BV solutions of the Jin-Xin model

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We consider the (special) Jin-Xin relaxation model [Jin-Xin '95]

$$\begin{cases} u_t + v_x = 0 \\ v_t + \Lambda^2 u_x = \frac{1}{\epsilon}(\mathcal{F}(u) - v) \end{cases} \qquad u, v \in \mathbb{R}^n,\ \Lambda \in \mathbb{R}, \tag{1}$$

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Diagonalizing $2F^- = u - v$, $2F^+ = u + v$, we obtain the BGK model

$$\begin{cases} F^-_t - F^-_x = \frac{1}{\epsilon}(M^-(u) - F^-) \\ F^+_t + F^+_x = \frac{1}{\epsilon}(M^+(u) - F^+) \end{cases} \qquad F^-, F^+ \in \mathbb{R}^n, \tag{2}$$

where $u = F^- + F^+$, $M^-(u) = \frac{u - \mathcal{F}(u)}{2}$, $M^+(u) = \frac{u + \mathcal{F}(u)}{2}$.

General settings

Equation (1) can be written as

$$u_t + A(u)u_x = \epsilon(u_{xx} - u_{tt}), \qquad u \in \mathbb{R}^n, \tag{3}$$

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1) $A(u)$ strictly hyperbolic and

$$-1 + c \le \lambda_i(u) \le 1 - c, \qquad c > 0;$$

2) the initial data $(u_0, \epsilon u_{0,t})$ are sufficiently smooth and with total variation less than $\delta_0 \ll 1$:

$$\|u_0\|_{L^\infty}, \|\epsilon u_{0,t}\|_{L^\infty} \le \delta_0, \quad \|u_{0,x}\|_{L^1}, \|\epsilon u_{0,tx}\|_{L^1} \le \delta_0.$$

Existence and stability theorem. *Under the above assumptions, there exists a global solution* (u, u_t) *of* (3), *defined for all* $t \geq 0$, *such that*

$$\|u(t)\|_{L^\infty}, \|\epsilon u(t)\|_{L^\infty} \leq C\delta_0, \quad \|u_x(t)\|_{L^1}, \|\epsilon u_{tx}(t)\|_{L^1} \leq C\delta_0. \quad (4)$$

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Moreover,

$$\begin{aligned} &\|u(t) - \widehat{u}(s)\|_{L^1} + \epsilon\|u_t(t) - \widehat{u}_t(s)\|_{L^1} \\ &\quad \leq L\Big(|t - s| + \Big\|(u_0 + \epsilon u_{0,t}) - (\widehat{u}_0 + \epsilon \widehat{u}_{0,t})\Big\|_{L^1}\Big) \\ &\qquad + Le^{-t/\epsilon}\epsilon\|u_{0,t} - \widehat{u}_{0,t}\|_{L^1} \\ &\qquad + L\Big(\epsilon^2\|u_{0,tx} - \widehat{u}_{0,tx}\|_{L^1} + \epsilon^3\|u_{0,txx} - \widehat{u}_{0,txx}\|_{L^1}\Big). \end{aligned} \tag{5}$$

Convergence theorem. *As* $\epsilon \to 0$*, the solution* $u^\epsilon(t)$ *with initial data* $(u_0, \epsilon u_{0,t})$ *converges to a unique limit* $u(t)$ *in* L^1_{loc}*.*

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The function $u(t)$ has uniformly bounded total variation and generates a Lipschitz continuous semigroup $u(t) = \mathcal{S}_{t-s}u(s)$,

$$\|u(t) - \widehat{u}(s)\|_{L^1} \leq L\Big(|t-s| + \|u(\tau) - \widehat{u}(\tau)\|_{L^1}\Big), \ t, s \geq \tau > 0. \quad (6)$$

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This semigroup is defined on a domain $\mathcal{D}$ containing all the function with sufficiently small total variation, and can be uniquely identified by a relaxation limiting Riemann Solver, i.e. the unique Riemann solver compatible with (3).

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in BV estimates it is important not $u_t \in L^1$ but $u_{tx} \in L^1$.

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$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Gamma(t)*\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} + \int_0^t \underbrace{\Gamma(t-s)*\begin{pmatrix} 0 \\ \mathcal{F}(u(s)) - A(0)u(s) \end{pmatrix}}_{\approx G_x(t-s)*u(s)^2} ds$$

• The dependence w.r.t. $u_0 + \epsilon u_{0,t}$ can be easily seen with the example

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The hyperbolic limit $\epsilon \to 0$ has the "initial data"

$$\lim_{t \to 0+} u(t) = 1 = \lim_{\epsilon \to 0} u_0 + \epsilon u_{t,0}.$$

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Our aim:

$$\|f^\pm(0)\|_{L^1}, \|g^\pm(0)\|_{L^1} \le \delta_0 \quad \Longrightarrow \quad f^\pm(t), g^\pm(t) \in L^1(\mathbb{R}).$$

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$$p = v_i \tilde{r}_i(u, v_i, \sigma), \quad \tilde{\lambda}_i = \langle \tilde{r}_i, A(u)\tilde{r}_i \rangle, \quad |\tilde{r}_i(u)| = 1. \tag{11}$$

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We can parameterize by the the kinetic component f_i:

$$u_t = f^- - f^+ = -\sigma u_x = f^- + f^+$$

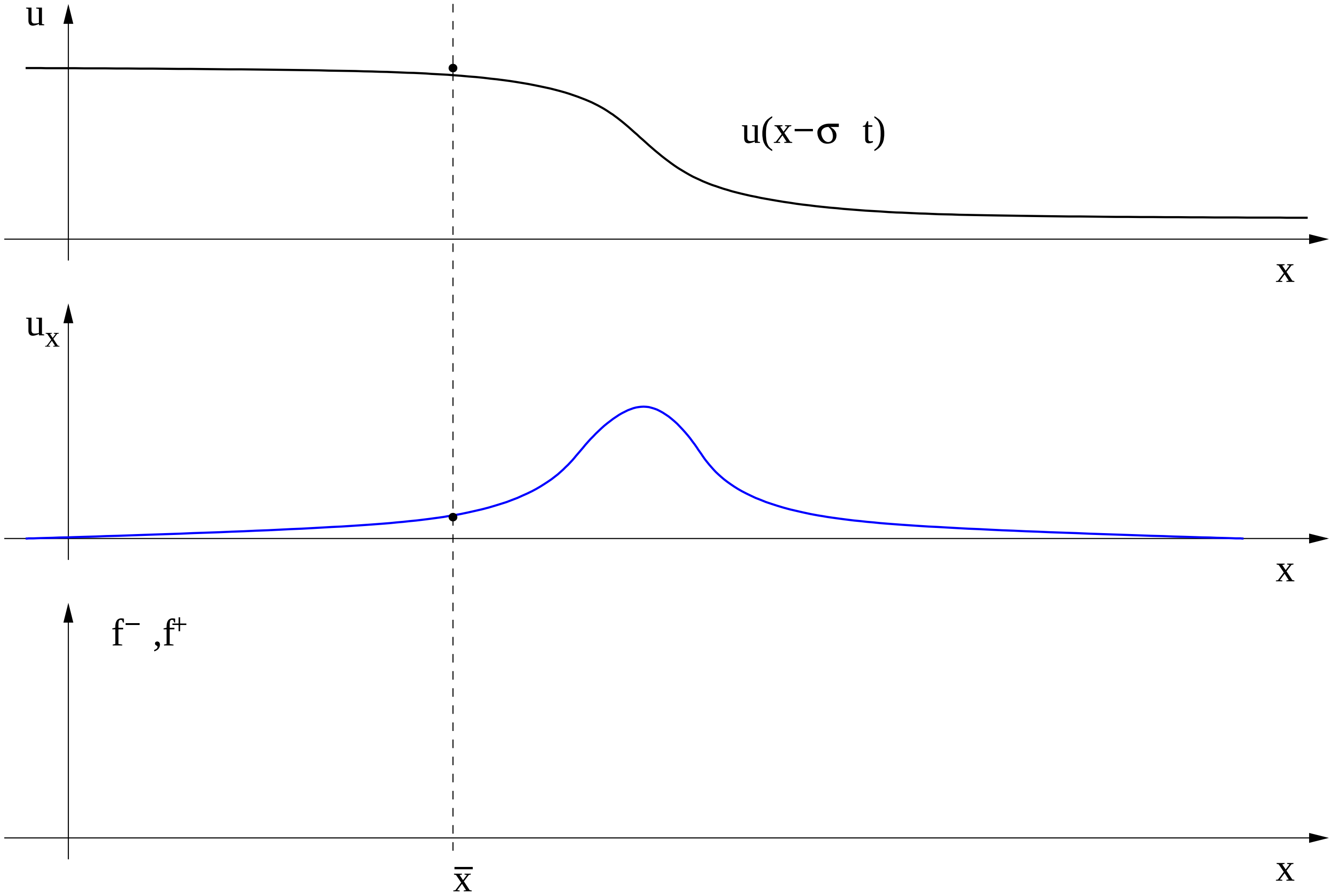
$$u_x = \frac{1}{1-\sigma} f^- = \frac{1}{1+\sigma} f^+.$$

