big| \bigg\}\,|dy|d\tau \\ &\le \int_0^\infty \int \bigg\{ 2|v_\tau| \;+\; \big|2vv_y^2 - v^2 f''\big| \;+\; \big|v^2 v_y f''\big| \bigg\}\,|dy|d\tau. \end{aligned} \tag{5.10}$$

3. Taking $\eta \doteq w_y = f' + v_y$ we obtain

$$\begin{aligned} E &= \int_0^\infty \int \Big\{ \big|V_\tau - vVw_{yy}\big| + |V|\,|v_{y\tau}| \Big\}\cdot\frac{1}{|v|}\,|dy|dt \\ &= \int_0^\infty \int \Big\{ |v^3 w_{yy}| + |v^2 v_{y\tau}| \Big\}\cdot\frac{1}{|v|}\,|dy|dt \\ &= \int_0^\infty \int \Big\{ |v_\tau| + |v|\,|v_{y\tau}| \Big\}\,|dy|dt. \end{aligned} \tag{5.11}$$

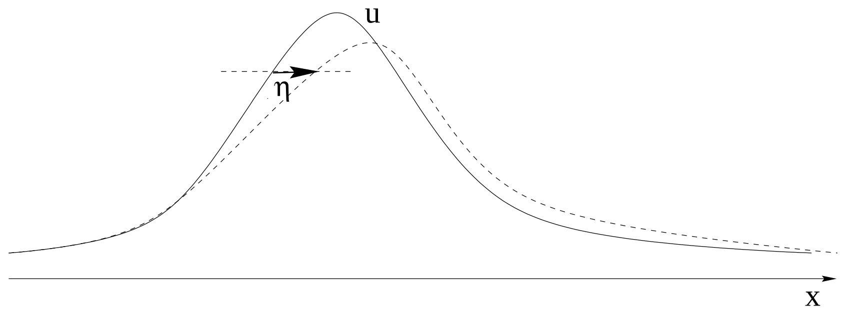

figure 3

Observe that in this last case we have $\eta = w_y = -u_t/u_x$. In other words (fig. 3), η is the speed of the intersection of the graph of $u(t,\cdot)$ with a horizontal line $u = \text{const}$.

6 - A viscous Glimm interaction functional

We summarize here the main results in [2]. Consider any polygonal line γ in $I\!R^2$, with vertices $A = P_0, P_1, \ldots, P_n = B$. Set $v_i \doteq P_i - P_{i-1}$ and consider the functional

$$Q(\gamma) \doteq \frac{1}{2}\sum_{i<j} |v_i \wedge v_j| \tag{6.1}$$

Let γ' be obtained from γ by replacing the two segments $P_{i-1}P_i$ and P_iP_{i+1} by one single segment $P_{i-1}P_{i+1}$, as in fig. 4. The area of the triangle $P_{i-1}P_iP_{i+1}$ is

$$\text{area } (P_{i-1}P_iP_{i+1}) = \frac{1}{2}|v_{i+1} \wedge v_i| \leq Q(\gamma) - Q(\gamma').$$

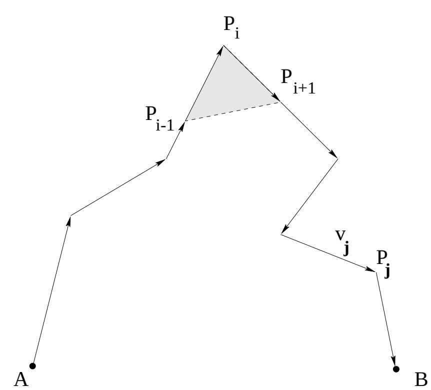

figure 4

A continuous version of the above estimate is the following. Let $\gamma = \gamma(t,\theta)$ be a parametrized curve moving in the plane, with fixed endpoints A, B. Call $\mathbf{v} \doteq \partial\gamma/\partial\theta$ the tangent vector and

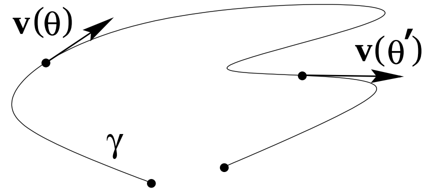

figure 5

define the functional (fig. 5)

$$Q\big(\gamma(t)\big) \doteq \frac{1}{2} \cdot \int_a^b \int_a^b \big|\mathbf{v}(t,\theta) \wedge \mathbf{v}(t,\theta')\big| \; d\theta d\theta'.$$

Assume that the motion of the curve is directed along the curvature. In other words, calling $\mathbf{n}$ the interior unit normal, assume that the inner product $\mathbf{n} \cdot \partial\gamma/\partial t$ is always non-negative. In this case, the evolution of γ can be uniformly approximated by a sequence of polygonals, each obtained from the previous one by replacing two consecutive edges by a single segment, as in fig. 4. Therefore, as proved in [2], for any $t < t'$, the area of the region (fig. 6) bounded by the curves $\gamma \doteq \gamma(t)$ and $\gamma' \doteq \gamma(t')$ is bounded by $Q(\gamma) - Q(\gamma')$. In particular, the total area swept by the curve during its motion is $\leq Q\big(\gamma(0)\big)$.

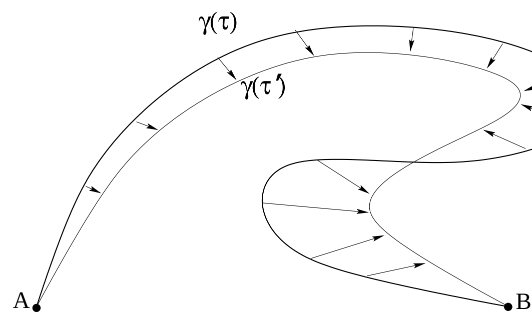

figure 6

As a special case, let $w = w(\tau, y)$ satisfy

$$w_\tau = \varphi(\tau, y) w_{yy}, \quad w(0, y) = \bar{w}(y)$$

with $\varphi \geq 0$. Applying the previous argument to the graph of w,

$$\gamma(t, y) \doteq \big(y, \; w(t, y)\big)$$

we obtain

$$\int_0^\infty \int \big|w_\tau(\tau, y)\big| \, d\tau |dy| \leq Q\big(\gamma(0)\big) = \frac{1}{2} \iint_{y<z} \big|\bar{w}_y(y) - \bar{w}_y(z)\big| \; |dy| dz. \tag{6.2}$$

If $w = w(\tau, y)$ is a solution of (5.6) and $u = u(t, x)$ is the corresponding solution of (5.1), the functional Q in (6.2) takes the form

$$Q(u) = \frac{1}{2} \iint_{x<x'} \big|u_x(x)\, u_x(x')\big| \cdot \Big|\eta(x) - \eta(x')\Big| \; dx dx', \tag{6.3}$$

where

$$\eta(x) \doteq f'\big(u(x)\big) - \frac{u_{xx}(x)}{u_x(x)} \tag{6.4}$$

is the speed of the intersection of the graph of a solution of (5.1) with a horizontal line (fig. 3). Observe that, for a BV solution of (5.1), the corresponding functional $Q(u)$ is finite at all times $t > 0$. Indeed,

$$Q(u) = \mathcal{O}(1) \cdot \big(\text{Tot.Var.}\{u\} + \text{Tot.Var.}\{u_x\}\big). \tag{6.5}$$

7 - Bounds on higher derivatives

In connection with the variable transformation (5.4), for any function φ one has

$$\frac{\partial \varphi}{\partial y} = \frac{1}{u_x} \cdot \frac{\partial \varphi}{\partial x}$$

