Asymptotic Homologies in Sub-Riemannian Geometry: Two Cases Study

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I. Local story: a no-Goh singular curve.

We consider a system:

$$\dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M, \ u_i \in \mathbb{R},$$

with the initial condition $q(0) = q_0$. The endpoint map:

$$F_t : L^2([0,t];\mathbb{R}^k) \to M, \quad F_t(u) = q(t),$$

where $u = (u_1(\cdot), \ldots, u_k(\cdot)) \in \mathbb{R}^k$,

$$\dot{q}(\tau) = f_0(q(\tau)) + \sum_{i=1}^{k} u_i(\tau) f_i(q(\tau)), \quad 0 \leq \tau \leq t.$$

We are interested in the level sets of the endpoint map and, in particular, in the local structure of these level sets.

Digression: Let $\mathcal{U}$ be a finite-dimensional manifold, $\varphi : \mathcal{U} \to M$ a smooth map.

If $\tilde{u} \in \mathcal{U}$ is a regular point of φ, $D_{\tilde{u}}\varphi(T_{\tilde{u}}\mathcal{U}) = T_{\tilde{q}}M$, where $\tilde{q} = \varphi(\tilde{u})$, then $\varphi^{-1}(\tilde{q}) \cap O_{\tilde{u}}$ is a ball,

$$\varphi^{-1}(\tilde{q}) \cap O_{\tilde{u}} \setminus \{\tilde{u}\} \cong \mathbb{S}^m,$$

where $m = \dim U - \dim M - 1$

If $\tilde{u}$ is a critical point of φ, i. e, $\lambda D_{\tilde{u}}\phi = 0$ for some $\lambda \in T^*_{\tilde{q}},\ \lambda \neq 0$, then we have to study the Hessian:

$$\lambda D^2_{\tilde{u}}\varphi : \ker D_{\tilde{u}}\varphi \times \ker D_{\tilde{u}}\varphi \to \mathbb{R}. \qquad (*)$$

Let σ be the signature of the quadratic form (*). If λ is unique and (*) is nondegenerate, then

$$\varphi^{-1}(\tilde{q}) \cap O_{\tilde{u}} \setminus \{\tilde{u}\} \cong \mathbb{S}^{\frac{1}{2}(m+\sigma)} \times \mathbb{S}^{\frac{1}{2}(m-\sigma)}.$$

$F_t : L^2([0,t];\mathbb{R}^k) \to M$ is defined on the infinite dimensional space.

If $\tilde{u}$ is a regular point, then $F_t^{-1}(\tilde{q}(t)) \cap O_{\tilde{u}}$ is a Hilbert ball and $F_t^{-1}(\tilde{q}(t)) \cap O_{\tilde{u}} \setminus \{\tilde{u}\}$ is contractible.

Let $\tilde{u}$ be a critical point, $\lambda_t D_{\tilde{u}} F_t = 0$. We set:

$$P^t : M \to M, \quad P^t : q(0) \mapsto q(t),$$

where $\dot{q}(\tau) = f_0(q(\tau)) + \sum_{i=1}^{k} \tilde{u}_i(\tau) f_i(q(\tau))$,

$$g_i^\tau = (P_*^\tau)^{-1} f_i, \quad \lambda_0 = P^{t*}\lambda_t,\ \lambda_0 \in T_{q_0}^* M.$$

Then:

$$D_{\tilde{u}}F_t(v) = P_*^t \int_0^t \sum_{i=0}^k v_i(\tau) g_i^\tau(q_0)\, d\tau,$$

$$\lambda_t D^2_{\tilde{u}}F_t(v,v) = \int_0^t \int_0^\tau \sum_{i,j=0}^k v_i(\theta) v_i(\tau) \langle \lambda_0, [g_i^\theta, g_j^\tau](q_0)\rangle \, d\theta d\tau.$$

Theorem 1 (Goh condition)**.** *If* $\exists\, \tau, i, j$ *such that*

$$\langle \lambda_0, [g_i^\tau, g_j^\tau](q_0)\rangle \neq 0,$$

then both positive and negative inertia indices of $\lambda_t D^2_{\tilde{u}} F_t$ *are infinite.*

Theorem 2. *Under conditions of Theorem 1, the pointed neighborhood* $F_t^{-1}(\tilde{q}(t)) \cap O_{\tilde{u}} \setminus \{\tilde{u}\}$ *is contractible.*

What about finite dimensional sections?

