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Sub-Riemannian metrics on $\mathbb{R}^3$.

A. A. AGRACHEV, EL-H. CHAKIR EL-A.
AND J. P. GAUTHIER

Abstract

This paper is a continuation of a series of papers of the authors dealing with sub-Riemannian metrics on $\mathbb{R}^3$ in the contact case. Our purpose here in is twofold :

1. We prove the smoothness of a certain normal form, which is the analog of "normal coordinates" in Riemannian geometry. This normal form is crucial for the purpose of studying singularities of the exponential mapping. In our previous papers, it was a "formal" normal form only.

2. We finish with the generic classification for the singularities of the exponential mapping of a germ of a contact sub-Riemannian metric.

1. Introduction.

1. A sub-Riemannian metric on a 3-dimensional manifold X is a couple $\Sigma = (\Delta, g)$ of a two dimensional vector subbundle Δ of TX, and an Euclidian metric $g : \Delta \to \mathbb{R}_+$ on Δ. When Δ is completely non-integrable, such a couple (Δ, g) defines a metric d on X ([24]). In most cases, we will consider that Δ is a contact structure on X. The set of the contact sub-Riemannian metrics on a given manifold X is denoted by $SubR$.

In previous papers of the authors ([2], [4], [9]), these metrics were studied in details in the contact case. The main purpose of these papers was the study of the conjugate locus of a point x_0, $CL(\Sigma)$, the cut-locus of a point x_0, $CutL(\Sigma)$ and the spheres of small radius centered at x_0. In contrast with Riemannian geometry, the study of $CL(\Sigma)$, $CutL(\Sigma)$ is partly a **local** problem : both of these sets have the point x_0 in their closure. As a consequence, even spheres of small radius are not smooth.

The most simple sub-Riemannian metric is the right invariant metric on the Heisenberg group : Δ is the (unique up to conjugation) left invariant 2-dimensional contact distribution and g is left invariant. This particular metric plays a very special role in the theory, since any contact sub-Riemannian metric can be treated as a perturbation of this Heisenberg metric.

2. Precisely, we have shown that a formal contact sub-Riemannian metric (specified for instance by a couple of formal vector fields (F, G), that are a formal orthonormal basis for the metric) has the following normal form :

$$(NF) \quad \begin{cases} \widetilde{F} = (1 + y^2\,\beta)\,e_1 - x\,y\,\beta\,e_2 + \frac{y}{2}(1+\gamma)\,e_3 \\ \quad = \frac{\partial}{\partial x} + y\left(\beta\,(y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}) + \frac{1}{2}(1+\gamma)\,\frac{\partial}{\partial w}\right) \\ \widetilde{G} = (1 + x^2\,\beta)\,e_2 - x\,y\,\beta\,e_1 - \frac{x}{2}(1+\gamma)\,e_3 \\ \quad = \frac{\partial}{\partial y} - x\left(\beta\,(y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}) + \frac{1}{2}(1+\gamma)\,\frac{\partial}{\partial w}\right) \end{cases} \tag{1.1}$$

where $F(0) = \widetilde{F}(0) = e_1$, $G(0) = \widetilde{G}(0) = e_2$, $\left[\widetilde{F}, \widetilde{G}\right](0) = e_3$, and where (x, y, w) are the linear coordinates on T_0X, dual to (e_1, e_2, e_3), β, γ are power series, uniquely determined, and meeting the "boundary" conditions $\beta(0, 0, w) = \gamma(0, 0, w) = \dfrac{\partial\gamma}{\partial x}(0, 0, w) = \dfrac{\partial\gamma}{\partial y}(0, 0, w) = 0$.

(When $\beta = \gamma = 0$ this is just the Heisenberg metric).

In our section 2, we will mainly show that this normal form (NF) is not only formal but C^∞ or C^ω (depending on either we consider C^∞ or C^ω metrics).

This is the main theorem of section 2, Theorem (2.7).

We also show (Theorem (2.6)) that this normal form can be obtained in all cases (not only in the contact case). But, in these other cases, the boundary conditions are not reached, and the normal form does not allow to define the invariants (tensor fields) of the metric structure Σ, that we define in section 2.4. (The construction of these invariants was already done in our previous paper ([9])).

In the Martinet case, the same methodology can be applied to obtain another normal form, which also defines functional invariants along the Martinet surface (see section 2.6).

For a similar study of spheres, wave fronts and conjugate loci in this Martinet case, see the papers ([3], [6], [7]).

3. Our section 3 is devoted to complements (of our previous papers [2], [4], [9]) about the conjugate locus of a point for a germ of a contact metric, and the finite determinacy of the exponential mapping in a neighbourhood of its singular locus.

Theorems 3.3, 3.4 are two main results of section 3. In these theorems, we examine the situation where our first principal invariant Q_2 vanishes. (Q_2 is a component of a certain tensor field defined by means of the normal form). This degenerate situation happens generically on a curve $\mathcal{C}$.

Theorem 3.3 deals with generic points of this curve, Theorem 3.4 deals with isolated points on this curve, that can appear generically. They can be of two types. In all cases, we describe precisely the conjugate locus (some strange cases appear) and we conclude with the fact that the exponential mapping is determined by a certain finite jet of the metric, in a neighbourhood of its singular locus.

The coordinates (x, y, w) of the smooth normal form (NF) (normal coordinates) are, in the contact case, unique up to a rotation in $\Delta(0)$.

The conjugate locus of 0 splits in two parts CL^+ and CL^- corresponding to $w > 0$ or $w < 0$ in these normal coordinates :

$$CL^+ = CL \cap \{w > 0\}, \quad CL^- = CL \cap \{w < 0\}.$$

Let us call CL^+, CL^- a "semi conjugate locus", typically denoted by $CL^\pm$.

As we know from our previous papers [2], [4], [9], for $|w|$ small enough, the intersection $CL^\pm_w$ of $CL^\pm$ with the planes w =constant, $w \neq 0$, is a closed curve γ, presenting 6 cusp points.

As we shall show, for generic cases, this curve will present only transversal self-intersections, and we will define the symbol S of $CL^\pm$ as follows:

S will be an ordered sequence of 6 (rational) numbers, $S = (s_1, \ldots, s_6)$, modulo cyclic permutations and reflexion. We follow the curve γ, starting from a cusp point. s_i is $\frac{1}{2}$ the number of self intersections between the i^{th} and $(i+1)^{th}$ cusp point.

Our result describing the shape of possible "semi conjugate loci" is :

THEOREM 1.1. *1) At generic points of the curve $\mathcal{C}$, the possible symbols for generic "semi conjugate loci" are:*

$$S_1 = (2, 1, 1, 2, 1, 0), \qquad S_2 = (2, 1, 1, 1, 1, 1),$$

$$S_3 = (0, 1, 1, 1, 1, 1),$$

2) At isolated points of the first type of $\mathcal{C}$:

$$S_1^* = S_1 = (2, 1, 1, 2, 1, 0), \qquad S_2^* = S_2 = (2, 1, 1, 1, 1, 1),$$

But in that case, the exponential mapping is determined by a higher order jet of the metric (7^{th} jet) than at generic points of $\mathcal{C}$ (5^{th} jet).

3) At isolated points of the second type, the possible symbols are :

$$S_4 = (\tfrac{1}{2}, \tfrac{1}{2}, 1, 0, 0, 1), \qquad S_5 = (1, \tfrac{1}{2}, \tfrac{1}{2}, 1, 1, 1),$$

$$S_6 = (\tfrac{3}{2}, \tfrac{1}{2}, 1, 1, 0, 1), \qquad S_7 = (2, \tfrac{1}{2}, \tfrac{1}{2}, 2, 0, 0).$$

Some corresponding pictures of semi conjugate loci (not all of them because it is not that easy to draw them) are given in section 3. On the contrary, it is easy to give all pictures for the semi singular sets at the source of the exponential mapping, as follows:
We draw a circle, with six sectors of size $\frac{\pi}{3}$, corresponding to the positions of the cusp points. In each sector, there are stars, corresponding to preimages of double points. Both preimages p_1, p_2 of a double point $d \in CL_w^{\pm}$, are such that p_2 is close to $p_1 + \pi$. All configurations are shown on figure 1.1.

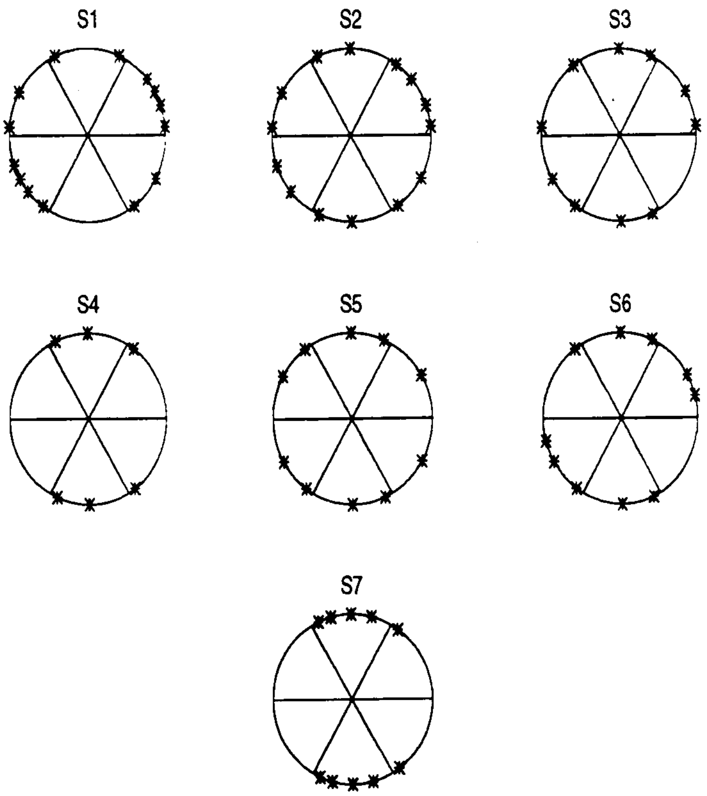

Figure 1.1: Semi singular sets at the source.

Also, it follows from the results of our section 3.5 that the symbols S_1 to S_7 give a complete classification of germs (in $I\!R^3$) of semi conjugate loci, with respect to germs of (origin preserving) homeomorphisms of $I\!R^3$, that are smooth together with their inverse, outside the origin.

Let us point out the fact that in this paper, some computations were made using a formal calculus language : Namely, Mathematica. If a posteriori, **all the computations in the paper can be done by hand**, it is clear to us that it would have been very difficult to obtain the results without using such a formal program. For the interested reader, we give (see Appendix A_3) the complete program allowing to compute all the expansions needed for our purposes (mainly, the expansion of the exponential mapping and conjugate loci at order 7).

2. Normal coordinates.

Convention : All along the paper, the notation $O(\psi)$ means a function of the form $\psi\, F$, where F is a smooth function of all variables under consideration. We allow to write $O(\psi)$, $O(\varphi)$ in the same expression. This doesn't mean that the F's are the same. Also, $O^1(\psi,\, \varphi)$ means $O(\psi)+O(\varphi)$, $O^2(\psi,\, \varphi)$ means $O(\psi^2)+O(\psi\,\varphi)+O(\varphi^2)$.

2.1. Preliminaries.

Let us consider a sub-Riemannian metric $(\Delta,\, g)$ on X, but let us not assume that Δ is a contact structure (it can be integrable, or Martinet, or other). Let us consider a parametrized smooth curve $\Gamma :]-\epsilon,\, \epsilon[\to X$, $\Gamma(0) = 0$, which is transversal to Δ :

$$\mathbb{R}\,\frac{d\Gamma}{dt} + \Delta(\Gamma(t)) = T_{\Gamma(t)}X.$$

If Δ is contact, there is, up to orientation, a natural choice for such a curve :

– The orientation being given, there is a unique 1-form ω on X such that

1. $\Delta = \ker\, \omega$,

2. $d\omega$ restricted to Δ is the area form.

– There is a unique vector field ν on X such that :

$$i(\nu)\,(\omega \wedge d\omega) = d\omega \;\text{ or equivalently }\; \omega(\nu) = 1,\; i(\nu)\, d\omega = 0.$$

The integral curve through zero of ν, $\exp t\,\nu(0)$ is such a curve Γ, in that case.

Let us consider the Hamiltonian H of the metric. It is defined as follows : associated to $(\Delta,\, g)$, there is a cometric on T^*X :

$$H(\psi) = \frac{1}{2} \sup_{v \in \Delta\setminus\{0\}} \left(\frac{\psi(v)}{\|v\|_g}\right)^2.$$

On fibers of $\pi_X : T^*X \to X$, H is a positive semi-definite quadratic form, the kernel of which is the annihilator of Δ.

Locally, we can assume that the metric is specified by an orthonormal frame field $(F,\, G)$. Then,

$$H = \frac{1}{2}\left((\psi(F))^2 + (\psi(G))^2\right).$$

Let $\vec{H}$ denote the corresponding Hamiltonian vector field on T^*X.

The curve Γ defines also a codistribution A_Γ along Γ :

$$A_\Gamma(t) = \text{Annihilator of } \frac{d\Gamma}{dt} \text{ in } T^*_{\Gamma(t)}X.$$

The cometric $H|_{A_\Gamma(t)}$, H restricted to $A_\Gamma(t)$ is a positive definite quadratic form $g'(t)$.

Then, $A_\Gamma = \bigcup_{t\in]-\epsilon,\,\epsilon[} A_\Gamma(t)$ carries the structure of a metric bundle over Γ, $\pi_X : A_\Gamma \to \Gamma$, and this structure can be trivialized as follows :

We take along Γ a coordinate system $\xi = (x, y, w)$ such that : $\Gamma(t) = (0, 0, t)$, and at $x = y = 0$, the distribution $\Delta(\Gamma(t))$ is $ker\, dw$ and the metric g is $dx^2 + dy^2$. If ϵ is small, we can identify the coordinates ξ in a neighbourhood U of Γ with coordinates in the tangent bundle. Dual coordinates in the cotangent bundle T^*X are (p, q, r).

In these coordinates, $A_\Gamma = \{(0, 0, w, p, q, 0) \,|\, w \in]-\epsilon, \epsilon[\}$, $A_\Gamma(t) =$ is the vector space $\{(0, 0, t, p, q, 0)\}$. The cometric H is $\frac{1}{2}(p^2 + q^2)$.

With these coordinates, the structural group $SO(2)$ of this trivial bundle acts now on the fibers of A_Γ in the trivial way :

$$\alpha(w, p, q) = (w, p\cos\alpha - q\sin\alpha, p\sin\alpha + q\cos\alpha) = (w, e^{-J\alpha}(p, q)) \qquad (2.1)$$

Any smooth curve $\delta :]-\epsilon, \epsilon[\to SO(2)$ determines a change of coordinates

$$(\widetilde{w}, \widetilde{p}, \widetilde{q}) = (w, e^{-J\delta(w)}(p, q))$$

of this structure (of a trivial metric bundle).

Such coordinates and changes of coordinates on A_Γ we call coordinates (changes) "adapted to Γ".

Set $C_\Gamma = A_\Gamma \cap H^{-1}(\frac{1}{2})$. In these coordinates, C_Γ is the cylinder $\{p^2 + q^2 = 1\}$.

Associated to these cartesian coordinates "adapted to Γ", there are the corresponding cylindrical coordinates "adapted to Γ", (w, ρ, φ), $\rho^2 = p^2 + q^2$, $p = \rho\cos\varphi$, $q = \rho\sin\varphi$. In these cylindrical coordinates, C_Γ is the cylinder $\{\rho = 1\}$, and "adapted to Γ" coordinate changes are : $(w, \rho, \varphi) \to (w, \rho, \varphi - \delta(w))$.

2.2. Γ-Normal coordinates.

The mapping

$$\Phi : \quad A_\Gamma \to X$$
$$\Phi(\lambda) = \pi_X \circ \exp \overrightarrow{H}(\lambda).$$

is a smooth diffeomorphism, (provided that ϵ is small) from a cylindrical neighbourhood N_Γ of $\{p = q = 0\}$ (the zero section) in A_Γ, to U_Γ a neighbourhood of Γ in X.

Given an "adapted to Γ" coordinate system (w, p, q), consider the coordinates system on $U_\Gamma \subset X$:

$$(w,\, p,\, q) \circ \Phi^{-1}. \tag{2.2}$$

DEFINITION 2.1. *These coordinates on $U_\Gamma \subset X$ we call Γ-normal (cartesian) coordinates.*

An important observation is the following : if $(w, \tilde{\rho}, \tilde{\varphi})$ are the corresponding cylindrical Γ-normal coordinates, they can also be defined as follows.

Set $\tilde{\Phi} : C_\Gamma \times [0, a[\to U_\Gamma$, for $a > 0$, sufficiently small:

$$\tilde{\Phi}(c,\, \rho) = \pi_X \circ \exp \rho \overrightarrow{H}(c). \tag{2.3}$$

Then, if c has adapted cylindrical coordinates (w, φ) in C_Γ,

$$(w,\, \tilde{\rho},\, \tilde{\varphi})\,(\mu) = (w,\, \rho,\, \varphi) \circ \tilde{\Phi}^{-1}(\mu). \tag{2.4}$$

Then :

CLAIM 2.2. *In these Γ-normal (cartesian or cylindrical) coordinates, geodesics through Γ, satisfying the transversality conditions w.r.t. Γ, are straight lines contained in the planes $\{w = constant\}$.*

Also, the following lemma is a series of consequences of Pontriagin's maximum principle ([23]).

Let C_s be the set of points x of X that are at distance s of Γ. (If Δ is not maximally non-integrable, we don't have a distance. In that case, we just mean, similarly to the distance case, that the inf of the length of curves tangent to Δ and joining x to Γ is s).

LEMMA 2.3. *If a and ϵ are small enough, then for all $s < a$:*

i) C_s is the smooth cylinder $\tilde{\Phi}(C_\Gamma, s)$ ($\{\rho = s\}$ in Γ-normal coordinates),

ii) For $c \in C_\Gamma$, $\exp s \overrightarrow{H}(c)$ satisfies the transversality conditions w.r.t. C_s, i.e. $\exp s \overrightarrow{H}(c)$ vanishes on $T_{\tilde{\Phi}(c,s)} C_s$.

REMARK 1. *In particular, abnormal curves (see [22]) are never optimal in the problem with free initial condition on Γ : the adjoint vector has to be nonzero, and transversal to both Γ and Δ. This is impossible.*

Conversely, assume that $(w,\ x,\ y)$ is a coordinate system (with associated cylindrical coordinate system $(w,\ \tilde{\rho},\ \tilde{\varphi})$) such that

$i)$ $\Gamma(t) = (t,\ 0,\ 0)$,

$ii)$ For $\psi_{\gamma(t)} \in T^*_{\gamma(t)}X$, $H(\psi_{\gamma(t)}) = \frac{1}{2}$, $\psi_{\gamma(t)} \frac{d\gamma}{dt} = 0$, then

$\pi_X \circ \exp s \overrightarrow{H}(\psi_{\gamma(t)}) = (w,\ s,\ \tilde{\varphi})$, for a certain $\tilde{\varphi}$.

Then, it defines an adapted to Γ coordinate system, and it is the corresponding Γ-normal coordinate system.

2.3. Γ-normal forms.