Using the uniform bounds on $u_x, u_{xx}, u_{xxx} \ldots$, by successive differentiations we obtain

$$v_y = -\frac{u_{xx}}{u_x} = \mathcal{O}(1) \cdot \frac{1}{|v|}, \tag{7.1}$$

$$v_{yy} = \frac{1}{u_x}\left(-\frac{u_{xxx}}{u_x} + \frac{u_{xx}^2}{u_x^2}\right) = \mathcal{O}(1) \cdot \left(\frac{1}{v^2} + \frac{v_y^2}{|v|}\right), \tag{7.2}$$

$$\begin{aligned} v_{yyy} &= \frac{1}{u_x}\left(-\frac{u_{xxxx}}{u_x^2} + \frac{4u_{xxx}u_{xx}}{u_x^3} - \frac{3u_{xx}^3}{u_x^4}\right) \\ &= \mathcal{O}(1) \cdot \left(\frac{1}{|v^3|} + \frac{|v_y|}{|v^3|} + \frac{|v_y^3|}{v^2}\right). \end{aligned} \tag{7.3}$$

$$\begin{aligned} v_{yyyy} &= \frac{1}{u_x}\left(-\frac{u_{xxxxx}}{u_x^3} + \frac{7u_{xxxx}u_{xx}}{u_x^4} + \frac{4u_{xxx}^2}{u_x^4} - \frac{25u_{xxx}u_{xx}^2}{u_x^5} + \frac{15u_{xx}^4}{u_x^6}\right) \\ &= \frac{-4v_{yy}^2}{v} + \mathcal{O}(1) \cdot \left(\frac{1}{v^4} + \frac{|v_y^3|}{v^4} + \frac{v_y^4}{v^3}\right). \end{aligned} \tag{7.4}$$

The last estimate was obtained observing that

$$\frac{u_{xxx}^2}{u_x^4} = \left(\frac{v_y^2}{v} + v_{yy}\right)^2.$$

Differentiating (5.6) we find

$$w_\tau = v^2 w_{yy}, \tag{7.5}$$

$$w_{y\tau} = 2vv_y w_{yy} + v^2 w_{yyy}, \tag{7.6}$$

$$w_{yy\tau} = 2v_y^2 w_{yy} + 4vv_y w_{yyy} + 2vv_{yy}w_{yy} + v^2 w_{yyyy}. \tag{7.7}$$

On regions where v_y remains uniformly bounded, from (7.1)–(7.3) it thus follows

$$w_\tau = \mathcal{O}(1), \qquad\qquad w_{y\tau} = \mathcal{O}(1) \cdot \frac{1}{v}. \tag{7.8}$$

In the case where the flux is convex, it is interesting to derive a time dependent lower bound on w_{yy} which resembles an Oleinik-type estimate for the gradient w_y. Assume $f'' \geq 0$ and consider a solution with $v > 0$. Define

$$Z \doteq w_{yy} + \frac{1}{2\tau v}\,.$$

With successive differentiations we find

$$\begin{aligned}
Z_y &= w_{yyy} - \frac{v_y}{2\tau v^2}\,,\\
Z_{yy} &= w_{yyyy} - \frac{v_{yy}}{2\tau v^2} + \frac{2v_y^2}{2\tau v^3}\,,\\
Z_\tau &= \left[2v_y^2 w_{yy} + 4vv_y w_{yyy} + 2vv_{yy}w_{yy} + v^2 w_{yyyy}\right] - \frac{1}{2\tau^2 v} - \frac{w_{yy}}{2\tau}\,.
\end{aligned}$$

A lengthy but straightforward computation now yields

$$Z_\tau - v^2 Z_{yy} - 4vv_y Z_y + \left(\frac{1}{\tau} - 2v_y^2 - 2vv_{yy}\right) Z = \frac{f''}{2\tau} > 0. \tag{7.9}$$

This allows us to use the maximum principle: if $Z \geq 0$ on the parabolic boundary $\partial^-\Omega$ of a domain Ω in the τ-y plane, then $Z \geq 0$ on the whole set Ω. As an application, assume that

$$u_x(0,x) = \bar u_x(x) \leq 0 \qquad \text{for all} \ \ x \in I\!\!R.$$

By the strong maximum principle this implies

$$u_x(t,x) < 0 \qquad \text{for all} \ \ t > 0,\ \ x \in I\!\!R,$$

so that the solution is strictly monotone decreasing as a function of x, for all $t > 0$. From (7.9) it now follows the inequality

$$w_{yy} \leq \frac{1}{2\tau v} \tag{7.10}$$

for all y, τ with $\tau > 0$.

8 - Estimates related to graph length

Let $\varphi = \varphi(\xi)$ be a smooth, convex scalar function which admits asymptotes as $\xi \to \pm\infty$. More precisely, assume that there exists constants κ, β such that

$$\lim_{\xi\to\infty} \left|\varphi(\xi) - \kappa\xi - \beta\right| = 0 = \lim_{\xi\to-\infty} \left|\varphi(\xi) + \kappa\xi + \beta\right|. \tag{8.1}$$

For example, the function $\varphi(\xi) = \sqrt{1 + (\xi - \beta)^2}$ satisfies the above conditions with $\kappa = 1$. Consider a BV solution $u = u(t,x)$ of (5.1) and let w, v be the corresponding solutions of (5.6), (5.7) respectively, so that the bounds (7.1)–(7.4) hold. Introduce the functional

$$\Phi(w) \doteq \int \varphi(w_y)\,|dy|, \tag{8.2}$$

where the integral ranges over all branches of the (possibly multivalued) function w. Observe that, by (8.1), the function φ has sublinear growth, hence $\Phi(w)$ can be bounded in terms of the length of the graph of w. We now compute the time derivative of $\Phi(w)$, and show that it is non-positive. Let $a < b$ be the locations of two consecutive zeroes of v, so that

$$v\big(\tau,\, a(\tau)\big) = v\big(\tau,\, b(\tau)\big) = 0, \qquad\qquad v\big(\tau, y\big) \neq 0 \qquad \text{for all} \qquad y \in \big]a(\tau),\ b(\tau)\big[\,.$$

We claim that

$$\frac{d}{d\tau}\int_{a(\tau)}^{b(\tau)} \varphi(w_y)\, dy = -\int_{a(\tau)}^{b(\tau)} \varphi''(w_y) v^2 w_{yy}^2\, dy \mp \Big[\big(\beta + \kappa f'(a)\big)\dot a + \big(\beta + \kappa f'(b)\big)\dot b\Big], \tag{8.3}$$

where the minus sign is taken if $v > 0$ and the plus sign if $v < 0$ for $a < y < b$. Observe that (8.3) cannot be obtained from (7.5) by a straightforward integration by parts, because one may have $w_y, w_{yy} \to \pm\infty$ as $y \to a, b$. The boundary terms must therefore be handled with care. To fix the ideas, assume $v > 0$ for $a < y < b$. Fix $\varepsilon > 0$ and consider the points $a^\varepsilon > a$, $b^\varepsilon < b$ such that

$$v\big(a^\varepsilon(\tau)\big) = \varepsilon = v\big(b^\varepsilon(\tau)\big).$$

The time derivatives of these points are easily computed as

$$\dot a^\varepsilon = -\frac{v_\tau}{v_y}(a^\varepsilon), \qquad\qquad \dot b^\varepsilon = -\frac{v_\tau}{v_y}(b^\varepsilon). \tag{8.4}$$

This yields the relations

$$\begin{aligned} w_\tau(a^\varepsilon) &= v_\tau(a^\varepsilon) = -\dot a^\varepsilon v_y(a^\varepsilon) = -\dot a^\varepsilon\big(w_y(a^\varepsilon) - f'(a^\varepsilon)\big),\\ w_\tau(b^\varepsilon) &= v_\tau(b^\varepsilon) = -\dot b^\varepsilon v_y(b^\varepsilon) = -\dot b^\varepsilon\big(w_y(b^\varepsilon) - f'(b^\varepsilon)\big). \end{aligned} \tag{8.5}$$

