Dynamical Planimetry

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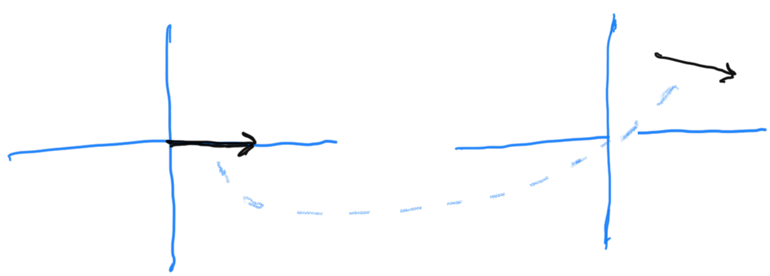
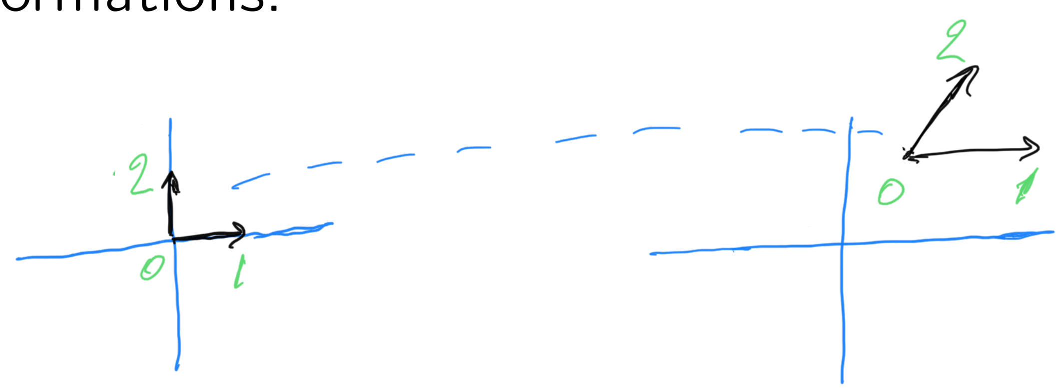
Simplest planimetric objects: couples of points (segments) and triples of points (triangles). We consider segments and triangles with ordered vertices and allow to continuously change them according to certain simple rules.

(3d) The segments with ordered vertices are "free vectors". The space of free vectors of a fixed length is the group of rigid motions.



(4d) The space of free vectors of an arbitrary length is the group of similitudes.

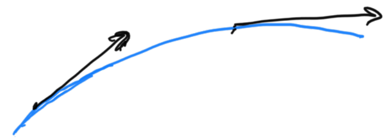
(5d) Triangles with ordered vertices are "free frames". The space of free frames of a fixed area is the group of area preserving affine transformations.



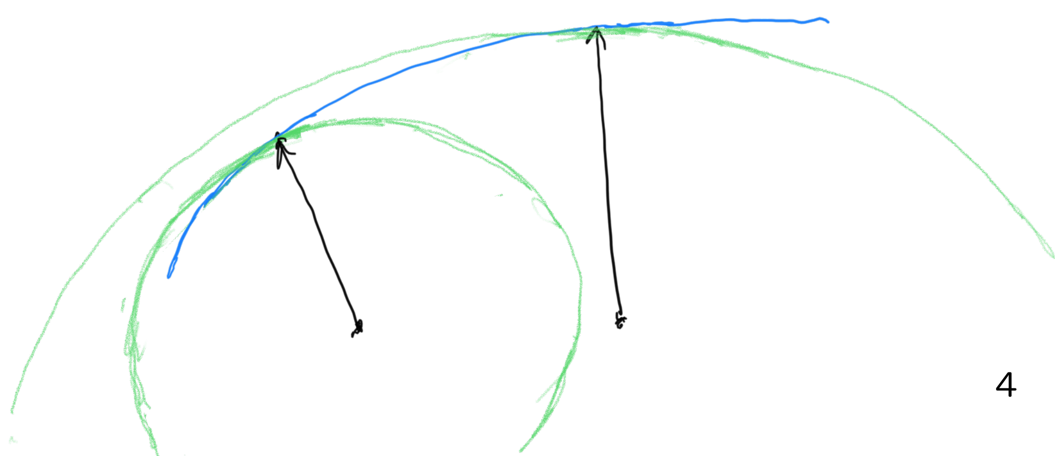
(6d) The space of all free frames is the group of all affine transformations.

Dynamics

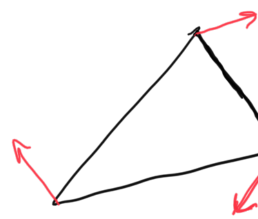
(3d) Bicycle: the origin of the vector is moving in the direction of the vector.