We consider on a neighbourhood N of Γ, a cartesian Γ-normal coordinate system $\xi = (w,\ x,\ y)$, $\xi_c = (w,\ \rho,\ \varphi)$ is the corresponding cylindrical system.

On $N\backslash\{\Gamma\}$, we set $F = \frac{\partial}{\partial \rho}$, and we chose G such that $(F,\ G)$ is an orthonormal positive frame for the metric on $N\backslash\{\Gamma\}$ (positive orientation on Δ is given by Γ). G is unique.

A trivial reasonning using Lemma 2.3 (ii) and the Pontriagin's maximum principle, shows that G has to be tangent to the cylinders C_s:

Let $\delta(s) = (w, s\cos(\varphi), s\sin(\varphi))$ be one of our geodesics (starting from Γ with transversality conditions). Let $\Lambda(s) = (r(s), p(s), q(s))$ be the corresponding adjoint vector along this geodesic. By Lemma 2.3 (ii), $\Lambda(s)$ vanishes on $T_{\delta(s)}C_s$. Hence:

$i)$ $r(s) = 0$,

$ii)$ $(0, p(s), q(s))$ vanishes on $\frac{\partial}{\partial \varphi}$

Otherwise, since the geodesic $\delta(s)$ is a trajectory of F, by the PMP, $\Lambda(s).G = 0$.

Therefore G is a linear combination of $\frac{\partial}{\partial w}, \frac{\partial}{\partial \varphi}$.

This shows that, there are smooth functions a, b on $N\backslash\{\Gamma\}$ such that

$$G = a\,\frac{\partial}{\partial w} + (b + \frac{1}{\rho})\,\frac{\partial}{\partial \varphi}.$$

We will need the following trivial technical lemma :

LEMMA 2.4. *Let $f : U \subset R^p \times R \times R \to R$, $f : (w,x,y) \to f(w,x,y)$, be any function, U some neighbourhood of zero. Assume that $x\,f$ and $y\,f$ are C^∞ (resp. C^ω) on U. Then, f is C^∞ (resp. C^ω) on U.*

PROOF. Left to the reader. □

COROLLARY 2.5. *$\bar{a} = \frac{a}{\rho}$ and $\bar{b} = \frac{b}{\rho}$ are smooth functions on N.*

PROOF. Let (p, q, r) be the dual coordinates of our normal coordinates (x, y, w). The expression of the Hamiltonian in these coordinates is a smooth function of (x, y, w, p, q, r).

Computing H on our orthonormal basis (F, G), on $N \backslash \{\Gamma\}$, we get :

$$2H = (p^2+q^2)+\langle (p, q),\ J(x, y)\rangle^2 \frac{b}{\rho}(2+b\rho)+2r\langle (p, q),\ J(x, y)\rangle (1+\rho b)\frac{a}{\rho}+r^2 a^2, \tag{2.5}$$

where J is as in formula 2.1. (Once the orientation is chosen, J can also be understood intrinsically as the complex structure on Δ defined by :

$$ar_g(u, v) = g(J(u), v),$$

where ar_g is the area form associated to g on Δ).

Therefore, setting $p = q = 0$, we obtain that a^2 is a smooth function.

Also, we know that

$$2H(p, q, r, 0, 0, w) = p^2 + q^2,$$

therefore

$$a(0, 0, w) = 0.$$

We consider now the 2×2 Gramm matrix $\mathcal{G}(x, y, w)$, associated to our coordinates and our frame (F, G) on $N \backslash \{\Gamma\}$,

$$\mathcal{G}(x, y, w) = \begin{pmatrix} \langle F, F\rangle & \langle F, G\rangle \\ \langle F, G\rangle & \langle G, G\rangle \end{pmatrix},$$

$\mathcal{G}$ depends on the frame (F, G), but its determinant and trace do not. Therefore, its determinant $\det \mathcal{G}(x, y, w)$ is a smooth function.

A straightforward computation shows that

$$\det \mathcal{G} = (1 + b\rho)^2 + a^2.$$

Besides, $\det \mathcal{G}(0, 0, w) = 1$. Therefore, $b\rho$ is a smooth function, and $(b\rho)(0, 0, w) = 0$.

Also,

$$\frac{\partial H}{\partial r} = \langle (p, q),\ J(x, y)\rangle (1 + \rho b)\frac{a}{\rho} + r a^2$$

is smooth. Applying lemma 2.4, we get that $\frac{a}{\rho}$ is smooth.

In the expression 2.5 above, the first, third and fourth terms are smooth, as we have just shown.

Hence, the second term also is smooth. This implies that $x^2 \frac{b}{\rho}$, $xy \frac{b}{\rho}$, $y^2 \frac{b}{\rho}$ are smooth. Applying lemma 2.4 three times, we get that $\frac{b}{\rho}$ is smooth. □

Now, let us make the following (singular along Γ) change of orthonormal frame field :

set :

$$\begin{cases} \widetilde{F} = \frac{x}{\rho} F - \frac{y}{\rho} G, \\ \widetilde{G} = \frac{y}{\rho} F + \frac{x}{\rho} G. \end{cases} \tag{2.6}$$

We get :

$$\begin{cases} \widetilde{F} = \frac{\partial}{\partial x} - y\frac{b}{\rho}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - y\frac{a}{\rho}\frac{\partial}{\partial w}, \\ \widetilde{G} = \frac{\partial}{\partial y} + x\frac{b}{\rho}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + x\frac{a}{\rho}\frac{\partial}{\partial w} \end{cases} \tag{2.7}$$

Setting $\beta = \dfrac{b}{\rho}$, $(1+\gamma) = -2\,\dfrac{a}{\rho}$, we obtain finally :

$$\begin{cases} \widetilde{F} = \frac{\partial}{\partial x} - y\,\beta\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{y}{2}(1+\gamma)\frac{\partial}{\partial w}, \\ \widetilde{G} = \frac{\partial}{\partial y} + x\,\beta\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \frac{x}{2}(1+\gamma)\frac{\partial}{\partial w}. \end{cases}$$

This is exactly our normal form (NF) of section 1.

Therefore, we have shown :

THEOREM 2.6. *(Δ, g) being given, not necessarily contact, together with a parametrized curve $\Gamma :]-\epsilon, \epsilon[\to X$, $\Gamma(0) = 0$, transversal to Δ, there is a smooth coordinate system $\xi = (w, x, y)$, $\xi_c = (w, \rho, \varphi)$ around 0, such that :*

1. $\Gamma(t) = (t, 0, 0)$,

2. *In these coordinates, (Δ, g) has the normal form (NF). (without the boundary conditions)*

Moreover, this coordinate system is unique up to a change of coordinates of the form :

$(\widetilde{w}, \widetilde{\rho}, \widetilde{\varphi}) = (w, \rho, \varphi - \delta(w))$, *where $\delta :]-\epsilon, \epsilon[\to SO(2)$ is a smooth mapping*

2.4. Contact case.

In this case, as we already know, we have two natural candidates for $\Gamma : t \to \exp \epsilon\, t\, \nu(0)$, where ν is the vector field defined in section 2.1, and $\epsilon = \pm 1$. Let us chose $\epsilon = 1$.

Also, in that case, $\dfrac{a}{\rho} \neq 0$ for $x = y = 0$.

Setting $\widetilde{\varphi} = \varphi - \delta(w)$, we see that, since $F = \dfrac{\partial}{\partial \rho}$, $G = a\,\dfrac{\partial}{\partial w} + (b + \dfrac{1}{\rho})\,\dfrac{\partial}{\partial \varphi}$, $\dfrac{a}{\rho}$ is unchanged and $\beta = \dfrac{b}{\rho}$ is changed for $\dfrac{b}{\rho} - \dfrac{\partial \delta}{\partial w}\,\dfrac{a}{\rho}$.

This shows, since $\frac{a}{\rho}(w,\, 0,\, 0) \neq 0$, that we can chose δ for $\beta(w,\, 0,\, 0) = 0$. This choice is **unique up to** $\delta(0)$.

Second, if ω denotes the 1-form of the section 2.1, the expression of ω in normal coordinates is

$$\omega = f\,(dw + \tau\,(x\,dy - y\,dx)),$$

with $\tau = -\frac{\overline{a}}{1+\beta\rho^2}$ and,

$$f = -\frac{1+\beta\,\rho^2}{2\,\overline{a} + \rho\frac{\partial\overline{a}}{\partial\rho} + \rho^3(\beta\frac{\partial\overline{a}}{\partial\rho} - \overline{a}\frac{\partial\beta}{\partial\rho})}, \text{ where } \overline{a} = \frac{a}{\rho}. \tag{2.8}$$

On Γ, $(x = y = 0)$, $\omega(\nu) = 1$, $\nu = \frac{\partial}{\partial w}$, $\omega = f\,dw$.

This shows that $f|_{\rho=0} = 1$, which implies that $\frac{a}{\rho}|_{\rho=0} = -\frac{1}{2}$, or

$$\gamma|_{\rho=0} = 0. \tag{2.9}$$

Also, ν has to be such that $i(\nu)\,(d\omega) = 0$. It is easily computed that, along Γ (at $x = y = 0$),

$$i(\nu)\,(d\omega)|_{\rho=0} = -\frac{\partial f}{\partial x}\,dx - \frac{\partial f}{\partial y}\,dy \tag{2.10}$$

This implies that, $\frac{\partial f}{\partial x}(w,\, 0,\, 0) = \frac{\partial f}{\partial y}(w,\, 0,\, 0) = 0$.

This shows that, by (2.8), $\frac{\partial\overline{a}}{\partial x}(w,\, 0,\, 0) = \frac{\partial\overline{a}}{\partial y}(w,\, 0,\, 0) = 0$, or

$$\frac{\partial\gamma}{\partial x}(w,\, 0,\, 0) = \frac{\partial\gamma}{\partial y}(w,\, 0,\, 0) = 0.$$

We have obtained the boundary conditions for (NF).

THEOREM 2.7. *A germ at 0 of contact sub-Riemannian metric $(\Delta,\, g)$ being given, there is, up to orientation and up to the action of $SO(2)$, a unique coordinate system in which the metric $(\Delta,\, g)$ has the normal form (NF), with boundary conditions.*

REMARK 2. *Observe that, in this section 2, the results are available in the C^∞ or C^ω category : just replace everywhere smooth by C^∞ (resp. C^ω).*

Any choice of the value of $\delta(0) \in SO(2)$ determines completely the normal form, and the normal coordinates, (up to orientation). $\delta(0)$ being chosen, let $(w,\, x,\, y)$ be these coordinates. In these coordinates, the derivatives

$$d_i = \frac{\partial^i\beta}{\partial w^i}|_{w=0},\; d_i' = \frac{\partial^i\gamma}{\partial w^i}|_{w=0}$$

are functions of (x, y) only.

The k^{th} differentials $D_i^k(x,\, y)$, $D_i'^k(x,\, y)$, at $(x,\, y) = 0$, of these functions d_i, d_i' are homogeneous polynomials of degree k in $(x,\, y)$.

The pull backs on $\Delta(0)$ of the D_i^k, $D_i'^k$, by the mapping $(dx,\, dy) \to (x,\, y)$ allow to define covariant symmetric tensors on $\Delta(0)$, that we denote by $\beta_{i,k,\delta(0)}$, $\gamma_{i,k,\delta(0)}$.

PROPOSITION 2.8. *The typical point of X is now denoted by χ (in the previous section, in coordinates, it was $x = y = w = 0$). The tensors $\beta_{i,k,\delta(0)_\chi,\chi}$, $\gamma_{i,k,\delta(0)_\chi,\chi}$, are independent on the choice of $\delta(0)_\chi$ and, if i is even, of the orientation chosen on Δ. We denote them by $\beta_{i,k,\chi}$, $\gamma_{i,k,\chi}$, and the corresponding tensor fields by $\beta_{i,\,k}$, $\gamma_{i,\,k}$.*

PROOF. If we consider $\beta_{i,k,\delta(0)_\chi,\chi}$, $\gamma_{i,k,\delta(0)_\chi,\chi}$, $\beta_{i,k,\delta'(0)_\chi;\chi}$, $\gamma_{i,k,\delta'(0)_\chi,\chi}$ and the corresponding normal coordinates in $\Delta(\chi)$, $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})|_\chi$, and $(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'})|_\chi$, then the element $A = (\delta'(0)_\chi - \delta(0)_\chi) \in SO(2)$ maps $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})|_\chi$ to $(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'})|_\chi$. But, by construction,

$$\beta_{i,k,\delta'(0)_\chi,\chi} = \beta_{i,k,\delta(0)_\chi,\chi} \circ A,\; \gamma_{i,k,\delta'(0)_\chi,\chi} = \gamma_{i,k,\delta(0)_\chi,\chi} \circ A.$$

If we reverse the orientation on Δ, the normal frame

$$(F,\, G) = \left(\frac{\partial}{\partial \rho},\; -\frac{\rho}{2}(1+\gamma)\frac{\partial}{\partial w} + (\rho\,\beta + \frac{1}{\rho})\frac{\partial}{\partial \varphi}\right)$$

is changed for $(F,\, -G)$, but φ changes for $-\varphi$, and $\nu = \frac{\partial}{\partial w}$ at $x = y = 0$, changes for $-\frac{\partial}{\partial w}$. □

Therefore, these tensor fields $\beta_{2i,k}$ (resp. $\beta_{2i+1,k}$), are invariants of the metric structure $(\Delta,\, g)$, (resp. modulo orientation).

This was already stated in our paper [9]. The invariants defined here are just certain components of those we defined in [9]. (We will use them again in the next section).

2.5. Notations.

At this point, we need to introduce some notations, relative to the tensor fields $\beta_{i,\,j}$, $\gamma_{i,\,j}$, that will be used **extensively** in the second part of this paper. These notations are the same as those we used in the paper [9]. They concern contact metrics only.

Let us set :

$$\beta_k^{2\,n+k} = \beta_{n,\,k}, \qquad \gamma_k^{2\,n+k} = \gamma_{n,\,k}. \tag{2.11}$$

This notation is extremely important because, in the approximation at order n (in the sense of our paper [9]) of the exponential mapping and the conjugate locus CL, only the tensor fields β_j^{l-1}, γ_j^{l+1}, $l \leq n$, play a role.

Also, if $\bigodot^k \Delta^*$ denotes the tensor bundle of k-symmetric covariant tensors over Δ, the structural group $SO(2)$ of Δ acts on its typical fiber, allowing to decompose $\bigodot^k \Delta^*$ into isotypic components,

$$\bigodot^k \Delta^* = \bigoplus \left(\bigodot^k \Delta^*\right)_j ,$$

where $\left(\bigodot^k \Delta^*\right)_j$ is the component relative to the j^{th} power of the basic character $e^{i\varphi}$,($i = \sqrt{-1}$.)

$$\text{If} \quad \beta_j^n \in \bigodot^k \Delta^*, \quad \text{then} \quad \beta_j^n = \sum \beta_{j,i}^n, \quad \beta_{j,i}^n \in \left(\bigodot^k \Delta^*\right)_i . \tag{2.12}$$

Some of these tensors played a basic role in our paper [9], hence, we introduced special notations for them :

$$Q_2 = \gamma_{2,2}^2, \quad \text{and} \quad Q_0 = \gamma_{2,0}^2. \tag{2.13}$$

γ_2^2 is a quadratic form on Δ, we denote its trace with respect to g by $tr_g(\gamma_2^2)$. It is easy to see that

$$Q_0 = \gamma_{2,0}^2 = \frac{1}{2} tr_g(\gamma_2^2)\, g.$$

Also,

$$\gamma_3^3 = \gamma_{3,1}^3 + \gamma_{3,3}^3,$$

We set,

$$V_3 = \gamma_{3,3}^3, \quad V_1 = \gamma_{3,1}^3. \tag{2.14}$$

In normal coordinates, (x, y, w), at the typical point $\chi = 0$ of X, we set :

$V_1 = Re\left(\overline{a}\,(dx + i\,dy)\,(dx^2 + dy^2)\right)$, $V_3 = Re\left((b_1 + i\,b_2)\,(dx + i\,dy)^3\right)$, with

$a = a_1 + i\,a_2 = \widetilde{a}\,(-\sin\omega_a + i\,\cos\omega_a)$.

Let us set also $b = (b_1, b_2) = \widetilde{b}\,(\sin\omega_b, -\cos\omega_b)$.

Also, depending on the context, it can happen that $b = b_1 + i\,b_2$.

($\overline{a}$ denotes the complex conjugate of a).

2.6. Martinet case.

Of course, in the Martinet case, the previous normal form works, but there is an other one, based on the same idea, which is better. Curiously, it is also much easier to obtain.

We will assume that the Martinet surface M (the set of points where $[\Delta, \Delta] \subset \Delta$) is smooth, and that the distribution Δ is transversal to M. (In particular, this method doesn't work for points at which the distribution is tangent to M. But these points appear generically as isolated points in M, hence, there is still some work to do in that case).

We replace the curve $\Gamma(t)$ of 2.1 to 2.4 by the Martinet surface M. We consider geodesics that verify the transversality conditions w.r.t. M, and we apply exactly the same method. It leads to the normal form :

$$(NF_1) \quad F = \frac{\partial}{\partial x}, \; G = (1 + x\,a)\frac{\partial}{\partial y} + x^2\,b\frac{\partial}{\partial w}, \tag{2.15}$$

with the boundary conditions $\dfrac{\partial a}{\partial x}(0,\,0,\,w) = 0,\;\; b(0,\,0,\,w) = 1$.

These "normal" coordinates and these functions a and b are uniquely determined up to orientation, hence they define functional invariants.
The curve $(0,\,0,\,t)$ is a trajectory of the vector field $\dfrac{1}{2}[F,[F,G]]$.

We leave the details to the reader.

In the beautiful papers ([3], [6], [7]) the Martinet case is adressed. A similar normal form is obtained in the analytic case. A study of spheres, wave fronts, conjugate loci is done in the so called "flat case" ($a = b = 0$), for some perturbations of this flat case, and at the end in the integrable case.

3. Generic conjugate loci.

All along this section, the notations of section 2.5 will be in force.

3.1. Some preliminary results.

In the contact case, consider the principal invariant $Q_2 = \gamma_{2,2}^2$ (2.13).

For an open dense subset of $SubR$, Q_2 vanishes on a curve $\mathcal{C}$ (possibly empty) ([9]). Let us fix such a contact sub-Riemannian metric Σ, with $\mathcal{C} \neq \emptyset$, and $x_0 \in \mathcal{C}$. We consider a sufficiently small neighbourhood N of x_0, with normal coordinates $(x,\,y,\,w)$, $x_0 = (0,0,0)$, and we restrict the metric to N. The dual coordinates on T^*N are $(p,\,q,\,r)$ and the set $\mathrm{C}_{x_0} = \Pi_N^{-1}(x_0) \cap H^{-1}(\frac{1}{2})$ is the cylinder $\{p^2 + q^2 = 1\}$. If N is small enough, points of CL in N appear only along geodesics $\{\pi_X \circ \exp s\overrightarrow{H}(\lambda_{x_0})\}$ for λ_{x_0} in a certain neighbourhood of infinity $N^* \subset \mathrm{C}_{x_0}$. We will show in our appendix 4.2 that it does exists a suitable coordinate system $(h,\,\varphi)$ on N^* such that the conjugate locus mapping $CL : N^* \to N$ has the following expansion with respect to h ("suspended form"):

$$CL(\varphi,\,h) = (x(\varphi,\,h),\,y(\varphi,\,h),\,h = \sqrt{\epsilon\,\frac{w}{\pi}}) = (\sum_{i=4}^{7} f_i(\varphi)\,h^i + O(h^8),\,h), \tag{3.1}$$

where ϵ is ± 1 according to the sign of w and $\sqrt{a}$ denotes the positive square root of the positive real number a. (As we know, the conjugate locus splits in two parts, CL^+ and CL^-, corresponding to $w > 0$ and $w < 0$. Also, let us emphasize that the coordinates (x, y, w) are just the normal coordinates).