Using (8.5) we now compute

$$\begin{aligned} \frac{d}{d\tau}\int_{a(\tau)}^{b(\tau)} \varphi(w_y)\, dy &= \lim_{\varepsilon\to 0}\ \frac{d}{d\tau}\int_{a^\varepsilon(\tau)}^{b^\varepsilon(\tau)} \varphi(w_y)\, dy\\ &= \lim_{\varepsilon\to 0}\ \left[\int_{a^\varepsilon}^{b^\varepsilon} \varphi'(w_y)\big(v^2 w_{yy}\big)_y\, dy + \dot b^\varepsilon\cdot\varphi\big(w_y(b^\varepsilon)\big) - \dot a^\varepsilon\cdot\varphi\big(w_y(a^\varepsilon)\big)\right]\\ &= -\lim_{\varepsilon\to 0}\int_{a^\varepsilon}^{b^\varepsilon} \varphi''(w_y) v^2 w_{yy}^2\, dy + \lim_{\varepsilon\to 0}\Big[-\varphi'\big(w_y(b^\varepsilon)\big)\,\dot b^\varepsilon\big(w_y(b^\varepsilon) - f'(b^\varepsilon)\big)\\ &\qquad + \varphi'\big(w_y(a^\varepsilon)\big)\,\dot a^\varepsilon\big(w_y(a^\varepsilon) - f'(a^\varepsilon)\big) + \dot b^\varepsilon\cdot\varphi\big(w_y(b^\varepsilon)\big) - \dot a^\varepsilon\cdot\varphi\big(w_y(a^\varepsilon)\big)\Big]\\ &= -\int_a^b \varphi''(w_y)v^2 w_{yy}^2\, dy - \Big[\big(\beta + \kappa f'(a)\big)\dot a + \big(\beta + \kappa f'(b)\big)\dot b\Big]\,. \end{aligned} \tag{8.6}$$

Indeed, if $a(\tau)$ is the height of a local minimum for $u(\tau,\cdot)$, we then have $w_y(a^\varepsilon) \to \infty$ as $a^\varepsilon \to a+$, hence

$$\begin{aligned} \lim_{\varepsilon\to 0}&\Big[\varphi'\big(w_y(a^\varepsilon)\big)\,\big(w_y(a^\varepsilon) - f'(a^\varepsilon)\big) - \varphi\big(w_y(a^\varepsilon)\big)\Big]\\ &= \lim_{\xi\to\infty}\ \big[\varphi'(\xi)\,\big(\xi - f'(a)\big) - \varphi(\xi)\big]\\ &= -\beta - \kappa f'(a). \end{aligned} \tag{8.7}$$

Similarly, if $b(\tau)$ is the height of a local maximum of $u(\tau,\cdot)$, we then have $w_y(b^\varepsilon)\to-\infty$ as $b^\varepsilon\to b-$, hence

$$\begin{aligned}\lim_{\varepsilon\to 0}&\Big[-\varphi'\big(w_y(b^\varepsilon)\big)\,\big(w_y(b^\varepsilon)-f'(b^\varepsilon)\big)+\varphi\big(w_y(b^\varepsilon)\big)\Big]\\ &=\lim_{\xi\to-\infty}\big[-\varphi'(\xi)\,\big(\xi-f'(b)\big)+\varphi(\xi)\big]\\ &=-\beta-\kappa f'(a).\end{aligned}\tag{8.8}$$

In the case where $v<0$, one has $w_y(a^\varepsilon)\to-\infty,\ \ w_y(b^\varepsilon)\to\infty$ as $a^\varepsilon\to a+,\ \ b^\varepsilon\to b-$, hence the signs in the limits (8.7)-(8.8) are reversed. This yields (8.3).

To complete the proof we observe that, in cases where

$$a(\tau)=\lim_{x\to\pm\infty}u(\tau,x)\qquad\text{or}\qquad b(\tau)=\lim_{x\to\pm\infty}u(\tau,x),$$

the limits $w_y\to\pm\infty$ may fail. However, in these cases one trivially has $\dot a\equiv 0$ or $\dot b\equiv 0$, and the formula (8.3) again holds.

Summing (8.3) over all branches of the multivalued function w, and observing that boundary terms cancel each other, we now obtain

$$\frac{d}{d\tau}\Phi\big(w(\tau)\big)=\frac{d}{d\tau}\int\varphi\big(w_y(\tau)\big)\,|dy|=-\int\varphi''(w_y)v^2w_{yy}^2\,dy\ \le 0.\tag{8.9}$$

In terms of the original solution $u=u(t,x)$ of (5.1), this yields

$$\frac{d}{dt}\int_{-\infty}^{\infty}\varphi\Big(f'(u)-\frac{u_{xx}}{u_x}\Big)|u_x|\,dx\le 0.\tag{8.10}$$

In the special case $\varphi(\xi)\doteq\sqrt{1+\xi^2}$, from (8.9) it follows

$$\int_0^\infty\int(1+w_y^2)^{-3/2}v^2w_{yy}^2\,|dy|d\tau\le\Phi\big(w(0)\big)=\int\sqrt{1+\bar w_y^2(y)}\,|dy|\,.\tag{8.11}$$

The left hand side of (8.11) describes the shortening of the graph of w. For any fixed κ, restricted to regions where $|w_y|\le\kappa$, the first factor in the integrand is uniformly bounded. Hence

$$\int_0^\infty\int_{|w_y|<\kappa}v^2w_{yy}^2\,|dy|d\tau=\mathcal{O}(1).\tag{8.12}$$

By an approximation argument, the smoothness assumption on φ can be dropped. If φ is any convex function satisfying (8.1), then the map

$$t\mapsto\int\varphi\big(w_y(\tau,y)\big)\,|dy|$$

is non-increasing. For example, taking $\varphi(\xi)\doteq|\xi|$ we see that the total variation of the map $w(\tau,\cdot)$ is non increasing in time.

We conclude this section observing that, by similar techniques, one can easily obtain new Lyapunov functionals described by multiple integrals. For example, consider a symmetric, convex function φ satisfying (8.1). Let us define

$$\Psi(w)\doteq\frac14\int\!\!\int\varphi\big(w_y(y)-w_y(y')\big)\ |dy|\,|dy'|.\tag{8.13}$$

Notice that the function $\varphi(\cdot - c)$ still satisfies (8.1) for every value of the constant c. For a solution $w = w(\tau, y)$ of (5.6) we can thus compute the time derivative of Ψ as in (8.3)–(8.9):

$$\frac{d}{d\tau}\Psi\big(w(\tau)\big) = -\frac{1}{2}\iint \varphi''\big(w_y(y) - w_y(y')\big)\, v^2(y)\, w_{yy}^2(y)\ |dy|\,|dy'|. \tag{8.14}$$

In the special case where

$$\varphi_\varepsilon(\xi) \doteq (\varepsilon + \xi^2)^{-1/2}, \qquad \varphi''_\varepsilon(\xi) = \varepsilon(\varepsilon + \xi^2)^{-3/2},$$

the corresponding functional Ψ_ε satisfies

$$\frac{d}{d\tau}\Psi_\varepsilon\big(w(\tau)\big) = -\frac{1}{2}\iint \varepsilon\Big(\varepsilon + \big(w_y(y) - w_y(y')\big)^2\Big)^{-3/2} v^2(y)\, w_{yy}^2(y)\ |dy|\,|dy'|. \tag{8.15}$$

We are particularly interested in the limit $\Psi_\varepsilon \to \Psi$ as $\varepsilon \to 0$. To study the time derivative $d\Psi\big(w(\tau)\big)/d\tau$, let y be a point where $w_{yy}(y) \neq 0$. Using the mean value theorem and the change of variable $z = w_{yy}(y)\varepsilon^{-1/2}y'$, for $\rho > 0$ suitably small we compute the limit

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{y-\rho}^{y+\rho} \varepsilon\Big(\varepsilon + \big(w_y(y') - w_y(y)\big)^2\Big)^{-3/2} dy' &= \lim_{\varepsilon\to 0}\int_{y-\rho}^{y+\rho} \varepsilon^{-1/2}\left(1 + \frac{w_{yy}^2(y^*)(y'-y)^2}{\varepsilon}\right)^{-3/2} dy' \\
&= \lim_{\varepsilon\to 0}\int_{-\rho w_{yy}(y)/\sqrt{\varepsilon}}^{\rho w_{yy}(y)/\sqrt{\varepsilon}} \frac{1}{\big|w_{yy}(y)\big|}\cdot\left(1 + \frac{w_{yy}^2(y^*)}{w_{yy}^2(y)}z^2\right)^{-3/2} dz \\
&= \frac{1}{\big|w_{yy}(y)\big|}\cdot\int_{-\infty}^{\infty}(1+z^2)^{-3/2}dz \\
&= \frac{2}{\big|w_{yy}(y)\big|}\,.
\end{aligned} \tag{8.16}$$

Recalling (5.6), from (8.15) and (8.16) one obtains the estimate

$$\frac{d}{d\tau}\Psi\big(w(\tau)\big) = \frac{d}{d\tau}\iint \big|w_y(y) - w_y(y')\big|\ |dy|\,|dy'| \leq -\int \big|u_\tau(\tau, y)\big|\ |dy|\,. \tag{8.17}$$

This provides an alternative derivation of the basic formula (6.2).