The center manifold for the kinetic components f^-, f^+ is

$$f^- = (1-\sigma) v_i \tilde{r}_i(u, v_i, \sigma) = f_i^- \tilde{r}_i \left(u, \frac{f_i^-}{1-\sigma}, \sigma \right) = f_i^- \tilde{r}_i^-(u, f_i^-, \sigma) \tag{12}$$

$$f^+ = (1+\sigma) \tilde{r}_i \left(u, \frac{(1+\sigma) v_i}{1+\sigma}, \sigma \right) = f_i^+ \tilde{r}_i^+(u, f_i^+, \sigma) \tag{13}$$

Identification of a travelling profile: $u(\bar{x})$, σ and $v_i(\bar{x})$, Identification of a travelling profile: $u(\bar{x})$, σ and $v_i(\bar{x})$,



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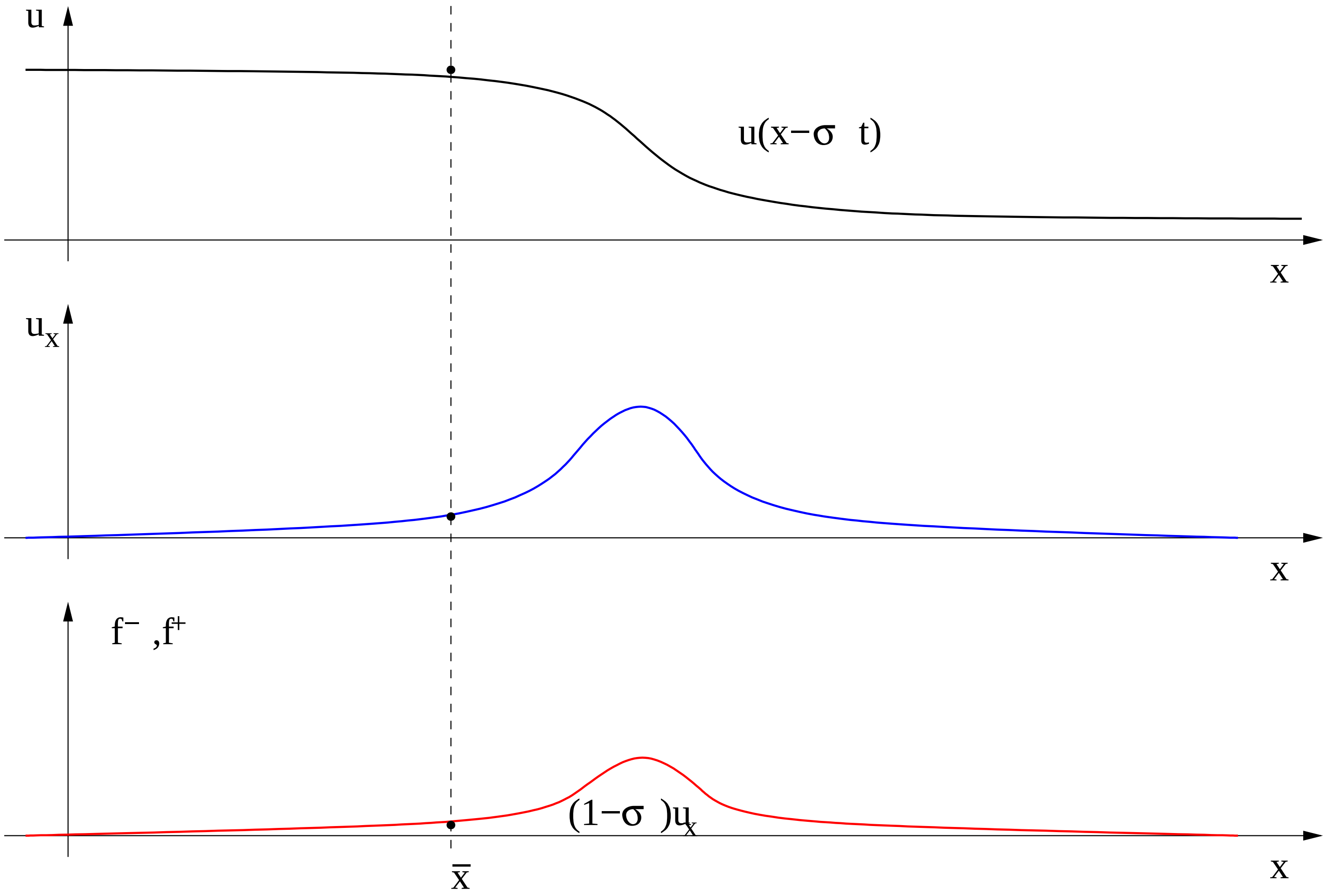

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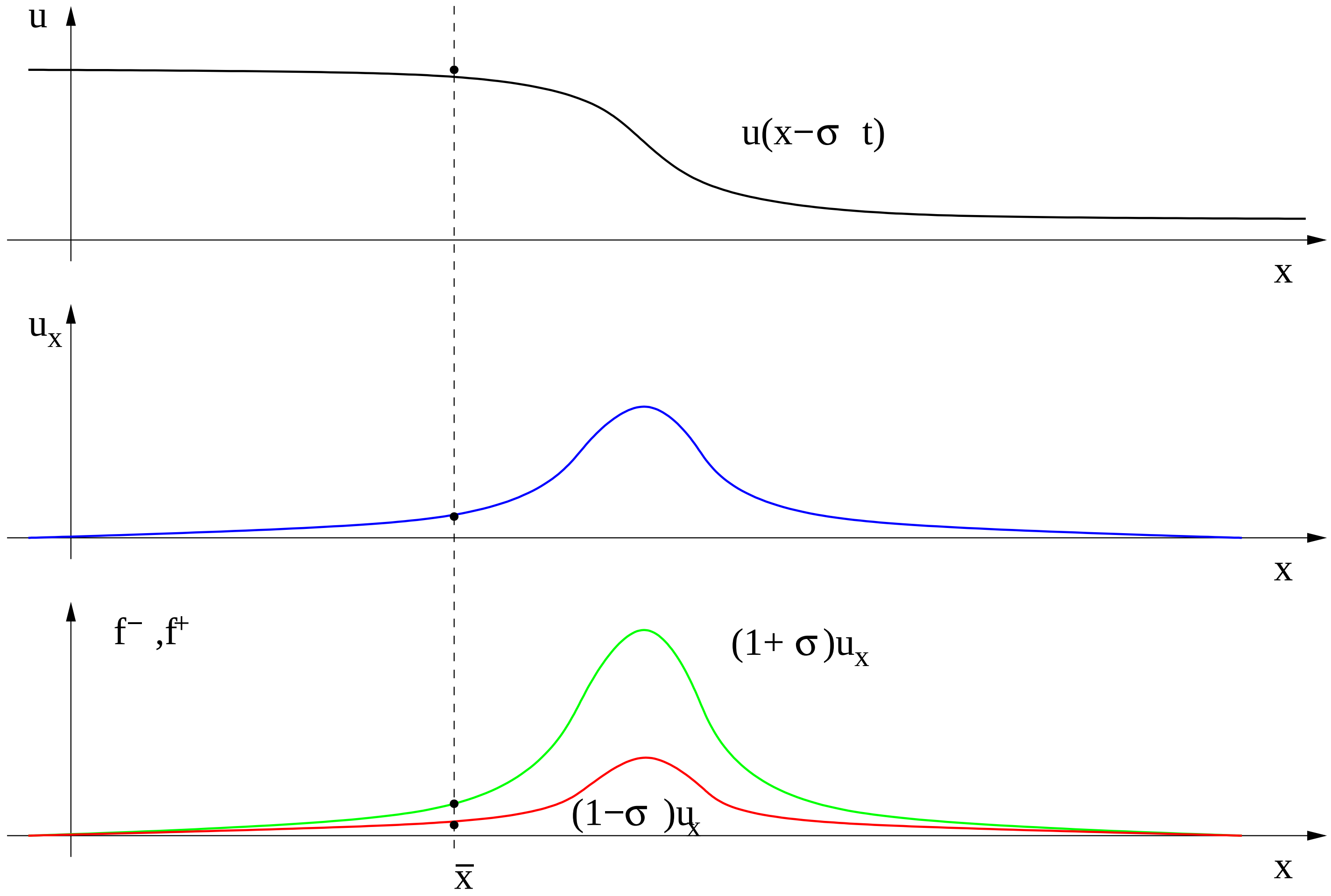