We first summarize the properties of these maps f_i, that will be crucial for our purposes. These properties will be shown in Appendix A_2, together with the complete expression of the $f_i's$ when possible.

1. $f_4(\varphi) = (-5\pi\tilde{a}\cos\omega_a + 15\pi\tilde{b}\cos(2\varphi + \omega_b) + \frac{15}{2}\pi\tilde{b}\cos(4\varphi + \omega_b),$

$$-5\pi\tilde{a}\sin\omega_a - 15\pi\tilde{b}\sin(2\varphi + \omega_b) + \tfrac{15}{2}\pi\tilde{b}\sin(4\varphi + \omega_b))$$

$f_4(\varphi)$ does not depend on ϵ.

2. $f_5(\varphi + \pi) = -f_5(\varphi),\ f_6(\varphi + \pi) = f_6(\varphi),\ f_7(\varphi + \pi) = -f_7(\varphi).$

3. $\frac{df_4(\varphi)}{d\varphi} \wedge \frac{df_5(\varphi)}{d\varphi} = 0,$ and

4. $\frac{df_4(\varphi)}{d\varphi} \wedge f_5(\varphi) = -20\,\pi^2\,\tilde{b}\sin(3\varphi + \omega_b)\,P(\varphi) = \Psi(\varphi)$

$$= -20\,\pi^2\,(b_1\cos 3\varphi - b_2\sin 3\varphi)\,P(\varphi).$$

5. $\frac{df_4(\varphi)}{d\varphi} \wedge f_7(\varphi) = -20\,\pi^3\,\tilde{b}\sin(3\varphi + \omega_b)\,Q(\varphi),$

6. $\frac{d^2 f_4(\varphi)}{d\varphi^2} \wedge \frac{d^2 f_5(\varphi)}{d\varphi^2} = -20\,\pi^2\,\tilde{b}\,(\sin(3\varphi + \omega_b)\frac{d^3P}{d\varphi^3} - 3\cos(3\varphi + \omega_b)\frac{d^2P}{d\varphi^2}) = \tilde{Q}(\varphi),$ (3.2)

where :

- $Q(\varphi) = \overline{Q}_1(\varphi) + \overline{Q}_2(\varphi)$, $\overline{Q}_1$ (resp. $\overline{Q}_2$) depends non trivially (resp. does not depend) on the coefficients of our invariants γ_j^6. ($\overline{Q}_1$ is in fact a general trigonometric polynomial in $e^{\pm 2i\varphi}$, $e^{\pm 4i\varphi}$, $e^{\pm 6i\varphi}$ with no constant term, the coefficients of which are independent linear combinations of the coefficients of γ_6^6, γ_4^6, γ_2^6).
- $P(\varphi) = A_\epsilon\cos(2\varphi) + B_\epsilon\sin(2\varphi) + C\cos(4\varphi) + D\sin(4\varphi)$, where A_+, A_-, B_+, B_-, C, D are independent linear combinations of the coefficients of our invariants γ_4^4, γ_2^4 of section 2.

(3.2).2 is a consequence of proposition 3.1, and its corollary in ([2]).

Knowing f_4, f_5 (that are given in appendix A_2) one can easily check (3.2).3, (3.2).4, (3.2).6.

Using these relations (3.2) and no more, we will be able to analyze precisely the nature of the conjugate locus, and to derive a theorem of **finite determinacy** of the exponential mapping along its singular set at the source. This theorem is similar to the theorem we got for generic points (out of $\mathcal{C}$) in our previous papers [4], [9].

To get this result, we need to show the "stability" of certain jets of the exponential mapping (in suspended form, where it is the suspension of a mapping between 2-dimensional manifolds). For this purpose, following the basic results of Whitney ([28], [27], [13]), we have to compute the self-intersections of the conjugate locus, and to show that they are "in general position".

3.2. A basic lemma.

Let $Self$ denote the set of self-intersections of CL (i.e. the set of $(h, \varphi_1, \varphi_2)$ such that $h > 0$ and 1) $\varphi_1 \neq \varphi_2$ 2) $CL(\varphi_1, h) = CL(\varphi_2, h)$). We are interested with the germ of $Self$ along $\{h = 0\}$. A value of φ is said "adherent to $Self$" if $(0, \varphi, \varphi')$ lies in the closure of $Self$ for some φ'. The set of such φ is denoted by $A\text{-}Self$. It follows from (3.2).1 and from our "local stability" result along the conjugate locus at the source ([4], [9]), that if $\varphi \in A\text{-}Self$, $\varphi' = \varphi + \pi \in A\text{-}Self$, and no other value of φ' is possible.

LEMMA 3.1. $A\text{-}Self \subset \{\varphi \,|\, \Psi(\varphi) = 0\}$.

PROOF. Assume that $(h, \varphi, \varphi + \pi + \delta) \in Self$, for $h > 0$, h and δ small. Then

$$f_4(\varphi + \pi + \delta) + h\, f_5(\varphi + \pi + \delta) = f_4(\varphi) + h\, f_5(\varphi) + O(h^2)$$

By (3.2).1, (3.2).2, $f_4(\varphi + \pi) = f_4(\varphi)$, $f_5(\varphi + \pi) = -f_5(\varphi)$. Then

$$f_4(\varphi + \delta) - f_4(\varphi) - h\,(f_5(\varphi + \delta) - f_5(\varphi)) - 2\,h\, f_5(\varphi) + O(h^2) = 0,$$

$$\delta\,\frac{df_4}{d\varphi} - 2\,h\, f_5(\varphi) + O^2(h, \delta) = 0.$$

This shows that, if $\varphi^* \in A\text{-}Self$, $\frac{df_4}{d\varphi}(\varphi^*) \wedge f_5(\varphi^*) = 0$. □

As a consequence of this lemma, $A\text{-}Self$ splits in two subsets :

$$A\text{-}Self = A\text{-}Self_1 \cup A\text{-}Self_c.$$

$A\text{-}Self_1$ is the set of roots of the trigonometric polynomial $P(\varphi)$, $A\text{-}Self_c = \{k\,\frac{\pi}{3} - \frac{\omega_b}{3}\}$ is the set of cuspidal angles. Set : $\nu = \frac{A_\epsilon - i\,B_\epsilon}{2}$, $\mu = \frac{C - i\,D}{2}$.

$$P(\varphi) = \nu\, e^{2\,i\,\varphi} + \overline{\nu}\, e^{-2\,i\,\varphi} + \mu\, e^{4\,i\,\varphi} + \overline{\mu}\, e^{-4\,i\,\varphi}.$$

Set :

$$\widetilde{P}(z) = \mu\, z^4 + \nu\, z^3 + \overline{\nu}\, z + \overline{\mu}.$$

$\varphi \in A\text{-}Self_1$ iff $e^{2\,i\,\varphi}$ is a root of $\widetilde{P}(z)$.

Therefore, these polynomials will be the most important objects in the remaining of the paper.

Let us list several very elementary facts for these polynomials. Assume $\mu \neq 0$. Then,

F0 $\widetilde{P}$ has a root on the unit circle.

F1 (i) If z is a root, $\frac{1}{\overline{z}}$ is a root. (ii) If z is a double root, then $\frac{1}{\overline{z}}$ is a double root. Hence, by (F0) the number of roots on the unit circle (counted with multiplicity) is 2 or 4.

F2 $\widetilde{P}$ has a triple root on the unit circle iff $F_2(\mu, \nu) = 4\,\mu^2\,\overline{\nu} + \nu^3 = 0$.

F3 $\widetilde{P}$ has a double root on the unit circle iff $F_3(\mu, \nu) = 27\,Re^2(\,\mu\,\overline{\nu}^2) - (4\,|\mu|^2 - |\nu|^2)^3 = 0$.

F4 $\widetilde{P}$ has not two double roots on the unit circle.

F5 If $\widetilde{P}$ has a double root on the unit circle, then the two other roots are on the unit circle.

PROOF.

(F0), (F4), (F5) are direct consequences of the fact that the integral over its period of the trigonometric polynomial P is zero.

(F1) : (i) Obvious. (ii) assume that z is a double root

$$\begin{cases} \mu\,z^4 + \nu\,z^3 + \overline{\nu}\,z + \overline{\mu} = 0 \\ 4\,\mu\,z^3 + 3\,\nu\,z^2 + \overline{\nu} = 0 \end{cases}$$

Set $z = \frac{1}{\overline{\widetilde{z}}}$.

$$\begin{cases} (1) \quad \mu\,\widetilde{z}^4 + \nu\,\widetilde{z}^3 + \overline{\nu}\,\widetilde{z} + \overline{\mu} = 0 \\ (2) \quad 4\overline{\mu} + 3\,\overline{\nu}\,\widetilde{z} + \nu\,\widetilde{z}^3 = 0 \end{cases}$$

$4 \times (1) - (2)$ gives :

$$4\,\mu\,\widetilde{z}^4 + 3\,\nu\,\widetilde{z}^3 + \overline{\nu}\,\widetilde{z} = 0.$$

z cannot be zero because $\mu \neq 0$, hence $\widetilde{z}$ cannot be zero, $4\,\mu\,\widetilde{z}^3 + 3\,\nu\,\widetilde{z}^2 + \overline{\nu} = 0$ shows that $\widetilde{z}$ is a double root.

(F2) : If z is a triple root on the unit circle, z is nonzero and

$$\begin{cases} (1) \; \mu\,z^4 + \nu\,z^3 + \overline{\nu}\,z + \overline{\mu} = 0 \\ (2) \; 4\,\mu\,z^3 + 3\,\nu\,z^2 + \overline{\nu} = 0 \\ (3) \; 12\,\mu\,z^2 + 6\,\nu\,z = 0 \\ (4) \; z\,\overline{z} = 1 \end{cases} \Rightarrow (5) \;\; \nu = -2\,\mu\,z$$

Plugging this in (2) gives :

$$4\,\mu\,z^3 - 6\,\mu\,z^3 + \overline{\nu} = 0$$

$$-\overline{\nu} = -2\,\mu\,z^3 = \frac{(-2\mu z)^3}{4\,\mu^2} = \frac{\nu^3}{4\,\mu^2}. \tag{3.3}$$

This gives (F2). Plugging (5), $z = -\dfrac{\nu}{2\,\mu}$ in (4) gives $\mu\,\overline{\mu} = 4\,\nu\,\overline{\nu}$, which also a direct consequence of (3.3). Conversely, it is obvious to check that if (F2) is satisfied, then (1), (2), (3), (4) are also satisfied by $z = -\dfrac{\nu}{2\,\mu}$.

(F3) : First notice that $z = e^{i\,\varphi}$ is a double root of $\widetilde{P}(z)$ iff z is a simple root of $\dfrac{d\widetilde{P}}{dz}$: If it is so : $\mu = \dfrac{-3\nu\,z^2 - \overline{\nu}}{4\,z^3} = \dfrac{\overline{z}^3}{4}(-3\dfrac{\nu}{\overline{z}^2} - \overline{\nu})$, $\overline{\mu} = -\dfrac{3}{4}\overline{\nu}\,z - \dfrac{1}{4}\nu\,z^3$. Replacing in $\widetilde{P}(z) = 0$ gives :

$$\widetilde{P}(z) = (-3\,\nu\,z^2 - \overline{\nu})\frac{z}{4} + \nu\,z^3 + \overline{\nu}\,z - \frac{3}{4}\overline{\nu}\,z - \frac{1}{4}\nu\,z^3 = 0.$$

Then, (F3) is only a matter of trivial computations on the polynomial $z^3\,\frac{d\widetilde{P}}{dz}(\frac{1}{z}) = 4\mu + 3\nu z + \overline{\nu}\,z^3 = R(z)$. □

3.3. Transversality.

1. Let $\mathcal{R}$ denote the set of orthonormal frames of contact sub-Riemannian metrics on some open set $U \subset X$.

Let us denote by $J^n(\mathcal{R})$ the vector bundle of n-jets of elements of $\mathcal{R}$: $J^n(\mathcal{R})$ is nothing but the fiber product over U,

$$J^n(\mathcal{R}) = J^n(V\mathcal{F}) \times_U J^n(V\mathcal{F}),$$

$J^n(V\mathcal{F})$ is the vector bundle of n-jets of sections of TU.

Let p^n denote the set of real polynomials in the two variables (x, y) that are homogeneous of degree n.

Set

$$\mathcal{P}^n = \bigoplus \{p^i \,|\, 1 \le i \le n,\ i = n\,mod\ 2\},\ \mathcal{P}^0 = \{0\},$$

$$\mathcal{Q}^n = \bigoplus \{p^i \,|\, 2 \le i \le n,\ i = n\,mod\ 2\}.$$

(F, G) denotes a typical element of $\mathcal{R}$, and χ_0 a typical element of U.

For $n > 0$ let us define a mapping

$$\Pi^n_{\mathcal{N}} : J^{n+2}(\mathcal{R}) \to \mathcal{P}^{n-1} \times \mathcal{Q}^{n+1},$$

by its restriction to the fiber $\Pi^{-1}(\chi_0)$ of the bundle $J^{n+2}(\mathcal{R})$:

$$\Pi^n_{\mathcal{N}}(j^{n+2}_{\chi_0}(F),\ j^{n+2}_{\chi_0}(G)) = (p^{n-1},\ q^{n+1}) \in \mathcal{P}^{n-1} \times \mathcal{Q}^{n+1},$$

with $p^{n-1} = B_i^{n-1}$, $q^{n+1} = \Gamma_j^{n+1}$, where B_i^{n-1}, Γ_j^{n+1} are the representative of our tensors β_{i,χ_0}^{n-1}, γ_{j,χ_0}^{n+1} in the **unique** normal coordinate system at χ_0 such that

$$(F(\chi_0), G(\chi_0)) = (\widetilde{F}(\chi_0), \widetilde{G}(\chi_0)),$$

$(\widetilde{F}, \widetilde{G})$ is the normal form of (F, G) at χ_0.

It is stated in our paper ([9]) that this map, $\Pi_{\mathcal{N},\chi_0}^n$ is a surjective submersion. For (F, G) fixed, we define

$$\Pi_{\mathcal{N}}^n(F, G) : U \to \mathcal{P}^{n-1} \times \mathcal{Q}^{n+1},$$

by

$$\Pi_{\mathcal{N}}^n(F, G)(\chi_0) = \Pi_{\mathcal{N}}^n(j_{\chi_0}^{n+2}(F), j_{\chi_0}^{n+2}(G)).$$

2. Let us define several subsets S^i, of $\mathcal{P}^4 \times \mathcal{Q}^6$, and the corresponding subsets $\widetilde{S}^i$, $\overline{S}^i$ of $J^7(\mathcal{R})$, $\mathcal{R}$:

$$\widetilde{S}^i = (\Pi_{\mathcal{N}}^5)^{-1}(S^i),$$

$$\overline{S}^i = \{(F, G) \mid j^7(F, G) \pitchfork \widetilde{S}^i\}.$$

Since all the $\widetilde{S}^i$ will be Whitney stratified sets, it follows from the transversality theorems ([5], [14]), that all the $\overline{S}^i$ are open dense for the Whitney topology on $\mathcal{R}$.

Moreover, since in all cases, the S^i are either manifolds, or Whitney-stratified sets **of codimension** 3 **at least**, it will follow that for $(F, G) \in S = \cap \overline{S}^i$, (an open dense set), the sets

$$(\Pi_{\mathcal{N}}^n(F, G))^{-1}(S^i)$$

are smooth submanifolds of U of the same codimension as the S^i.

3. Definition of the S^i.

$$S^1 = \{Q_2 = \gamma_{2,2}^2 = 0\},$$

$$S^2 = S^1 \cap (\{V_3 = \gamma_{3,3}^3 = 0\} \cup \{\mu = 0\}),$$

$$S^3 = S^1 \cap (\{F_3(\mu, \nu) = 0\} \cup \{Res(\mu, \nu, b) = 0\}),$$

where :

i) F_3, μ, ν have been defined in section 3.2. Remember that (section 2.5) b is such that :

$$V_3 = \gamma_{33}^3 = Re((b_1 + i\, b_2)\,(x + i\, y)^3), \ b = (b_1, b_2).$$

ii) $Res(\mu, \nu, b)$ is the resultant polynomial of the two polynomials $\widetilde{P}(z)$ (defined in 3.2) and $T(z) = (b_1 + i\, b_2)\, z^3 + b_1 - i\, b_2$.

$$S^4 = S^1 \cap \{F_2(\mu, \nu) = 0\},$$

$$S^5 = S^1 \cap (\{\overline{\nu}\, b - \mu\, \overline{b} = 0\} \cup \{27\, \nu^3\, b\, \overline{b}^2 + (\overline{\nu}\, b - 4\, \mu\, \overline{b})^3\}).$$

Let $\widehat{S}^6 \subset \mathcal{P}^4 \times \mathcal{Q}^6 \times S_1$ (S_1, the circle) be defined by :

$$Q_2 = 0,\ P(\varphi) = 0,\ \frac{dP}{d\varphi}(\varphi) = 0,\ Q(\varphi) = 0 \text{ or } \widetilde{Q}(\varphi) = 0.$$

Q, $\widetilde{Q}$ have been defined in (3.2).5, (3.2).6 respectively.

It follows from our section 3.1, that $\widehat{S}^6$ is an analytic subset of $\mathcal{P}^4 \times \mathcal{Q}^6 \times S_1$, of codimension 5. The projection $S^6 = \Pi_1(\widehat{S}^6)$ on $\mathcal{P}^4 \times \mathcal{Q}^6$ is therefore a subanalytic subset of $\mathcal{P}^4 \times \mathcal{Q}^6$, of codimension 4 at least. (In fact, S^6 is semialgebraic.)

S^1, $\widetilde{S}^1$ are smooth submanifolds of codimension 2. S^2, $\widetilde{S}^2$ are the union of two smooth manifolds of codimension 4. S^3, S^4, S^5 are algebraic subsets of $\mathcal{P}^4 \times \mathcal{Q}^6$ of codimension 3, 4, 4 respectively.

4. Since $(F, G) \in S$, which is open dense, the following holds :

- The set of points of X where $Q_2 = 0$ is a smooth curve $\mathcal{C}$,
- On $\mathcal{C}$, V_3 is nonzero, μ is nonzero, $F_2(\mu, \nu)$ is nonzero, $\overline{\nu}\, b - \mu\, \overline{b}$ is nonzero. Moreover $F_3(\mu, \nu)$ and $Res(\mu, \nu, b)$ do not vanish simultaneously on $\mathcal{C}$.
- On $\mathcal{C}$, $F_3(\mu, \nu)$, $Res(\mu, \nu, b)$ are nonzero, except at isolated points. (This last case, $Res(\mu, \nu, b) = 0$, will be the most surprizing).
- It is easily seen that the condition $\overline{\nu}\, b - \mu\, \overline{b} = 0$ is equivalent, when $b \neq 0$ (equivalently $V_3 \neq 0$), to the fact that $\widetilde{P}(z)$ and $T(z)$ have 2 distinct common roots (on the unit circle) (which is also equivalent to the fact that they have 3 common roots). This does not happen on $\mathcal{C}$.
- Also it is easily seen that, when $b \neq 0$, the condition $27\, \nu^3\, b\, \overline{b}^2 + (\overline{\nu}\, b - 4\, \mu\, \overline{b})^3 = 0$, in the definition of S^5, is equivalent to the fact that a root of $T(z)$ is equal to a double root of $\widetilde{P}(z)$. Hence, for these values, a cusp point should be a triple root of $\Psi(\varphi)$. This does not happen on $\mathcal{C}$.