Let $E_n \subset L^2([0,t];\mathbb{R}^k)$ be the space of vector trigonometric polynomials of degree not greater than n (with rescaled frequencies $\frac{2\pi m}{t}$, $0 \le m \le n$).

Assume that the system and control $\tilde{u}$ are real-analytic and let $\tilde{u}(0) = 0$.

Theorem 3. *If the matrices* $\left\{\langle\lambda_0, [g_i^\tau, g_j^\tau]\rangle\right\}_{i,j=1}^k$, $\tau \ge 0$, *and* $\left\{\langle\lambda_0, [[f_0, f_i], f_j]]\rangle\right\}_{i,j=1}^k$ *are nondegenerate, then there exists a piecewise constant integral-valued function* $t \mapsto \sigma(t)$, $t \ge 0$, *such that for any continuity point* t *of* $\sigma(\cdot)$, $\exists\, N > 0$ *such that* $\forall\, n \ge N$ *the form* $Q_t^n \doteq \lambda_0 D_{\tilde{u}}^2 F_t\Big|_{E_n}$ *is nondegenerate and* $sgn(Q_t^n) = \sigma(t)$.

Corollary 1. *Under conditions of Theorem 3,*

$$F_t^{-1}(\tilde{q}_t) \cap (\tilde{u} + E_n) \cap O_{\tilde{u}} \setminus \{\tilde{u}\} \cong \mathbb{S}^{\frac{1}{2}(m+\sigma(t))} \times \mathbb{S}^{\frac{1}{2}(m-\sigma(t))},$$

where $m = \dim E_n - \dim M - 1$.

Proposition 1. $\sigma(t)$ *is a gauge invariant, i.e. it depends only on the affine distribution* $f_0(q) + span\{f_1(q), \ldots, f_k(q)\}, q \in M$, *trajectory* $\tilde{q}(\cdot)$ *and* λ_0.

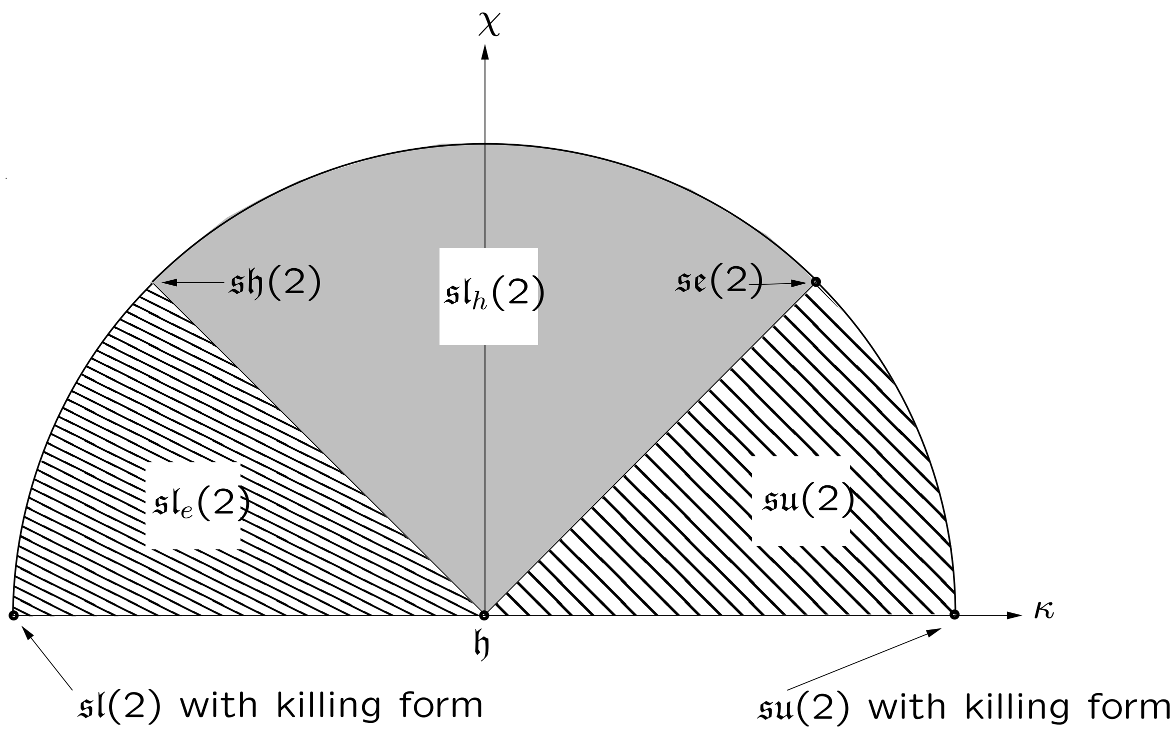
Example. Let $\Delta = span\{f_1, \ldots, f_k\}$ be a contact distribution, f_0 a contact vector field and the matrix $[[f_0, f_i], f_j]$ is nondegenerate; then $\tilde{u} = 0$ satisfies conditions of Theorem 3.