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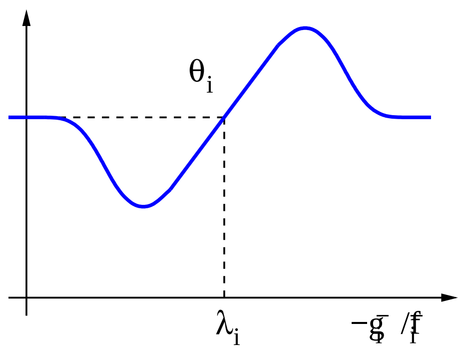

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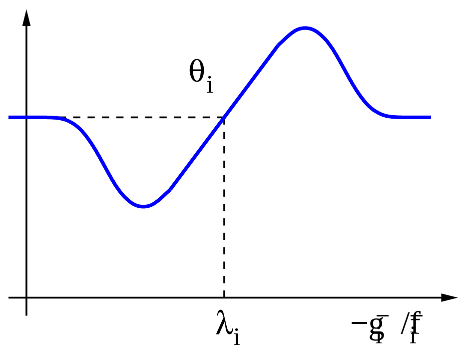

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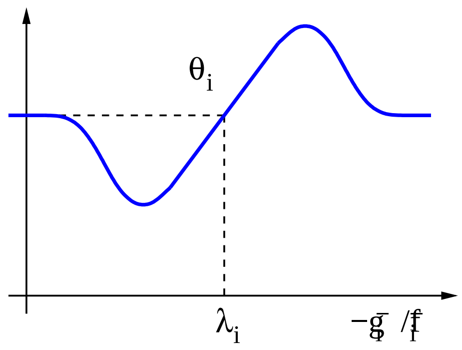
where θ_i is the cutoff function 

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Similarly for (f^+, g^+):

$$\begin{cases} f^+ &= \sum_i f_i^+ \tilde{r}_i^+(u, f_i^+, \sigma_i^+) \\ g^+ &= \sum_i g_i^+ \tilde{r}_i^+(u, f_i^+, \sigma_i^+) \end{cases} \qquad \sigma_i^+ = \theta_i\left(-\frac{g_i^+}{f_i^+}\right), \tag{15}$$

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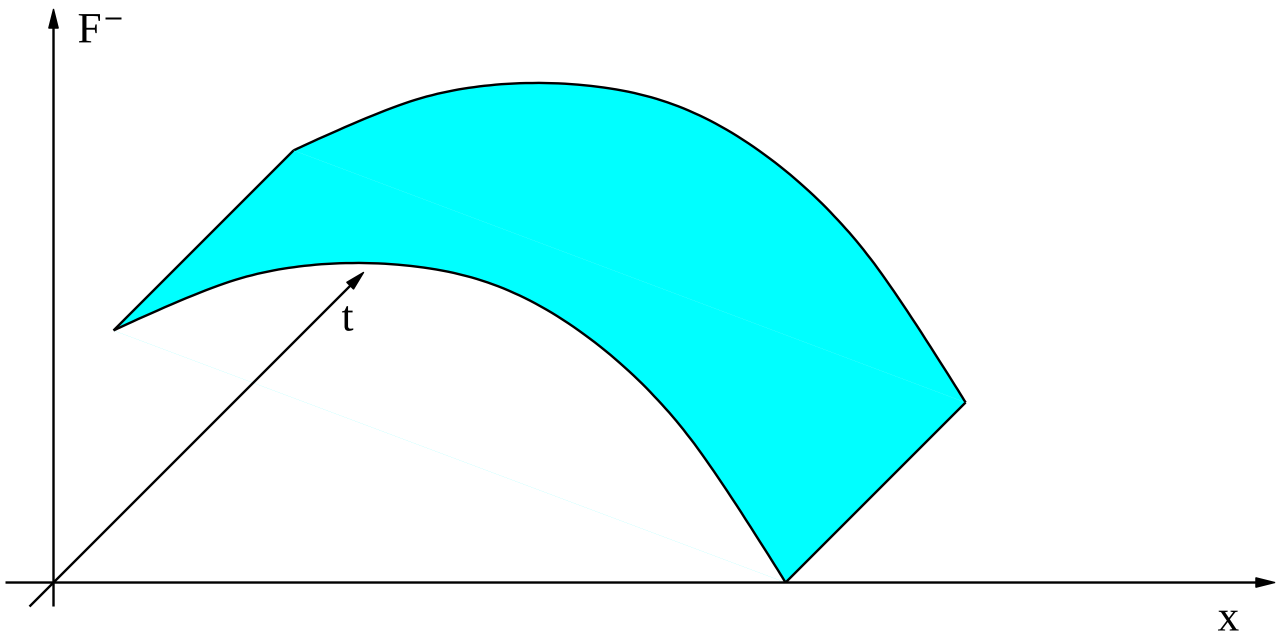

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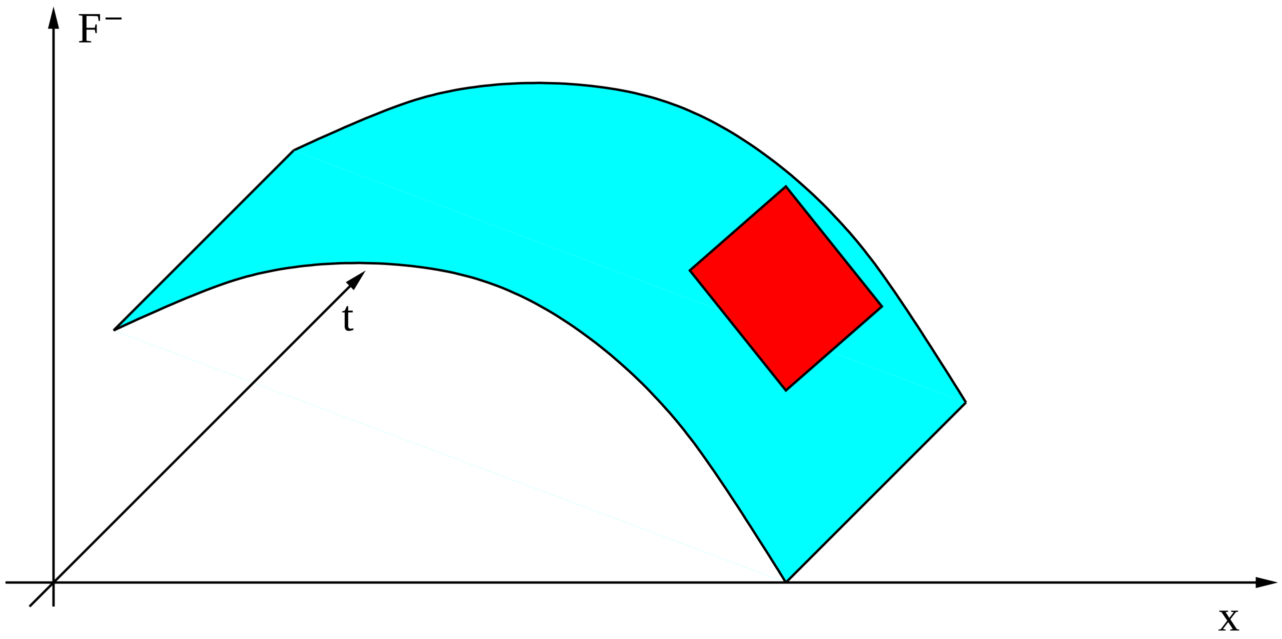