9 - Viscous rarefaction waves

In this section we make the additional assumption that f is convex, so that $f'' \geq 0$. We will show that, on regions where $v < 0$ (i. e. $u_x > 0$), the alternative choice $\eta = f'(u)$ also yields a bounded value for E in (5.10). In other words, one can trace viscous rarefaction waves by choosing η equal to their characteristic speed. This result is not needed toward the proof of the main theorem, but we believe it has some interest by itself.

Let $\big[a(\tau),\ b(\tau)\big]$ be an interval where $v < 0$, with $v \to 0$ for $y \to a+$ and for $y \to b-$. Recalling (5.7), we compute

$$\begin{aligned}
\frac{d}{d\tau}\int_{a\tau()}^{b(\tau)} |v|\, dy &= -\int_{a(\tau)}^{b(\tau)} v^2(v_{yy} + f'')\, dy \\
&= -\int_{a(\tau)}^{b(\tau)} 2|v|v_y^2\, dy - \int_{a(\tau)}^{b(\tau)} v^2 f''(y)\, dy.
\end{aligned} \tag{9.1}$$

Observe that in the above integration by parts the boundary terms vanish because of (7.1). Since both terms on the right hand side of (9.1) are ≤ 0, we conclude

$$\int_0^\infty \int_{a(\tau)}^{b(\tau)} 2|v|v_y^2 \, dy d\tau + \int_0^\infty \int_{a(\tau)}^{b(\tau)} v^2 f'' \, dy d\tau \leq \int_{a(0)}^{b(0)} \big|v(0,y)\big| \, dy. \tag{9.2}$$

Observing that

$$v^2 \, |v_y| \leq \frac{|v| \, v_y^2}{2} + \frac{|v^3|}{2} \qquad\qquad |v| = \mathcal{O}(1),$$

this provides an a-priori estimate of the right hand side of (5.8), on regions where $v \leq 0$. This corresponds to rarefaction waves, with $u_x \geq 0$.

10 - Regions with large gradient

Assume $f'(u) \in [-\eta^*, \eta^*]$ for all $u \in \big[a(t), \ b(t)\big]$, and let w be a solution of (5.6) defined for $y \in [a,b]$, with $w(a) = f(a)$, $w(b) = f(b)$.

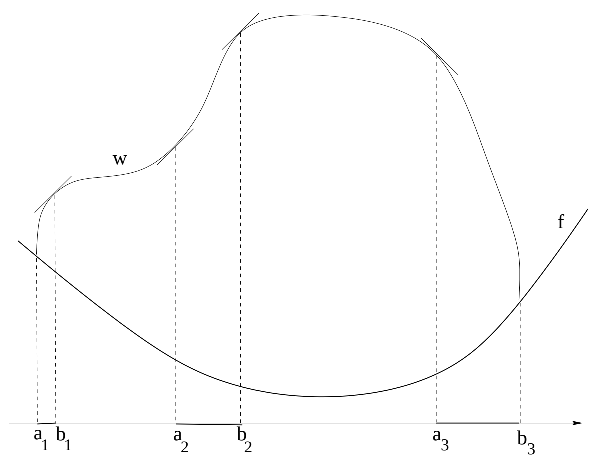

figure 7

Consider the region where the gradient of w is large (fig. 7).

$$J \doteq \Big\{ y \in [a,b] \, ; \quad v \neq 0, \quad \big|w_y(y)\big| \geq \eta^* \Big\} = \bigcup_{j=0}^{\nu} \,]a_j, \, b_j[. \tag{10.1}$$

Consider the speed

$$\eta \doteq \begin{cases} \eta^* & \text{if} \quad w_y \geq \eta^*, \\ -\eta^* & \text{if} \quad w_y \leq -\eta^*. \end{cases}$$

Observe that the above choice implies

$$v_y(w_y - \eta) = (w_y - f')(w_y - \eta) \geq 0 \qquad\qquad \text{for all} \;\; y \in J.$$

Recalling (5.9) we now compute

$$
\begin{aligned}
E_J &\doteq 2\int_J \big|v_\tau + vv_y(w_y-\eta)\big|\,|dy| \\
&\le 2\int_J |w_\tau|\,|dy| + 2\sum_j \int_{a_j}^{b_j} \big|vv_y(w_y-\eta)\big|\,dy \\
&= 2\int_J |w_\tau|\,|dy| + \sum_j \left|\int_{a_j}^{b_j} 2vv_y(w_y-\eta)\,dy\right| \\
&= 2\int_J |w_\tau|\,|dy| + \sum_j \int_{a_j}^{b_j} \big|v^2(w_y-\eta)_y\big|\,dy \\
&\le 3\int_J |w_\tau|\,|dy|
\end{aligned}
\tag{10.2}
$$

We observe that, in the integration by parts over each interval $[a_j, b_j]$ the boundary terms play no role. As $y \to a_j$ or $y \to b_j$ either $w_y \to \pm\eta^*$, or else $v \to 0$ and hence by (7.1) we have $v^2(w_y \pm \eta^*) \to 0$. Thanks to (6.2), the quantity E_J in in (10.2) is thus uniformly bounded.

11 - Parabolic estimates

Define the rectangles (fig.8)

$$
\begin{aligned}
R &\doteq [-3/4,\ 3/4] \times [1,\ 10], \\
R' &\doteq [-1/2,\ 1/2] \times [5,\ 6].
\end{aligned}
\tag{11.1}
$$

$$
\begin{aligned}
\Gamma &\doteq [-1,\ 1] \times [1,\ 10], \\
\Gamma' &\doteq [-1/4,\ 1/4] \times [5,\ 6], \\
\Gamma'' &\doteq [-2,\ 2] \times [1,\ 10].
\end{aligned}
\tag{11.2}
$$

Consider the equation

$$
W_t = a(Y,W)W_{YY}\,. \tag{11.3}
$$

We assume that (11.3) is uniformly parabolic with smooth coefficients:

$$
a_* \le a(Y,W) \le a^* \qquad \text{for all} \ \ (Y,W) \in \Gamma'', \tag{11.4}
$$

$$
\|a\|_{\mathcal{C}^3(\Gamma'')} \le \kappa \tag{11.5}
$$

for some constants $a^* > a_* > 0$ and $\kappa > 0$. The next lemma, based on the Schauder interior estimates for parabolic equations [7], provides bounds for the mixed second derivative W_{Yt}, in terms of the total variation of the boundary data.