Special case: M is a 3-dimensional unimodular Lie group, f_0, Δ are left-invariant. Then f_0 is a Reeb field for a sub-Riemannian structure on Δ. Let χ, κ be basic invariants of this structure, $\langle \lambda_0, f_0 \rangle > 0$.

We have:

$$\sigma(t) = \begin{cases} \frac{\kappa}{|\kappa|}\left(2 + 4\left[\frac{t\sqrt{\kappa^2 - \chi^2}}{2\pi}\right]\right), & \text{if } |\kappa| > \chi; \\ 0, & \text{if } |\kappa| < \chi. \end{cases}$$

Here $[\cdot]$ is the integral part of a real number.



χ
$\mathfrak{sh}(2)$
$\mathfrak{sl}_h(2)$
$\mathfrak{se}(2)$
$\mathfrak{sl}_e(2)$
$\mathfrak{su}(2)$
κ
$\mathfrak{h}$
$\mathfrak{sl}(2)$ with killing form
$\mathfrak{su}(2)$ with killing form

II. Global story: a step two Carnot group.

A step two Lie algebra and group:

$$\mathfrak{g} = V \oplus W,\ [V,V] = W,\ [\mathfrak{g}, W] = 0, \quad \mathfrak{G} = e^{\mathfrak{g}}.$$

To any $\omega \in W^*$ we associate an operator $A_\omega \in \mathrm{so}(V)$ by the formula:

$$\langle A_\omega \xi, \eta \rangle = \langle \omega, [\xi, \eta] \rangle, \quad \xi, \eta \in V.$$

It is easy to see that $\omega \mapsto A_\omega,\ \omega \in W^*$ is an injective linear map. Moreover, any injective linear map from W^* to $\mathrm{so}(V)$ defines a structure of step two Carnot Lie algebra on the space $V \oplus W$ by the same formula. Hence step two Carnot Lie algebras are in the one-to-one correspondence with linear systems of anti-symmetric operators. Sometimes we need non-resonance generic assumptions on the linear system, which I won't specify.

An H^1-curve $\gamma : [0,1] \to \mathfrak{G}$ is called *horizontal* if $\dot{\gamma}(t) \in V_{\gamma(t)}$ for a. e. $t \in [0,1]$.

The following multiplication in $V \times W$ gives a simple realization of $\mathfrak{G}$ with the origin in $V \times W$ as the unit element:

$$(v_1, w_1) \cdot (v_2, w_2) = \left(v_1 + v_2, w_1 + w_2 + \frac{1}{2}[v_1, v_2] \right).$$

Starting from the origin horizontal curves are determined by their projection to the first level and have a form:

$$\gamma(t) = \left(\xi(t), \frac{1}{2} \int_0^t [\xi(t), \dot{\xi}(t)]\, dt \right), \quad 0 \le t \le 1,$$

where $\xi(\cdot) \in H^1([0,1]; U)$, $\xi(0) = 0$.

We set:

$$\varphi(\xi) = \frac{1}{4\pi}\int_0^1 |\dot{\xi}(t)|^2\,dt.$$

We focus on the horizontal curves corresponding to closed curves ξ; they connect the origin with the second level. Given $w \in W\setminus 0$, let Ω_w be the space of horizontal curves connecting $(0,0)$ with $(0,w)$; then

$$\Omega_w = \left\{\xi \in H^1([0,1];V) : \xi(0) = \xi(1) = 0,\ \frac{1}{2}\int_0^1 [\xi(t),\dot{\xi}(t)]\,dt = w\right\}.$$

For any $s > 0$, we set: $\Omega_w^s = \{\xi \in \Omega_w : \varphi(\xi) \le s\}$. Note that central reflection $\xi \mapsto -\xi$ preserves Ω_w^s. We denote by $\bar{\Omega}_w^s$ the image of Ω_w^s under the factorization $\xi \sim (-\xi)$.

