Regularity of solutions to Hamilton-Jacobi and Hyperbolic Conservation Laws

L. Caravenna, C. De Lellis, M. Gloyer, R. Robyr, S. B.

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Outline

Structure of solutions to Hamilton-Jacobi

Consider the Hamilton-Jacobi equation

$$u_t + H(\nabla u) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^m,$$

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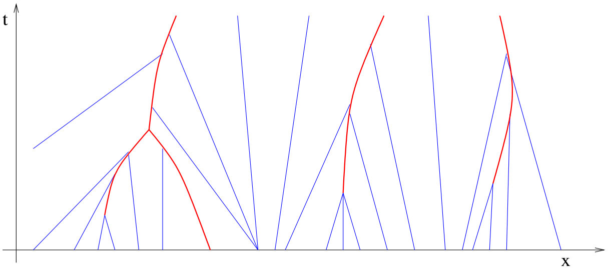
with uniformly convex Hamiltonian H and Lipschitz initial data. We expect a smooth function outside countably many regular hypersurfaces of codimension 1.



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For strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R},\ u \in \mathbb{R}^n,$$

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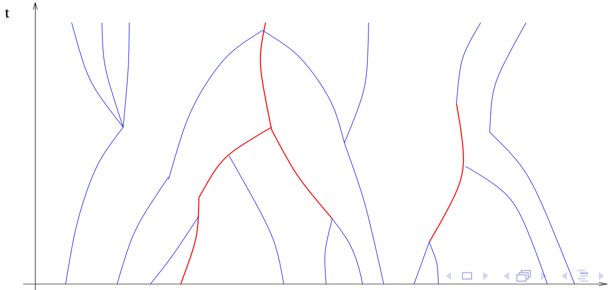

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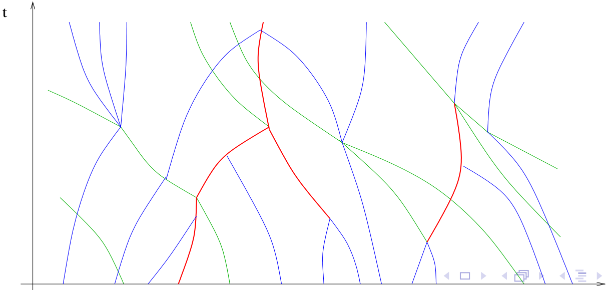

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The first question is an easy application of a well known rectifiability criteria.

Euler-Lagrange equation for singular variational problems

The Euler-Lagrange equation for the functional

$$\int_\Omega \big(\mathbb{1}_D(\nabla u) + u\big)\mathcal{L}^m, \quad u \in u_0 + W_0^{1,\infty}(\Omega),$$

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$$1 - (\mathbb{1}_D)^*(\nabla u) = 0, \quad u \llcorner_{\partial\Omega} = u_0.$$

Optimal transportation

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can be written as transportation problems along a set of geodesic, and the dynamical interpretation of the transport correspond to solve the PDE (in the sense of currents)

$$\mathrm{div}(d\rho) = \mu - \nu,$$

where d is the “direction” of the geodesics.

Precise decay estimates

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3. the continuous part of v_i satisfies the equation

$$(v_i)_t + (\lambda_i v_i)_x = J_i, \quad J_i \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}).$$

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Area estimate

The solution to HJ equation

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is given by

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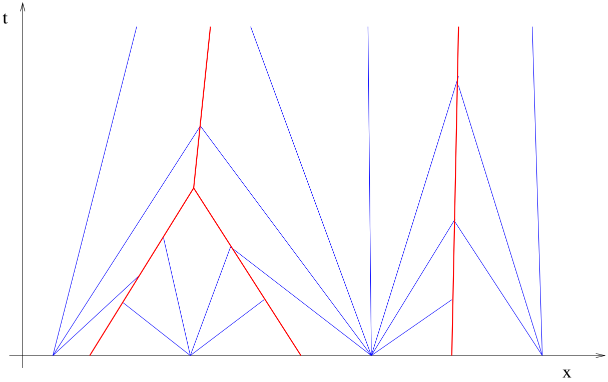
In particular, it is uniformly approximated by the sequence of functions

$$u_n(t,x) = \min \left\{ u(0,y) + tL\left(\frac{x-y}{t}\right), y \in \{y_1, \dots, y_n\} \right\}. \quad (1)$$

These solutions have a very simple structure:



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By (1), one can show that this is the worst case, i.e.

$$c(t,s,x) \begin{cases} \leq \left(\frac{t-s}{t}\right)^m & t \leq s \\ \geq \left(\frac{t-s}{t}\right)^m & t \geq s \end{cases}$$

