Lagrangian representation for conservation laws

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A paradigm for Lagrangian representation: linear transport

The model for Eulerian-Lagrangian formulations is the transport equation

$$\partial_t u + b(t,x) \cdot \nabla u = 0,$$

$$\partial_t \rho + \mathrm{div}\big(\rho b(t,x)\big) = 0,$$

and the associated ODE for characteristics

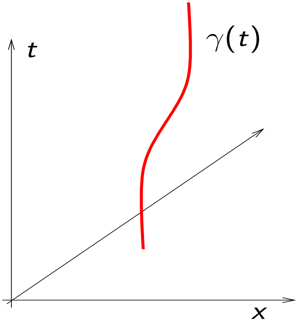
$$\frac{d}{dt} \mathtt{X}(t) = b(t, X(t)).$$

For smooth vector fields, classical formulas yield

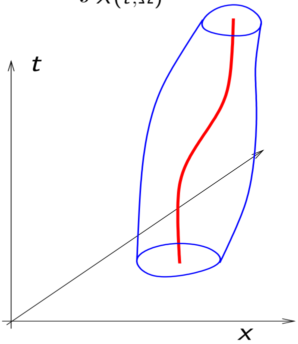
$$\frac{d}{dt} u(t, X(t,y)) = 0, \qquad \frac{d}{dt}\big(J(t,y)\rho(t, X(t,y))\big) = 0,$$

where $J(t,y) = \det \nabla X(t,y)$.

$$\frac{d}{dt}u(t,\gamma(t)) = 0$$



$$\frac{d}{dt}\int_{X(t,\Omega)} \rho(t,x)dx = 0$$



In the non smooth case (e.g. $b \in L^\infty, \mathrm{div} b = 0$), the equivalence between the two formulation is more subtle.

Eulerian: This is the distributional formulation, which is the conservative case is

$$\int \rho\big(\partial_t \phi + b \cdot \nabla \phi\big)\mathcal{L}^{d+1} = 0$$

for a test function ϕ.

Lagrangian: One needs to consider probability measures η concentrated on the set of a.c. curves $\gamma(t)$ solving the ODE for characteristics

$$\frac{d}{dt}\gamma = b(t, \gamma).$$

For $\rho(t)\mathcal{L}^d$ probability,

$$\int \phi(t, x)\rho(t, x)\mathcal{L}^d(dx) = \int \phi(t, \gamma(t))\eta(d\gamma).$$

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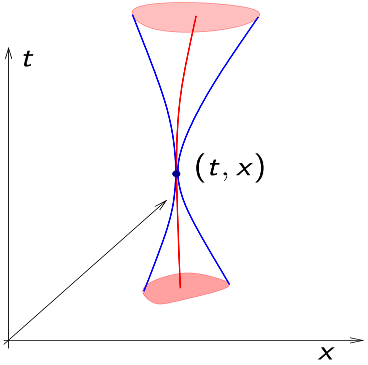

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The fundamental theorem is

Theorem (L. Ambrosio)

If $t \mapsto \mu_t$ is a family of probability measures in $\mathbb{R}^d$ solving

$$\partial_t \mu_t + \mathrm{div}\big(b(t,x)\mu_t\big) = 0,$$

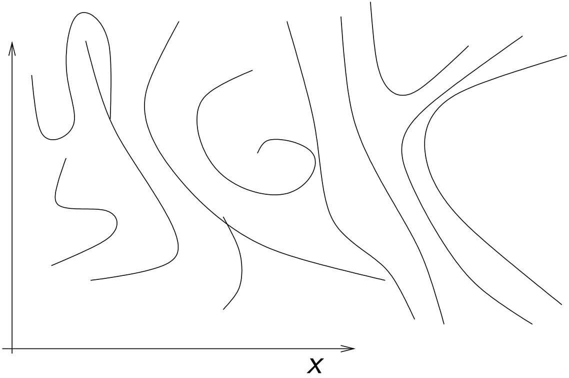
then there is η probability concentrated on the set of solutions of the ODE $\frac{d\gamma}{dt} = b(t,\gamma)$ such that

$$\int \phi(t,x)\mu_t(dx) = \int \phi(t,\gamma(t))\eta(d\gamma).$$

The fact that sufficiently many characteristics exist is a consequence of the fact that μ_t is a weak solution.