As a consequence of these facts, and of the results of sections 3.2, 3.3 1–3, we can state the following :

THEOREM 3.2. *There is an open dense subset G of SubR (for the Whitney topology) such that :*

i) The set of points of X on which $Q_2 = 0$ is a smooth curve $\mathcal{C}$ (possibly empty). On $\mathcal{C}$, μ and b (or V_3) don't vanish.

ii) On the complement of a discrete subset of $\mathcal{C}$,

a) $\widetilde{P}$ has either 2 or 4 distinct simple roots on the unit circle,

b) $\widetilde{P}(z)$ and $T(z)$ have no common root.

iii) At isolated points of $\mathcal{C}$, either

a) $\widetilde{P}(z)$ has a double root on the unit circle, which is not triple, and which is not a root of T. In that case :

a.1) The other roots are distinct, are on the unit circle, do not coincide with roots of T,

a.2) If φ^* denotes this double root of $P(\varphi)$, $Q(\varphi^*) \neq 0$, $\widetilde{Q}(\varphi^*) \neq 0$, or,

b) $\widetilde{P}(z)$ and $T(z)$ have one common simple root and one only (on the unit circle).

The other roots of $\widetilde{P}$ are simple.

REMARK 3. ($\epsilon = \pm 1$). *Since A_+, A_-, B_+, B_-, C, D, can be any (by our section 3.1) it follows that different configurations can occur for $\epsilon = 1$ and $\epsilon = -1$ (any combination among the possibilities given in Theorem 3.2).*

3.4. The conjugate loci.

Let us consider a generic $\Sigma = (F, G)$, in the sense of our previous Theorem 3.2.

3.4.1. Case 1. Generic points of the curve $\mathcal{C}$.

(Points on $\mathcal{C}$ with simple roots for $\widetilde{P}$, and such that $\widetilde{P}$ and T have no common root. ((*ii*) of Theorem 3.2)).

We already know, by lemma 3.1 , that self-intersections can appear only either close to cusp points or to points of A-$Self_1$ (simple roots of P or $\widetilde{P}$). These two distinct parts of the self intersection set $Self$, we denote by $Self_c$ and $Self_1$ respectively.

Characterization of $Self_1$.

We have to solve the equation :

$$f_4(\varphi + \pi + \delta) + f_5(\varphi + \pi + \delta)\, h = f_4(\varphi) + h\, f_5(\varphi) + O(h^2),$$

for φ close to zero, where zero is a simple root of $P(\varphi)$, δ and h small.

This equation rewrites (denoting $'$ for $\frac{d}{d\varphi}$) :

$$0 = \delta\, f_4'(\varphi) + O(\delta^2) - h\, f_5(\varphi+\delta) - h\, f_5(\varphi) + O(h^2)$$

$$0 = \delta\, f_4'(\varphi) - 2\,h\, f_5(\varphi) \; - h\,(\, f_5(\varphi+\delta) - \; f_5(\varphi)) + O(h^2) + O(\delta^2)$$

$$0 = \delta\, f_4'(0) + \delta\,\varphi\, f_4''(0) + O(\delta\varphi^2) - 2\,h\, f_5(0) - 2\,h\,\varphi\, f_5'(0) + O(h^2) + O(\delta^2)$$

$$+O(h\varphi^2) + O(h\delta), \text{ rewritten, } \; 0 = T_{IV}. \tag{3.4}$$

An important fact is the following, easy to check from formula (3.2).1

$$\frac{df_4}{d\varphi} \wedge \frac{d^2 f_4}{d\varphi^2} = 3600\, \widetilde{b}^2\, \pi^2\, \sin^2(3\,\varphi + \omega_b).$$

Since b is nonzero by Theorem 3.2 (i), and ω_b is nonzero by (ii), $\frac{df_4}{d\varphi}(0)\wedge\frac{d^2 f_4}{d\varphi^2}(0) \neq 0$.

As a consequence, it is equivalent to solve :

$$T_{IV} = 0 \text{ or to solve } T_{IV} \wedge \frac{df_4}{d\varphi}(0) = 0, \; T_{IV} \wedge \frac{d^2 f_4}{d\varphi^2}(0) = 0$$

$$0 = T_{IV} \wedge \tfrac{d^2 f_4}{d\varphi^2}(0) = \delta\, f_4'(0) \wedge f_4''(0) - 2\,h\, f_5(0) \wedge f_4''(0) + O(h^2) + O(\delta^2) + O(\varphi\, h)$$

$$+O(h\,\delta) + O(\delta\,\varphi).$$

Also, $f_5(0) \wedge f_4''(0)$ is nonzero because :

$$\frac{d}{d\varphi}(f_4' \wedge f_5) = f_4'' \wedge f_5 + f_4' \wedge f_5'.$$

By equation (3.2).3, $f_4' \wedge f_5' = 0$. Since zero is not a double root of P and ω_b is nonzero, zero is not a double root of $f_4' \wedge f_5$.

Therefore, we can solve the equation $T_{IV} = 0$, to get :

$$\delta = 2\frac{f_5(0) \wedge f_4''(0)}{f_4'(0) \wedge f_4''(0)}\, h + h\, O^1(\varphi,\, h) = A\, h + h\, O^1(\varphi,\, h), \; A \neq 0 \tag{3.5}$$

Replacing in (3.4), we can divide by h ($h > 0$). After this, we rewrite the equation $T_{IV} \wedge \frac{df_4}{d\varphi}(0) = 0$:

$$A\,\varphi\, f_4''(0) \wedge f_4'(0) - 2\, f_5(0) \wedge f_4'(0) + O(h) + O(\varphi^2) = 0$$

But, $f_5(0) \wedge f_4'(0) = 0$. Hence, we can solve in φ, to get :

$$\begin{cases} \varphi = \varphi(h) \\ \delta = A\,h + h\,O(h) \end{cases} \tag{3.6}$$

This gives the equation of the self-intersection (with $h > 0$). It is a smooth curve starting from 0.

Let us show that this self-intersection is now transversal.

To prove this, it is sufficient to show that the following expression $\mathcal{T}(\varphi, h, \delta)$ is nonzero on the self-intersection.

$$\mathcal{T}(\varphi, h, \delta) = (f_4'(\varphi + \delta) + h\,f_5'(\varphi + \delta + \pi)) \wedge (f_4'(\varphi) + h\,f_5'(\varphi) + O(h^2)).$$

$$\mathcal{T}(\varphi, h, \delta) = (f_4'(\varphi) - h\,f_5'(\varphi) + \delta\,f_4''(\varphi) + O(\delta^2) + O(\delta\,h)) \wedge (f_4'(\varphi) + h\,f_5'(\varphi) + O(h^2)).$$

But $\delta = A\,h + O(h^2)$, $f_4' \wedge f_5' = 0$,

$$\mathcal{T}(\varphi, h, \delta) = \delta\,f_4''(\varphi) \wedge f_4'(\varphi) + O(h^2) = A\,f_4''(\varphi) \wedge f_4'(\varphi)\,h + O(h^2). \tag{3.7}$$

This quantity is nonzero for $0 < h$, h small, φ close to zero.

Characterization of $Self_c$.

We assume now that $\omega_b = 0$. We are looking again for solutions close to zero of the equation :

$$f_4(\varphi+\delta) - f_4(\varphi) + h\,(f_5(\varphi+\delta+\pi) - f_5(\varphi+\pi)) + h\,(f_5(\varphi+\pi) - f_5(\varphi)) + O(h^2) = 0,$$

but now, $f_4'(0) = 0$ (cusp points).

$$\begin{aligned} 0 \;=\; & \delta\,f_4'(\varphi) + \tfrac{\delta^2}{2} f_4''(\varphi) + \tfrac{\delta^3}{6} f_4'''(\varphi) - h\,\delta\,f_5'(\varphi) - 2h\,f_5(\varphi) + O(\delta^4) + O(h\,\delta^2) \\ & + O(h^2) = T_{IVc}. \end{aligned}$$

Now, the point is that $f_4'''(0) \wedge f_4''(0) = -64800\,\tilde{b}^2\,\pi^2$. Hence, to solve $T_{IVc} = 0$, it is equivalent to solve :

$$\begin{cases} T_{IVc} \wedge f_4'''(0) = 0 \\ T_{IVc} \wedge f_4''(0) = 0 \end{cases}$$

$$\begin{aligned} T_{IVc} = {} & \delta\,f_4'(0) + \delta\,\varphi\,f_4''(0) + \tfrac{\delta}{2}\varphi^2 f_4'''(0) + O(\delta\varphi^3) + \tfrac{\delta^2}{2}\,f_4''(0) \\ & + \tfrac{\delta^2}{2}\varphi\,f_4'''(0) + O(\delta^2\varphi^2) + \tfrac{\delta^3}{6} f_4'''(0) + O(\delta^3\varphi)h\delta\,f_5'(0) \\ & + O(h\delta\varphi) - 2hf_5(0) - 2\,h\,\varphi\,f_5'(0) + O(h\varphi^2) + O(\delta^4) + O(h\,\delta^2) + O(h^2) = 0. \end{aligned}$$

$f_4'(0) = 0$, hence :

$$T_{IVc} \wedge f_4''(0) = (\tfrac{\delta}{2}\varphi^2 + \tfrac{\delta^3}{6} f_4'''(0) \wedge f_4''(0) + \tfrac{\delta^2}{2}\varphi f_4'''(0) \wedge f_4''(0) - 2h\, f_5(0) \wedge f_4''(0)$$
$$+O(h\delta) + O(h\,\varphi) + O(\delta\varphi^3) + O(\delta^4) + O(\delta^3\varphi)O(\varphi^2\,\delta^2) + O(h^2) = 0. \tag{3.8}$$

$$T_{IVc} \wedge f_4'''(0) = \delta\,\varphi\, f_4''(0) \wedge f_4'''(0) + \tfrac{\delta^2}{2} f_4''(0) \wedge f_4'''(0) - 2h f_5(0) \wedge f_4'''(0) +$$
$$O(h\varphi) + O(\delta\varphi^3) + O(\delta^4) + O(h^2) + O(h\delta) + +O(\delta^3\varphi) + O(\delta^2\varphi^2) = 0. \tag{3.9}$$

It follows from the implicit function theorem applied to (3.8) that :

$$h = \frac{f_4'''(0) \wedge f_4''(0)}{2\, f_5(0) \wedge f_4''(0)} \left(\frac{\delta}{2}\varphi^2 + \frac{\delta^3}{6} + \frac{\delta^2}{2}\varphi\right) + \delta^2\, O^2(\delta,\, \varphi) + O(\delta\,\varphi^3). \tag{3.10}$$

(Again, $f_5(0) \wedge f_4''(0)$ has to be nonzero :

$$\frac{d}{d\varphi}(f_4' \wedge f_5)(0) = f_4''(0) \wedge f_5(0),$$

and zero is not a root of $P(\varphi)$ by Theorem (3.2), *ii*), *b*).

We can replace (3.10) in (3.8), (3.9) to get :

$$\begin{cases} \varphi = -\frac{\delta}{2} + O(\delta^2), \\ h = \frac{f_4'''(0) \wedge f_4''(0)}{48\, f_5(0) \wedge f_4''(0)}\delta^3 + O(\delta^4). \end{cases} \tag{3.11}$$

Note : Since h has to be positive, we see that one half only of this curve works : either $\delta > 0$ or $\delta < 0$.

We have now to show the transversality of the self-intersection in that case. For this, again, the following expression $\mathcal{T}(\varphi, h, \delta)$ has to be nonzero on the self-intersection.

$$\mathcal{T}(\varphi, h, \delta) = (f_4'(\varphi + \delta) + h\, f_5'(\varphi + \delta + \pi)) \wedge (f_4'(\varphi) + h\, f_5'(\varphi) + O(h^2)). \tag{3.12}$$

Again, expanding this expression and taking into account (3.11), we obtain, after some straightforward computations :

$$\mathcal{T}(\varphi, h, \delta) = -\frac{\delta^3}{8} f_4'''(0) \wedge f_4''(0) + O(\delta^4) \tag{3.13}$$

This shows that this self-intersection is always transversal.

It follows now from standard arguments of singularity theory that, since the conjugate locus CL is now "in general position", the map $h^4\, f_4(\varphi) + h^5\, f_5(\varphi) + O(h^6)$ is R.L-equivalent, for $h > 0$, small enough, to its jet $h^4\, f_4(\varphi) + h^5\, f_5(\varphi)$. Therefore, we can now state our first result (in a neighbourhood of the singular locus "at the source" $\mathcal{S}$) :

THEOREM 3.3. *There is an open dense subset E of SubR (in the Whitney topology), for which :*

On the smooth curve $\mathcal{C}$ where the fundamental invariant Q_2 vanishes, there is an open dense set O (complement of a discrete subset) on which :

$$Self = Self_c \cup Self_1,$$

$Self_c$ is the union of three curves, each of them satisfying equation (3.11). $Self_1$ is the union of 2 curves or 4 curves, each of them satisfying equation (3.6). All self-intersections are transversal. $Self_+$ can be different from $Self_-$ (among the above possibilities), and there is a neighbourhood U of $\mathcal{S} \cap \{0 < h = \sqrt{\epsilon \frac{w}{\pi}} < a, \epsilon = \pm 1\}$, (a sufficiently small), such that the restriction of the exponential mapping $\mathcal{E}|_U$ is 5-determined in h. (It is also determined by the 5-jet of the metric).

We show now the 2 pictures : figure (3.1), figure 3.2, showing the generic conjugate locus on an open dense set of $\mathcal{C}$.

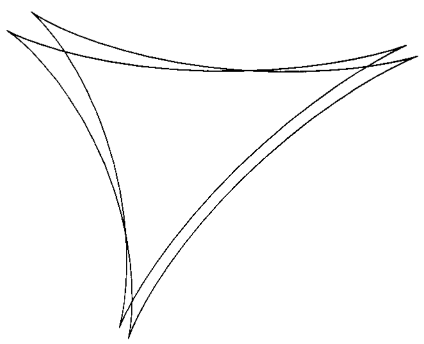

Figure 3.1: 5 self-intersection lines, $S_3 = (0, 1, 1, 1, 1, 1)$.

3.4.2. Case 2. Isolated points on $\mathcal{C}_2$ corresponding to double roots of $\widetilde{P}$ or common simple roots of $\widetilde{P}$ and T (collision between $Self_c$ and $Self_1$).

Points of $Self_1$ corresponding to double roots of $\widetilde{P}(z)$.(The most complicated case).

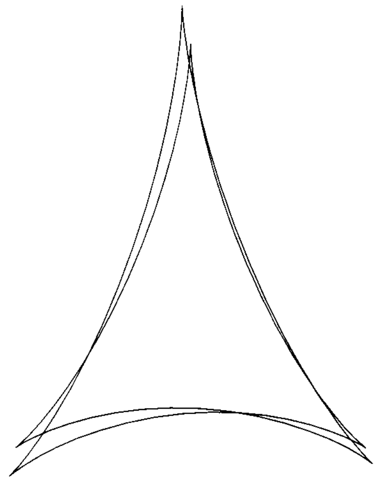

Figure 3.2: 7 self-intersection lines, $S_2 = (2, 1, 1, 1, 1, 1)$.

We already know that the two other roots are simple and on the unit circle. No root coincides with a cusp point.

The equation for the self-intersections is :

$$0 = f_4(\varphi+\delta) - f_4(\varphi) + h\,(f_5(\varphi+\delta+\pi) - f_5(\varphi+\pi)) - 2\,h\,f_5(\varphi)$$
$$+h^2(f_6(\varphi+\pi+\delta) - f_6(\varphi)) + +h^3\,(f_7(\varphi+\pi+\delta) \tag{3.14}$$
$$-f_7(\varphi+\pi)) + h^3\,(f_7(\varphi+\pi) - f_7(\varphi)) + h^4\,R(\varphi,\,\delta,\,h).$$

But now, by (3.2).2 section 3.1, we also know that $f_6(\varphi+\pi) = f_6(\varphi)$ (this is the reason why this term will be unefficient , and that we have to consider a high order jet) and $f_7(\varphi+\pi) = -f_7(\varphi)$. Using this, (3.14) can be rewritten :

$$0 = f_4(\varphi+\delta) - f_4(\varphi) - h\,(f_5(\varphi+\delta) - f_5(\varphi)) - 2\,h\,f_5(\varphi) + h^2(f_6(\varphi+\delta) - f_6(\varphi)) -$$
$$h^3\,(f_7(\varphi+\delta) - f_7(\varphi)) - 2\,h^3\,f_7(\varphi) + O(h^4). \tag{3.15}$$

Expanding in δ gives :

$$0 = \delta\,f_4'(\varphi) + \tfrac{\delta^2}{2}\,f_4''(\varphi) + \tfrac{\delta^3}{6}\,f_4'''(\varphi) + O(\delta^4) - h\,\delta\,f_5'(\varphi) - h\tfrac{\delta^2}{2}\,f_5''(\varphi) + O(h\,\delta^3) -$$
$$2\,h\,f_5(\varphi) + h^2\,\delta\,f_6'(\varphi) + O(h^2\delta^2) + O(h^3\delta) - 2\,h^3 f_7(\varphi) + O(h^4) = E_{IV}. \tag{3.16}$$

Now, as consequences of the fact that $\varphi = 0$ is a double root of $P(\varphi)$ (not triple by Theorem 3.2, *iii*), *a*.1), we get :

$$f_5(0) = 0, \tag{3.17}$$

(because $f_4'(0) \wedge f_5(0) = 0$ and $0 = f_4''(0) \wedge f_5(0) = \frac{d}{d\varphi}(f_4' \wedge f_5)(0)$ and $f_4'(0) \wedge f_4''(0) \neq 0$).

$$\begin{cases} 1. \;\; f_4''(0) \wedge f_5''(0) \neq 0, \\ 2. \;\; f_4'(0) \wedge f_7(0) \neq 0, \;\; \text{(Theorem 3.2, } iii), \; a.2). \end{cases} \tag{3.18}$$

(See also conditions 3.2 5–6, and definition of $\widehat{S}^6$, S^6 in section 3.3.3).