Proposition 2. *There exists a finite-dimensional subspace* $E \subset H^1([0,1];V)$ *such that* $\bar{\Omega}^s_w \cap \bar{E}$ *is a deformation retract of* $\bar{\Omega}^s_w$*, where* $\bar{E}$ *is the projectivization of* E *.*
Corollary 2. $\bar{\Omega}^s_w$ *has homotopy type of a semi-algebraic set.*

We introduce the notation $\bar{E}^s_w = \bar{\Omega}^s_w \cap \bar{E}$ and consider the homology $H.(\bar{E}^s_w;\mathbb{Z}_2)$ and its image in $H.(\bar{E};\mathbb{Z}_2)$ by the homomorphism induced by the imbedding $\bar{E}^s_w \subset \bar{E}$. We have:

$$\text{rank}\Big(H_i(\bar{E}^s_w;\mathbb{Z}_2)\Big) = \beta_i(\bar{E}^s_w) + \varrho_i(\bar{E}^s_w),$$

where $\beta_i(\bar{E}^s_w)$ is rank of the kernel of the homomorphism from $H_i(\bar{E}^s_w;\mathbb{Z}_2)$ to $H_i(\bar{E};\mathbb{Z}_2)$ induced by the imbedding $\bar{E}^s_w \subset \bar{E}$ and $\varrho_i(\bar{E}^s_w) \in \{0,1\}$ is the rank of the image of this homomorphism.

Now we build two positive atomic measures on the half-line $\mathbb{R}_+$, the "Betti distributions":

$$\mathfrak{b}(\bar{E}_w^s) \doteq \frac{1}{s} \sum_{i \in \mathbb{Z}_+} \beta_i(\bar{E}_w^s) \delta_{\frac{i}{s}}, \quad \mathfrak{r}(\bar{E}_w^s) \doteq \frac{1}{s} \sum_{i \in \mathbb{Z}_+} \varrho_i(\bar{E}_w^s) \delta_{\frac{i}{s}}.$$

Assume that $\dim W = 2$; it appears that there exist limits of these families of measures:

$$\lim_{s \to \infty} \mathfrak{b}(\bar{E}_w^s), \quad \lim_{s \to \infty} \mathfrak{r}(\bar{E}_w^s)$$

in the weak topology. Moreover, the limiting measures are absolutely continuous with explicitly computed densities!

Let $\alpha : \Delta \to \mathbb{R}$ be an absolutely continuous function defined on an interval Δ. We denote by $|d\alpha|$ a positive measure on Δ such that

$$|d\alpha|(S) = \int_S \left|\frac{d\alpha}{dt}\right| dt, \quad S \subset \Delta.$$

The operators A_ω, $\omega \in W^*$, have purely imaginary eigenvalues. Let $0 \leq \alpha_1(\omega) \leq \cdots \leq \alpha_m(\omega)$ are such that $\pm i\alpha_j m\ j = 1, \ldots, m$, are all eigenvalues of A_ω counted according the multiplicities.

Let $\bar{W}^* = (W \setminus 0)/\Big(w \sim cw, \forall c \neq 0\Big)$ be the projectivization of W^*, $\bar{W}^* = \mathbb{RP}^1$.

Given $w \in W \setminus 0$, we take the line $w^{\perp} \in W^*$ and consider the affine line

$$\ell_w = \bar{W}^* \setminus \bar{w}^{\perp} \subset \bar{W}^*.$$

Moreover, we define functions

$$\lambda_j^w : \ell_w \to \mathbb{R}_+, \ j = 1, \ldots, m, \qquad \phi^w : \ell_w \to \mathbb{R}_+$$

by the formulas:

$$\lambda_j^w(\bar{\omega}) = \frac{\alpha_j(\omega)}{\langle \omega, w \rangle}, \qquad \phi^w(\bar{\omega}) = \sum_{j=1}^{m} \lambda_j^w(\omega).$$

Theorem 4. *Assume that there exists* $\omega \in W^*$ *such that the matrix* A_ω *has simple spectrum (a non-resonance assumption). Then, for any* $w \in W \setminus 0$*, there exist the following limits in the weak topology of the space of positive measures on* $\mathbb{R}_+$*:*

$$\mathfrak{b}_w = \lim_{s\to\infty} \mathfrak{b}(\bar{E}^s_w), \qquad \mathfrak{r}_w = \lim_{s\to\infty} \mathfrak{r}(\bar{E}^s_w).$$

Moreover,

$$\mathfrak{b}_w = \phi^w_*\Big(\sum_{j=1}^{m} |d\lambda^w_j|\Big), \qquad \mathfrak{r}_w = \chi_{[0,\min \phi^w]} dt,$$

where dt *is the Euclidean measure.*