Application: 2-dimensional autonomous $\mathrm{BV} \cap L^\infty$ nearly incompressible vector fields,

$$\partial_t \rho + \mathrm{div}(\rho b) = 0, \quad \rho \in L^\infty_{\mathrm{loc}}.$$



[Bonicatto-Gusev & al. '15]

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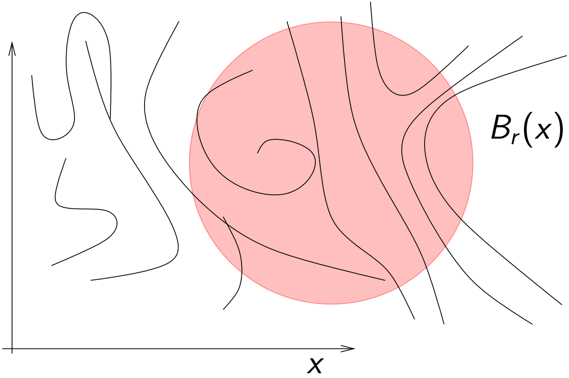

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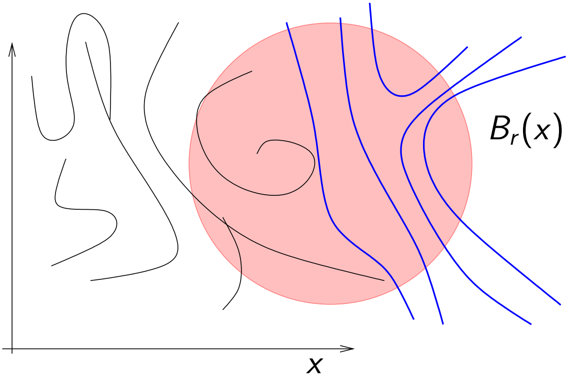
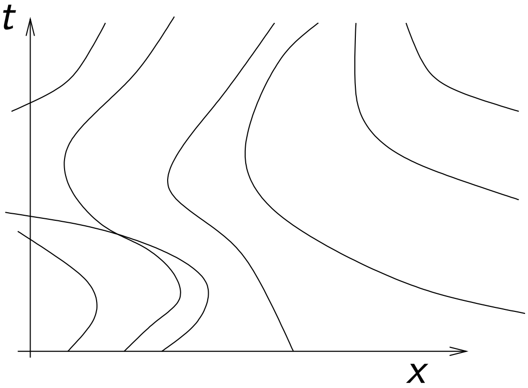


[Bonicatto-Gusev & al. '15]

Application: local ODE theory for $b \in L^1_{\mathrm{loc}}$



[Ambrosio-Colombo-Figalli '15]

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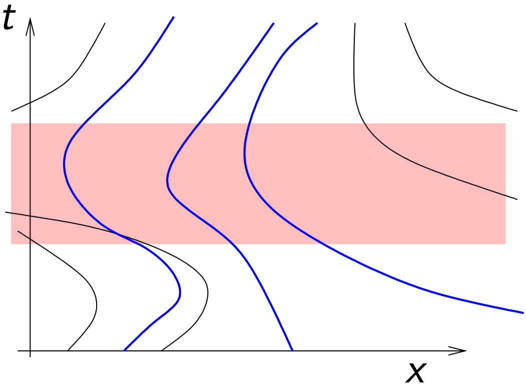


[Ambrosio-Colombo-Figalli '15]

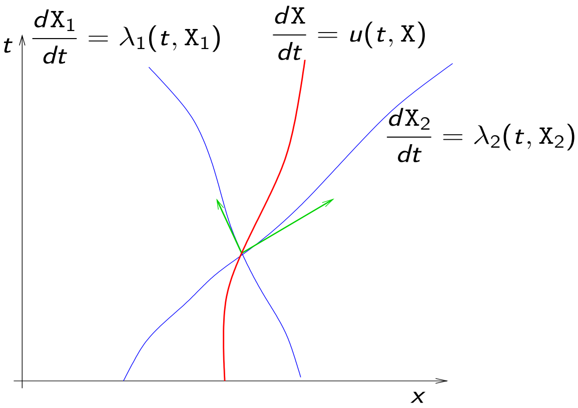
Gas dynamics

The Eulerian formulation of isentropic gas dynamics in 1-d

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P(\rho)) = 0. \end{cases}$$

Under the change of coordinates $x = \mathtt{X}(t, y)$, $\frac{d\mathtt{X}}{dt} = u \circ \mathtt{X}$,