By (3.18).1, to solve equation (3.16) : $E_{IV} = 0$, it is equivalent to solve :

$$\begin{cases} 1. \;\; E_{IV} \wedge f_4''(0) = 0, \\ 2. \;\; E_{IV} \wedge f_5''(0) = 0. \end{cases} \tag{3.19}$$

Let us consider (3.19).1. It gives, using $f_5(0) = 0$ (3.17) :

$$0 = \delta\, f_4'(0) \wedge f_4''(0) - 2\, h\, \varphi\; f_5'(0) \wedge f_4''(0) + O(h^2) + O(\delta\varphi^2) + O(\delta^2) + O(h\,\delta) + O(h\,\varphi^2).$$

Because $\varphi = 0$ is not a triple root of $f_4' \wedge f_5(\varphi)$, it follows that :

$$f_5'(0) \wedge f_4''(0) \neq 0 : \;\; \frac{d}{d\varphi}(f_4' \wedge f_5)(\varphi) = f_4'' \wedge f_5(\varphi),$$

(because $(f_4' \wedge f_5')(\varphi)$ is identically zero),

$$\frac{d^2}{d\varphi^2}(f_4' \wedge f_5)(0) = (f_4''' \wedge f_5)(0) + (f_4'' \wedge f_5')(0) \neq 0,$$

$f_5(0) = 0$ (by (3.17)), hence :

$$(f_4'' \wedge f_5')(0) \neq 0. \tag{3.20}$$

Then, the implicit function theorem implies that :

$$\begin{cases} \delta = 2\, \frac{f_5'(0) \wedge f_4''(0)}{f_4'(0) \wedge f_4''(0)}\, h\, \varphi + O(h\,\varphi^2) + O(h^2), \\ \delta = \chi\, h\, \varphi \; + O(h^2) + O(h\,\varphi^2). \end{cases} \tag{3.21}$$

Hence, let us set :

$$\delta = \chi\, h\, \varphi + \xi\, h^2 + h\,(a\,\varphi^2 + b\,h^2 + c\,h\,\varphi) + h\,O^3(h,\, \varphi) \tag{3.22}$$

Let us plug this in (3.16) and divide by h (using $f_5(0) = 0$) :

$$0 = (\chi\,\varphi + \xi\, h)\, f_4'(\varphi) + (a\,\varphi^2 + b\,h^2 + c\,h\,\varphi)\, f_4'(0) - \chi\, h\, \varphi\, f_5'(0) - 2\varphi f_5'(0) -$$

$$h^2\, \xi\, f_5'(0) - \varphi^2\, f_5''(0) - 2\, h^2\, f_7(0) + O^3(\varphi,\, h) = E_{IV}' \tag{3.23}$$

$E'_{IV} \wedge f''_4(0) = 0$ gives :

$$\xi h\, f'_4(\varphi) \wedge f''_4(0) + (a\,\varphi^2 + b\,h^2 + c\,h\,\varphi)\, f'_4(0) \wedge f''_4(0) - 2\frac{(f'_5(0)\wedge f''_4(0))^2}{f'_4(0)\wedge f''_4(0)} h\,\varphi -$$

$$h^2\,\xi\, f'_5(0) \wedge f''_4(0) - \varphi^2 f''_5(0) \wedge f''_4(0) - 2h^2 f_7(0) \wedge f''_4(0) + O^3(\varphi, h).$$

As a consequence,

$$\xi = 0,\ b = 2\,\frac{f_7(0) \wedge f''_4(0)}{f'_4(0) \wedge f''_4(0)},\ a = \frac{f''_5(0) \wedge f''_4(0)}{f'_4(0) \wedge f''_4(0)},\ c = 2\frac{(f'_5(0) \wedge f''_4(0))^2}{(f'_4(0) \wedge f''_4(0))^2}. \tag{3.24}$$

Now $E'_{IV} \wedge f''_5(0) = 0$ gives :

$$\chi\,\varphi\, f'_4(0) \wedge f''_5(0) + \varphi^2(\chi\, f''_4(0) \wedge f''_5(0) + a\, f'_4(0) \wedge f''_5(0)) + h^2(b\, f'_4(0) \wedge f''_5(0)$$

$$-2\, f_7(0) \wedge f''_5(0)) + h\,\varphi\,(c\, f'_4(0) \wedge f''_5(0) - \chi\, f'_5(0) \wedge f''_5(0)) - 2\,\varphi\, f'_5(0) \wedge f''_5(0)$$

$$+O^3(\varphi, h) = 0.$$

Let us now use the two following facts :

$$\begin{cases} 1.\ \ f'_5(\varphi) = r(\varphi)\, f'_4(\varphi), \quad \text{and} \\ 2.\ \ f'_4(\varphi) \wedge f''_5(\varphi) = f'_5(\varphi) \wedge f''_4(\varphi) = r(\varphi)\, f'_4(\varphi) \wedge f''_4(\varphi). \end{cases}$$

This gives : $\chi = 2\, r(0)$ in (3.21), and

$$\varphi^2\,(2\, r(0)\, f''_4(0) \wedge f''_5(0) + a\, r(0) f'_4(0) \wedge f''_4(0)) + h^2\,(b\, r(0)\, f'_4(0) \wedge f''_4(0) -$$

$$2 f_7(0) \wedge f''_5(0)) + h\,\varphi\,(c\, r(0)\, f'_4(0) \wedge f''_4(0) - 2r^3(0)\, f'_4(0) \wedge f''_4(0)) + O^3(\varphi, h) = 0.$$

Also, $c = 2\, r^2(0)$, therefore

$$\varphi^2\ r(0)\, f''_4(0) \wedge f''_5(0) + 2\, h^2\,(f_7(0) \wedge f''_4(0)\, r(0) - f_7(0) \wedge f''_5(0)) + O^3(\varphi, h) = 0.$$

Finally,

$$\varphi^2\ r(0)\, f''_4(0) \wedge f''_5(0) - 2\, h^2\, f_7(0) \wedge (f''_5(0) - r(0)\, f''_4(0)) + O^3(\varphi, h) = 0. \tag{3.25}$$

Also, $r(0) = \dfrac{f'_5(0) \wedge f''_4(0)}{f'_4(0) \wedge f''_4(0)}$. By (3.20), $r(0) \neq 0$.

Also,

$$f''_5(\varphi) = r(\varphi)\, f''_4(\varphi) + r'(\varphi)\, f'_4(\varphi).$$

$r'(0)$ is nonzero :

$$f''_5(0) \wedge f''_4(0) = r'(0)\, f'_4(0) \wedge f''_4(0),$$

and we know, by (3.18).1 that

$$f''_5(0) \wedge f''_4(0) \neq 0.$$

Therefore, our equation is

$$-\varphi^2 r(0) r'(0) f_4'(0) \wedge f_4''(0) - 2h^2 r'(0) f_7(0) \wedge f_4'(0) + O^3(h, \varphi) = 0,$$

or

$$A(\varphi, h) = \varphi^2 r(0)\ f_4'(0) \wedge f_4''(0) + 2h^2\ f_7(0) \wedge f_4'(0) + O^3(h, \varphi) = 0. \tag{3.26}$$

$f_7(0) \wedge f_4'(0)$ is also nonzero by (3.18).2.

We can apply the Malgrange preparation theorem to this equation :

$$A(\varphi, h) = U(\varphi, h)\, (\varphi^2 + a_0(h)\, \varphi + a_1(h)), \tag{3.27}$$

where U is a unit, $a_0(0) = a_1(0) = 0$.

$$\frac{\partial A}{\partial h} = \frac{\partial U}{\partial h} (\varphi^2 + a_0(h)\, \varphi + a_1(h)) + U\, (a_0'(h)\, \varphi + a_1'(h)).$$

But $\dfrac{\partial A}{\partial h}(0, 0) = 0$. Hence $a_1'(0) = 0$.

$$\frac{\partial^2 A}{\partial h \partial \varphi}(0, 0) = U(0, 0)\, a_0'(0).$$

But $\dfrac{\partial^2 A}{\partial h \partial \varphi}(0, 0) = 0$. This implies that $a_0'(0) = 0$. Hence,

$$a_0(h) = h^2\, \overline{a}_0(h).$$

$\dfrac{\partial^2 A}{\partial h^2}(0, 0) = U(0, 0)\, a_1''(0)$. But,

$$\frac{\partial^2 A}{\partial h^2}(0, 0) = 4\, f_7(0) \wedge f_4'(0).$$

Hence, $U(0, 0)\ a_1''(0) = 4\, f_7(0) \wedge f_4'(0)$.

$$\frac{\partial A}{\partial \varphi} = \frac{\partial U}{\partial \varphi} (\varphi^2 + a_0(h)\, \varphi + a_1(h)) + U\, (2\, \varphi + a_0(h)),$$

and

$$\frac{\partial^2 A}{\partial \varphi^2}(0, 0) = 2\, U(0, 0) = 2\, r(0)\, f_4'(0) \wedge f_4''(0).$$

Therefore, as a conclusion :

$$\begin{cases} a_0(h) = h^2\, \overline{a}_0(h), \\ a_1(h) = 2\, \frac{f_7(0) \wedge f_4'(0)}{r(0)\, f_4'(0) \wedge f_4''(0)} h^2 + O(h^3) = \theta\, h^2 + O(h^3). \end{cases}$$

It follows that the solutions of our equation are given by :

$$\varphi = \tfrac{1}{2}(-h^2\,\overline{a}_0(h) \pm \sqrt{h^4\,\overline{a}_0^2(h) - 4\theta\,h^2 + O(h^3)}),$$

$$\varphi = \tfrac{1}{2}(-h^2\,\overline{a}_0(h) \pm h\,\sqrt{-4\theta + O(h)}).$$

Therefore, if $\theta > 0$ there is no solution, and if $\theta < 0$,

$$\left\{\begin{array}{l} \varphi = \pm h\,\sqrt{-\theta} + O(h^2), \quad \theta = 2\,\frac{f_7(0)\wedge f_4'(0)}{r(0)\,f_4'(0)\wedge f_4''(0)}, \\ \delta = \pm 2\,r(0)\,\sqrt{-\theta}\,h^2 + O(h^3). \end{array}\right. \tag{3.28}$$

REMARK 4. *Since, if there is a double root for $\widetilde{P}$, the two other roots are on the unit circle (see F5, section 3.2), this shows that, in this case, the number of self-intersection lines of CL is again 5 or 7 (cannot be 3).*

To check transversality of the self-intersection, we have again to show that the following $\mathcal{T}(h)$ is nonzero, for small h :

$$\mathcal{T} = (f_4'(\varphi+\delta) + h\,f_5'(\varphi+\pi+\delta) + h^2 f_6'(\varphi+\delta) + h^3 f_7'(\varphi+\pi+\delta) + O(h^4))\wedge$$
$$(f_4'(\varphi) + h\,f_5'(\varphi) + h^2 f_6'(\varphi) + h^3 f_7'(\varphi)).$$

A straightforward computation shows that (taking 3.28 into account),

$$\mathcal{T} = -\,\delta\;f_4' \wedge f_4''(0) + O(h^3).$$

(The most strange case) Collision between $Self_c$ and $Self_1$.

In that case, we already know that, generically, all the other roots of $\widetilde{P}$ are simple. (There are 1 or 3 other roots on the unit circle). Moreover, (generically) there is one collision at most between roots of $\widetilde{P}$ and cusp points (Theorem 3.2, *iii*), *b*).

The equation of the self-intersection is given by :

$$f_4(\varphi+\delta) - f_4(\varphi) + h\,(f_5(\varphi+\delta+\pi) - f_5(\varphi+\pi)) + h\,(f_5(\varphi+\pi) - f_5(\varphi)) +$$
$$h^2\,(f_6(\varphi+\delta) - f_6(\varphi)) + O(h^3) = 0,$$

$$F_{IV} = \delta\,f_4'(\varphi) + \tfrac{\delta^2}{2} f_4''(\varphi) + \tfrac{\delta^3}{6} f_4'''(\varphi) + O(\delta^4) - h\,\delta\,f_5'(\varphi) - h\tfrac{\delta^2}{2} f_5''(\varphi) + O(h\,\delta^3) -$$
$$2\,h\,f_5(\varphi) + h^2\delta\,f_6'(\varphi) + O(h^3) + O(h^2\,\delta^2) = 0. \tag{3.29}$$

Now, $\varphi = 0$ is a cusp point and a root of $P(\varphi)$. $f_4'(0) = 0$. Also, 0 is not a double root of $P(\varphi)$ (by Theorem 3.2, *iii*, b). Therefore, $f_4''' \wedge f_5(0) + f_4'' \wedge f_5'(0) \neq 0$, but, $f_4' \wedge f_5'(\varphi) = 0$, $f_4'' \wedge f_5'(\varphi) + f_4' \wedge f_5''(\varphi) = 0$, hence, $f_4'' \wedge f_5'(0) = 0$. Hence, $f_4''' \wedge f_5(0) \neq 0$. As a conclusion:

$$\begin{cases} f_4'(0) = 0,\ f_4'' \wedge f_5(0) = 0,\ f_4'' \wedge f_5'(0) = 0, \\ f_4''' \wedge f_5(0) \neq 0. \end{cases} \tag{3.30}$$

In particular $f_5(0) \neq 0$.

Again, it is easily seen that $f_4'''(0) \wedge f_4''(0) \neq 0$.

Hence, it is equivalent to solve :

$$\begin{cases} F_{IV} \wedge f_4'''(0) = 0, \\ F_{IV} \wedge f_4''(0) = 0. \end{cases}$$

$$0 = F_{IV} \wedge f_4'''(0) = (\delta\varphi + \tfrac{\delta^2}{2}) f_4''(0) \wedge f_4'''(0) - h\,\delta\, f_5(0) \wedge f_4'''(0)$$

$$-2\,h\,f_5(0) \wedge f_4'''(0) - 2\,h\,\varphi\, f_5'(0) \wedge f_4'''(0) + O(h^2\delta) + O(h^3) + O(h\,\delta\varphi)$$

$$+O(\delta\varphi^2) + O(\delta^3) + O(h\,\delta^2) + O(h\,\varphi^2).$$

But, since $f_4''(0) \wedge f_5(0) = 0$ and $f_5(0) \neq 0$, then for some real $\lambda \neq 0$, $f_5(0) = \lambda f_4''(0)$:

$$0 = (\delta\varphi + \tfrac{\delta^2}{2} - 2\lambda h)\ f_4''(0) \wedge f_4'''(0) - 2 f_5'(0) \wedge f_4'''(0)\, h\,(\varphi + \tfrac{\delta}{2}) +$$

$$O(h^2\delta) + O(h^3) + O(h\,\delta\varphi) + O(\delta\varphi^2) + O(\delta^3) + O(h\,\delta^2) + O(h\,\varphi^2).$$

We can solve this last equation in h :

$$h = \frac{1}{2\,\lambda\, f_4''(0) \wedge f_4'''(0)} (\delta\,(\varphi + \frac{\delta}{2}) f_4''(0) \wedge f_4'''(0) - 2\,h\,(\varphi + \frac{\delta}{2})\, f_5'(0) \wedge f_4'''(0) + \delta O^2 + h\,O'^2).$$

$$h = \frac{1}{2\lambda}\delta\,(\varphi + \frac{\delta}{2}) + \delta\, O^2(\varphi, \delta) = \delta\, F(\varphi, \delta),\ F(0,0) = 0. \tag{3.31}$$

Let us replace this last expression in $F_{IV} \wedge f_4''(0) = 0$.

$$0 = \delta f_4'(\varphi) \wedge f_4''(0) + \tfrac{\delta^2}{2} f_4''(\varphi) \wedge f_4''(0) + \tfrac{\delta^3}{6}\, f_4'''(\varphi) \wedge f_4''(0)$$

$$-\delta^2\, F(\varphi, \delta)\, f_5'(\varphi) \wedge f_4''(0) - 2\,\delta\, F(\varphi, \delta)\, f_5(\varphi) \wedge f_4''(0) + \delta^3\, O(\varphi, \delta).$$

δ factors out, to give :

$$0 = \tfrac{\varphi^2}{2} f_4'''(0) \wedge f_4''(0) + \tfrac{\delta\varphi}{2} f_4'''(0) \wedge f_4''(0) + \tfrac{\delta^2}{6} f_4'''(0) \wedge f_4''(0) -$$

$$\delta F(\varphi, \delta)\, f_5'(0) \wedge f_4''(0) - 2\, F(\varphi, \delta)\, \varphi\, f_5'(0) \wedge f_4''(0) + O^3(\varphi, \delta).$$

and since $f_5^{'}(0) \wedge f_4^{''}(0) = 0$,

$$0 = \tfrac{1}{2}(\varphi^2 + \delta\varphi + \tfrac{\delta^2}{3})\, f_4^{'''}(0) \wedge f_4^{''}(0) + O^3(\varphi, \delta)$$

$$= \tfrac{1}{2}((\varphi + \tfrac{\delta}{2})^2 + \tfrac{\delta^2}{12})\, f_4^{'''}(0) \wedge f_4^{''}(0) + O^3(\varphi, \delta)$$

Dividing by $(\varphi + \dfrac{\delta}{2})^2 + \dfrac{\delta^2}{12}$ shows that $f_4^{'''}(0) \wedge f_4^{''}(0) = 0$, which is not true.

Hence, there is no self-intersection.

THEOREM 3.4. *Under the notations of Theorem 3.3, there are two types of isolated points on the curve $\mathcal{C}$:*

1. Points at which 1 root of $\widetilde{P}$ is double, the two others being simple (and on the unit circle). Then :

– $Self_c$ is again the union of 3 curves, each of them meeting equation (3.11).

–$Self_1$ is the union of 4 curves or 2 curves, two of them satisfy equation (3.6),

2 of them satisfy equation (3.28) (if $\theta > 0$, they vanish).

All self-intersections are transversal. This can happen either for $Self_+$ or for $Self_-$ and one of them only. There is a neighbourhood U of $\mathcal{S} \cap \{a > h = \sqrt{\epsilon \frac{w}{\pi}} > 0\}$, a small, such that the restriction of the exponential mapping $\mathcal{E}|_U$ is 7-determined in h. (It is also determined by the 7-jet of the metric).

2. Other points are such that $Self_+$ or $Self_-$ (one only among them) are as follows :

– $Self_c$ is the union of 2 curves, each of them meeting the equation (3.11).

–$Self_1$ is the union of 3 or 1 curve, each of them meeting the equation (3.6).

All self-intersections are transversal, $\mathcal{E}|_U$ is 5-determined in h (It is also determined by the 5^{th} jet of the metric).

The corresponding pictures are shown below, figure 3.3–7, showing the generic conjugate loci at isolated points of $\mathcal{C}$.

- Double roots of $\widetilde{P}$.
  - Figure (3.3), figure (3.4) : 7^{th} jet in h of the conjugate locus.
  As already noticed, the symbol $S_2^* = S_2 = (2, 1, 1, 1, 1, 1)$.
  The pictures (3.2), (3.3) are smoothly equivalent. But they are not equivalent in a metric sense.
  - Figure (3.5) : 6^{th} jet in h of the conjugate locus. The 6^{th} jet in h of the metric does not determine $\mathcal{E}|_U$, one checks that self-intersections are not "in general position".
- Collision between cusps and roots of $\widetilde{P}$.
  - Figure (3.6) : 5 self-intersection lines.
  - Figure (3.7) : 3 self-intersection lines.

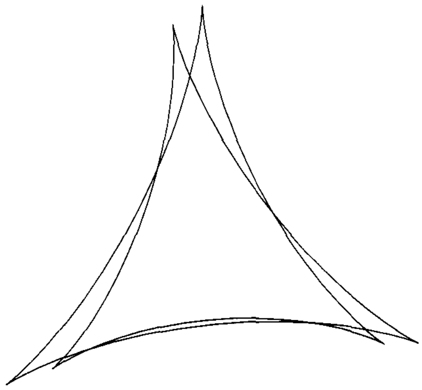

Figure 3.3: 7 self-intersection lines, $S_2^* = S_2 = (2, 1, 1, 1, 1, 1)$.