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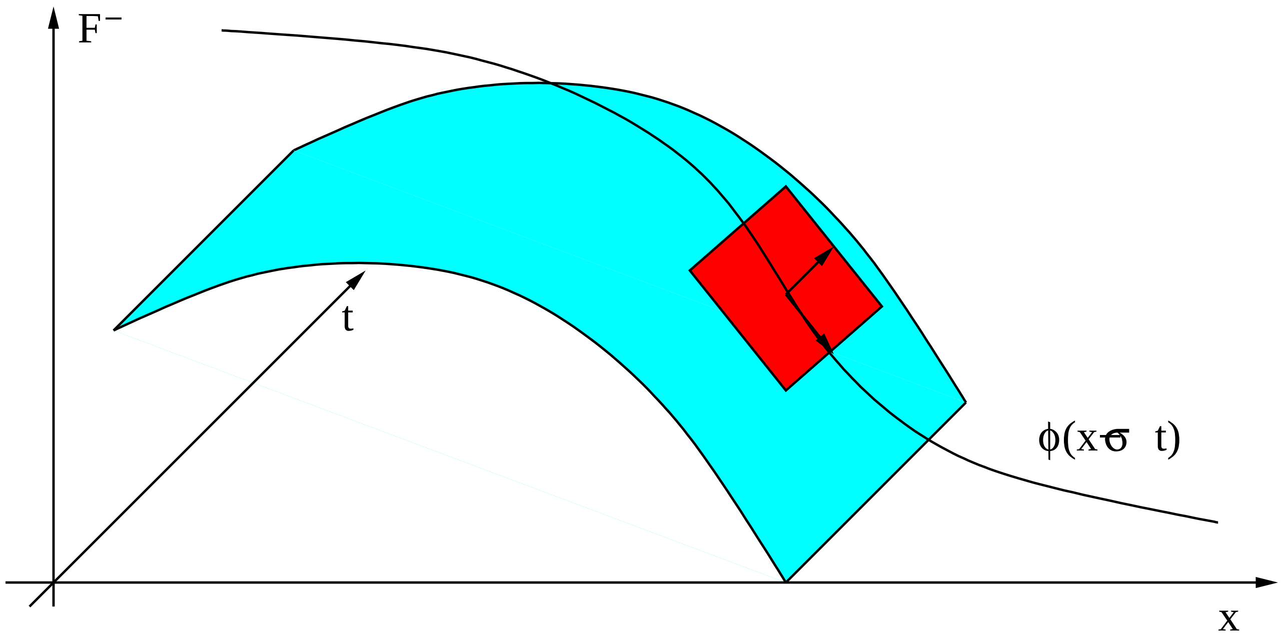

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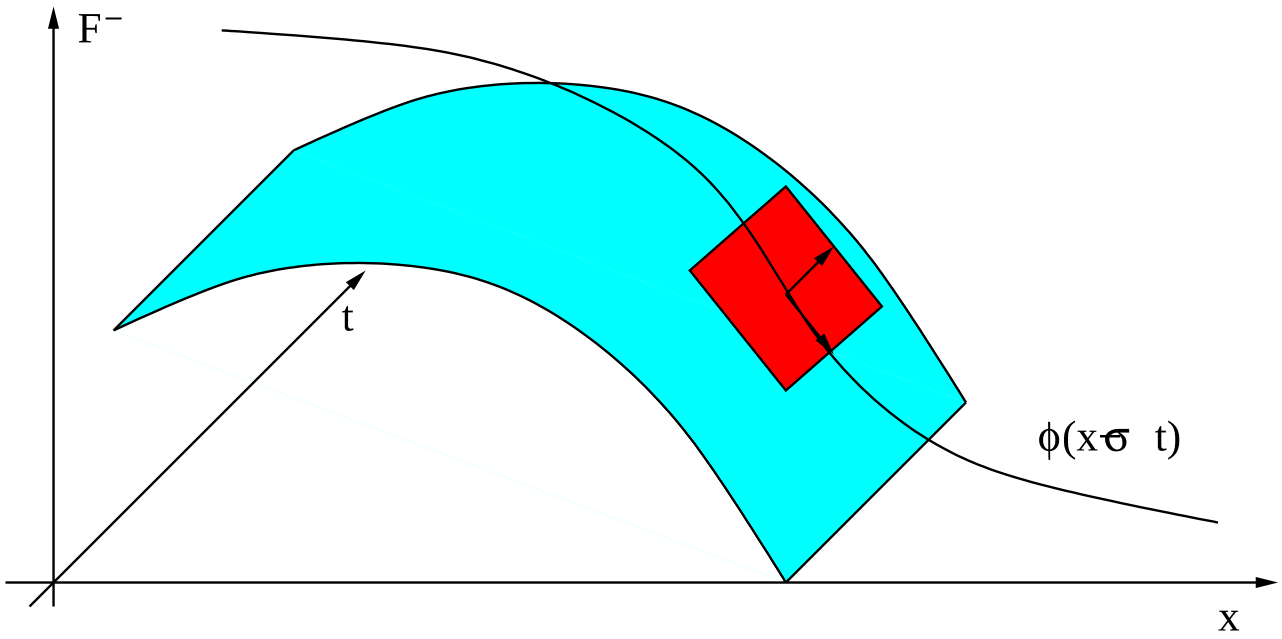

We obtain thus $2n$ *travelling waves:* n *for* F^- *and* n *for* F^+*.*

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$$\tilde{\lambda}^-_i = \tilde{\lambda}_i\left(u, \frac{f^-_i}{1-\sigma^-_i}, \sigma^-_i\right), \quad \tilde{\lambda}^+_i = \tilde{\lambda}_i\left(u, \frac{f^+_i}{1+\sigma^-_i}, \sigma^+_i\right).$$

After some computations, one obtains the source terms of the form

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\begin{aligned}
|\varsigma^{\pm}_{f,i}|, |\varsigma^{\pm}_{g,i}| \leq\ & C \sum_{j\neq k} (|f_j^-| + |g_j^-|)(|f_k^+| + |g_k^+|) + C \sum_j |g_j^- f_j^+ - f_j^- g_j^+| \\
& + C \sum_j \Big(|f_j^- + f_j^+|^2 + |g_j^- + g_j^+|^2 \Big) \chi \left\{ \frac{f_j^+}{f_j^-} \not\approx 1 \right\} \\
& + C \sum_j (\|f_j^-\|^2_{L^1} + \|f_j^+\|^2_{L^1}) |f_j^- - f_j^+| \chi\{f_j^- \cdot f_j^+ < 0\} \\
& + C \sum_j (\|f_j^-\|^2_{L^1} + \|f_j^+\|^2_{L^1}) |g_j^- - g_j^+| \chi\{g_j^- \cdot g_j^+ < 0\}.
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Prove that the source terms are quadratic w.r.t. $\|f^{\pm}\|_{L^1}$, $\|g^{\pm}\|_{L^1}$.

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- L^1 decay terms:

$$\sum_j |f_j^- - f_j^+|\chi\{f_j^- \cdot f_j^+ < 0\} + \sum_j |g_j^- - g_j^+|\chi\{g_j^- \cdot g_j^+ < 0\}.$$

Interaction of the same family

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Consider the 2×2 system

$$\left\{ \begin{array}{lcl} F_t^- - F_x^- & = & \frac{1-\mathcal{F}(u)}{2} - F^- \\ F_t^+ - F_x^+ & = & \frac{1+\mathcal{F}(u)}{2} - F^+ \end{array} \right. \qquad u = F^- + F^+, |\mathcal{F}'(u)| \leq 1 - c. \tag{19}$$

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Construct a functional which bounds

$$\int_0^{+\infty} \int_{\mathbb{R}} |f^-(t,x) g^+(t,x) - g^-(t,x) f^+(t,x)| dx dt.$$

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For simplicity we assume in the following $\lambda = 0$.