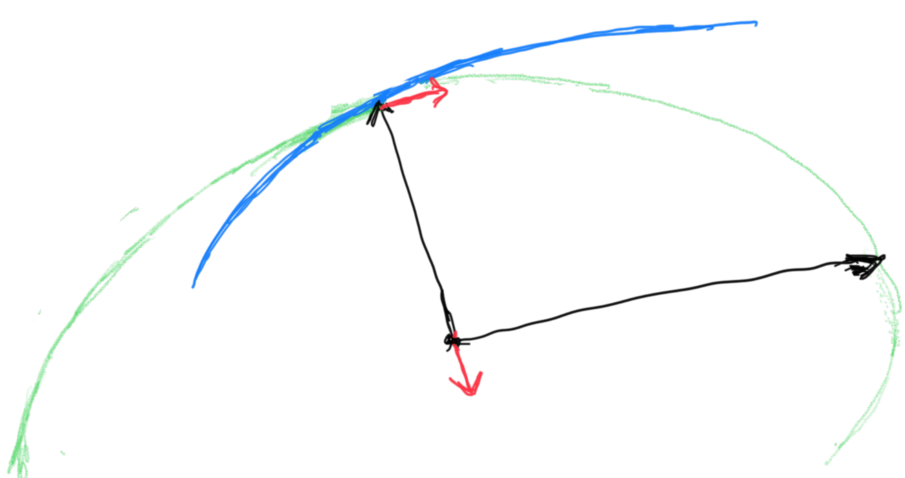
(4d) Osculating circle: The endpoint of the vector is a point of the curve, while the origin of the vector is the curvature center of this curve.



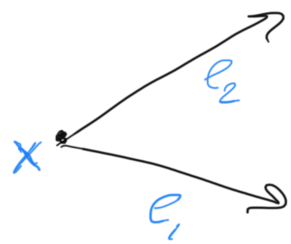
(5d) Three ants: Any vertex moves parallel to the opposite side of the triangle.



(6d) Osculating ellipse: The endpoint of the first vector of the frame is a point on the curve, while the origin of the frame is the center of the osculating ellipse. The second vector is parallel to the tangent line to the curve at the point and the endpoint of the second vector belongs to the ellipse.



Our rules impose linear nonholonomic constraints on the velocities of the vertices. In any of four cases the configuration is described by a free frame:



In the 3d and 4d cases the frame is orthogonal, while in the 5d and 6d cases the frame is arbitrary. In the osculating circle and ellipse cases, point x is the center of the circle or ellipse and $x + e_1$ is the point of the curve.

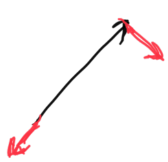
In the 5d case, the distribution of admissible velocities has rank 3 and is spanned by three vector fields where any of three fields moves only one vertex.

In all other cases, the distribution of admissible velocities has rank 2 and contains a vector field that "rotates" the vector $x + e_1$ around x:

$$\dot{e}_1 = -e_2, \quad \dot{e}_2 = e_1, \quad \dot{x} = 0$$

The second field is as follows:

$$\dot{e}_1 = 0, \quad \dot{e}_2 = 0, \quad \dot{x} = -e_1 \qquad (3d)$$



$$\dot{e}_1 = e_1, \quad \dot{e}_2 = e_2, \quad \dot{x} = -e_1 \qquad (4d)$$

$$\dot{e}_1 = e_1, \quad \dot{e}_2 = 2e_2, \quad \dot{x} = -e_1 \qquad (6d)$$

We can present the configurations in the matrix form

$$Q = \begin{pmatrix} e_1 & e_2 & x \\ 0 & 0 & 1 \end{pmatrix},$$

as elements of the group GL(3); then $\dot{Q} = QA$, where

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for the first vector field, and

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (3d)$$

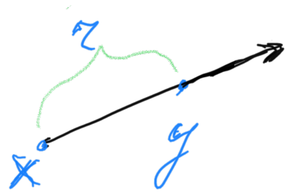
$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (4d)$$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (6d)$$

for the second vector field.

Geodesics.

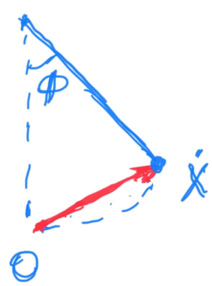
In the 3d case, we are minimizing the length of the path of the point $y = x + re_1$ where r is a nonzero constant.