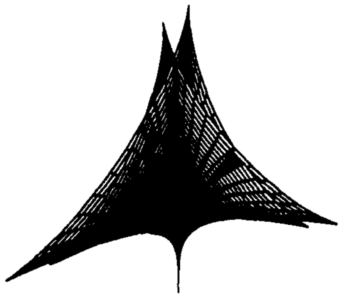

Figure 3.4: 7 self-intersection lines.

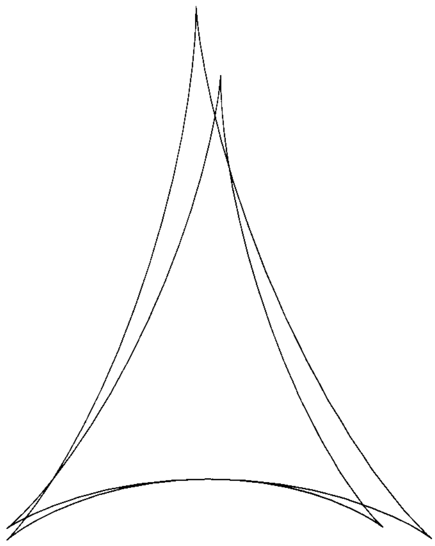

Figure 3.5:

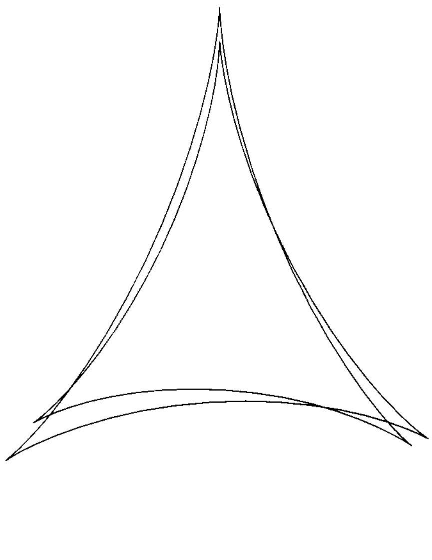

Figure 3.6: 5 self-intersection lines, $S_5 = (1, \frac{1}{2}, \frac{1}{2}, 1, 1, 1)$.

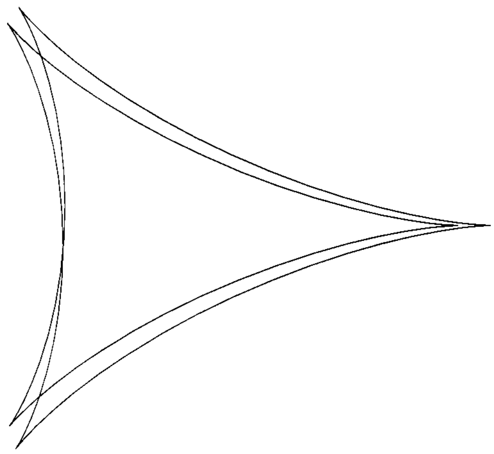

Figure 3.7: 3 self-intersection lines,$S_4 = (\frac{1}{2}, \frac{1}{2}, 1, 0, 0, 1)$.

3.5. Possible symbols for "semi conjugate loci".

3.5.1. Preliminaries.

We know, from section 3.4, that generically, there are two types of cusp points:

- Regular cusp points: they are adherent to the connected component of infinity of the complement of $CL_w^{\pm}$ (see the introduction for the definition of $CL_w^{\pm}$),
- Interior cusp points: they are not adherent to this connected component.

Moreover, there is **one interior cusp point at most**.

CLAIM 3.5. *a- The number of the self-intersections between two successive regular cusp points is even,*
b- Let α_1, α_2 be the two preimages of a point $d \in Self_c$. Let c_1, c_2 be the two preimages of the corresponding cusp points. α_1 and α_2 belong to the same half circle defined by c_1 and c_2.

PROOF. (b) follows from formulas (3.2).2 and 3.11.
(a) is obvious. □

In the two next sections 3.5.2, 3.5.3, we will show that the distributions of roots of the polynomials $\widetilde{P}$ between cusp points are not arbitrary. We will give a list of all possible distributions.
A simple reasonning (left to the reader) using claim 3.5, shows that it corresponds a unique symbol to each of these possible distributions of roots.

3.5.2. Codimension 2 cases (complement of a discrete subset of $\mathcal{C}$).

$\widetilde{P}$ has two roots on the unit circle.

CLAIM 3.6. *If P has only two roots b_1, b_2 on the unit circle, then* $|b_1 - b_2| \geq \frac{2\pi}{3}$.

PROOF. $q(z) = e^{i\omega}(z - \lambda e^{i\alpha})(z - \frac{1}{\lambda} e^{i\alpha})(z - e^{ib_1})(z - e^{ib_2})$. For $q(z)$ be of the form $\mu z^4 + \nu z^3 + c z^2 + \overline{\nu} z + \overline{\mu}$, it is necessary and sufficient that

$$\omega = \alpha + \frac{b_1 + b_2}{2} + k\pi \quad \text{(trivial computations)}.$$

Now, $c = 0$ if and only if

$$\lambda + \frac{1}{\lambda} = -\frac{\cos(\alpha - \frac{b_1+b_2}{2})}{\cos(\frac{b_1-b_2}{2})} \quad \text{(trivial computations again)}.$$

Assume that $|b_1 - b_2| < \frac{2\pi}{3}$, then $\frac{1}{2} < \cos(\frac{b_1-b_2}{2}) \leq 1$ and $\lambda + \frac{1}{\lambda} < 2$. This is impossible for $\lambda > 0$. □

Hence, the only possible symbol is S_3 is that case. Actually, it is the case of figure 3.1.

$\widetilde{P}$ has 4 roots on the unit circle.

CLAIM 3.7. *If P has four roots α_1, α_2, α_3, α_4, then for any $\alpha \in S_1$ (the circle) :*
a) $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \not\subset]\alpha, \alpha + \frac{2\pi}{3}[$,
b) $\{\alpha_1, \alpha_2\} \subset]\alpha, \alpha + \frac{2\pi}{3}]$ *and* $\{\alpha_3, \alpha_4\} \subset [\alpha + \frac{2\pi}{3}, \alpha + \frac{4\pi}{3}[$, *is impossible.*

PROOF. a) Assume that $-\frac{\pi}{3} < \alpha_1 \leq \alpha_2 \leq \alpha_4 \leq \alpha_3 < \frac{\pi}{3}$, $e^{i\alpha_k}$, $k = 1, \ldots, 4$ are roots of $\widetilde{P}$. In that case, a direct computation as in claim 3.6 shows that α_4 is determined by the relation :

$$e^{i\alpha_4} = -\frac{e^{i(\alpha_1+\alpha_2)} + e^{i(\alpha_1+\alpha_3)} + e^{i(\alpha_2+\alpha_3)}}{e^{i\alpha_1} + e^{i\alpha_2} + e^{i\alpha_3}} = \frac{N}{D} \tag{3.32}$$

Clearly, D never vanishes.

Let us leave α_1, α_2 fixed and decrease continuously α_3. Then α_4 moves continuously. A collision happens producing a double root. We can now perturb this double root into two roots out of the unit circle. The two other roots remain in the interval $]-\frac{\pi}{3}, \frac{\pi}{3}[$. But this is impossible by our claim 3.6.

b) Assume now that the four roots are $e^{i\,\alpha_k}$, $-\frac{2\,\pi}{3} < \alpha_1 \leq \alpha_2 \leq 0 \leq \alpha_4 \leq \alpha_3 < \frac{2\,\pi}{3}$. Again, in 3.32, D cannot vanish (D vanishes only for α_1, α_2, $\alpha_3 = \alpha_0 + \frac{2\,k\,\pi}{3}$).

First, let us leave α_1, α_2 fixed and decrease α_3 up to the moment when : either α_3 meets α_4 or $\alpha_4 = 0$. In the first case, proceeding exactly as in (a) above shows that this cannot happen. Now, on the same way, let us leave α_3, $\alpha_4 = 0$ fixed and increase α_1. For the same reason as in (a), it is impossible that α_2 meets α_1 or α_3. Then α_1 reaches α_4 and α_2, $\alpha_3 \in [0, \frac{2\pi}{3}[$. This is impossible. □

This shows that the two only possible symbols in that case are S_1, S_2. Actually, S_2 corresponds to figure 3.2. It is easily seen that S_1 can be obtained by a small perturbation of the polynomial

$$\widetilde{P}_0(z) = \frac{i}{2}\,z^4 + z^3 + z - \frac{i}{2}.$$

Even by computer, it is not easy to draw it.

3.5.3. Codimension 3 cases (Isolated points on $\mathcal{C}$).

Double roots of $\widetilde{P}$. In that case, clearly, it cannot happen that the two other roots are between the same two cusps, because, by small perturbation, we go to a situation forbidden by claim 3.6. Hence the only possibilities are :

$$S_1^* = S_1, \qquad S_2^* = S_2.$$

S_2 corresponds to figures 3.3, 3.4, 3.5. S_1 also can be obtained by considering a small perturbation of $\widetilde{P}_0(z)$.

Collision between cusps and roots of $\widetilde{P}$. a) $\widetilde{P}$ has four roots . In that case, the distribution of roots could be as follows (assuming that the cusp point is $\alpha_1 = 0$):

1. $\alpha_2 \in]-\frac{2\pi}{3}, 0[$, α_3, $\alpha_4 \in]\frac{2\pi}{3}, \frac{4\pi}{3}[$,

2. $\alpha_2 \in]0, \frac{2\pi}{3}[$, α_3, $\alpha_4 \in]\frac{2\pi}{3}, \frac{4\pi}{3}[$.

But this is impossible by claim 3.7 because $(\alpha_1, \alpha_2) \in]-\frac{2\pi}{3}+\epsilon, \epsilon[$, $(\alpha_3, \alpha_4) \in]\frac{2\pi}{3}+\epsilon, \frac{4\pi}{3}-\epsilon[$ for a small ϵ.

3. $\alpha_2 \in]0, \frac{2\pi}{3}[$, α_3, $\alpha_4 \in]-\frac{2\pi}{3}, 0[$.

Again, by claim 3.7, this is impossible, considering, for a small ϵ, the intervals $]-\frac{2\pi}{3}-\epsilon, -\epsilon[$, $]-\epsilon, \frac{2\pi}{3}-\epsilon[$.

4. α_2, α_3, $\alpha_4 \in]-\frac{2\pi}{3}, 0[$ or α_2, α_3, $\alpha_4 \in]0, \frac{2\pi}{3}[$.

By small perturbation of the interval, the 4 roots are between the same two cusps. This is impossible by claim 3.7.

5. $\alpha_2 \in]0, \frac{2\pi}{3}[$, $\alpha_3 \in]\frac{2\pi}{3}, \frac{4\pi}{3}[$, $\alpha_4 \in]-\frac{2\pi}{3}, 0[$.

This is the case corresponding to symbol S_5 and to figure 3.6.

$$6.\ \alpha_2 \in]\frac{2\pi}{3}, \frac{4\pi}{3}[,\ \alpha_3,\ \alpha_4 \in] - \frac{2\pi}{3}, 0[\text{ or }]0, \frac{2\pi}{3}[.$$

It corresponds to symbol S_6, and again can be obtained by mean of a small perturbation of the polynomial $\widetilde{P}_0(z)$.

$$7.\ \alpha_2,\ \alpha_3,\ \alpha_4 \in]\frac{2\pi}{3}, \frac{4\pi}{3}[.$$

It corresponds to the symbol S_7 and can be obtained in the same way as S_6.

b) $\widetilde{P}$ has two roots. If the cusp point is $\alpha_1 = 0$, then, α_2 belongs to $]\frac{2\pi}{3}, \frac{4\pi}{3}[$. If not, perturbing the interval contradicts claim 3.6. Therefore, the only possible symbol is S_4, and it corresponds to figure 3.7.

Theorem 1.1 is proved.

3.6. Possible full conjugate loci : $CL^+ \cup CL^-$.

The position of the cusp points is determined by the equation

$$\sin(3\,\varphi + \omega_b) = 0, \tag{3.33}$$

ω_b is related to $V_3 = \gamma^3_{3,3}$ and has been defined in section 2.5. Hence, the position of the cusp points is the same for CL^+ and CL^-.

Otherwise,

$$\widetilde{P}^+(z) = \mu\, z^4 + \nu^+\, z^3 + \overline{\nu}^+\, z + \overline{\mu},$$

$$\widetilde{P}^-(z) = \mu\, z^4 + \nu^-\, z^3 + \overline{\nu}^-\, z + \overline{\mu},$$

where $Re(\mu)$, $Im\,(\mu)$ and $Re(\nu^+)$, $Im(\nu^+)$, $Re(\nu^-)$, $Im(\nu^-)$ are independent linear combinations of the coefficients of γ^4_4, γ^4_2.

So, the problem of classification of possible full conjugate loci reduces to the problem of describing all possible couples of distribution of (unit length) roots of the polynomials $\widetilde{P}^+$, $\widetilde{P}^-$ in the 3 sectors $\left]0, \frac{2\pi}{3}\right[$, $\left]\frac{2\pi}{3}, \frac{4\pi}{3}\right[$, $\left]-\frac{2\pi}{3}, 0\right[$.

We don't adress this problem in its full generality.

What is clear, anyway, is that the full conjugate loci CL will be such that CL^+ is any among the codimension 2 possibilities (S_1, S_2, S_3), and CL^- is any among the codimension 2 and codimension 3 possibilities.

4. Appendices.

4.1. Appendix A_1: Computing the exponential mapping.

For details, we refer to our previous papers [9], [4].

Geodesics are trajectories of the Hamiltonian vector field $\overrightarrow{H}$ associated to the Hamiltonian $H(\psi)$ on T^*X ($\pi_X : T^*X \to X$)

$$H(\psi) = \frac{1}{2}\left(\psi(F)^2 + \psi(G)^2\right).$$

The metric (F, G) is in normal form (coordinates in $X : \xi = (x, y, w) = (z, w)$)

$$F = (1 + y^2\beta)\tfrac{\partial}{\partial x} - xy\beta\tfrac{\partial}{\partial y} + \tfrac{y}{2}(1+\gamma)\tfrac{\partial}{\partial w},$$

$$G = (1 + x^2\beta)\tfrac{\partial}{\partial y} - xy\beta\tfrac{\partial}{\partial x} - \tfrac{x}{2}(1+\gamma)\tfrac{\partial}{\partial w},$$

$$\beta(0, w) = \gamma(0, w) = \tfrac{\partial\gamma}{\partial z}(0, w) = 0.$$

Coordinates in the cotangent bundle are $(\widetilde{p}, \widetilde{q}, r) = (\widetilde{\zeta}, r)$. $(x, y, w, \widetilde{p}, \widetilde{q}, r)$ have weight $1, 1, 2, -1, -1, -2$ respectively.

$$H^{-1}(\frac{1}{2}) \cap \pi_X^{-1}(0) = \{\widetilde{p}^2(0) + \widetilde{q}^2(0) = 1\}.$$

We set $\widetilde{p}(0) = \cos\varphi$, $\widetilde{q}(0) = \sin\varphi$. For $r(0) \neq 0$, we set $\rho = \dfrac{1}{r(0)}$, $p = \dfrac{\widetilde{p}}{r}$, $q = \dfrac{\widetilde{q}}{r}$, $\zeta = (p, q)$. s denotes the arclength and t the new time : $dt = r(s)\,ds$. p, q have weight 1.

One has, for $r(0) \neq 0$ and s small :

$$\frac{dz}{ds} = r(s)\,\frac{\partial H}{\partial \widetilde{\zeta}}|_{r=1}, \quad \text{and} \quad \frac{d}{ds}(\frac{\widetilde{\zeta}}{r}) = -r(s)\,\frac{\partial H}{\partial z}|_{r=1} + r(s)\,\frac{\widetilde{\zeta}}{r}\,\frac{\partial H}{\partial w}|_{r=1}. \tag{4.1}$$

Or,

$$\tfrac{dz}{dt} = \tfrac{\partial H}{\partial \widetilde{\zeta}}|_{r=1}, \qquad \tfrac{d\zeta}{dt} = -\tfrac{\partial H}{\partial z}|_{r=1} + \zeta\,\tfrac{\partial H}{\partial w}|_{r=1}. \tag{4.2}$$

For all k, (4.2) can be rewritten :

$$\frac{d(z, \zeta)}{dt} = A(z, \zeta) + \sum_{i=3}^{k} F_i(z, \zeta, w) + O^{k+1}(z, \zeta, w), \tag{4.3}$$

where F_i is homogeneous of degree i, where A is a linear operator (corresponding to the Heisenberg sub-Riemannian metric), and where $O^{k+1}(z, \zeta, w)$ has order $(k+1)$ with respect to the gradation :

$$\deg x = \deg y = \deg p = \deg q = 1, \ \deg w = 2.$$

Also, $\dfrac{dw}{ds} = \dfrac{\partial H}{\partial r} = r(s)\,\dfrac{\partial H}{\partial r}|_{r=1}$,

$$\frac{dw}{dt} = \frac{\partial H}{\partial r}|_{r=1}. \tag{4.4}$$

This can be rewritten :

$$\frac{dw}{dt} = G_2(z,\,\zeta) + \sum_{i=4}^{k} G_i(z,\,\zeta,\,w) + O^{k+1}(z,\,\zeta,\,w),, \tag{4.5}$$

where G_i are homogeneous of degree i, where G_2 corresponds to the Heisenberg sub-Riemannian metric, and where $O^{k+1}(z,\,\zeta,\,w)$ has order $(k+1)$ w.r.t. the gradation.

