

Abnormal geodesics and related topics¹

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1 Definitions and notations

Let M be a smooth n -dimensional manifold and denote by $TM = \bigcup_{q \in M} T_q M$, $\dim TM = 2n$ the tangent bundle of M . Let

$$\Delta = \bigcup_{q \in M} \Delta_q, \quad \Delta_q \in T_q M$$

be a vector distribution of a constant rank k on M : $\dim \Delta_q = k$ for all $q \in M$.

Definition 1 *The distribution Δ is called integrable if at any point $q \in M$ there exists an immersed sub-manifold $N_q \subset M$ such that $\Delta_{\hat{q}} = T_{\hat{q}} N_q$ for any $\hat{q} \in N_q$. The distribution Δ is called completely non-holonomic if it is not tangent to any sub-manifold $N \subset M$.*

If $k = 1$, then Δ defines a prescribed single direction at any point q of M . By existence and uniqueness theorem for systems of ordinary differential equations at any point $q \in M$ there is a single trajectory passing in this direction, thus the distribution Δ is integrable. The first nontrivial situation occurs when $k = 2$. Already in this case a generic distribution is completely non-holonomic. Let us discuss in more detail.

Let us consider two complete vector fields $f_1, f_2 \in TM$ such that $\Delta_q = \text{span}\{f_1(q), f_2(q)\}$ at any $q \in M$. These vector fields define a pair of ODE on M :

$$\dot{q} = f_i(q), \quad i = 1, 2.$$

Denote by $e^{tf_i} : M \mapsto M$ the flow degenerated by f_i on M . By definition,

$$\frac{d}{dt} e^{tf_i}(q) = f_i(e^{tf_i}(q)).$$

Fix some $q_0 \in M$ and $t \in \mathbb{R}$. Consider a trajectory which starts at q_0 and consists of four pieces of duration t each and such that the whole trajectory is organized as follows: first we follow the integral curve of f_1 starting at q_0 , then

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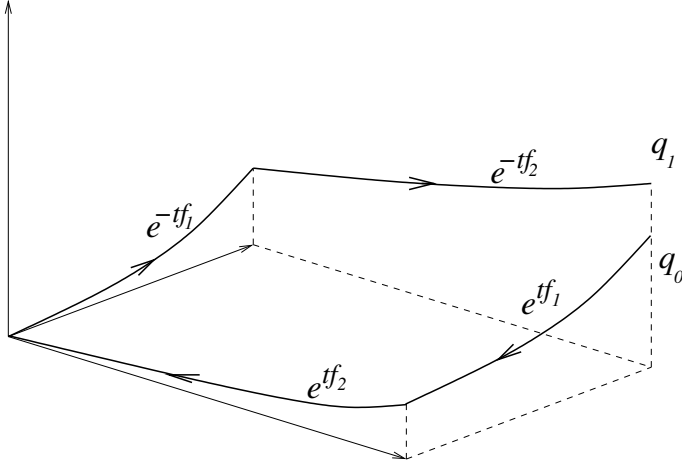


Figure 1: The Lie bracket of the fields f_1 and f_2

we switch the direction to f_2 , and finally we come back following $-f_1$ and $-f_2$ subsequently. In such a way we come to the point

$$q_1 = e^{-tf_2} \circ e^{-tf_1} \circ e^{tf_2} \circ e^{tf_1}(q_0) = q_0 + t^2 (f_1 \circ f_2 - f_2 \circ f_1)(q_0) + o(t^2).$$

It not hard to see that in general $q_1 \neq q_0$ and $q_1 = q_0$ only if the vector fields f_1 and f_2 commute, i.e., if the differential operator $[f_1, f_2] = f_1 \circ f_2 - f_2 \circ f_1$, called the *Lie bracket* of the fields f_1 and f_2 , is zero.

That is, the Lie bracket is a kind of measure of the non-integrability of the vector distribution Δ .

Using the coordinate representation of the fields f_1, f_2 in some local coordinates on M one can see that the Lie bracket of a pair a vector fields is a first order differential operator:

$$f_1 = \sum_{i=1}^n a_i \frac{\partial}{\partial q_i}, \quad f_2 = \sum_{i=1}^n b_i \frac{\partial}{\partial q_i},$$

$$[f_1, f_2] = \sum_{i,j=1}^n \left(a_j \frac{\partial b_i}{\partial q_j} - b_j \frac{\partial a_i}{\partial q_j} \right) \frac{\partial}{\partial q_i} = \frac{df_2}{dq} f_1 - \frac{df_1}{dq} f_2,$$

i.e., it is again a vector field. Iterating this procedure one can construct the vector fields $[f_1, [f_1, f_2]]$, $[f_2, [f_1, f_2]]$ and so on.

The importance of the commutation properties of vector fields spanning the distribution Δ is shown by the following theorem. Denote

$$\Delta^m = [\Delta, \Delta^{m-1}], \quad \Delta^1 = \Delta, \quad m = 1, \dots$$

Theorem 1 (Rashevski-Chow) *If for any $q \in M$ there exists an integer $m_q \in \mathbb{N}$ such that $\Delta_q^{m_q} = T_q M$, then the distribution Δ is completely non-holonomic and any two points of M can be connected by a path tangent to Δ at*

any point. The integer m_q is called the degree of non-holonomy at q .