Consider the system (20), and construct the scalar variables

$$
\begin{aligned}
P^{--}(t,x,y) &= f^{-}(t,x)g^{-}(t,y) - f^{-}(t,y)g^{-}(t,x)\\
P^{-+}(t,x,y) &= f^{+}(t,x)g^{-}(t,y) - f^{-}(t,y)g^{+}(t,x)\\
P^{+-}(t,x,y) &= f^{-}(t,x)g^{+}(t,y) - f^{+}(t,y)g^{-}(t,x)\\
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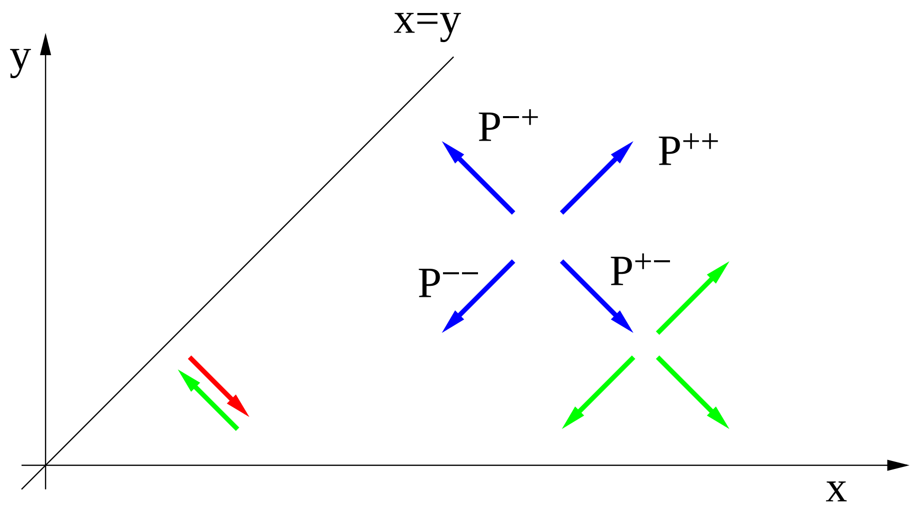
\end{aligned}
$$

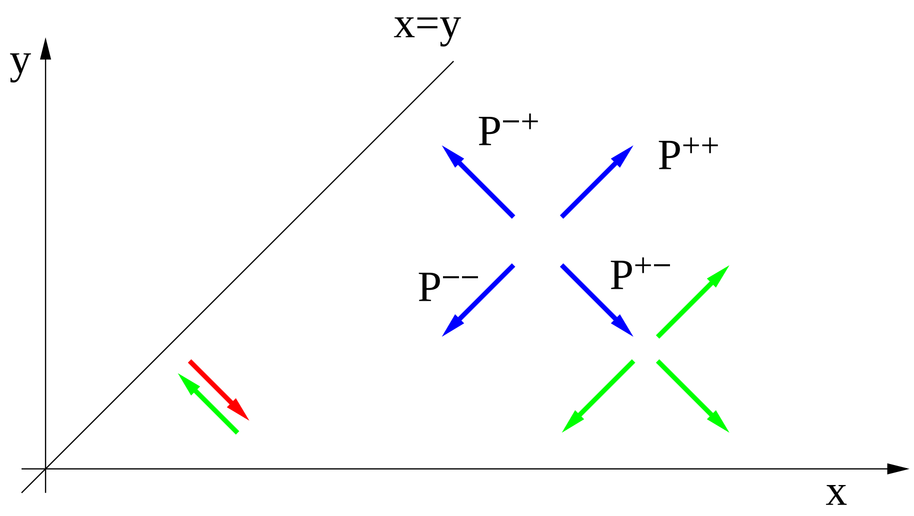
which satisfy the system

$$
\begin{cases}
P_t^{--} + \mathrm{div}((-1,-1)P^{--}) &= \frac{P^{+-}+P^{-+}}{2} - P^{--} \\
P_t^{-+} + \mathrm{div}((-1,1)P^{-+}) &= \frac{P^{--}+P^{++}}{2} - P^{-+} \\
P_t^{+-} + \mathrm{div}((1,-1)P^{+-}) &= \frac{P^{--}+P^{++}}{2} - P^{+-} \\
P_t^{++} + \mathrm{div}((1,1)P^{++}) &= \frac{P^{+-}+P^{-+}}{2} - P^{++}
\end{cases} \qquad (21)
$$

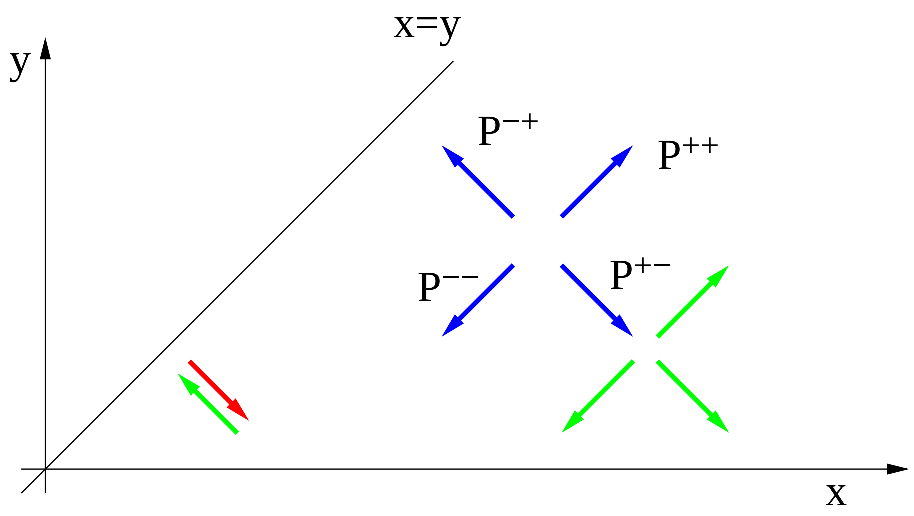
for $x \geq y$ and the boundary conditions

$$
P^{-+}(t,x,x) + P^{+-}(t,x,x) = 0, \qquad P^{++}(t,x,x) = P^{--}(t,x,x) = 0.
$$


x=y
y
P−+
P++
P−−
P+−
x



We may read the boundary conditions as follows: a particle P^{-+} hits the boundary and bounce back as P^{+-} but with opposite sign.



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We are interested in an estimate of the flux of P^{-+} through the boundary $\{x = y\}$, which is given by

$$\int_0^{+\infty} \int_{\mathbb{R}} |P^{-+}(t, x, x)| dx dt = \int_0^{+\infty} \int_{\mathbb{R}} |f^- g^+ - g^- f^+| dx dt.$$

Flux through the boundary

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A very simple situation is the 2×2 system

$$\left\{ \begin{array}{lcl} f_t^- - f_x^- & = & \frac{f^+ - f^-}{2} \\ f_t^+ + f_x^+ & = & \frac{f^- - f^+}{2} \end{array} \right. \qquad x \geq 0,$$

with boundary condition $f^+(x=0) + f^-(x=0) = 0$.

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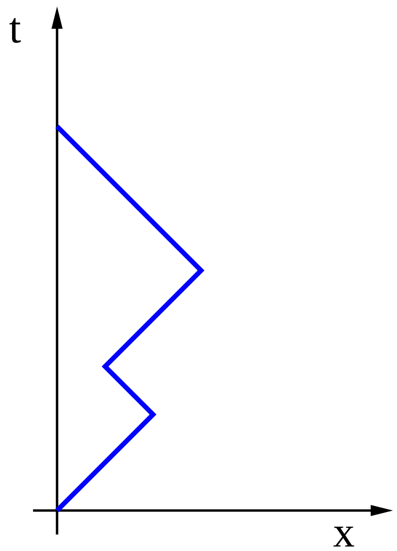
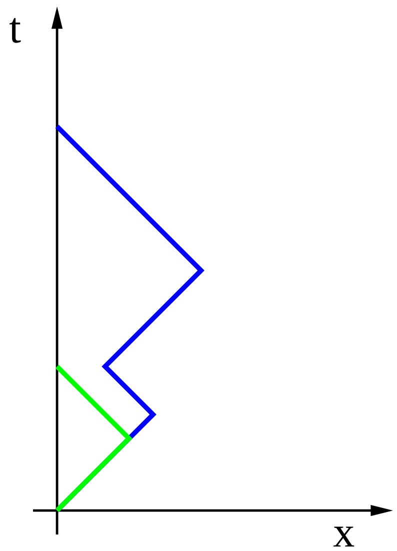
with boundary condition $f^+(x=0) + f^-(x=0) = 0$.
We want to estimate

$$\int_0^\infty |f^-(t,0)|dt, \tag{22}$$

i.e. the total amount of particles which hit the boundary and bounce back with the opposite sign.