Since along optimal rays we have the dual solution

$$u(s,x) = \max\left\{u(t,y) - (t-s)L\left(\frac{y-x}{t-s}\right)\right\},$$

we obtain the bound on the Jacobian

$$\min\left\{\left(\frac{t-s}{t}\right)^m, \left(\frac{T-s}{T-t}\right)^m\right\} \leq c \leq \max\left\{\left(\frac{t-s}{t}\right)^m, \left(\frac{T-s}{T-t}\right)^m\right\},$$

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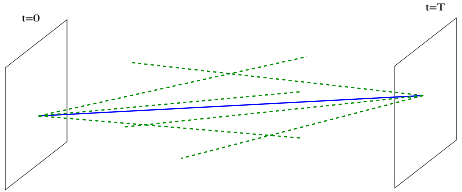

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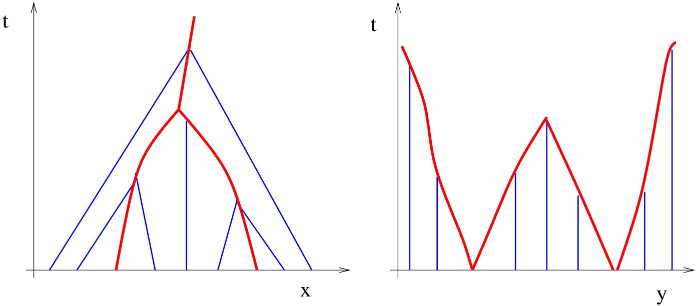
The above estimate implies that the change of variable

$$\big(t, y\big) \mapsto \big(t, y + td(y)\big)$$

has an integrable Jacobian with Lipschitz regularity in t.



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$$\mathcal{L}^{d+1}_{\mathbb{R}^+\times\mathbb{R}^m}=\int c(t,y)dtm(dy),$$

i.e. $\forall\phi\in C^c(\mathbb{R}^+\times\mathbb{R}^m)$

$$\int\phi\mathcal{L}^{d+1}=\int\bigg(\int\phi(t,y+td(y))c(t,y)dt\bigg)m(dy).$$

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Remark. The proof depends only on the convexity of H.

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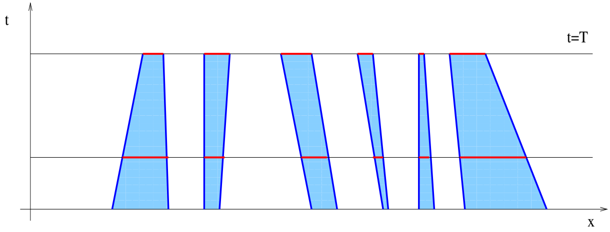

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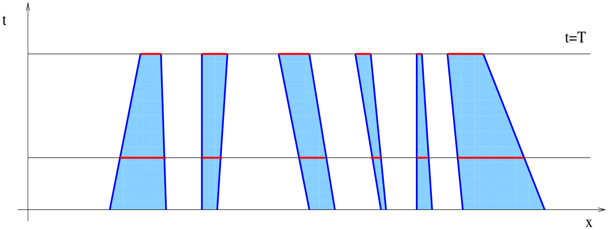

In particular, if there is a Cantor part (hence single rays), the area is strictly positive.

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Since $d(t,x) = D^2 H(\nabla u) D^2 u$, we obtain that $\mathrm{tr}(D^2 u)$ has not Cantor parts, hence $D^2 u$ has not Cantor parts.

A measure for shock creation

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$$|J_i^s|\big((s,t] \times \mathbb{R}\big) \le \mathrm{Tot.Var.}(v_v - v_i^s(s)) - \mathrm{Tot.Var.}(v_v - v_i^s(t)) + C\big(Q(s) - Q(t)\big).$$

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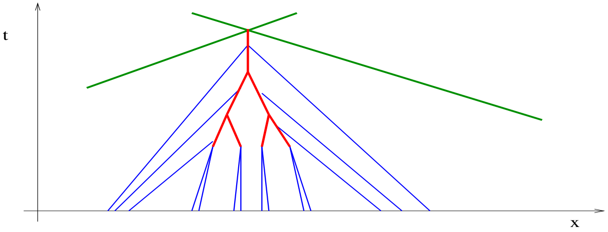

SBV regularity

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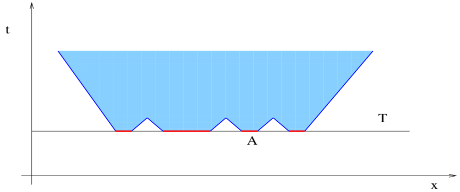

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Outline

L. Ambrosio and C. De Lellis.
A note on admissible solutions of 1d scalar conservation laws and 2d Hamilton-Jacobi equations.

S.B. and L. Caravenna.
SBV regularity for hyperbolic systems.

S.B. and F. Cavalletti.
The Monge problem for distance cost in geodesic spaces

S.B., C. De Lellis and R. Robyr.
SBV regularity for Hamilton-Jacobi equations in $\mathbb{R}^n$.

S.B. and M. Gloyer.
Euler equation for a singular variational problem.