Lemma 2. *Let $W = W(t,Y)$ be a smooth solution of (11.3) defined for $t \in [t_0, T]$, $Y \in [-3/4,\ 3/4]$, taking values inside the interval $[1,\ 10]$, so that (11.4)-(11.5) apply.*

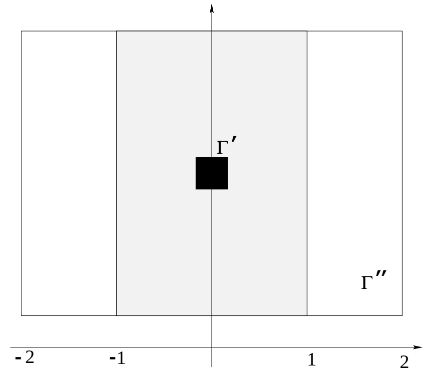

figure 8

a) For every $c_1 > 0$ *there exists a constant* C *depending only on* a_*, a^*, κ, c_1 *such that*

$$\int_{t_0+c_1}^{T}\int_{-1/2}^{1/2} \big|W_{Yt}(t,Y)\big|\,dY\,dt \le C\cdot\left(1+\int_{t_0}^{T}\Big(\big|W_t(t,\,-3/4)\big|+\big|W_t(t,\,3/4)\big|\Big)\,dt\right). \tag{11.6}$$

b) If in addition

$$\big|W_Y(t_0,Y)\big| \le 2 \qquad \textit{for all}\ \ Y\in[-3/4,\ 3/4], \tag{11.7}$$

then there exists a constant C *depending only on* a_*, a^*, κ *such that, for every* $t_1 \in\,]t_0,\ t_0+1]$, *one has*

$$\int_{t_1}^{T}\int_{-1/2}^{1/2} \big|W_{Yt}(t,Y)\big|\,dY\,dt \le C\cdot\left(1+\big|\ln(t_1-t_0)\big|+\int_{t_0}^{T}\Big(\big|W_t(t,\,-3/4)\big|+\big|W_t(t,\,3/4)\big|\Big)\,dt\right). \tag{11.8}$$

Applying Lemma 2 to the rectangles

$$R_\theta \doteq [\theta-3/4,\ \theta+3/4]\times[1,\ 10],$$
$$R' \doteq [\theta-1/2,\ \theta+1/2]\times[5,\ 6],$$

integrating over $\theta\in[-1/4,\ 1/4]$ and observing that

$$\Gamma = \bigcup_{|\theta|\le 1/4} R_\theta, \qquad\qquad \Gamma' = \bigcap_{|\theta|\le 1/4} R'_\theta,$$

we obtain

Lemma 3. *Let* $W = W(t,Y)$ *be a smooth solution of (11.3) defined for* $t\in[t_0,\,T]$, $Y\in[-1,\ 1]$, *taking values inside the interval* $[1,\ 10]$, *so that (11.4)-(11.5) apply.*

a) For every $c_1 > 0$ *there exists a constant* C *depending only on* a_*, a^*, κ, c_1 *such that*

$$\int_{t_0+1}^{T}\int_{-1/4}^{1/4} \big|W_{Yt}(t,Y)\big|\,dY\,dt \le C\cdot\left(1+\int_{t_0}^{T}\int_{-1}^{1}\big|W_t(t,Y)\big|\,dy\,dt\right). \tag{11.9}$$

b) If in addition

$$\big|W_Y(t_0,Y)\big| \le 2 \qquad \textit{for all } \ Y\in[-1,\ 1], \tag{11.10}$$

then there exists a constant C depending only on a_*, a^*, κ such that, for every $t_1 \in]t_0,\ t_0+1]$, one has

$$\int_{t_1}^T \int_{-1/2}^{1/2} \big|W_{Yt}(t,Y)\big|\, dY\,dt \le C\cdot \left(1+\big|\ln(t_1-t_0)\big| + \int_{-1}^1 \big|W_t(t,Y)\big|\,dy\,dt\right). \tag{11.11}$$

The constant C depends only on a_*, a^*, κ.

12 - Regions with small gradient

It now remains to estimate the integral (5.11) restricted to regions where the gradient $w_y = f'(u) - (u_{xx}/u_x)$ is small:

$$\begin{aligned} E^* &\doteq \iint_{|w_y|<\eta^*} |vw_{y\tau}|\,|dy|\,d\tau \\ &= \iint_{|w_y|<\eta^*} \big|v^3 w_{yyy} + 2v^2 v_y w_{yy}\big|\,|dy|\,d\tau \end{aligned} \tag{12.1}$$

.

An a priori bound for (12.1) should be provided in terms of the "area" functional Q in (6.3) and the "graph length" functional Φ in (8.11). By the assumption of small total variation, is not restrictive to assume

$$\big|f'(y)\big| \le \eta^* \le \frac{1}{4} \tag{12.2}$$

for every y in the range of our solution u.

We start by constructing countably many rectangles which are rescaled copies of the rectangles in (11.2):

$$\Gamma_\alpha \doteq \big(y_\alpha,\ f(y_\alpha)\big) + v_\alpha \Gamma,$$

$$\Gamma'_\alpha \doteq \big(y_\alpha,\ f(y_\alpha)\big) + v_\alpha \Gamma',$$

$$\Gamma''_\alpha \doteq \big(y_\alpha,\ f(y_\alpha)\big) + v_\alpha \Gamma'',$$

such that, for some integer N, the following property holds (fig. 9):

(P) Every point P in the open region where $w \neq f(y)$ is contained in some Γ'_α and in no more than N distinct rectangles Γ''_α

Observe that, by (12.2) and the definition of the rectangles Γ''_α, every point $(y,w)\in\Gamma''$ satisfies

$$w - f(y) \in \left[\frac{1}{2}v_\alpha,\ \frac{21}{2}v_\alpha\right]. \tag{12.3}$$

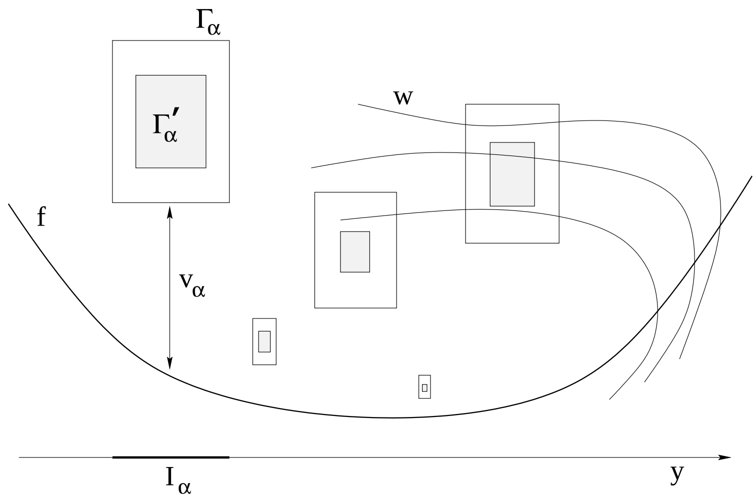

figure 9

We now define the intervals and the sets

$$I_\alpha \doteq [y_\alpha - v_\alpha,\ y_\alpha + v_\alpha], \qquad \Omega_\alpha \doteq \Big\{(\tau, y)\,;\quad \big(y,\ w(\tau,y)\big) \in \Gamma_\alpha\Big\},$$

$$I'_\alpha \doteq [y_\alpha - v_\alpha/4,\ y_\alpha + v_\alpha/4], \qquad \Omega'_\alpha \doteq \Big\{(\tau, y)\,;\quad \big(y,\ w(\tau,y)\big) \in \Gamma'_\alpha\Big\},$$

$$I''_\alpha \doteq [y_\alpha - 2v_\alpha,\ y_\alpha + 2v_\alpha], \qquad \Omega''_\alpha \doteq \Big\{(\tau, y)\,;\quad \big(y,\ w(\tau,y)\big) \in \Gamma''_\alpha\Big\}.$$

Clearly, we have the inclusions

$$\Gamma'_\alpha \subset \Gamma_\alpha \subset \Gamma''_\alpha, \qquad \Omega'_\alpha \subset \Omega_\alpha \subset \Omega''_\alpha, \qquad I'_\alpha \subset I_\alpha \subset I''_\alpha.$$

Observe that by (12.3) the equation (5.6) is uniformly parabolic restricted to each domain Ω''_α. For each α, it is convenient to rescale the variables v, w, y by setting

$$V = \frac{v}{v_\alpha}, \qquad W = \frac{w - f(y_\alpha)}{v_\alpha}, \qquad Y = \frac{y - y_\alpha}{v_\alpha}. \tag{12.4}$$

When the point (y, w) falls inside Γ''_α, the corresponding point (Y, W) falls inside the rectangle Γ''. In particular, we have

$$Y \in [-2,\, 2], \qquad W \in [1,\, 10], \qquad V \in \left[\frac{1}{2},\ \frac{21}{2}\right]. \tag{12.5}$$