Let $\langle \cdot, \cdot \rangle_q$ be an inner product on Δ_q . By definition, for any vector $v \in \Delta_q$ we have $|v| = \sqrt{\langle v, v \rangle_q}$. The following function

$$\delta(q_0, q_1) = \inf \left\{ \int_0^1 |\dot{\gamma}(t)| dt, \gamma : [0, 1] \mapsto M, \dot{\gamma}(t) \in \Delta_{\gamma(t)} \right\}, \quad q_0, q_1 \in M$$

is called the *Sub-Riemannian* or *Carnot-Carathéodory distance* on M . By the Rashevski-Chow theorem, if the distribution Δ is completely non-holonomic, then $\delta(q_0, q_1) < +\infty$ for any $q_0, q_1 \in M$.

2 Isoperimetric problem on the plane

Let $\mathbb{R}^2 = \{x = (x^1, x^2)\}$ and consider a 1-form $\mu = a_1 dx^1 + a_2 dx^2$, where a_1 and a_2 are two smooth functions on M . We want to find

$$(1) \quad \inf \left\{ \int_0^1 |\dot{x}(t)| dt : x(0) = x_0, x(1) = x_1 \text{ such that} \right. \\ \left. \int_{x(\cdot)} \mu = \int_0^1 a(x(t)) \dot{x}^1(t) + a_2(x(t)) \dot{x}^2(t) dt = \text{const} = y_1 \right\}$$

Let us formulate this problem as the problem of minimizing of the Sub-Riemannian distance. First of all we consider the extended state space $M = \mathbb{R}^3 = \{q = (x, y), x \in \mathbb{R}^2, y \in \mathbb{R}\}$. The coordinate y is the current value of the cost function. Denote

$$\Delta_q = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ a_1(q) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ a_2(q) \end{pmatrix} \right\},$$

and set

$$\left| \begin{pmatrix} v^1 \\ v^2 \\ a_1(q)v^1 + a_2(q)v^2 \end{pmatrix} \right| = ((v^1)^2 + (v^2)^2)^{1/2}.$$

Then we can rewrite problem (1) as follows:

$$\inf \left\{ \int_0^1 |\dot{\gamma}(t)| dt, \gamma : [0, 1] \mapsto M, \text{ s.t } \dot{\gamma}(t) \in \Delta_{\gamma(t)}, \gamma(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \gamma(1) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right\}.$$

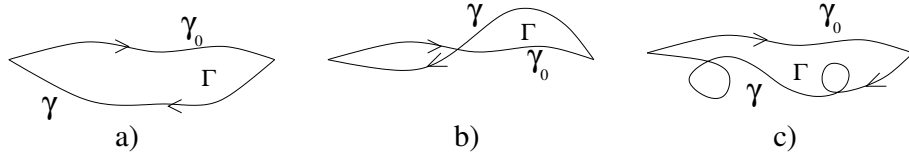


Figure 2: Paths connecting two points

Exercise 1 Let

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ a_1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 \\ 0 \\ a_2 \end{pmatrix}.$$

Show that

$$f_1 \wedge f_2 \wedge [f_1, f_2] \neq 0 \quad \iff \quad d\mu \neq 0.$$

Example 1 (*Charged particle in the magnetic field on the plane*) The isoperimetric problem is equivalent to the Least Action Principle of Classical Mechanics applied to the plane motion of the charged particle in the magnetic field on the plane. Indeed, let $b : \mathbb{R}^2 \mapsto \mathbb{R}$ be the curl of the vector potential of the magnetic field, and let $x = (x^1, x^2) \in \mathbb{R}^2$ be the coordinate of a charged particle of unit mass. Then $|\ddot{x}(t)| = c|b(x)|$, where $c \in \mathbb{R}$ is a scalar constant, and $d\mu = b(x)dx^1 \wedge dx^2$. We are interested in the trajectories $x(t)$ such that $b(x(t)) \equiv 0$. More precisely, we would like to understand whether these trajectories could be the minimizers of the isoperimetric problem³.

Assume that $db \neq 0$ and consider a smooth curve $\gamma_0 = b^{-1}(0)$. Let γ be another trajectory of the particle with the same terminal points: $\gamma(0) = \gamma_0(0)$, $\gamma(1) = \gamma_0(1)$, and let Γ be the domain enclosed between the curves γ and γ_0 (see Fig.1, a)). We have

$$(2) \quad \int_{\gamma(\cdot)} \mu - \int_{\gamma_0(\cdot)} \mu = \int_{\Gamma} d\mu = \int_{\Gamma} b(x)dx^1 \wedge dx^2 > 0.$$

This argument works also if the curves γ and γ_0 have some intermediate intersection points (Fig.1, b)), but it fails if one of this curves has self-intersections (Fig.1, c)). In the latter case one can choose $\gamma(t)$ such that the integral in (2) becomes zero. Therefore, γ_0 is not isolated in the H^1 -topology, though it is always a local minimizer (the proof of this fact is rather delicate, below we will state a general result).