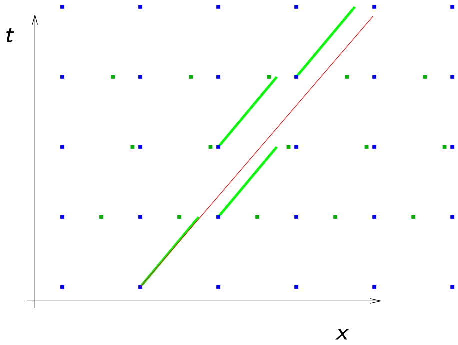
$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x P(v^{-1}) = 0. \end{cases}$$



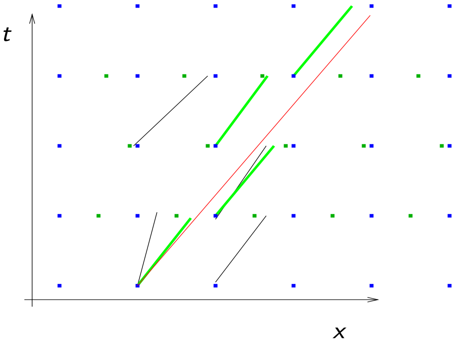
There is only one family of curves $t \mapsto \mathtt{X}(t, y)$, and the new system is a system of PDE.

Liu's wave tracing

The first clue to the possibility to use characteristics to transport the solution comes from [Liu '75].

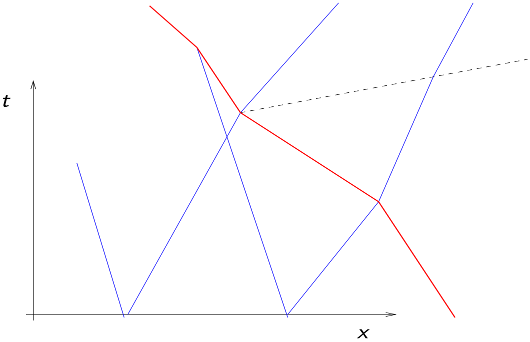


In the case of a real solution, one first has to show that (locally) the speed of the i-waves is locally almost constant (controlled by Glimm interaction estimates), and then trace these waves across the grid.



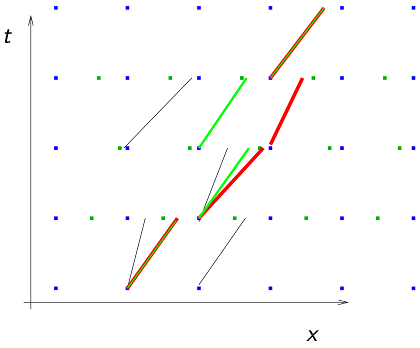
Wavefront tracking

In some sense, the wavefront tracking approximations are a sort of "Lagrangian half-representation": one traces the wave position and their sizes, and at each Riemann problem the size changes and the new speeds can be computed.



Refined interaction estimates

[Ancona-Marson '10] introduced a refined wave tracing similar to Liu: now the fact that waves may separates in the future forces to split jumps into parts, and trace all these pieces.

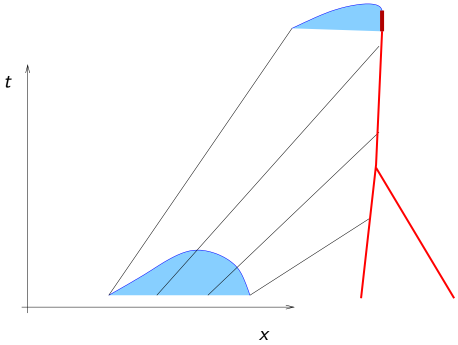


Another case where some type of Eulerian transport occurs is for SBV estimates of conservation laws [Caravenna & al. '10].
It is shown that the components of the derivative satisfies

$$\partial_t v_i + \partial_x(\tilde{\lambda}_i v_i) = \mu_i,$$

and μ_i is is a measure which is controlled by the interaction and cancellations occurring in the evolution.
The speed $\tilde{\lambda}(t, x)$ is defined everywhere by Rankine-Hugoniot conditions.