t
x


t
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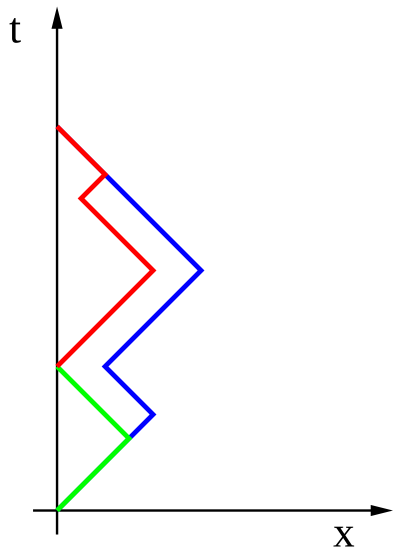
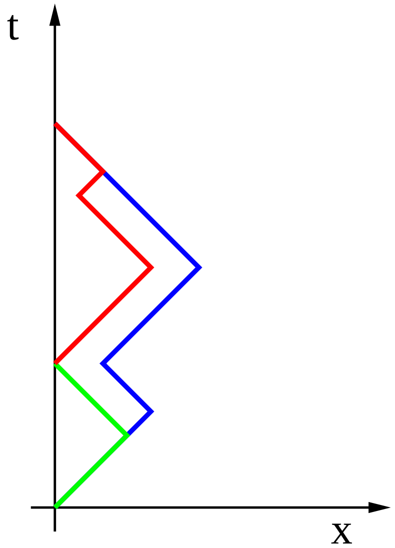
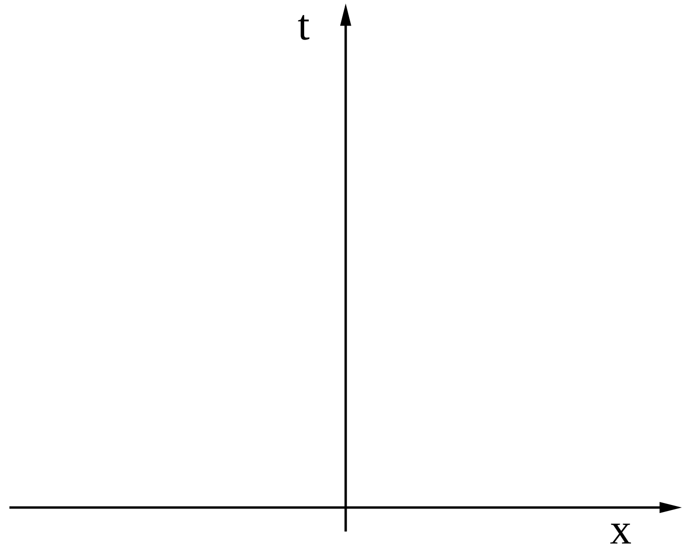

t
x


t
x


t
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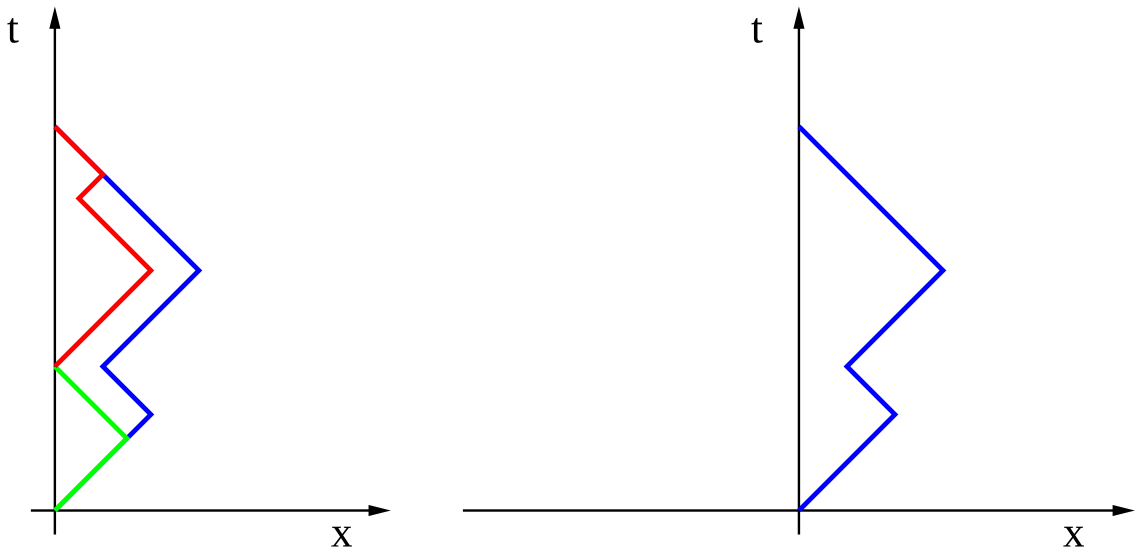
t
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t
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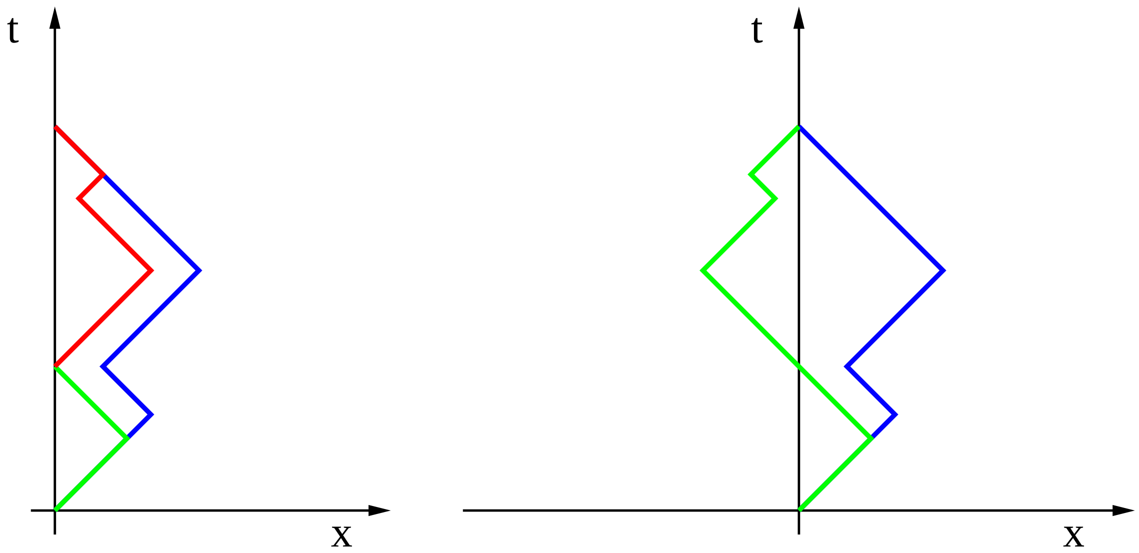
t
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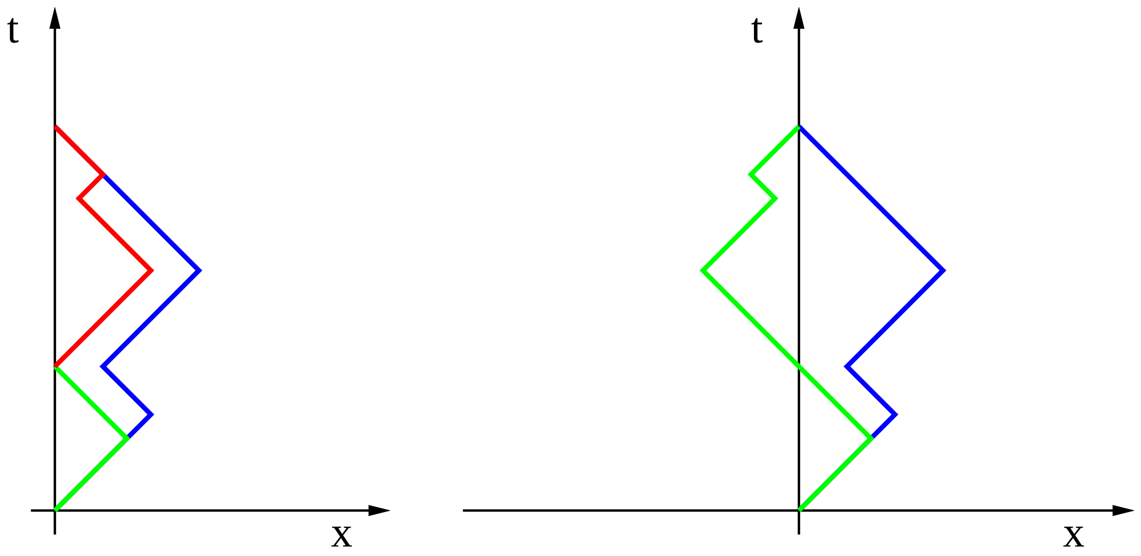
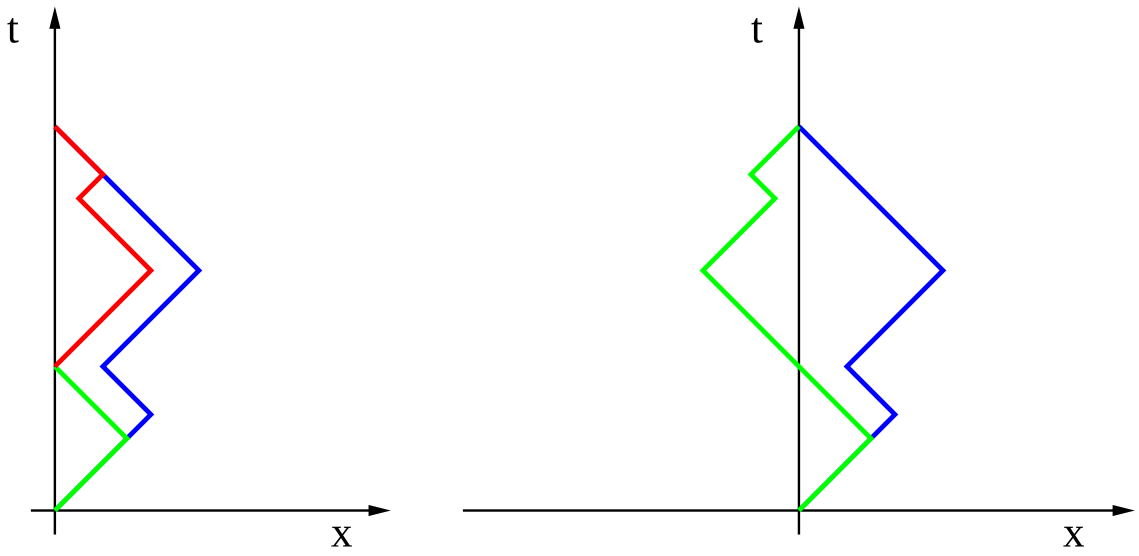
t
x