Exercise 2 Consider the case $b(x) = x^1x^2$. Here $b^{-1}(0)$ consists of the coordinate axes and $db|_0 = 0$. Let $\gamma_0 = (0, \beta)$ and $\gamma_1 = (\alpha, 0)$. Show that the trajectory that pass through the origin is not a minimizer.

³This question was studied by R. Montgomery

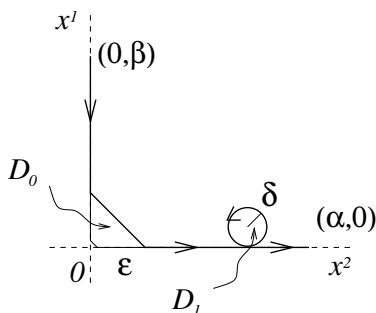


Figure 3: Example of an alternative path

(Hint: Consider the path $\hat{\gamma}$, which contains a circle loop of radius δ , as at shown in Fig. 2 Then $\Gamma = D_0 \cup D_2$,

$$\int_{\Gamma} x^1 x^2 dx^1 dx^2 = \int_{D_0} x^1 x^2 dx^1 dx^2 + \int_{D_1} x^1 x^2 dx^1 dx^2.$$

Show that

$$\int_{D_0} x^1 x^2 dx^1 dx^2 \sim \varepsilon^4, \quad \int_{D_1} x^1 x^2 dx^1 dx^2 \sim \delta^3,$$

so that by an accurate choice of ε and δ one can make the right-hand side of (2) to be zero while the length of the new curve will be shorter than the length of the broken line passing through the origin.)

Open problem Does there always exist a smooth minimizer?

3 Singular curves

Let us come back to the general situation. For the moment we assume

$$M = \mathbb{R}^n, \quad \Delta_q = \text{span}\{f_1(q), \dots, f_k(q)\}, \quad q \in M, \quad k < n.$$

Denote

$$\Omega = \left\{ \gamma(\cdot) : [0, 1] \mapsto M, \text{ s.t } \gamma \in H^1([0, 1], \mathbb{R}^n), \dot{\gamma}(\cdot) \in \Delta_{\gamma(\cdot)} \right\}.$$

Since $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for any $t \in [0, 1]$ we have

$$\dot{\gamma}(t) = \sum_{i=1}^k u_i(t) f_i(\gamma(t)).$$

We assume that the coordinate functions $u_i(\cdot) \in L^2([0, 1])$ so that for any fixed initial data $\gamma(0) = \gamma_0$ we have $\gamma(\cdot) \sim (\gamma_0, u(\cdot))$, where $u = (u_1, \dots, u_k)$. Hence $\Omega \simeq \mathbb{R}^n \times L_2^k([0, 1])$.

In the case of an arbitrary smooth manifold M the space Ω is a Hilbert manifold modeled on $\mathbb{R}^n \times L_2^k([0, 1])$.

Let $\gamma^0 \in \Omega$ be a fixed curve. Along this curve we have

$$\Delta_{\gamma^0(t)} = \text{span}\{f_1^t(\gamma^0(t)), \dots, f_k^t(\gamma^0(t))\},$$

where $\{f_i^t\}_{i=1}^k$ is a basis of vector fields possibly depending on t , which is well defined along the curve γ^0 . For any curve $\gamma(\cdot) \in \Omega$, which is uniformly close to γ^0 , we have

$$\dot{\gamma}(t) = \sum_{i=1}^k u_i(t) f_i^t(\gamma(t)).$$

Definition 2

$$\partial : \Omega \mapsto M \times M : \quad \partial(\gamma(\cdot)) = (\gamma(0), \gamma(1)).$$

The mapping ∂ is well defined C^∞ -mapping from the Hilbert manifold Ω to the space of terminal points. The critical points of ∂ are called *singular curves* or *abnormal geodesics*.

Remark The constant curves $\gamma(t) = \text{const}$ are automatically critical points of ∂ .

Our next goal will be to characterize the singular curves.

3.1 The Lagrange multipliers method

Let $F_t : \gamma(\cdot) \mapsto \gamma(t)$ denote the evaluation mapping associated to the curve $\gamma(\cdot)$. Then $\partial(\gamma(\cdot)) = (F_0, F_1)$. By the classical Lagrange multipliers rule if the curve $\gamma(\cdot) \in \Omega$ is a singular trajectory then there exists a non-trivial pair of co-vectors (λ_0, λ_1) , $\lambda_i \in T_{\gamma(\cdot)}^* M$, $i = 1, 2$ such that

$$\lambda_0 D_\gamma F_0 = \lambda_1 D_\gamma F_1.$$

It turns out that any piece of a singular trajectory is singular. More precisely, we have the following lemma:

Lemma 2 *If γ is a singular trajectory, then for any $t \in [0, 1]$ it is a critical point for the pair (F_0, F_t) .*

Proof Let us use the special coordinates in Ω defined above⁴: along $\gamma(\cdot)$ we choose the frame of vector fields $\{f_i^t\}_{i=1}^k$ such that for any $q_t \in \mathcal{O}_{\gamma(t)}$

$$(3) \quad \dot{q}_t = \sum_{i=1}^k u_i(t) f_i^t(q_t),$$

⁴We would like to stress out that we do not require any coordinates on M .

$$\Delta_{q_t} = \text{span}\{f_1^t(q_t), \dots, f_k^t(q_t)\}.$$

Admissible trajectories are solutions of control system (3) with

$$u(\cdot) = (u_1(\cdot), \dots, u_k(\cdot)) \in L_2^k([0, 1])$$

being control functions. Denote by u_γ the control, which generates the singular trajectory γ . Locally we have the following splitting

$$\Omega = M \times L_2^k([0, 1]) = \{(q_t, u(\cdot)) : q_t \in \mathcal{O}_{\gamma(t)}, u(\cdot) \in L_2^k([0, 1]) : u(\cdot) \in \mathcal{O}_{u_\gamma}\}.$$

Let us first show that λ_0 uniquely defines λ_t . Indeed, if there would be two co-vectors λ_t and $\hat{\lambda}_t$ such that $\lambda_0 D_\gamma F_0 = \lambda_t D_\gamma F_t$ and $\lambda_0 D_\gamma F_0 = \hat{\lambda}_t D_\gamma F_t$, then $(\lambda_t - \hat{\lambda}_t) D_\gamma F_t = 0$, which is not possible because F_t is a submersion.

Let us choose some specific control $\tilde{u}(\cdot)$ and consider the corresponding curve

$$\dot{\gamma}(\tau) = \sum_{i=1}^k \tilde{u}_i(\tau) f_i^\tau(\gamma(\tau)), \quad t \leq \tau \leq 1.$$

Denote by

$$P_{t,\tau} : \mathcal{O}_{\gamma(t)} \mapsto \mathcal{O}_{\gamma(\tau)}$$

the flow that transforms the neighborhood of $\gamma(t)$ into the neighborhood of $\gamma(\tau)$. By definition,

$$\frac{\partial}{\partial \tau} P_{t,\tau}(q) = \sum_{i=1}^k \tilde{u}_i(\tau) f_i^\tau(P_{t,\tau}(q)), \quad P_{t,t}(q) = q.$$

Let

$$\Omega_t(\gamma) = \left\{ q(\cdot) \in \Omega : q(\tau) \in P_{t,\tau}(q(t)), t \leq \tau \leq 1 \right\}.$$

If the curve γ is a critical point for the pair (F_0, F_1) , then it is critical for $(F_0, F_1)|_{\Omega_t(\gamma)}$. Essentially we have

$$F_1|_{\Omega_t(\gamma)} = P_{t,1} \circ F_t,$$

where $P_{t,1}$ is a fixed diffeomorphism that does not depend on the curve any more!

Let us compute the restriction of $D_\gamma F_1$ to $\Omega_t(\gamma)$. We have

$$\lambda_0 D_\gamma F_0 = \lambda_1 P_{t,1*} D_\gamma F_t, \quad \forall t.$$

On the other hand, by definition of the adjoint mapping,

$$(4) \quad \lambda_t = \lambda_1 P_{t,1*} = P_{t,1}^* \lambda_1.$$

Thus

$$\lambda_0 D_\gamma F_0 = \lambda_t D_\gamma F_t$$

which proves the lemma.

Observe that for any t

$$(5) \quad \langle \lambda_t, \Delta_{\gamma(t)} \rangle = 0.$$

It is not hard to show now that conditions (4) and (5) are sufficient to characterize the Lagrange multipliers.

Definition 3 *A curve $t \mapsto \lambda_t$, satisfying (4) and (5), is called the singular extremal.*

3.2 Hamiltonian setting

Here we recall briefly some classical facts of Hamiltonian Dynamics. Denote by

$$\pi : T^*M \mapsto M$$

the canonical projection of the cotangent bundle onto its base manifold. The linear mapping

$$\pi_* : T_\lambda(T^*M) \mapsto T_{\pi(\lambda)}M$$

is the differential of the projector π . Let

$$s_\lambda : T_\lambda(T^*M) \mapsto \mathbb{R}, \quad \lambda \in T^*M$$

be a one form on T^*M such that

$$s_\lambda = \lambda \circ \pi_*.$$

This form is called *the tautological* or *Liouville* form. Its differential $\sigma = ds$ is a non-degenerate closed 2-form and it defines *the canonical symplectic structure* on T^*M .