$$\partial_t v_i + \partial_x(\tilde{\lambda}_i v_i) = \mu_i$$



Continuous solutions to balance laws

[Bigolin-Caravenna-Serra Cassano '12] shows that if u is a continuous solution to

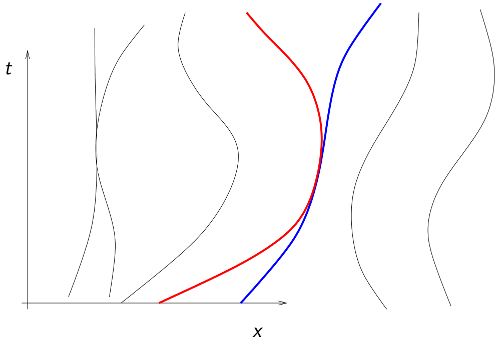
$$\partial_t u + \partial_x (u^2/2) \in L^\infty,$$

then along characteristics (whose existence is assured by Peano's Theorem) $\frac{d\gamma}{dt} = u(t,\gamma)$ it holds

$$\frac{d}{dt} u(t,\gamma(t)) \in L^\infty.$$

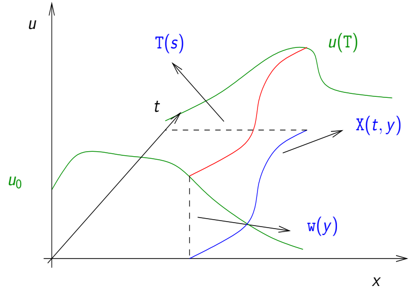
It can extended to general fluxes F, where a continuous monotone flow $\mathtt{X}$ of characteristics is constructed [Alberti-Caravenna & al. '16].

$$\partial_t u + \partial_x F(u) \in L^\infty, \qquad \begin{cases} \frac{d\gamma}{dt} = u(t,\gamma(t)), \\ \frac{d}{dt} u(t,\gamma(t)) \in L^\infty. \end{cases}$$



Lagrangian representation in the scalar case

Aiming at more refined interaction estimates, [Modena & al. '13] proved first that in the scalar case a representation exists.



$$\frac{d\mathtt{X}}{dt} = \tilde{\lambda}$$

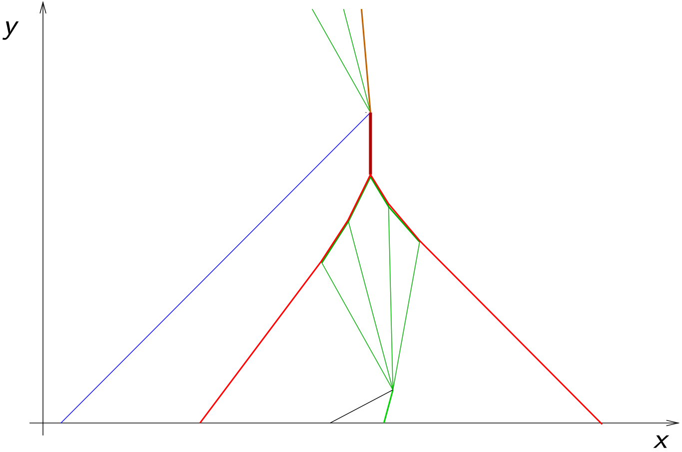
$y \mapsto \mathtt{X}(t, y)$ monotone

$y \mapsto \mathtt{w}(y)$ Lipschitz

$u(t, x) = w(\mathtt{X}^{-1}(t, x))$ $\mathcal{L}^1$-a.e.

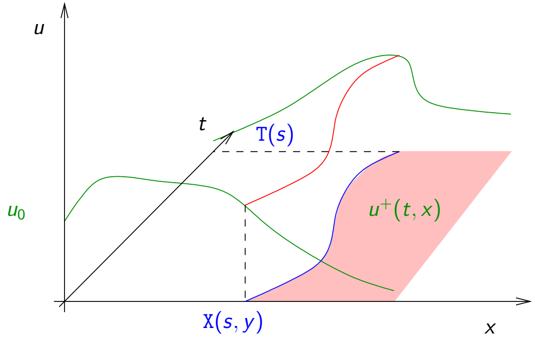
$y \mapsto \mathtt{T}(y)$ cancellation time

Example



Concentration of entropy

For entropy solution in L^∞, a Lagrangian representation can be found by passage to the limit: the compactness by interpreting the characteristics of the Lagrangian representation as admissible boundaries [Marconi at al. '16].