Direct computations yield

$$W_Y = w_y, \qquad W_{YY} = v_\alpha\, w_{yy}, \qquad W_{YYY} = v_\alpha^2 w_{yyy}, \tag{12.6}$$

$$V_\tau = W_\tau = V^2 W_{YY}, \qquad V_{\tau Y} = W_{\tau Y} = 2V V_Y W_{YY} + V^2 W_{YYY}, \tag{12.7}$$

$$\int_{I_\alpha} |w_\tau|\, dy = v_\alpha^2 \int_{-1}^{1} |W_\tau|\, dY. \tag{12.8}$$

$$\int_{I'_\alpha} |vw_{y\tau}|\, dy = v_\alpha^2 \int_{-1/2}^{1/2} |VW_{Y\tau}|\, dY, \tag{12.9}$$

$$\int_{I''_\alpha} v^2 w_{yy}^2\, dy = v_\alpha \int_{-2}^{2} V^2 W_{YY}^2\, dY. \tag{12.10}$$

Thanks to the bounds (7.2)–(7.4), on regions where

$$|W_Y| = |w_y| \leq 2$$

we have

$$\max\Big\{|W_{YY}|,\ |W_{YYY}|,\ |W_\tau|,\ |W_{Y\tau}|\Big\} \leq \frac{C_1}{v_\alpha} \tag{12.11}$$

for some constant C_1. For each α, define the set of times

$$\mathcal{T}_\alpha \doteq \Big\{\tau\,;\ \text{there exist } y', y'' \text{ such that } \big(y',\, w(\tau, y')\big) \in \Gamma'_\alpha,$$
$$\big(y'',\, w(\tau, y'')\big) \in \Gamma''_\alpha,\quad \big|w_y(\tau, y')\big| \leq \eta^*,\quad \big|w_y(\tau, y'')\big| \geq 1\Big\}.$$

The double integral in (12.1) can now be estimated as

$$\begin{aligned} E^* &\doteq \iint_{|w_y|<\eta^*} |vw_{y\tau}|\, |dy|\, d\tau \\ &\leq \sum_\alpha \iint_{(\tau,y)\in\Omega'_\alpha,\ |w_y|\leq\eta^*,\ \tau\in\mathcal{T}_\alpha} \big|vw_{y\tau}\big|\, |dy|\, d\tau \\ &\qquad + \sum_\alpha \iint_{(\tau,y)\in\Omega'_\alpha,\ |w_y|\leq\eta^*,\ \tau\notin\mathcal{T}_\alpha} \big|vw_{y\tau}\big|\, |dy|\, d\tau \\ &\doteq E^\sharp + E^\flat. \end{aligned} \tag{12.12}$$

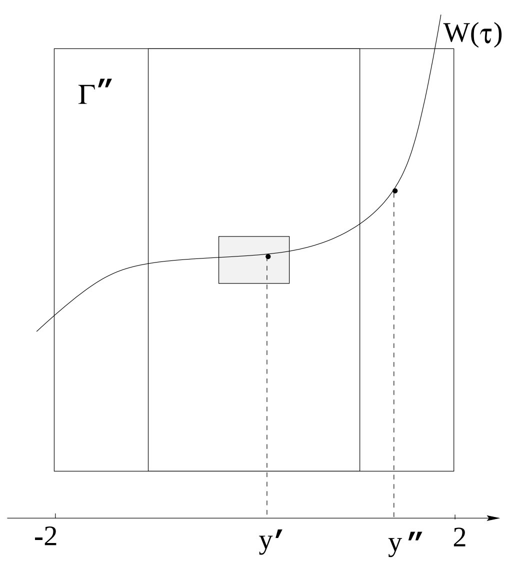

figure 10

We begin with an estimate of $E^\sharp$. Observe that (fig. 10), for $\tau \in \mathcal{T}_\alpha$, the derivative $W_Y(\tau, \cdot)$ changes from a small value $\leq \eta^*$ to some value ≥ 1 within the interval $[-2, 2]$. Moreover, by (12.5) the factor V^2 remains bounded away from zero. Therefore, for $\tau \in \mathcal{T}_\alpha$ we have

$$\int_{-2}^{2} V^2 W_{YY}^2 \, dY \geq c_0 \tag{12.13}$$

for some constant $c_0 > 0$. On the other hand, by (12.5) and (12.11) it follows

$$\int_{|Y|\leq 1/2,\ |W_Y|\leq \eta^*} \big|V W_{Y\tau}(\tau, Y)\big| \, dY \leq \frac{C_2}{v_\alpha}. \tag{12.14}$$

with $C_2 \doteq 21\, C_1/2$. Using (12.13)-(12.14), the rescaling properties (12.9)-(12.10) and finally (6.4) we obtain

$$\begin{aligned}
E^\sharp &\doteq \sum_\alpha \iint_{(\tau,y)\in\Omega'_\alpha,\ \tau\in\mathcal{T}_\alpha} \big|v w_{y\tau}\big| \, dy d\tau \\
&= \sum_\alpha v_\alpha^2 \int_{\mathcal{T}_\alpha} \int_{|Y|\leq 1/2,\ |W_Y|\leq\eta^*} |V W_{Y\tau}| \, dY \, d\tau \\
&\leq \sum_\alpha v_\alpha^2 \int_{\mathcal{T}_\alpha} \frac{C_2}{v_\alpha} \, d\tau \\
&\leq \sum_\alpha \frac{C_2 v_\alpha}{c_0} \int_{\mathcal{T}_\alpha} \int_{|Y|\leq 2,\ |W_Y|\leq 1} V^2 W_{YY}^2 \, d\tau \\
&\leq \sum_\alpha \frac{C_2}{c_0} \int_{\mathcal{T}_\alpha} \int_{y\in I''_\alpha,\ |w_y|\leq 1} v^2 w_{yy}^2 \, dy d\tau \\
&\leq \frac{N\, C_2}{c_0} \iint_{|w_y|\leq 1} v^2 w_{yy}^2 \, dy d\tau \\
&= \mathcal{O}(1).
\end{aligned} \tag{12.15}$$

Next, we provide an estimate on $E^\flat$. For each α we consider the open set of times

$$\mathcal{T}'_\alpha \doteq \Big\{\tau > 0\,;\quad 1 < W(\tau, Y) < 10 \qquad \text{for all } Y \in [-1, 1]\Big\}. \tag{12.16}$$

We write $\mathcal{T}'_\alpha$ as a disjoint union of open intervals

$$\mathcal{T}'_\alpha = \bigcup_j J_{\alpha,j}\,, \qquad J_{\alpha,j} =]t^-_{\alpha j},\ t^+_{\alpha j}[\,.$$

For a given interval $J_{\alpha,j}$, call

$$t_{\alpha,j} \doteq \inf\Big\{t \in J_{\alpha,j}\,;\quad \big|W_Y(t, Y)\big| \leq 1 \qquad \text{for all}\ \ Y \in [-1,\ 1]\Big\}, \tag{12.17}$$

$$t'_{\alpha,j} \doteq \inf\Big\{t \in J_{\alpha,j}\,;\quad \big|W_Y(t, Y)\big| \leq \eta^* \qquad \text{for all}\ \ Y \in [-2,\ 2]\Big\}, \tag{12.18}$$

$$\begin{aligned}
t''_{\alpha,j} \doteq \inf\Big\{t \in J_{\alpha,j}\,;\quad \big|W_Y(t, Y)\big| \leq \eta^* \qquad &\text{for all}\ \ Y \in [-2,\ 2] \\
\text{and } W(t, Y^*) \in [5, 6] \text{ for some } Y^* &\in [-1/4,\ 1/4]\Big\},
\end{aligned} \tag{12.19}$$

Recalling that $\eta^* \leq 1/4$, it is clear that $t_{\alpha,j^-} \leq t_{\alpha,j} \leq t'_{\alpha,j} \leq t''_{\alpha,j}$. We shall only consider pairs of indices (α, j) for which the set of times in (12.19) is nonempty. In the other case there is nothing to prove. The set $\mathcal{A}$ of all relevant pairs of indices will be split into four subsets: $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$, according to the following cases.