In what follows we will call the smooth functions on T^*M *Hamiltonians*. To any Hamiltonian $h \in C^\infty(T^*M)$ there corresponds a unique vector field \vec{h} on T^*M , called *the Hamiltonian vector field associated to the Hamiltonian h* :

$$d_\lambda h = \sigma(\cdot, \vec{h}(\lambda)), \quad \lambda \in T^*M.$$

If x_1, \dots, x_n are some local coordinates on M , then $\lambda = \sum_{i=1}^n \xi_i dx_i$, where (ξ, x) are the canonical coordinates on T^*M . In these coordinates the Liouville form and the symplectic structure have the following canonical representation:

$$ds_{(\xi, x)} = \sum_{i=1}^n \xi_i dx_i, \quad \sigma = \sum_{i=1}^n d\xi_i \wedge dx_i.$$

The Hamiltonian system $\dot{\lambda} = \vec{h}(\lambda)$ defined by the vector field \vec{h} reads:

$$\begin{cases} \dot{\xi}_i &= -\frac{\partial h}{\partial x_i} \\ \dot{x}_i &= \frac{\partial h}{\partial \xi_i} \end{cases} \quad i = 1, \dots, n.$$

We can apply the Hamiltonian language to the description of singular extremals. Let us consider the curve $\lambda_t = P_{t,1}^* \lambda_1$. By definition this curve is the lifting of some curve “downstairs” on M : $q(t) = \pi(P_{t,1}^* \lambda_1)$ such that $\dot{q}(t) = f_t(q(t))$. The flow $P_{t,1} : q(t) \mapsto q(1)$ is a Hamiltonian flow corresponding to the Hamiltonian

$$h_t = \langle \lambda, f_t(\pi(\lambda)) \rangle$$

In other words, $\lambda_t = P_{t,1}^* \lambda_1$ if and only if it satisfies the Hamiltonian equation $\dot{\lambda}_t = \vec{h}_t(\lambda_t)$ and $\dot{x} = f_t(x)$, where $x(t) = \pi(\lambda_t)$.

Let

$$\Delta^\perp = \{ \lambda \in T^*M : \langle \lambda, \Delta_{\pi(\lambda)} \rangle = 0 \}$$

be the annihilator of the distribution Δ .

Proposition 3 *The curve $t \mapsto \lambda_t \in \Delta^\perp$ is a singular extremal if and only if*

$$\sigma(\dot{\lambda}_t, T_{\lambda(t)} \Delta^\perp) = 0,$$

or, equivalently, if and only if

$$\dot{\lambda} \in \text{Ker} \sigma|_{\Delta^\perp}.$$

Proof For simplicity we assume that the vector fields f_t^i are autonomous. Let $\Delta = \{f_1, \dots, f_k\}$, $u \in \mathbb{R}^{2n}$ and $f_u = \sum_{i=1}^k u_i f_i$. Denote by

$$h_i(\lambda) = \langle \lambda, f_i(\pi(\lambda)) \rangle$$

the Hamiltonians associated to the vector fields f_i . Take some $\lambda \in \Delta^\perp$ and let $\xi \in T_\lambda(T^*M)$. We claim that $\sigma(\xi, T_\lambda \Delta^\perp) = 0$ if and only if $\xi = \sum_{i=1}^k u_i \vec{h}_i(\lambda)$.

Indeed, by definition we have

$$(6) \quad \sigma(T_\lambda \Delta^\perp, \vec{h}_i(\lambda)) = \langle d_\lambda h_i, T_\lambda \Delta^\perp \rangle.$$

On the other hand, h_i are constant on Δ^\perp :

$$\Delta^\perp = \{ \lambda : h_i(\lambda) = 0, i = 1, \dots, k \}.$$

Hence the right hand side of (6) is zero. So there exists $u(t)$ such that $\dot{\lambda} = \sum_{i=1}^k u_i(t) \vec{h}_i(\lambda)$. But as we have already seen, this is possible if and only if $\lambda_t = P_{t,1}^* \lambda_1$, where the flow $P_{t,1}$ is generated by $\dot{q} = \sum_{i=1}^k u_i(t) f_i(q)$.

Example 2 (*Co-dimension one distribution*)

Let ω be a 1-form on M such that

$$\Delta_q = \{v \in T_q M : \langle \omega_q, v \rangle = 0\}, \quad q \in M.$$

Then

$$\Delta_q^\perp = \{u \omega_q, u \in \mathbb{R}\}.$$

We have

$$\begin{aligned} s|_{\Delta^\perp} &= u \omega, \\ \sigma|_{\Delta^\perp} &= d(u\omega) = du \wedge \omega + u d\omega. \end{aligned}$$

Since $\text{Ker}(du \wedge \omega) = \Delta$ it follows that $\dot{\lambda} \in \text{Ker} \sigma|_{\Delta_q^\perp}$ if and only if $\dot{\lambda} \in \text{Ker} d\omega|_{\Delta_q}$. If the manifold M is of even dimension, i.e. if $n = 2m$ for some $m \in \mathbb{N}$, then $\sigma|_{\Delta^\perp}$ has a kernel and hence through each point of M passes a singular extremal. Moreover, in this case singular extremals foliate Δ^\perp .