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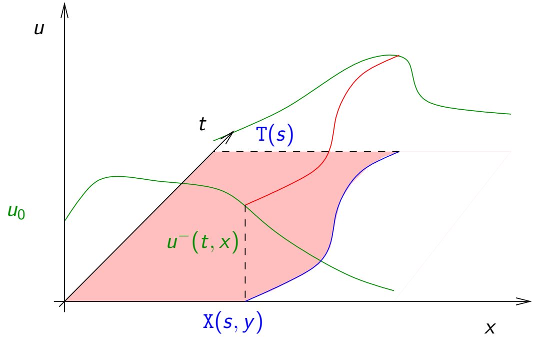

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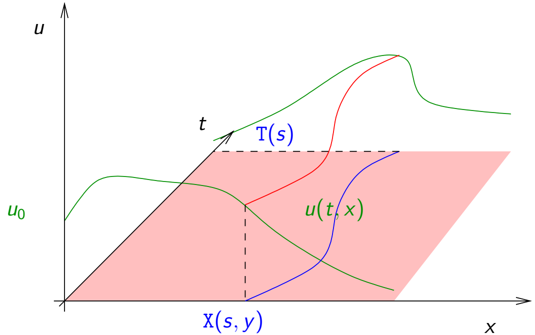

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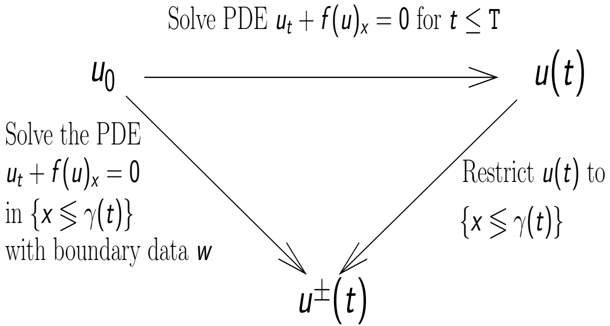
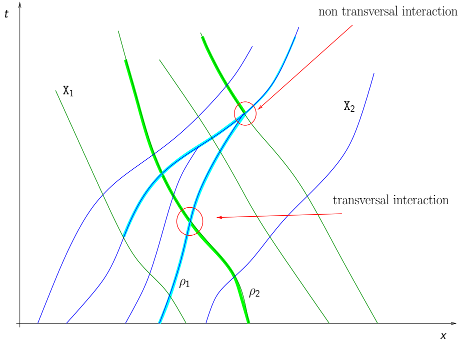


Lagrangian representation for systems

For system, each family i generates its own flux $\mathtt{X}_i$, and has an associated density ρ_i.



$y_i \mapsto X_i(t, y_i)$ monotone

$$\frac{d\mathtt{X}_i}{dt} = \tilde{\lambda}_i(t, \mathtt{X}_i)$$

$\rho_i(t, y_i) \in L^\infty$, $\partial_t \rho_i \in \mathcal{M}$ equivalent to

$$\partial_t \rho_i + \partial_x(\tilde{\lambda}_i \rho_i) = \mu_i$$

Lagrangian representation for systems

For system, each family i generates its own flux $\mathtt{X}_i$, and has an associated density ρ_i:

- $y_i \mapsto X_i(t, y_i)$ monotone
- $\dfrac{d\mathtt{X}_i}{dt} = \tilde{\lambda}_i(t, \mathtt{X}_i)$
- $\rho_i(t, y_i) \in L^\infty$, $\partial_t \rho_i \in \mathcal{M}$,

and the solution can be reconstructed by

$$D_x u = \sum_i (\mathtt{X}_i)_\sharp \big(\rho_i \tilde{r}_i[\{\mathtt{X}_i, \rho_i\}_i]\big),$$

where the vectors $\tilde{r}_i$ are reconstructed by knowing $\mathtt{X}_i, \rho_i$.