Initial conditions are

$$z(0) = 0,\ w(0) = 0,\ \zeta(0) = (\rho\cos\varphi,\ \rho\sin\varphi). \tag{4.6}$$

Therefore,

$$\begin{cases} (z,\,\zeta) = \rho\,(z_1(t,\,\varphi),\,\zeta_1(t,\,\varphi)) + \sum\limits_{i=3}^{k} \rho^i\,(z_i(t,\,\varphi),\,\zeta_i(t,\,\varphi)) + O(\rho^{k+1}), \\ w = \rho^2\,w_2(t,\,\varphi) + \sum\limits_{i=4}^{k} \rho^i\,w_i(t,\,\varphi) + O(\rho^{k+1}). \end{cases} \tag{4.7}$$

$$(z_1,\,\zeta_1)(t,\,\varphi) = e^{A\,t}\,(z(0),\,\zeta(0)),\ w_2(t,\varphi) = \int_0^t G_2(z_1(\tau,\varphi),\,\zeta_1(\tau,\varphi))\,d\tau. \tag{4.8}$$

These last expressions can be easily computed. They give the exponential mapping for the Heisenberg metric :

$$\begin{cases} z_1(t,\,\varphi) = 2\,\sin(\tfrac{t}{2})\,\left(\cos(\varphi - \tfrac{t}{2}),\,\sin(\varphi - \tfrac{t}{2})\right), \\ \zeta_1(t,\,\varphi) = \cos(\tfrac{t}{2})\,\left(\cos(\varphi - \tfrac{t}{2}),\,\sin(\varphi - \tfrac{t}{2})\right), \\ w_2(t) = \tfrac{1}{2}(t - \sin t). \end{cases} \tag{4.9}$$

Also F_3 and G_4 don't depend on w, hence, setting $\Lambda = (z,\,\zeta)$,

$$\begin{cases} \Lambda_3(t,\varphi) = (z_3,\zeta_3)(t,\varphi) = \int_0^t e^{A\,(t-\tau)}\,F_3(\Lambda_1(\tau,\varphi))\,d\tau, \\ w_4(t,\varphi) = \int_0^t \left(\frac{\partial G_2}{\partial\Lambda}(\Lambda_1(\tau,\varphi))\,.\,\Lambda_3(\tau,\,\varphi) + G_4(\Lambda_1(\tau,\varphi))\right)\,d\tau. \end{cases} \tag{4.10}$$

The following terms are easily computed, in the same way. We give the expressions that we shall need. F_4, G_5 don't depend on w,

$$\begin{cases} \Lambda_4(t,\varphi) = (z_4,\zeta_4)(t,\varphi) = \int_0^t e^{A\,(t-\tau)}\,F_4(\Lambda_1(\tau,\varphi))\,d\tau, \\ w_5(t,\varphi) = \int_0^t \left(\frac{\partial G_2}{\partial\Lambda}(\Lambda_1(\tau,\varphi))\,\Lambda_4(\tau,\,\varphi) + G_5(\Lambda_1(\tau,\varphi))\right)\,d\tau. \end{cases} \tag{4.11}$$

$$\begin{cases} \Lambda_5(t,\varphi) = \int_0^t e^{A\,(t-\tau)}\,(F_5(\Lambda_1(\tau,\varphi),\, w_2(\tau)) + \frac{\partial F_3}{\partial \Lambda}(\Lambda_1(\tau,\varphi))\;\Lambda_3(\tau,\,\varphi))\,d\tau, \\ w_6(t,\varphi) = \int_0^t \frac{1}{2}\frac{\partial^2 G_2}{\partial \Lambda^2}\,.\,(\Lambda_3(\tau,\,\varphi),\Lambda_3(\tau,\,\varphi)) + G_6(\Lambda_1(\tau,\varphi),w_2(\tau)) + \\ \qquad \frac{\partial G_2}{\partial \Lambda}(\Lambda_1(\tau,\varphi))\,.\,\Lambda_5(\tau,\,\varphi) \;+\; \frac{\partial G_4}{\partial \Lambda}(\Lambda_1(\tau,\varphi))\,.\,\Lambda_3(\tau,\,\varphi))d\tau. \end{cases} \tag{4.12}$$

$$\begin{cases} \Lambda_6(t,\varphi) = \int_0^t e^{A\,(t-\tau)}\,\big(F_6(\Lambda_1(\tau,\varphi),w_2(\tau)) + \frac{\partial F_3}{\partial \Lambda}(\Lambda_1(\tau,\varphi))\,\Lambda_4(\tau,\,\varphi) \\ \quad + \frac{\partial F_4}{\partial \Lambda}(\Lambda_1(\tau,\varphi))\,\Lambda_3(\tau,\varphi)\big)\,d\tau, \\ \Lambda_7(t,\varphi) = \int_0^t e^{A\,(t-\tau)}\,(F_7(\Lambda_1(\tau,\varphi),\,w_2(\tau)) + \frac{\partial F_3}{\partial \Lambda}(\Lambda_1(\tau,\varphi))\,\Lambda_5(\tau,\,\varphi) \\ \quad + \frac{\partial F_4}{\partial \Lambda}(\Lambda_1(\tau,\varphi))\,\Lambda_4(\tau,\varphi) + \\ \frac{1}{2}\frac{\partial^2 F_3}{\partial \Lambda^2}(\Lambda_1(\tau,\varphi))\,(\Lambda_3(\tau,\varphi),\Lambda_3(\tau,\varphi)) + \frac{\partial F_5}{\partial \Lambda}(\Lambda_1(\tau,\varphi),\,w_2(\tau))\,\Lambda_3(\tau,\varphi) \\ \quad + \frac{\partial F_5}{\partial w}(\Lambda_1(\tau,\varphi))\,w_4(\tau,\varphi))\,d\tau, \end{cases} \tag{4.13}$$

As we shall see, we will need to compute these values for $t = 2\,\pi$ only.

4.2. Appendix $\mathbf{A}_2$: The exponential mapping in suspended form, the conjugate locus :

$$\mathcal{E}(t,\,\rho,\,\varphi) : \begin{cases} z(t,\rho,\,\varphi) = \rho\, z_1(t,\,\varphi) + \sum_{i=3}^{7} \rho^i\, z_i(t,\,\varphi) + O(\rho^8), \\ w(t,\rho,\,\varphi) = \rho^2\, w_2(t) + \sum_{i=4}^{6} \rho^i\, w_i(t,\,\varphi) + O(\rho^7). \end{cases}$$

Integral expressions of z_i, w_i were computed in our appendix 4.1.

Let us consider the variable $\epsilon = \pm 1$, according to $w > 0$ or $w < 0$. Let us set $t = 2\pi\epsilon + \tau$ (the conjugate new time has an expansion $t_c = 2\pi\,\epsilon + O(\rho^2)$, as is shown in our previous papers [9], [4] and this will appear again here in). Therefore, τ_c, the conjugate time τ, has order ρ^2.

$$w_2(t) = \frac{1}{2}(t - \sin t).\;\; w_2(\tau) = w_2^{\tau}(\tau) = \pi\,\epsilon + \frac{1}{2}(\tau - \sin \tau),$$

hence, $w_2^{\tau}(0) = \pi\,\epsilon$, $w_2^{\tau\,\prime}(0) = 0$, $w_2^{\tau\,\prime\prime}(0) = 0$. Therefore, we obtain the following **important fact** :

$$w_2(\tau) = \pi\,\epsilon + O(\tau^3) = w_2^{\tau}(0) + \tau^3\,\psi(\tau), \tag{4.14}$$

for some smooth function ψ.

Let us set :

$$h=\sqrt{\frac{\epsilon\, w}{\pi}}. \tag{4.15}$$

$$w(\tau,\,\rho,\,\varphi)=\pi\,\epsilon\,\rho^2(1+O(\tau^3)+\frac{1}{\pi\,\epsilon}(\rho^2\,w_4(\tau,\varphi)+\rho^3\,w_5(\tau,\varphi)+\rho^4\,w_6(\tau,\varphi)+O(\rho^5)).$$

$$h=\rho\,(1+O(\tau^3)+\tfrac{1}{\pi\,\epsilon}(\rho^2\,w_4(\tau,\varphi)+\rho^3\,w_5(\tau,\varphi)+\rho^4\,w_6(\tau,\varphi)+O(\rho^5)))^{\frac{1}{2}}.$$

Using the implicit function theorem, we can solve this last equation in ρ. After straightforward computations, we get :

$$\begin{aligned}\rho=h\,(1-h^2\tfrac{w_4^{\tau}(0)}{2\pi\epsilon}-h^3\tfrac{w_5^{\tau}(0)}{2\pi\epsilon}+h^4(\tfrac{7w_4^{\tau}(0)^2}{8\pi^2}-\tfrac{w_6^{\tau}(0)}{2\pi\epsilon})-w_4^{\tau'}(0)\tfrac{h^2\tau}{2\pi\epsilon}+\\ O(\tau^3)+O(h^5)+O(h^2\tau^2)+O(h^3\tau)\,).\end{aligned} \tag{4.16}$$

Let us set now

$$\tau=\theta\,h^2+\sigma\,h^3, \tag{4.17}$$

for some constant θ.

$$\rho=h\,(1-h^2\epsilon\frac{w_4^{\tau}(0)}{2\pi}-h^3\epsilon\frac{w_5^{\tau}(0)}{2\pi}+h^4(\frac{7w_4^{\tau}(0)^2}{8\pi^2}-\epsilon\frac{w_6^{\tau}(0)}{2\pi}-w_4^{\tau'}(0)\frac{\epsilon\,\theta}{2\pi})+O(h^5)). \tag{4.18}$$

On the other hand,

$$z(\tau,\,\rho,\,\varphi)=\rho\,z_1(\tau,\,\varphi)+\sum_{i=3}^{7}\rho^i\,z_i(\tau,\,\varphi)+O(\rho^8). \tag{4.19}$$

We set

$$z_i^{(k)}=\frac{d^kz_i}{d\tau^k},\quad w_i^{(k)}=\frac{d^kw_i}{d\tau^k}.$$

We also denote $z_i^{'}$ for $z_i^{(1)}$ etc....

Replacing (4.17), (4.18), in this expression (4.19), we get, after tedious computations :

$$z(\sigma,h,\varphi)=\sum_{i=3}^{7}A_i\,h^i+O(h^8), \tag{4.20}$$

$$\begin{cases} A_3=(\theta\,z_1^{'}+z_3)|_{t=2\pi\,\epsilon},\quad A_4=(\sigma\,z_1^{'}+z_4)|_{t=2\pi\,\epsilon},\\ A_5=(\frac{-\epsilon}{2\pi}(\theta\,w_4\,z_1^{'}+3\,w_4\,z_3)+\frac{\theta^2}{2}z_1^{''}+\theta\,z_3^{'}+z_5)|_{t=2\pi\,\epsilon}\\ A_6=\frac{1}{2\pi}(-\epsilon\,\sigma\,w_4z_1^{'}-\epsilon\,\theta w_5\,z_1^{'}+2\pi\sigma\theta z_1^{''}-3\epsilon\,w_5z_3+2\,\pi\sigma z_3^{'}-\\ 4\epsilon\,w_4\,z_4+2\,\pi\,\theta\,z_4^{'}+2\,\pi\,z_6)|_{t=2\pi\,\epsilon}=\sigma\,A_6^1+A_6^0,\\ A_7=\sigma^2\,A_7^2+\sigma\,A_7^1+A_7^0,\\ A_7^0=z_7|_{t=2\pi\,\epsilon}+\text{ terms not depending on } z_7.\ A_7^1,\ A_7^2 \text{ do not depend on } z_7.\end{cases} \tag{4.21}$$

As we know from our previous papers, the conjugate time τ_c is obtained for

$$\theta = -6\,\pi\,\epsilon\; tr_g(\gamma_2^2) \tag{4.22}$$

(where γ_2^2 is our quadratic invariant defined in section 2.5) : to see this it is sufficient to compute θ for

$$0 = z_1^{\tau'}(0,\varphi) \wedge (\theta \tfrac{\partial z_1^{\tau'}(0,\varphi)}{\partial \varphi} + \tfrac{\partial z_3^{\tau}(0,\varphi)}{\partial \varphi})$$

$$= z_1^{t'}(2\pi\epsilon,\varphi) \wedge (\theta \tfrac{\partial z_1^{t'}(2\pi\epsilon,\varphi)}{\partial \varphi} + \tfrac{\partial z_3^{t}(2\pi\epsilon,\varphi)}{\partial \varphi}),$$

but, $z_1^{t'}(2\pi\epsilon,\varphi) \wedge \dfrac{\partial z_1^{t'}(2\pi\epsilon,\varphi)}{\partial \varphi} = 1,\;\; z_1^{t'}(2\pi\epsilon,\varphi) \wedge \dfrac{\partial z_3^{t}(2\pi\epsilon,\varphi)}{\partial \varphi} = 6\,\pi\,\epsilon\, tr_g(\gamma_2^2).$

Also, we assume that $\gamma_{2,2}^2 = 0$ (on the curve $\mathcal{C}$, by definition, this invariant vanishes). In that case,

$$-6\,\pi\,\epsilon\, tr_g(\gamma^2)\; z_1^{t'}(2\pi\epsilon,\varphi) + z_3^{t}(2\pi\epsilon,\varphi) = 0,$$

as it is easily checked.

Therefore, at points of $\mathcal{C}$, we have the expression of the exponential mapping, in suspended form, in a certain neighbourhood U of the conjugate locus at the source, for h (or ρ) small enough

$$z(\sigma,\, h,\, \varphi) = \sum_{i=4}^{7} A_i\, h^i + O(h^8), \tag{4.23}$$

where A_i are given in (4.21).

We have to compute the expression of the conjugate σ_c in terms of h and φ. For this, we have to solve the following equation in σ :

$$\frac{\partial z}{\partial \sigma}(\sigma,\, h,\, \varphi) \wedge \frac{\partial z}{\partial \varphi}(\sigma,\, h,\, \varphi) = 0 \tag{4.24}$$

Solving this equation in σ, with the implicit function theorem gives :

$$\sigma_c = -(z_1^{'}(2\pi\epsilon) \wedge \tfrac{\partial z_4}{\partial \varphi}) - h\,(z_1^{'}(2\pi\epsilon) \wedge \tfrac{\partial A_5}{\partial \varphi}) + h^2(-(z_1^{'}(2\pi\epsilon) \wedge \tfrac{\partial A_6^0}{\partial \varphi}) - (A_6^1 \wedge \tfrac{\partial z_4}{\partial \varphi}(2\pi\epsilon))$$

$$+(z_1^{'} \wedge \tfrac{\partial z_4}{\partial \varphi})(2\pi\epsilon)\,(\,(A_6^1 \wedge \tfrac{\partial z_1^{'}}{\partial \varphi}(2\pi\epsilon)) + z_1^{'}(2\pi\epsilon) \wedge \tfrac{\partial A_6^1}{\partial \varphi})) + h^3(-z_1^{'}(2\pi\epsilon) \wedge \tfrac{\partial A_7^0}{\partial \varphi} +$$

$$\text{terms not depending on } A_7^0) + O(h^4). \tag{4.25}$$

It remains to replace this expression in the expression of z (4.23), to get the expansion of the conjugate locus :

$$z_c(h,\varphi) = z(\sigma_c, h, \varphi) = \sum_{i=4}^{7} f_i(\varphi)\, h^i + O(h^8).$$

$f_4(\varphi) = -z_1'(2\pi\epsilon)\,(z_1' \wedge \dfrac{\partial z_4}{\partial\varphi})(2\pi\epsilon,\,\varphi) + z_4(2\pi\epsilon,\,\varphi)$. This expression we have already computed in our previous papers :

$$f_4(\varphi) = (-5\pi\,\widetilde{a}\cos\omega_a + 15\pi\,\widetilde{b}\,\cos(2\varphi+\omega_b) + \tfrac{15}{2}\pi\,\widetilde{b}\,\cos(4\varphi+\omega_b),$$
$$-5\pi\,\widetilde{a}\,\sin\omega_a - 15\pi\,\widetilde{b}\,\sin(2\varphi+\omega_b) + \tfrac{15}{2}\pi\,\widetilde{b}\,\sin(4\varphi+\omega_b)) \qquad (4.26)$$

The other expressions can be computed in the same way, just replacing (4.25) in (4.23). This has been done with Mathematica.

For f_5, we get the expressions below (appendix 4.3), for $\epsilon = \pm 1$. It is not that complicated and it can be obtained also by the hand.

It requires to compute only

$z_1(t)$, $w_2(t)$, $w_4(2\pi\epsilon)$, $\frac{\partial w_4}{\partial\varphi}(2\pi\epsilon)$, $z_3(2\pi\epsilon)$, $\frac{\partial z_3}{\partial\varphi}(2\pi\epsilon)$, $z_3'(2\pi\epsilon)$, $\frac{\partial z_3'}{\partial\varphi}(2\pi\epsilon)$, $z_5(2\pi\epsilon)$

and $\frac{\partial z_5}{\partial\varphi}(2\pi\epsilon)$.

For f_6, we compute the expression, but this is not necessary. The only thing we need is to know that

$$f_6(\varphi+\pi) = f_6(\varphi). \qquad (4.27)$$

But, actually, this holds by proposition 3.1. and its corollary in [2].

For f_7 the term z_7 appears in A_7^0 (formula (4.21)), but also, it comes through the term $A_4\,h^4$ of $z(\sigma,\,h,\,\varphi)$ in (4.21) : $A_4 = (\sigma\,z_1' + z_4)\,(2\,\pi\,\epsilon)$.

Then,the expression (4.25) produces also the term

$$T = -(z_1'(2\,\pi\,\epsilon) \wedge \frac{\partial A_7^0}{\partial\varphi})\,z_1'(2\,\pi\,\epsilon)\,h^7.$$

This term will not play any role : the only thing that we need is property (3.2).5 in section 3.1 for $\dfrac{\partial f_4}{\partial\varphi} \wedge f_7(\varphi)$.

$$\frac{\partial f_4}{\partial\varphi} = -10\,\widetilde{b}\,\pi\,\sin(3\,\varphi+\omega_b)\,(\cos\varphi,\,\sin\varphi), \quad z_1'(2\pi\epsilon) = (\cos\varphi,\,\sin\varphi),$$

therefore, $\dfrac{\partial f_4}{\partial\varphi} \wedge T = 0$. Hence, $\dfrac{\partial f_4}{\partial\varphi} \wedge f_7(\varphi) = \dfrac{\partial f_4}{\partial\varphi} \wedge z_7(2\,\pi\,\epsilon) +$ other terms that do not depend on the invariants β_i^4 and γ_j^6.

Also, by the expression (4.13) of our appendix 4.1,

$$\frac{\partial f_4}{\partial\varphi} \wedge z_7(2\,\pi\,\epsilon) = \frac{\partial f_4}{\partial\varphi} \wedge \int_0^{2\pi\epsilon} e^{A(t-\tau)}\,F_7(\Lambda_1(\tau,\,\varphi),\,w_2(\tau))\,d\tau +$$
$$\text{terms not depending on } \beta_i^4 \text{ and } \gamma_j^6. \qquad (4.28)$$

This last expression only has been computed by Mathematica, it is given in the next appendix, which shows that it is a general polynomial $Q(\varphi)$, in $e^{2i\varphi}$, of degree 3, the coefficients of which are linear independent combinations of the components of γ_j^6.

4.3. Appendix A_3: The program.

It does compute the expansion of the conjugate locus in terms of h, φ; following the method explained in our appendices A_1, A_2.