We can rewrite the integral (22) as

particles with speed -1 $-$ particles with speed 1.



We can rewrite the integral (22) as

$$\text{particles with speed -1} \; - \; \text{particles with speed 1.}$$

After some time we expect that the solution has almost forgotten the initial data so that

$$\text{particles with speed -1} \; \simeq \; \text{particles with speed 1.}$$

We consider the solution (f^{-}, f^{+}) with initial data $(0, \delta(x))$ as

$$\begin{pmatrix} f^{-}(t,x) \\ f^{+}(t,x) \end{pmatrix} = \begin{pmatrix} f^{-,0}(t,x) \\ f^{+,0}(t,x) \end{pmatrix} + \begin{pmatrix} f^{-,1}(t,x) \\ f^{+,1}(t,x) \end{pmatrix} + \begin{pmatrix} f^{-,2}(t,x) \\ f^{+,2}(t,x) \end{pmatrix},$$

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where

$$\begin{cases} f_t^{-,0} - f_x^{-,0} = -f^{-,0} \\ f_t^{+,0} + f_x^{+,0} = -f^{+,0} \end{cases} \qquad (0, \delta(x)),$$

We consider the solution (f^-, f^+) with initial data $(0, \delta(x))$ as

$$\begin{pmatrix} f^-(t,x) \\ f^+(t,x) \end{pmatrix} = \begin{pmatrix} f^{-,0}(t,x) \\ f^{+,0}(t,x) \end{pmatrix} + \begin{pmatrix} f^{-,1}(t,x) \\ f^{+,1}(t,x) \end{pmatrix} + \begin{pmatrix} f^{-,2}(t,x) \\ f^{+,2}(t,x) \end{pmatrix},$$

where

$$\begin{cases} f_t^{-,0} - f_x^{-,0} &= -f^{-,0} \\ f_t^{+,0} + f_x^{+,0} &= -f^{+,0} \end{cases} \qquad (0, \delta(x)),$$

$$\begin{cases} f_t^{-,1} - f_x^{-,1} &= \frac{f^{-,0} + f^{+,0}}{2} - f^{-,1} \\ f_t^{+,1} + f_x^{+,1} &= \frac{f^{-,0} + f^{+,0}}{2} - f^{+,1} \end{cases} \qquad (0, 0),$$

We consider the solution (f^-, f^+) with initial data $(0, \delta(x))$ as

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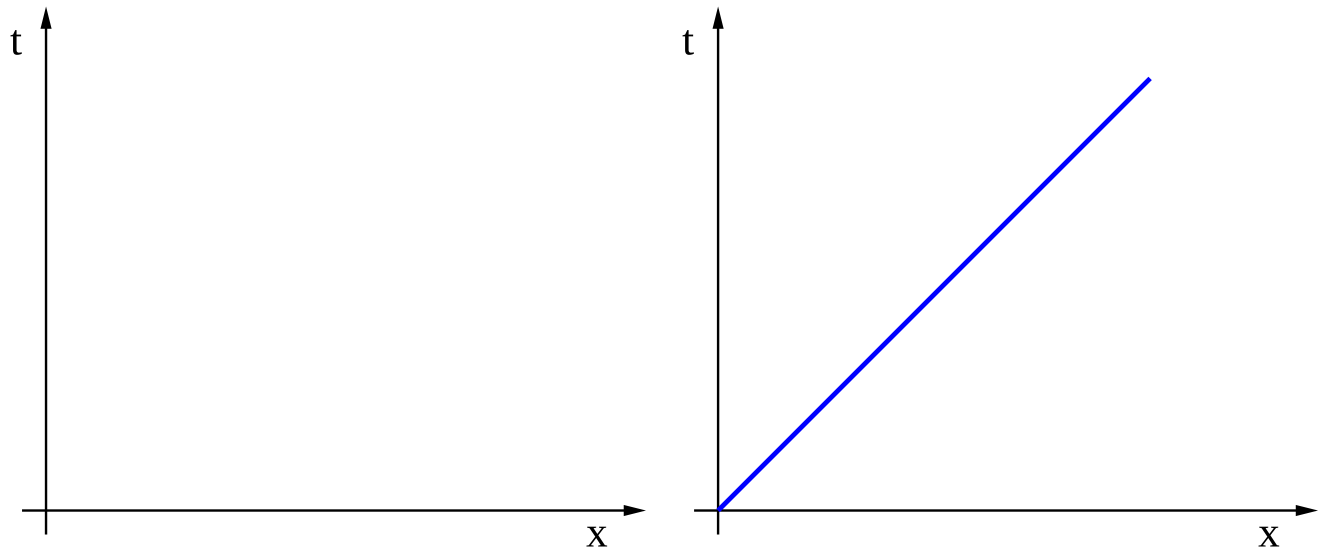
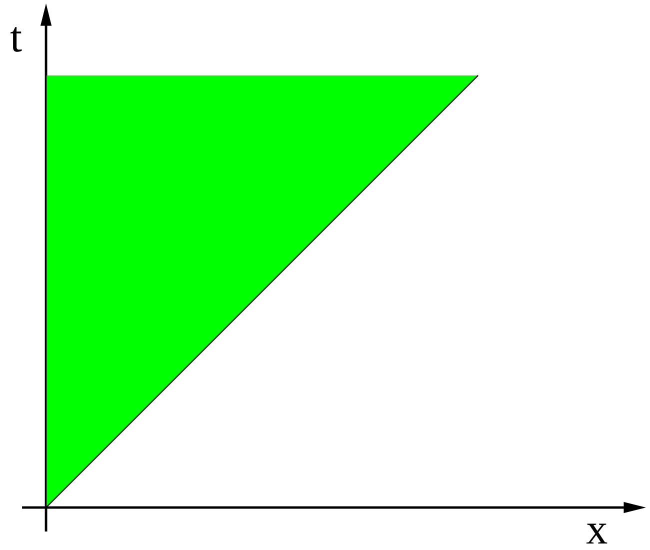
where

$$\left\{ \begin{array}{lcl} f_t^{-,0} - f_x^{-,0} & = & -f^{-,0} \\ f_t^{+,0} + f_x^{+,0} & = & -f^{+,0} \end{array} \right. \qquad (0, \delta(x)),$$