CASE 1: $t_\alpha^- = 0$, $t_{\alpha,j}^+ > 1$.

In this case, Lemma 3a yields the estimate

$$\int_1^{t_{\alpha,j}^+} \int_{-1/4}^{1/4} |W_{Y\tau}| \, dY \, d\tau \leq C \cdot \left(1 + \int_0^{t_{\alpha,j}^+} \int_{-1}^{1} |W_\tau| \, dY \, d\tau \right).$$

After a rescaling we obtain

$$\int_1^{t_{\alpha,j}^+} \int_{I'_\alpha} |v w_{y\tau}| \, dy d\tau \leq C' \left(v_\alpha^2 + \int_0^{t_{\alpha,j}^+} \int_{I_\alpha} |w_\tau| \, dy d\tau \right). \tag{12.19}$$

Observe that, by the property **(P)** of the covering,

$$\sum_{(\alpha,j)\in\mathcal{A}_1} v_\alpha \leq \frac{N}{2} \cdot \big[\text{total variation of } u(0,\cdot)\big] = \mathcal{O}(1).$$

Together with (6.2), this yields

$$\sum_{(\alpha,j)\in\mathcal{A}_1} \int_{t\in J_{\alpha,j}} \int_{y\in I'_\alpha, \ |w_y|<1} |v w_{y\tau}| \, dy d\tau = \mathcal{O}(1). \tag{12.20}$$

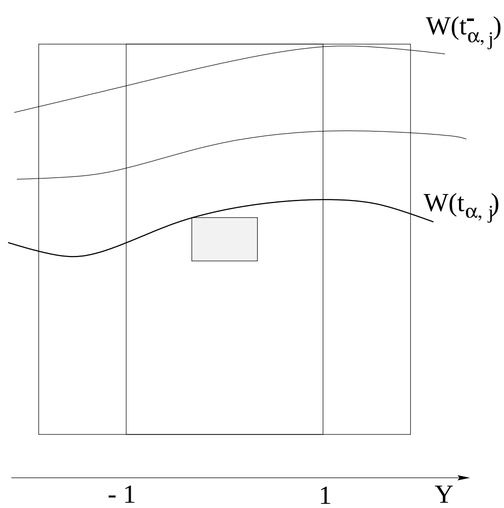

figure 11

CASE 2: $t_{\alpha,j} = t^-_{\alpha,j} > 0$.

The assumption implies

$$\left|W(t^-_{\alpha,j}, Y)\right| \leq 1 \qquad \text{for all } \; Y \in [-1,1].$$

Moreover, by continuity we must have (fig. 11)

$$W(t^-_{\alpha,j}, Y^\star) = 10 \quad \text{or} \quad W(t^-_{\alpha,j}, Y^\star) = 1$$

for some $Y^\star \in [-1,1]$. The bound on $|W_Y|$ implies that, for all $Y \in [-1,1]$, we have

$$W(t^-_{\alpha,j}, Y) > 8 \qquad \text{or} \qquad W(t^-_{\alpha,j}, Y) < 3$$

respectively. A simple comparison argument now shows that there exists a constant $c_1 > 0$ such that

$$t''_{\alpha,j} - t^-_{\alpha,j} > c_1.$$

Observing that

$$\int_{t^-_{\alpha,j}}^{t''_{\alpha,j}} \int_{-1}^{1} |W_\tau| \, dY \, d\tau > 2,$$

an application of Lemma 3a yields the estimate

$$\begin{aligned} \int_{t^-_{\alpha,j}+c_1}^{t^+_{\alpha,j}} \int_{-1/4}^{1/4} |W_{Y\tau}| \, dY \, d\tau &= \mathcal{O}(1) \cdot \left(1 + \int_{t^-_{\alpha,j}}^{t^+_{\alpha,j}} \int_{-1}^{1} |W_\tau| \, dY \, d\tau\right) \\ &= \mathcal{O}(1) \cdot \int_{t^-_{\alpha,j}}^{t^+_{\alpha,j}} \int_{-1}^{1} |W_\tau| \, dY \, d\tau \,. \end{aligned} \tag{12.21}$$

After a rescaling, this yields

$$\begin{aligned} &\sum_{(\alpha,j)\in\mathcal{A}_2} \int_{t\in J_{\alpha,j}} \int_{y\in I'_\alpha, \; |w_y|<1} |vw_{y\tau}| \, dy d\tau \\ &\qquad \leq \sum_{(\alpha,j)\in\mathcal{A}_2} \int_{t''_{\alpha,j}}^{t^+_{\alpha,j}} \int_{y\in I'_\alpha} |vw_{y\tau}| \, dy d\tau \\ &\qquad = \mathcal{O}(1) \cdot \sum_{(\alpha,j)\in\mathcal{A}_2} \int_{t^-_{\alpha,j}}^{t^+_{\alpha,j}} \int_{y\in I_\alpha} |w_\tau| \, dy d\tau \\ &\qquad = \mathcal{O}(1). \end{aligned} \tag{12.22}$$

CASE 3: $t^-_{\alpha,j} > 0$, $t''_{\alpha,j} - t^-_{\alpha,j} > 1$.

As in the previous case, Lemma 3a yields

$$\int_{t''_{\alpha,j}}^{t^+_{\alpha,j}} \int_{-1/4}^{1/4} |W_{Y\tau}| \, dY \, d\tau = \mathcal{O}(1) \cdot \left(1 + \int_{t^-_{\alpha,j}}^{t^+_{\alpha,j}} \int_{-1}^{1} |W_\tau| \, dY \, d\tau\right). \tag{12.23}$$

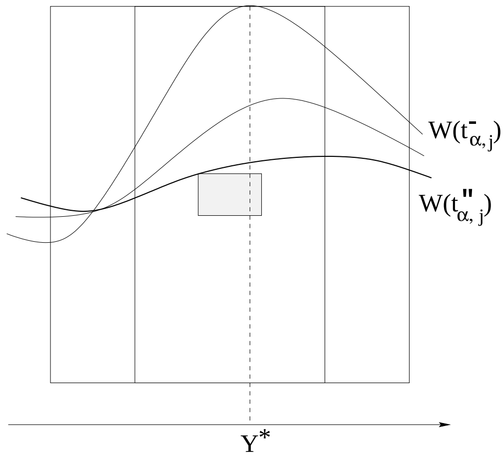

figure 12

If

$$\int_{t^-_{\alpha,j}}^{t^+_{\alpha,j}} \int_{-2}^{2} |W_\tau|\, dY\, d\tau \ \geq 1 \tag{12.24}$$

we can argue as in Case 2, and we are done. To handle the opposite case, assume that

$$W(t^-_{\alpha,j}, Y^\star) = 10 \tag{12.25}$$

for some $Y^\star \in [-1,1]$, the other case being similar. Consider the two squares

$$Q_1 \doteq [-2,-1] \times [7,8], \qquad\qquad Q_2 \doteq [1,2] \times [7,8]$$

Figure 12 shows a case where (12.24) holds. On the other hand, if (12.24) fails (fig. 13), there exist points

$$P_1 = (Y_1, W_1) \in Q_1, \qquad P_2 = (Y_2, W_2) \in Q_2$$

which never lie on the graph of W, as $t \in [t^-_{\alpha,j},\ t^+_{\alpha,j}]$. Hence

$$W(t, Y_1) < W_1 \leq 8, \qquad W(t, Y_2) < W_2 \leq 8 \qquad \text{for all} \quad t \in [t^-_{\alpha,j},\ t^+_{\alpha,j}].$$