In the case $n = 2m + 1$ the 2-form $d\omega$ can be non-degenerate. For example, if ω is a contact form, then $\text{Ker} d\omega|_{\Delta} = \emptyset$, i.e. there are no non-constant singular extremals and hence no non-constant singular trajectories.

Let us consider in detail the case of lowest possible dimension $n = 3$. Let ω be a generic 1-form on M . In local coordinates we have

$$\omega \wedge d\omega = b(x) dx_1 \wedge dx_2 \wedge dx_3.$$

If $b(x) \neq 0$ then $\omega \wedge d\omega \neq 0$ and ω is a contact form and vice versa.

Assume now that $b^{-1}(0)$ is a smooth 2-dimensional manifold.⁵ Out of $b^{-1}(0)$ there are no singular trajectories. The singular trajectories are the leaves of the line distribution $Tb^{-1}(0) \cap \Delta$. In general this foliation may have singularities of two types: saddle points or foci. Notice that any smooth piece of a singular trajectory out of singularities is a strong minimizer for any Sub-Riemannian distance.

Example 3 (*2-dimensional distribution in \mathbb{R}^n*)

⁵This manifold is called the Martinet surface.

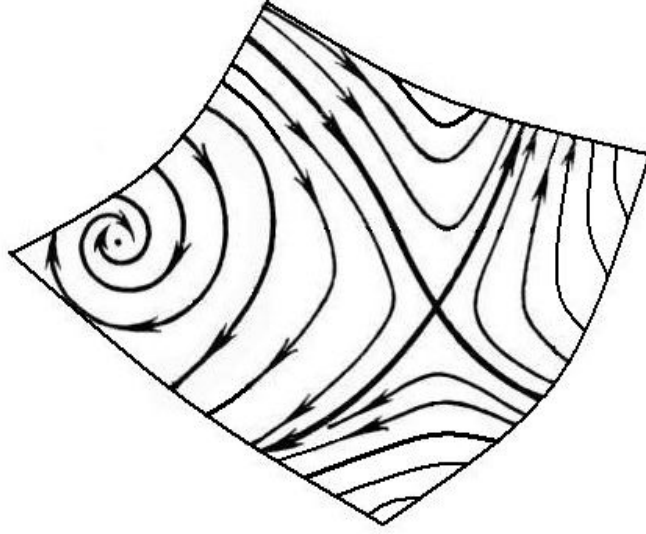


Figure 4: Martinet surface

Assume

$$\Delta_q = \{f_1(q), f_2(q)\}, \quad q \in M.$$

Then $\dim \Delta^\perp = 2n - 2$. Let

$$h_i(\lambda) = \langle \lambda, f_i(q) \rangle, \quad i = 1, 2,$$

$$h(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda).$$

The curve $t \mapsto \lambda_t$ is a singular extremal if and only if

$$\dot{\lambda}_t = u_1(t)\vec{h}_1(\lambda_t) + u_2(t)\vec{h}_2(\lambda_t), \quad \text{and} \quad h_1(\lambda_t) = h_2(\lambda_t) = 0.$$

Here the last condition means that $\lambda_t \in \Delta^\perp$. The vector-function $u = (u_1, u_2)$ is the unknown variable of the problem. Denote

$$\{h_i, h_j\}(\lambda) = \langle \lambda, [f_i, f_j](q) \rangle = h_{ij}.$$

We have

$$\frac{d}{dt}h_1(\lambda_t) = u_1(t)\underbrace{\{h_1, h_1\}(\lambda_t)}_{=0} + u_2(t)\{h_2, h_1\}(\lambda_t) = u_2(t)h_{12}(\lambda_t) = 0.$$

Similarly

$$u_1(t)h_{12}(\lambda_t) = 0.$$

Thus we obtain the following system:

$$\begin{cases} h_{12}(\lambda_t) = 0 \\ h_1(\lambda_t) = h_2(\lambda_t) = 0 \end{cases}$$

Differentiating again we get

$$u_1(t)h_{112}(\lambda_t) + u_2(t)h_{212}(\lambda_t) = 0$$

Then if $h_{112}(\lambda_t) \neq 0$ and $h_{221}(\lambda_t) \neq 0$ we get

$$\begin{cases} u_1(t) = h_{221}(\lambda_t) \\ u_2(t) = h_{112}(\lambda_t) \end{cases}$$

up to a multiplier. Hence the singular extremals satisfy the following ODE:

$$(7) \quad \dot{\lambda}_t = h_{221}(\lambda_t)\vec{h}_1(\lambda_t) + h_{112}(\lambda_t)\vec{h}_2(\lambda_t).$$

Exercise 3 Show that the sub-manifold $\{\lambda : h_1(\lambda) = h_2(\lambda) = h_{12}(\lambda) = 0\}$ is invariant with respect to the flow generated by (7).