$$\left\{ \begin{array}{lcl} f_t^{-,1} - f_x^{-,1} & = & \frac{f^{-,0}+f^{+,0}}{2} - f^{-,1} \\ f_t^{+,1} + f_x^{+,1} & = & \frac{f^{-,0}+f^{+,0}}{2} - f^{+,1} \end{array} \right. \qquad (0, 0),$$

$$\left\{ \begin{array}{lcl} f_t^{-,2} - f_x^{-,2} & = & \frac{f^{-,1}+f^{+,1}}{2} + \frac{f^{+,2}-f^{-,2}}{2} \\ f_t^{+,2} + f_x^{+,2} & = & \frac{f^{-,1}+f^{+,1}}{2} - \frac{f^{-,2}-f^{+,2}}{2} \end{array} \right. \qquad (0, 0).$$


t
x
t
x



t
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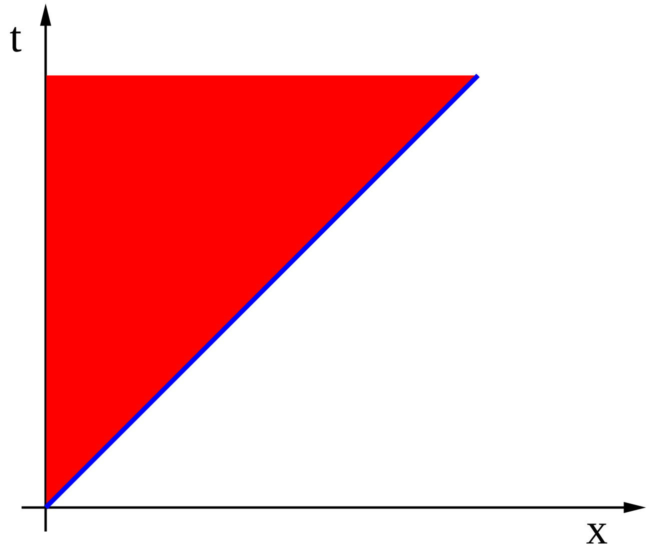
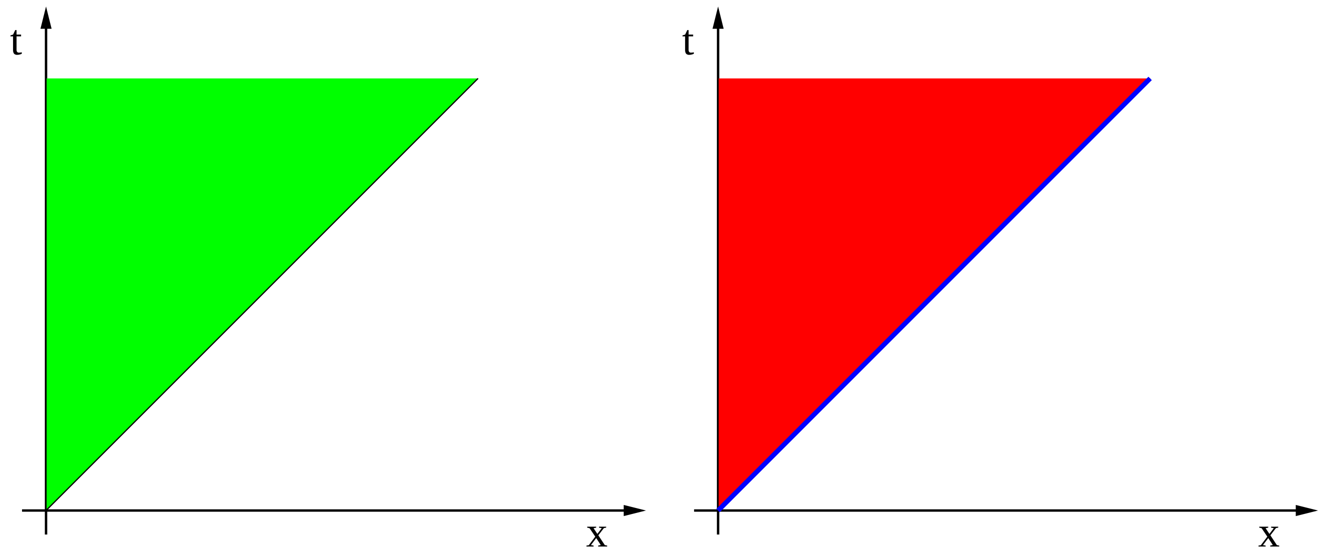
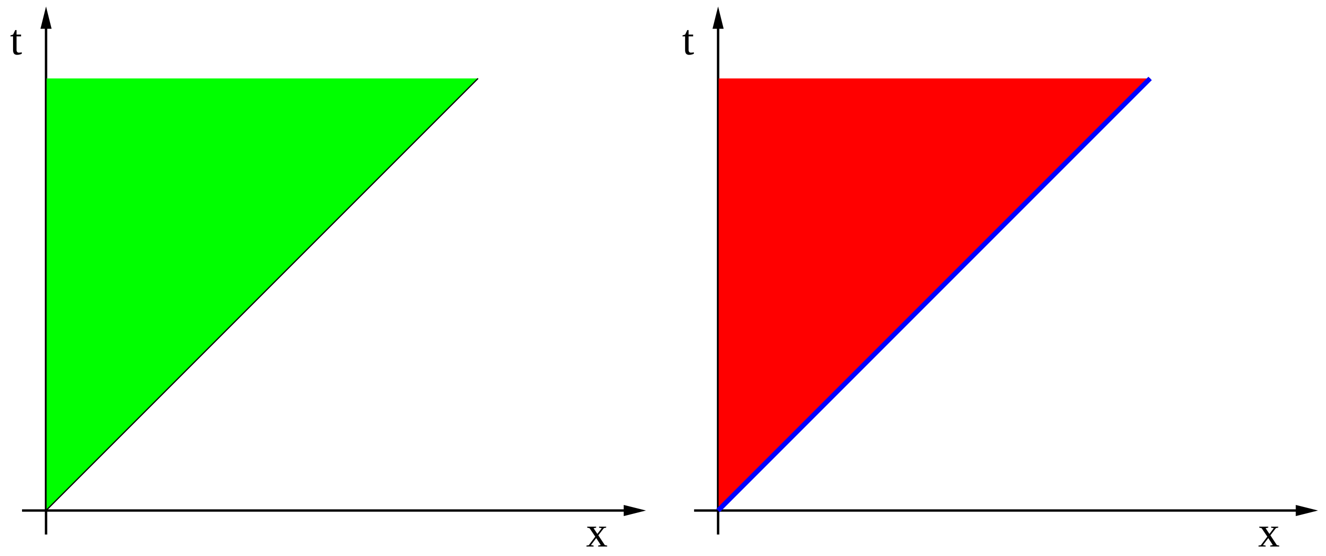


t
x



Explicitly

$$f^{-,0}(t,x) = 0, \qquad f^{+,0}(t,x) = e^{-t}\delta(x-t),$$



Explicitly

$$f^{-,0}(t,x) = 0, \qquad f^{+,0}(t,x) = e^{-t}\delta(x-t),$$

$$f^{-,1}(t,x) = \frac{e^{-t}}{2}\chi\{0 \le x \le t\},$$
$$f^{+,1}(t,x) = -\frac{e^{-t}}{2}\chi\{0 \le x \le t\} + \frac{t}{2}e^{-t}\delta(x-t).$$

The flux of $f^{\pm,0}$, $f^{\pm,1}$ at $x = 0$ is $1 + 1/2$, the source term for $f^{\pm,2}$ has total mass of $1/2$.

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We thus can estimate the flux as

$$\frac{\text{flux of } f^{\pm,0} + f^{\pm,1}}{\text{loss of } L^1 \text{ norm}} = \frac{1 + 1/2}{1 - 1/2} = 3.$$

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We conclude that

$$\int_0^\infty |f^-(t,0)|dt \leq 3(\|f^-(t=0)\|_{L^1} + \|f^+(t=0)\|_{L^1}). \qquad (23)$$

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$$\int_0^\infty |f^-(t,0)|dt \leq 3(\|f^-(t=0)\|_{L^1} + \|f^+(t=0)\|_{L^1}). \tag{23}$$

Similarly we can estimate

$$\int_0^{+\infty} \int_{\mathbb{R}} |f^- g^+ - g^- f^+| dxdt \leq 3 \sum_{\alpha,\beta=+-} \|P^{\alpha\beta}(t=0)\|_{L^1(\{x>y\})}.$$