Recalling (12.25), this implies that the height of a local max in W decreases from 10 down to 8, hence the total variation of W decreases by at least 4 units. After a rescaling, we obtain

$$\begin{aligned} &\sum_{(\alpha,j)\in\mathcal{A}_3} \int_{t\in J_{\alpha,j}} \int_{y\in I'_\alpha,\ |w_y|<1} |v w_{y\tau}|\, dy d\tau \\ &\qquad \leq \sum_{(\alpha,j)\in\mathcal{A}_3} \int_{t''_{\alpha,j}}^{t^+_{\alpha,j}} \int_{y\in I'_\alpha} |v w_{y\tau}|\, dy d\tau \\ &\qquad = \mathcal{O}(1) \cdot \sum_{(\alpha,j)\in\mathcal{A}_3} \int_{t^-_{\alpha,j}}^{t^+_{\alpha,j}} \int_{y\in I''_\alpha} |w_\tau|\, dy d\tau + \sum_{(\alpha,j)\in\mathcal{A}'_3} \mathcal{O}(1) \cdot v_\alpha^2 \\ &\qquad = \mathcal{O}(1). \end{aligned} \tag{12.26}$$

Here $\mathcal{A}_3'$ denotes the set of indices (α, j) for which the rescaled function W does not satisfy (12.24). The last estimate in (12.26) follows from (6.2) and the fact that, by the previous arguments, to each $(\alpha, j) \in \mathcal{A}_3'$ there corresponds a decrease of at least $4v_\alpha$ in the total variation of w. Since this total variation is non-increasing and initially bounded, there holds

$$\sum_{(\alpha,j)\in\mathcal{A}_3'} v_\alpha = \mathcal{O}(1).$$

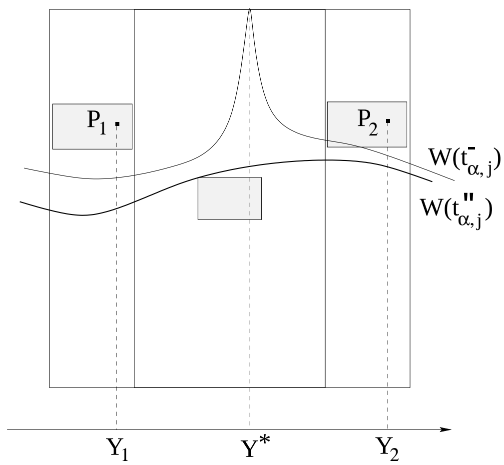

figure 13

CASE 4: $t_{\alpha,j} > t^-_{\alpha,j} > 0$, $t''_{\alpha,j} - t^-_{\alpha,j} \leq 1$.

We first observe that the bounds (12.11) imply

$$t''_{\alpha,j} - t'_{\alpha,j} \geq c_1 v_\alpha \tag{12.27}$$

for some constant $c_1 > 0$. Indeed, the derivative W_Y must change from ± 1 to a value inside $[-\eta^*, \eta^*]$ within the above time interval. We now consider two subcases. Assume first that

$$\int_{t^-_{\alpha,j}}^{t''_{\alpha,j}} \int_{-2}^{2} |W_\tau|\, dY\, d\tau = \int_{t^-_{\alpha,j}}^{t''_{\alpha,j}} \int_{-2}^{2} |V^2 W_{YY}|\, dY\, d\tau \geq 1, \tag{12.28}$$

In this case, since V is uniformly bounded above and below, by the Cauchy inequality we obtain the a priori bound

$$\int_{t^-_{\alpha,j}}^{t''_{\alpha,j}} \int_{-2}^{2} V^2 W_{YY}^2\, dY\, dt \geq c_2 \tag{12.29}$$

for some constant $c_2 > 0$. On the other hand, if (12.28) fails, then arguing as in Case 3 we conclude that the total variation of W decreases at least by 4 units within the time interval $[t^-_{\alpha,j},\ t''_{\alpha,j}]$. After a rescaling we see that, for some constant $c_3 > 0$, to each $(\alpha, j) \in \mathcal{A}_4$ there corresponds a decrease of at least $c_3 v_\alpha$ either in the length of the graph of w, or in the total variation of w. Hence

$$\sum_{(\alpha,j)\in\mathcal{A}_4} v_\alpha = \mathcal{O}(1). \tag{12.30}$$

Using Lemma 3b we now obtain

$$
\begin{aligned}
\int_{t''_{\alpha,j}}^{t^+_{\alpha,j}} \int_{-1/4}^{1/4} |W_{Y\tau}|\,dY\,d\tau &= \mathcal{O}(1)\cdot\left(1+\big|\ln(t''_{\alpha,j}-t'_{\alpha,j})\big| + \int_{t^-_{\alpha,j}}^{t^+_{\alpha,j}}\int_{-1}^{1}|W_\tau|\,dY\,d\tau\right)\\
&= \mathcal{O}(1)\cdot\left(1+|\ln v_\alpha| + \int_{t^-_{\alpha,j}}^{t^+_{\alpha,j}}\int_{-1}^{1}|W_\tau|\,dY\,d\tau\right).
\end{aligned}
\tag{12.31}
$$

After a rescaling one finds

$$
\begin{aligned}
&\sum_{(\alpha,j)\in\mathcal{A}_4}\int_{t\in J_{\alpha,j}}\int_{y\in I'_\alpha,\ |w_y|<1}|v w_{y\tau}|\,dyd\tau\\
&\qquad\leq \sum_{(\alpha,j)\in\mathcal{A}_4}\int_{t''_{\alpha,j}}^{t^+_{\alpha,j}}\int_{y\in I'_\alpha}|v w_{y\tau}|\,dyd\tau\\
&\qquad= \mathcal{O}(1)\cdot\sum_{(\alpha,j)\in\mathcal{A}_4}\int_{t^-_{\alpha,j}}^{t^+_{\alpha,j}}\int_{y\in I''_\alpha}|w_\tau|\,dyd\tau + \sum_{(\alpha,j)\in\mathcal{A}_4}\mathcal{O}(1)\cdot v_\alpha^2\big(1+|\ln v_\alpha|\big)\\
&\qquad= \mathcal{O}(1).
\end{aligned}
\tag{12.32}
$$

The four cases discussed above cover all possibilities. Together, the estimates (12.20), (12.22), (12.26) and (12.32) provide an a priori bound on the second integral $E^\flat$ in (12.12).

In view of Lemma 1, the existence of a front-tracing representation for the quantity $z^2_{1,x}$ in (3.4) implies a uniform bound on the total variation of the second Riemann coordinate z_2. This completes the proof of Theorem 1.

13 - The vanishing viscosity limit

Consider again the system (1.2), where A is the matrix defined at (3.2). Let an initial data $u(0,x)=\bar u\in\mathbf{L}^1$ be given, with small total variation. By the previous analysis, the total variation of the corresponding solutions u^ε remain uniformly bounded in time, as $\varepsilon\to 0$. By Helly's compactness theorem we can thus extract a subsequence, converging to some function $u=u(t,x)$ in $\mathbf{L}^1_{\mathrm{loc}}$. Since the convergence is strong, this limit function u provides an entropy admissible solution to the system of conservation laws (3.1). By the uniqueness theorem in [3, p.188], valid for BV solutions, we conclude that u coincides with the unique entropy weak solution with the given initial data.

Remark. The previous analysis has established global BV bounds for solutions of the viscous system (3.1). It remains an open problem to prove their $\mathbf{L}^1$ stability. Toward this goal, consider any solution $u=u(t,x)$, with suitably small total variation. Assume that we could show that every solution h of the linear variational system

$$
h_t + \big[A(u)h\big]_x = h_{xx}
\tag{13.1}
$$

satisfies an estimate of the form

$$
\big\|h(t)\big\|_{\mathbf{L}^1}\leq L\cdot\big\|h(0)\big\|_{\mathbf{L}^1},
\tag{13.2}
$$

for some constant L independent of t and u. As in [1], by a standard homotopy argument we could then conclude

$$\left\|u(t) - u'(t)\right\|_{\mathbf{L}^1} \leq L \cdot \left\|u(0) - u'(0)\right\|_{\mathbf{L}^1}$$

for every couple of solutions u, u' of (3.1). Observe that $h = u_x$ provides one particular solution of (13.1), for which the estimate (13.2) is known. We conjecture that the bound (13.2) can be proved by suitably extending the techniques used to estimate the total variation.

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