4 Properties of singular extremals

In this section consider the general situation: $\dim M = n$, $\Delta \subset TM$ and

$$\Delta_q = \text{span}\{f_1(q), \dots, f_k(q)\}, \quad q \in M.$$

As we have already seen, in the case $k = n - 1$ the situation depends on whether the integer n is even or odd. If $k = 2$, then generically through every $q \in M$ we have the $(n - 4)$ -dimensional family of abnormal geodesics. For instance, for $(2, 4)$ case we have exactly one geodesic through each point (see Example 6 below). In the case of a $(2, 5)$ -distribution at any $q \in M$ there is a one-parametric family of abnormal geodesics, whose velocities covers Δ_q .

4.1 Rigidity

Definition 4 Denote

$$\Omega = \{\gamma(\cdot) : [0, 1] \mapsto M, \dot{\gamma}(t) \in \Delta_{\gamma(t)}\}.$$

and consider the map

$$\partial : \gamma(\cdot) \mapsto (\gamma(0), \gamma(1)).$$

We say that the curve γ is rigid if and only if $\partial^{-1}(\gamma(0), \gamma(1))$ is isolated in $W^{1, \infty}$ -topology in Ω .

In other words, if γ is rigid we cannot deform it keeping fixed the end-points. The condition for γ to be a critical point of ∂ is the necessary condition for rigidity.

Above we have defined $\partial = (F_0, F_1)$, where F_0, F_1 are submersions. So, instead of ∂ we can consider the mapping $F = F_1|_{F_0^{-1}(q_0)}$. Note that the second variations of ∂ and F coincide. From now on we will work with the map

$$F : \gamma(\cdot) \mapsto \gamma(1), \quad \gamma(0) = q_0,$$

with q_0 fixed. We can write

$$\dot{\gamma}(t) = \sum_{i=1}^k u_i(t) f_i(\gamma(t)).$$

As soon as the basis $\{f_i\}_{i=1}^k$ is chosen, the control functions u_i , $i = 1, \dots, k$ are the coordinates of F_0^{-1} . By fixing $u_i(t)$, $i = 1, \dots, k$ and perturbing initial conditions we produce the flow

$$P_{t,1} \stackrel{\text{def}}{=} P_t : \mathcal{O}_{\gamma(t)} \mapsto \mathcal{O}_{\gamma(1)}$$

generated by the non-autonomous vector field

$$f_{u(t)}(q) = \sum_{i=1}^k u_i(t) f_i(q).$$

Exercise 4 Show that $D_\gamma F(v(\cdot)) = \int_0^1 P_{t*} f_{v(t)} dt (\gamma(1))$.

Denote $g_v^t = P_{t*} f_v$. We have

$$D_\gamma F(v(\cdot)) = \int_0^1 g_{v(t)}^t dt.$$

If we choose $v(\cdot)$ such that $\int_0^1 g_{v(t)}^t dt (\gamma(1)) = 0$, then

$$D_\gamma^2 F(v(\cdot)) = \int_0^1 \left[\int_0^1 g_{v(\tau)}^\tau d\tau, g_{v(t)}^t \right] dt (\gamma(1)).$$

We omit the details of this calculation here. The interested reader can find them in [1].

Since γ is a critical point the differential $D_\gamma F v$ is not onto, i.e., if

$$\text{Im} D_\gamma F = \text{span} \left\{ g_v^t, \quad t \in [0, 1], \quad v \in \mathbb{R}^k \right\},$$

then there exists $\lambda_1 \in T_{\gamma(1)}^* M$ such that

$$\langle \lambda_1, g_v^t \rangle = 0, \quad \langle \lambda_1, P_{t*} f_v \rangle = 0,$$

which implies

$$(8) \quad \langle P_t^* \lambda_1, f_v \rangle = 0.$$

Denote $\lambda_t = P_t^* \lambda_1$. Then (8) becomes

$$\langle \lambda_t, f_v \rangle = 0.$$

Consider now the following quadratic form:

$$(9) \quad \lambda_1 D_\gamma^2 F(v) = \left\langle \lambda_1, \int_0^1 \left[\int_0^t g_{v(\tau)}^\tau d\tau, g_{v(t)}^t \right] dt \right\rangle, \quad v \in \text{Ker} D_\gamma F.$$

Definition 5 *The index of the extremal $t \mapsto \lambda_t$ is the Morse index of the quadratic form (9)⁶*

$$\text{ind}(\lambda_t) \stackrel{\text{def}}{=} \text{ind}(\lambda_1 D_\gamma^2 F).$$

Definition 6

$$\text{corank}(\lambda_t) \stackrel{\text{def}}{=} \text{codim}(\text{span}\{g_v^t, t \in [0, 1], v \in \mathbb{R}^k\}).$$

The following two theorems illustrate the relation of the index of the extremals and the rigidity of the corresponding abnormal trajectories. We omit the proofs here.