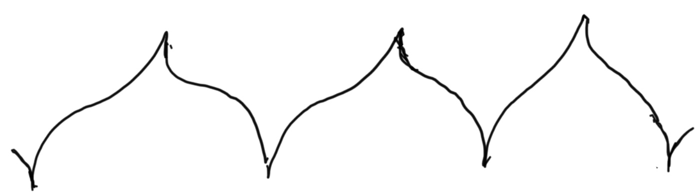
Let $\mathbb{R}^2 = \mathbb{C}$, $e_1 = e^{i\theta}$. Admissible paths:

$$\dot{\theta} = u, \quad \dot{x} = -ve^{i\theta}.$$

The length of the path of the point y is equal to $\int_0^1 (r^2u^2(t) + v^2(t))^{\frac{1}{2}}\,dt$. The geodesics are related to the dynamics of the mathematical pendulum $\ddot{\phi} = c \sin \phi$:



The shape of the curve $x(t)$ for the oscillating pendulum:



The shape of the curve $x(t)$ for the rotating pendulum:

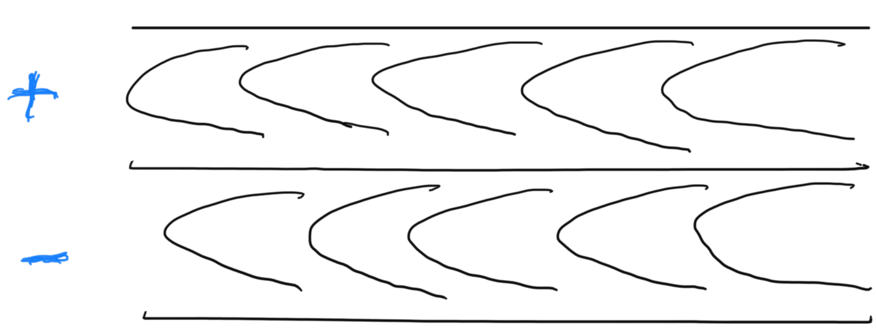
The 4d case. We set $e_1 = e^z$. Admissible paths:

$$\dot{z} = v + ui, \quad \dot{x} = -ve^z.$$

The distribution is projected one-to-one to the plane $\{z\}$, and a natural length of the admissible path is the length of the plane curve $z(\cdot)$. The geodesics are characterized by the conditions:

$$|\dot{z}| = const, \quad \ddot{z} = \langle a, e^z \rangle,$$

where a is a constant vector. This is the equation for the motion of a charged particle in the magnetic field $\langle a, e^z \rangle$. The level lines of the field are as follows:



Singular curves are critical points of the map $\gamma(\cdot) \mapsto (\gamma(0), \gamma(1))$ defined on the space of admissible curves.

Constant curves $\gamma(t) \equiv \gamma(0)$ are singular.

There are no other singular curves in the 3d case.

In the 4d case, the movement of z along the zero level of the magnetic field is a singular curve; this corresponds to the movement of the center of the osculating circle x along the line connecting x with the touching point of the circle and the curve.

(5d)

$$\dot{x}_i = u_i(x_{i-1} - x_{i+1}), \quad i \in \mathbb{Z}/3\mathbb{Z}.$$

All singular curves satisfy the condition $\sum_i u_i = 0$. The barycenter of the triangle $b = \frac{1}{3}\sum_i x_i$ is moving along a straight line; moreover, the barycenter of exactly one singular curve in the space of triangles with a given initial condition is moving along any straight line through $b(0)$.

For the straight lines that connect the barycenter with a vertex (i.e. for the medians), the vertex does not move while the opposite side is moving parallel to itself.

We separate the movements of the barycenter and around the barycenter in order to describe other singular curves and set $y_i = x_i - b$, $\sum_i y_i = 0$.

The appropriately parameterized matrix curve $Y = (y_1, y_2)$ is a solution of the Fuchsian system of the form:

$$3\dot{Y} =$$
$$Y\left(\frac{1}{\tau-1}\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix} + \frac{1}{\tau}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{1}{1+\tau}\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}\right).$$

(6d) The movement of the center of the osculating ellipse along the straght line connecting the center with the touching point of the ellipse and the curve is a singular curve of corang 2; in other words, the image of the differential of the map $\gamma(\cdot) \mapsto (\gamma(0), \gamma(1))$ in this point of the space of admissible curves has codimension 2.

A one-parametric family of singular curves goes out in any other direction of the plane of initial admissible velocities. All of them are described by the systems:

$$\dot{e}_1 = e_2 + ve_1,\ \dot{e}_2 = -e_1 + 2ve_2,\ \dot{x} = -ve_1;$$

$$\dot{v} = 2v(w - v) - 1, \quad \dot{w} = w(w - v) + 1. \quad (*)$$

Note that system (*) has no equilibria.

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