It gives $f_4(\varphi)$, $f_5(\varphi)$, $f_6(\varphi+\pi)-f_6(\varphi)$ and

$$\dot{z_1'}(2\pi\epsilon, \varphi)\wedge\int_0^{2\pi\epsilon} e^{A(t-\tau)}\, F_7(\Lambda_1(\tau, \varphi), w_2(\tau))\, d\tau.$$

```
*** Expressions of functions BBeta and GGamma of our normal form
                                              (case Q2=0 and wb = 0)****

GGamma = Gamma2 + Gamma3 +Gamma4 +Gamma5 +Gamma6;BBeta = Beta1 + Beta2 + Beta3
Beta4;Gamma4 =Gamma44+w Gamma42;Gamma5 = Gamma55 + w Gamma53; Gamma6 = Gamma66
w Gamma64 + w^2 Gamma62;Gamma2 = to/2 (x^2 + y^2 );Gamma3 = -b y^3 + 3 b x^2 y
(x^2 + y^2) (- vv1 x + vv2 y);Gamma44 =L44 (x^2+y^2)^2+a44 (x^4+y^4-6 x^2 y^2)
 4 b44 x y (x^2-y^2)+ c44 (x^4-y^4)-2 d44 x y (x^2+y^2);Gamma42= L42 (x^2+y^2)
 a42 (x^2 - y^2) -2 b42 x y ; Gamma53=g531 x^3+g532 y^3+g533 x y^2 + g534 x^2 y
Gamma55 = g551 x^5 + g552 y^5 + g553 x y^4+g554 x^4 y+g555 x^2 y^3+g556 x^3 y^2
Gamma66 = g661 x^6 + g662 x^5 y + g663 x^4 y^2 + g664 x^3 y^3 + g665 x^2 y^4 +
g666 x y^5 + g667 y^6; Gamma64 = g641 x^4+g642 x^3 y+ g643 x^2 y^2+ g644 x y^3
g645 y^4; Gamma62 = g621 x^2 + g622 x y + g623 y^2; Beta1 = l1 x + l2 y; Beta2
L22 (x^2 + y^2) + a22 (x^2 - y^2) - 2 b22 x y; Beta3 = w (b311 x + b312 y) +
b331 x^3 + b332 y^3 + b333 x y^2 + b334 x^2 y; Beta4 = b441 x^4 + b442 x^3 y +
b443 x^2 y^2 + b444 x y^3 + b445 y^4 + w (b421 x^2 + b422 x y + b423 y^2);

**************************** Hamiltonian H ***********************************

H=(p (1+y^2 BBeta)-q x y BBeta+y/2 r(1+GGamma))^2/2+(-p x y BBeta+q(1+x^2 BBeta
x/2 r (1+GGamma))^2/2; rl1 = {r -> 1}; weight = {x ->h x, y->h y, p->h p,q ->h
w->h^2 w}; suppr=Flatten[Table[h^i->0,{i,8,30}]];Hx=D[H,p] /. rl1;Hy = D[H, q]
rl1;Hp =(- D[H, x] + p D[H, w] ) /. rl1; Hq = (- D[H,y] + q D[H, w]) /. rl1; Hw
D[H, r] /. rl1;  F = Collect[({Hx, Hy, Hp, Hq} /. weight) /. suppr , h] /. supp
G = Collect[(Hw /. weight) /. suppr , h] /. suppr; G1 = D[G, h]; G2 = D[G1, h]/
G3 = D[G2, h]/3;G4 = D[G3, h]/4; G5 = D[G4, h]/5; G6 = D[G5, h]/6;{G1, G2,G3, G
G5, G6} = {G1, G2, G3, G4, G5, G6} /. h -> 0; F1 = D[F, h]; F2 = D[F1, h]/2; F3
D[F2, h]/3; F4 = D[F3, h]/4; F5 = D[F4, h]/5 ; F6 = D[F5, h]/6; F7 = D[F6, h]/7
{F1, F2, F3, F4, F5, F6, F7} = {F1, F2, F3, F4, F5, F6, F7} /. h -> 0;
matrixA = rb {{0, 1/2, 1, 0}, {-1/2, 0, 0, 1},{-1/4, 0, 0,1/2},{0,-1/4,-1/2, 0}
expA = {{Cos[rb/2]^2, Sin[rb]/2, Sin[rb], 2*Sin[rb/2]^2},{-Sin[rb]/2,Cos[rb/2]^
 -2*Sin[rb/2]^2, Sin[rb]}, {-Sin[rb]/4, - Sin[rb/2]^2/2, Cos[rb/2]^2, Sin[rb]/2
{Sin[rb/2]^2/2, -Sin[rb]/4, -Sin[rb]/2,Cos[rb/2]^2}};condinit ={0,0,rho Cos[phi
rho Sin[phi]}; rl2 = {t -> s}; rl3 = {t -> 2 Pi};
```

```
******* exp. mapping in Heisenberg case ********************************

{x1, y1, p1, q1} = ((expA /. rb -> t) . condinit ) // Simplify;heis = {x-> x1,
y-> y1, p-> p1, q-> q1},w2=Simplify[Integrate[((G2 /. heis) /. r12),{s, 0, t}]];
heis = {x-> x1, y-> y1, p-> p1, q-> q1, w-> w2};

******* Approximations of order 3 (in rho) of x, y, p, q ********************

F3s=Simplify[((F3 /. heis) /. r12)]; {x3,y3, p3,q3}=Simplify[Integrate[(expA /.
rb-> t-s) . F3s,{s,0,t}]];

******* Approximations of order 4 (in rho) of x, y, w, p, q *****************

F4s = ((F4 /. heis) /. r12)// Simplify;{x4,y4,p4,q4}=Simplify[Integrate[(expA /.
rb->t-s) . F4s,{s,0,t}]];w41s = ((D[G2, x] /. heis) x3 + (D[G2, y] /. heis) y3 +
(D[G2,p] /. heis) p3 + (D[G2,q] /. heis) q3) /. r12; w42s = (G4 /. heis) /. r12;
w4 = Simplify[Integrate[w41s + w42s, {s, 0, t}] ];

******* Approximations of order 5 (in rho) of x, y, w, p, q ******************

F51s = ((F5 /. heis) /. r12);F52s=(((Outer[D,F3,{x, y,p,q}]/. heis) . {x3, y3,
p3, q3})/. r12);{x5,y5,p5,q5} = Simplify[Integrate[(expA /. rb ->t-s) . (F51s +
F52s), {s, 0, t}]];w51s=((G5/. heis)/. r12);w52s=((D[G2,x] /. heis) x4 + (D[G2,
y]/. heis) y4+(D[G2, p]/. heis) p4+(D[G2,q] /. heis) q4) /. r12; w5 = Simplify[
Integrate[ w51s + w52s, {s, 0, t}]]

******* Approximations of order 6 (in rho) of x, y, w, p, q  (for t = + 2 pi) ***

F61s=((F6/. heis)/. r12);F62s=(((Outer[D, F3, {x, y, p, q}] /. heis) . {x4, y4,
p4,q4})/. r12);F63s=(((Outer[D,F4,{x, y, p, q}] /. heis) . {x3, y3, p3, q3}) /.
r12 );{x6, y6, p6, q6}=Simplify[Integrate[(expA/. rb->-s).(F61s + F62s + F63s),
{s,0,2 Pi}]];w62s=((D[G4,x] /. heis) x3 + (D[G4,y]/. heis) y3+(D[G4,p] /. heis)
p3 +(D[G4, q] /. heis) q3) /. r12; w63s= ((D[G2,x]/. heis) x5+(D[G2,y]/. heis)
y5+(D[G2,p]/. heis) p5+(D[G2,q]/. heis) q5)/. r12; mat ={{D[G2, x, x], D[G2, x,
y],D[G2, x, p], D[G2,x,q]},{ D[G2,x,y],D[G2,y,y], D[G2,y,p],D[G2,y,q]},{D[G2,x,
p], D[G2, p, y],D[G2,p,p],D[G2,p,q]},{D[G2,x,q], D[G2, q, y], D[G2, q, p],D[G2,
q,q]}};w64s=(1/2{x3,y3,p3,q3} . mat . {x3, y3, p3, q3} ) /. r12; w61s = ((G6 /.
heis) /. r12);w62pi=Simplify[Integrate[w61s+w62s + w63s + w64s, {s, 0, 2 Pi}]];

******* Approximations of order 7 (in rho) of x, y, p, q (for t = + 2 pi)  ***

F71s = ((F7 /. heis) /. r12);F72s=(((Outer[D,F3,{x,y,p,q}] /. heis) . {x5, y5,
p5, q5}) /. r12); F73s = (((Outer[D, F4,{x,y,p,q}]/. heis) . {x4,y4,p4,q4}) /.
r12) // Expand; For[i=1, i<=4, i++,tmp=F3[[i]];mat[i]={{D[tmp,x, x], D[tmp, x,
y], D[tmp, x,p], D[tmp, x, q]}, {D[tmp,x,y],D[tmp,y,y],D[tmp,y,p],D[tmp,y,q]},
{D[tmp, x,p],D[tmp,p,y],D[tmp,p,p],D[tmp, p, q]}, {D[tmp, x, q], D[tmp, q, y],
D[tmp, q,p],D[tmp,q,q]}};F74s[i]=(({x3,y3,p3,q3} . mat[i]) . {x3, y3, p3, q3})
/2];F74s =({F74s[1],F74s[2],F74s[3],F74s[4]}/.r12);F75s=(((Outer[D,F5,{x,y,p,
```

```
q}]/.heis).{x3,y3,p3,q3})/. rl2); F76s = (((D[F5, w] /. heis) w4 )/.rl2);
{x72pi,y72pi,p72pi,q72pi}=Simplify[Integrate[(expA/.rb->-s).(F71s+F72s+F73s+
F74s+F75s+F76s), {s,0,2 Pi}]]

****************** Suspended exp. mapping Zh (in a neighbourhood of +2 Pi)****

tet0 = -6 Pi to; rho = 1; z42pi = ({x4, y4} /. rl3); z1p2pi = (D[{x1, y1}, t]
/. rl3);w42pi = (w4 /. rl3); w4p2pi = (D[w4, t] /. rl3); z1s2pi = (D[{x1,y1},
{t, 2}] /. rl3);z1t2pi = (D[{x1, y1}, {t, 3}] /. rl3); z32pi =({x3,y3}/.rl3);
z3p2pi = (D[{x3, y3},t] /.rl3); z3s2pi = (D[{x3, y3}, {t, 2}] /. rl3); z42pi=
({x4,y4} /. rl3); z4p2pi = (D[{x4, y4},t] /. rl3); z52pi = ({x5, y5} /. rl3);
w52pi = w5 /. rl3; z5p2pi = (D[{x5,y5}, t] /. rl3); z62pi = {x6, y6}; AA3 = -
w42pi/(2 Pi); AA4 = - w52pi/(2 Pi); AA5 = 7/8 w42pi^2/Pi^2 - w62pi/(2 Pi) -
tet0 w4p2pi/(2 Pi); z72pi = {x72pi, y72pi}; A4 = rr z1p2pi+ z42pi; A5 =-tet0
/(2 Pi) w42pi z1p2pi+tet0^2 z1s2pi/2+tet0 z3p2pi+z52pi- 3/(2 Pi) w42pi z32pi;
A60 = - tet0/(2 Pi) w5t2pi z1p2pi+ z62pi + tet0 z4p2pi- 2/Pi w42pi z42pi-3/(2
Pi) w5t2pi z32pi;A61=-1/(2 Pi) w42pi z1p2pi+tet0 z1s2pi+z3p2pi;A6  = A60 + rr
A61; A70 = tet0^3/6 z1t2pi + AA3 tet0^2/2 z1s2pi + AA5 tet0 z1p2pi + tet0^2/2
z3s2pi +3 AA3 tet0 z3p2pi + 3 AA3^2 z32pi + 3 AA5 z32pi + 4 AA4 z42pi + tet0
z5p2pi+5 AA3 z52pi+ z72pi; A71 = AA4 z1p2pi + z4p2pi ; A72 = z1s2pi/2; A7 =
A70 + A71 rr + A72 rr^2; Zh = A4 h^4 + A5 h^5 + A6 h^6 + A7 h^7;

**************** Conjugate time  rrc  ****************************************

DD1 = Det[{z1p2pi, D[z42pi, phi]}]; DD2 = Det[{z1p2pi, D[A5, phi]}]; DD3 = Det
[{A61,D[z1p2pi, phi]}]; DD4 = Det[{z1p2pi, D[A61, phi]}];DD5=Det[{A61,D[z42pi,
phi]}];DD6 = Det[{z1p2pi, D[A60, phi]}]; DD7 = Det[{z1p2pi, D[A70, phi]}]; DD8
= Det[{A61, D[A5, phi]}]; DD9 = Det[{A71,D[z42pi, phi]}]; DD10 = Det[{z1p2pi,
D[z42pi, phi]}]^2( Det[{2 A72, D[z1p2pi, phi]}] + Det[{z1p2pi, D[A72, phi]}]);
DD11= Det[{z1p2pi, D[A5, phi]}] (Det[{A61, D[z1p2pi, phi]}]+Det[{z1p2pi,D[A61,
phi]}]);DD12=Det[{z1p2pi,D[z42pi,phi]}] (Det[{A71,D[z1p2pi,phi]}]+Det[{z1p2pi,
D[A71,phi]}]+Det[{2 A72,D[z42pi,phi]}]); rrc0 = - DD1; rrc1 = - DD2; rrc2 =
Expand[- (- DD1 (DD3 + DD4)+ DD5 + DD6)]; rrc3 = Expand[(- DD7 - DD8 - DD9 -
DD10 + DD11 + DD12)]; rrc = rrc0 + h rrc1 + h^2 rrc2 + h^3 rrc3;

**** Conjugate locus  Zhcp (resp. Zhcm) in a neighbourhood of +2 Pi(resp.-2 Pi)

f4p = rrc0 z1p2pi + z42pi; f5p = A5 + z1p2pi rrc1;f6p=A60+A61 rrc0+z1p2pi rrc2;
f7p = A70 + A71 rrc0 + A72 rrc0^2 + A61 rrc1 + z1p2pi rrc3;
Zhcp = f4p h^4 + f5p h^5 + f6p h^6 + f7p h^7

f4p ={(5*Pi*(-2*vv2 + 6*b*Cos[2*phi]+ 3*b*Cos[4*phi]))/2,(-5*Pi*(2*vv1+ 6*b*
Sin[2*phi] - 3*b*Sin[4*phi]))/2};

f5p = {(Pi*(-105*b42*Cos[phi] - 540*c44*Cos[phi] - 72*a42*Pi*Cos[phi]+ 720*a44
*Cos[3*phi]-35*b42*Cos[3*phi]-180*c44*Cos[3*phi] - 24*a42*Pi*Cos[3*phi] + 432*
a44*Cos[5*phi]- 105*a42*Sin[phi] + 540*d44*Sin[phi] + 72*b42*Pi*Sin[phi] - 35*
a42*Sin[3*phi] + 720*b44*Sin[3*phi] + 180*d44*Sin[3*phi]+24*b42*Pi*Sin[3*phi]+
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```
432*b44*Sin[5*phi]))/24,(Pi*(-105*a42*Cos[phi]+ 540*d44*Cos[phi] + 72*b42*Pi*
Cos[phi] + 35*a42*Cos[3*phi] + 720*b44*Cos[3*phi] - 180*d44*Cos[3*phi]- 24*b42*
Pi*Cos[3*phi] - 432*b44*Cos[5*phi] + 105*b42*Sin[phi] + 540*c44*Sin[phi]+72*
a42*Pi*Sin[phi] - 720*a44*Sin[3*phi] - 35*b42*Sin[3*phi] - 180*c44*Sin[3*phi]-
 24*a42*Pi*Sin[3*phi] + 432*a44*Sin[5*phi]))/24};

Simplify[(f6p /. phi -> phi + Pi) - f6p];

cond = {to->0,b->0,vv1->0,vv2->0,L44->0,a44->0,b44->0,c44->0,d44->0,L42->0,
a42->0,b42->0,g551->0,g552->0,g553->0,g554->0,g555->0,g556->0,g531->0,g532->0,
g533->0,g534->0,l1->0,l2->0,L22->0,a22->0,b22->0,b311->0,b312->0,b331->0,b332
->0,b333->0,b334->0}; z72pin={x72pi,y72pi}/. cond;f4phi=D[f4p,phi]//Simplify;

expr = Det[{f4phi, z72pin}] // Simplify

expr = -60*b*Pi*Sin[3*phi]*(Pi*(1645*g622*Cos[2*phi] - 5670*g641*Cos[2*phi]+
5670*g645*Cos[2*phi] - 12600*g662*Cos[2*phi] - 7560*g664*Cos[2*phi] - 12600*
g666*Cos[2*phi] - 2100*g621*Pi*Cos[2*phi] + 2100*g623*Pi*Cos[2*phi] - 2700*
g642*Pi*Cos[2*phi]- 2700*g644*Pi*Cos[2*phi] -960*g622*Pi^2*Cos[2*phi]+2646*
g641*Cos[4*phi] -2646*g643*Cos[4*phi] + 2646*g645*Cos[4*phi] + 10080*g662*
Cos[4*phi] - 10080*g666*Cos[4*phi] + 1080*g642*Pi*Cos[4*phi]-1080*g644*Pi*
Cos[4*phi] - 1080*g662*Cos[6*phi] + 1080*g664*Cos[6*phi] - 1080*g666*Cos[6*
phi] - 1645*g621*Sin[2*phi] + 1645*g623*Sin[2*phi] - 2835*g642*Sin[2*phi]-
2835*g644*Sin[2*phi] + 37800*g661*Sin[2*phi] + 2520*g663*Sin[2*phi] - 2520*
g665*Sin[2*phi]-37800*g667*Sin[2*phi] -2100*g622*Pi*Sin[2*phi] + 5400*g641*
Pi*Sin[2*phi] - 5400*g645*Pi*Sin[2*phi] + 960*g621*Pi^2*Sin[2*phi] - 960*
g623*Pi^2*Sin[2*phi] + 2646*g642*Sin[4*phi] - 2646*g644*Sin[4*phi] - 15120*
g661*Sin[4*phi] + 5040*g663*Sin[4*phi] + 5040*g665*Sin[4*phi]-15120*g667*
Sin[4*phi]- 1080*g641*Pi*Sin[4*phi] + 1080*g643*Pi*Sin[4*phi]-1080*g645*Pi*
Sin[4*phi]+1080*g661*Sin[6*phi]-1080*g663*Sin[6*phi]+1080*g665*Sin[6*phi] -
1080*g667*Sin[6*phi]))/720;

Zhcm = f4m h^4 + f5m h^5 + f6m h^6 + f7m h^7

f4m = - f4p;      Simplify[(f6m /. phi -> phi + Pi) - f6m];
f5m = {(Pi*(105*b42*Cos[phi]+540*c44*Cos[phi]-72*a42*Pi*Cos[phi] -720*a44*
Cos[3*phi] + 35*b42*Cos[3*phi]+180*c44*Cos[3*phi]-24*a42*Pi*Cos[3*phi]-432*
a44*Cos[5*phi] + 105*a42*Sin[phi] - 540*d44*Sin[phi] + 72*b42*Pi*Sin[phi] +
35*a42*Sin[3*phi] - 720*b44*Sin[3*phi] -180*d44*Sin[3*phi] + 24*b42*Pi*
Sin[3*phi] - 432*b44*Sin[5*phi]))/24, (Pi*(105*a42*Cos[phi]-540*d44*Cos[phi]
+ 72*b42*Pi*Cos[phi] - 35*a42*Cos[3*phi] -720*b44*Cos[3*phi]+ 180*d44*Cos[3*
phi] - 24*b42*Pi*Cos[3*phi] + 432*b44*Cos[5*phi] - 105*b42*Sin[phi]-540*c44*
Sin[phi]+72*a42*Pi*Sin[phi] + 720*a44*Sin[3*phi] +35*b42*Sin[3*phi]+180*c44*
Sin[3*phi] - 24*a42*Pi*Sin[3*phi] - 432*a44*Sin[5*phi]))/24};
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STEKLOV MATHEMATICAL INSTITUTE, 42 UL. VAVILOVA, GSP1, MOSCOW, 117966, RUSSIA. PARTIALLY SUPPORTED BY THE RUSSSIAN FUND OF FUNDAM. RESEARCH, GRANT 95-01-00310A,

INSA DE ROUEN, DEPARTMENT OF MATHEMATICS, LMI, URA CNRS D 1378, BP 08, PL. E. BLONDEL, 76131 MONT SAINT AIGNAN CEDEX, FRANCE,

UNIVERSITÉ DE BOURGOGNE, LABORATOIRE DE TOPOLOGIE UMR CNRS 5584, BP 400, 21004 DIJON CEDEX, FRANCE.