Theorem 4 *If γ is rigid, then its lift is an abnormal extremal $\lambda_t \in T_{\gamma(t)}^* M$ such that*

$$\text{ind}(\lambda_t) < \text{corank}(\lambda_t).$$

Theorem 5 *(Necessary and sufficient conditions for the finiteness of index) If $\text{ind}(\lambda) < +\infty$, then*

$$\langle \lambda_t, [f_{v_1}, f_{v_2}] \rangle(\gamma(t)) = 0 \quad (\text{Goh condition})$$

and for all $v \in \mathbb{R}^k$

$$\langle \lambda_t, [[f_u, f_v], f_v] \rangle(\gamma(t)) \geq 0, \quad t \in [0, 1]. \quad (\text{generalized Legendre condition})$$

In addition, if

$$\langle \lambda_t, [[f_{u(t)}, f_v], f_v] \rangle(\gamma(t)) \geq c|v|^2 \quad \forall v \perp u(t) \quad (\text{strong generalized Legendre condition}),$$

then

$$\text{ind}(\lambda_t) < +\infty.$$

We remark that just finiteness of the index is not enough for rigidity. Nevertheless, the strong generalized Legendre condition implies that the small enough pieces of γ are rigid and they are strong length minimizers for any Sub-Riemannian distance (i.e., local minima in the C^0 topology).

⁶i.e., the maximal dimension the subspaces where the quadratic form $\lambda_1 D_\gamma^2 F$ is negative definite.

4.2 Conjugate points

The points $\gamma(0)$ and $\gamma(1)$ are called *conjugate* if there exists a C^∞ -small perturbation of Δ such that λ_t remains a singular extremal associated to the same singular trajectory of the same corank and satisfying Goh and strong generalized Legendre conditions, but the index changes.

Theorem 6 *If $t \mapsto \lambda_t$ is such that $\gamma(0)$ and $\gamma(1)$ are not conjugate and such that $\text{ind}(\lambda_t) = 0$, then the curve $\gamma(t) = \pi(\lambda_t)$ is rigid and it is a strong length minimizer for any metric.*

Definition 7 *The singular extremal $t \mapsto \lambda_t$ is sharp if its index is finite.*

Example 4 Let $\Delta \in TM$, and denote

$$\Delta_q^2 = \text{span} \{[f_i, f_j](q), f_i, f_j \in \Delta\}, \quad q \in M.$$

From the Goh condition it follows that if the extremal λ is sharp, it must annihilate Δ^2 . Therefore if $\Delta^2 = TM$, then there are not sharp extremals.

Example 5 (Carnot groups) Assume $\dim \Delta_q = k$. The following situations are possible:

- i) if $n \leq k + (k-1)^2$, then a generic Carnot group does not admit sharp extremals;
- ii) if $n > k + (k-1)^2$, then there exists an open set of Carnot groups admitting sharp extremals;
- iii) if $n \gg k + (k-1)^2$, then a generic Carnot group admits sharp extremals.

Note that if $k = 2$ and λ_t is a singular extremal, then the Goh condition is satisfied automatically. Indeed,

$$\frac{\partial}{\partial t} \langle \lambda_t, f_v \rangle = 0,$$

and hence

$$\langle \lambda_t, [f_{u(t)}, f_v] \rangle = 0.$$

Example 6 ($k = 2$, generic germs of the distribution Δ)

- i) If $n = 3$, then $\Delta_q^2 = T_q M$ and this situation is not of interest for us.
- ii) Assume $n = 4$, $\dim \Delta_q^2 = 3$ and $\dim \Delta_q^3 = 4$. Such a distribution is called the Engel distribution. Let $\Delta = \text{span}\{f_1, f_2\}$ and consider the Lie bracket $[v_1 f_1 + v_2 f_2, \Delta^2]$. There exists a unique vector field $u_1 f_1 + u_2 f_2$ (singular direction) such that

$$[u_1 f_1 + u_2 f_2, \Delta^2] \in \Delta^2.$$

Therefore there is exactly one singular trajectory passing through every point of M . Without loss of generality we can assume that $u_1 f_1 + u_2 f_2 = f_1$. Then $[f_1, \Delta^2] \in \Delta^2$. We have

$$e_*^{tf_1} f_1 = f_1, \quad e_*^{tf_1} \Delta^2 = \Delta^2, \quad e_*^{tf_1} \Delta \neq \Delta.$$

Therefore the distribution $e_*^{tf_1} \Delta$ “rotates” around the direction of f_1 in Δ^2 . The points q_0 and $e_*^{tf_1}$ are conjugate if and only if $e_*^{tf_1} \Delta = \Delta$, i.e. the time t corresponds to a complete revolution of $e_*^{tf_1} \Delta$. The index of a singular extremal then equal to the number of complete revolutions.

Remark Let $\rho(q_1, q_0)$ denote the Carnot-Carathéodory distance between points q_1 and q_0 . Consider the sphere

$$S_{q_0}(r) = \{q \in M : \rho(q, q_0) \leq r\},$$

where r is sufficiently small. If γ is a singular geodesic starting at q_0 and $\gamma(1) \in S_{q_0}(l(\gamma))$, then the distance ρ is not C^1 and $d_{\gamma(1)}\rho$ is not defined. If the strong generalized Legendre condition holds, then $\gamma(1)$ belongs to the closure of the cut-locus of q_0 .

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