Докл. Акад. Наук СССР
Том 299 (1988), № 5

Soviet Math. Dokl.
Vol. 37 (1988), No. 2

HOMOLOGY OF INTERSECTIONS OF REAL QUADRICS

UDC 515.142+515.165

A. A. AGRACHEV

1. Let A be either the field of real or complex numbers, or the field of quaternions. Let $A\mathbf{P}^n$ be the quotient space of sphere $\{x \in A^{N+1} \mid |x| = 1\}$ by the equivalence relation $x \sim ax\ \forall a \in A$, $|a| = 1$. We denote by $\mathscr{P}_N^k(A)$ the space of all symmetric $\mathbf{R}$-bilinear maps $p: A^{N+1} \times A^{N+1} \to \mathbf{R}^{k+1}$ satisfying the condition $p(ax, ay) = |a|^2 p(x,y)\ \forall a \in A$. If $p \in \mathscr{P}_N^k(A)$ and $\omega \in \mathbf{R}^{k+1*}$, then $\omega p \in \mathscr{P}_N^0(A)$. In the case $A = \mathbf{R}$, ωp is an arbitrary symmetric bilinear form; in the case $A = \mathbf{C}$, it is the real part of an Hermitian form.

To any form $q \in \mathscr{P}_N^0(A)$ there correspond the inertia index, $\operatorname{ind} q$, equal to the maximal dimension of the subspace of A^{N+1} on which the quadratic form $x \mapsto q(x,x)$ is negative definite, and a number, $\operatorname{corank} q$, equal to the dimension of the kernel of form q. Accordingly, to any map $p \in \mathscr{P}_N^k(A)$ correspond two functions, $\operatorname{ind} p: \omega \mapsto \operatorname{ind}(\omega p)$ and $\operatorname{corank} p: \omega \mapsto \operatorname{corank}(\omega p)$, which assume nonnegative integer values. We denote by ωP the matrix corresponding to the form ωp; obviously, $(\omega P)^{\mathrm{T}} = \overline{\omega P}$.

If $p \in \mathscr{P}_N^k(A)$, $x \in A^{N+1}$, and $a \in A$, $|a| = 1$, then $p(ax, ax) = p(x,x)$. Consequently the map of the projective space $A\mathbf{P}^N$ into $\mathbf{R}^{k+1}$ given by $\hat{p}: \hat{x} \mapsto p(x,x)$, where $\hat{x} = \{ax, |a| = 1\} \in A\mathbf{P}^N$, is well-defined.

Let K be a convex closed cone in $\mathbf{R}^{k+1}$ with its vertex at the origin (we do not exclude the case $K = 0$), and let $K^0 = \{\omega \in \mathbf{R}^{k+1*} | \omega y \le 0\ \forall y \in K\}$ be its dual cone.

DEFINITION. A map $p \in \mathscr{P}_N^k(A)$ is called *degenerate relative to* K if, for some $\omega \in K^0 \backslash 0$ and $x \in A^{N+1} \backslash 0$ the relations $\omega P x = 0$ and $p(x,x) = 0$ are satisfied. In the contrary case, p is called *nondegenerate relative to* K.

The nondegenerateness of p relative to K is equivalent to the transversality of $\hat{p}: A\mathbf{P}^n \to \mathbf{R}^{k+1}$ to K (see [1]). If p is nondegenerate relative to K, then $\hat{p}^{-1}(K)$ is either 1) a topological manifold with boundary (the boundary, generally speaking, is nonsmooth; however, the complement of the boundary is a smooth manifold), or 2) the empty set.

Up to the end of this section we fix a convex closed cone $K \subset \mathbf{R}^{k+1}$ and a map $p \in \mathscr{P}_N^k(A)$, nondegenerate relative to K. Let S^k be the unit sphere in $\mathbf{R}^{k+1}$, and let $\Omega = S^k \cap K^0$. Assume that $\Omega_n = (\operatorname{ind} p + \operatorname{corank} p)^{-1}([0,n]) \cap \Omega$ for $n \ge 0$, and that $\Omega_j = \varnothing$ for $j < 0$.

The sets Ω_n determine a filtration of Ω by open subsets of Ω. In [1], the Euler characteristic of a manifold with boundary

$$\hat{p}^{-1}(K) = \{\hat{x} \in A\mathbf{P}^N | \omega\hat{p}(\hat{x}) \le 0\ \forall \omega \in \Omega\}$$

is expressed in terms of such filtration.

The purpose of this section is to describe the cohomology groups of the manifold $A\mathbf{P}^N \backslash \hat{p}^{-1}(K)$; however, the filtration Ω_n is inadequate for this purpose; additional information is required. In the following, without special mention, we shall consider for the case $A = \mathbf{R}$ only cohomology classes with coefficients in $\mathbf{Z}_2$, and for $A = \mathbf{C}$ and $A = \mathbf{H}$, those with coefficients in $\mathbf{Z}$. Furthermore, we introduce a number ν, with $\nu = 1$ if $A = \mathbf{R}$, $\nu = 2$ if $A = \mathbf{C}$, and $\nu = 4$ if $A = \mathbf{H}$.

Denote by $\mathscr{M}_N(A)$ the space of $(N+1) \times (N+1)$ matrices M over A satisfying the condition $M^{\mathrm{T}} = \overline{M}$, and let $\lambda_1(M) \le \cdots \le \lambda_{N+1}(M)$ be the ascending sequence of the

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55N45.

eigenvalues of a matrix $M \in \mathscr{M}_N(A)$ (they are reals). Then

$$\Omega_n = \{\omega \in \Omega | \lambda_{n+1}(\omega P) > 0\}.$$

For $0 \leq n \leq N-1$ we set

$$\Lambda_n(A) = \{M \in \mathscr{M}_N(A) | \lambda_{n+1}(M) \neq \lambda_{n+2}(M)\}.$$

The closed set $\mathscr{M}_N(A) \backslash \Lambda_n(A)$ is a pseudomanifold of codimension $\nu+1$ in $\mathscr{M}_N(A)$. For the cases $A = \mathbf{C}$ and $A = \mathbf{H}$, this pseudomanifold is orientable. We shall denote by $\gamma_n(A)$ the ν-dimensional cohomology class in $\Lambda_n(A)$ dual to this pseudomanifold; the value of $\gamma_n(A)$ at an arbitrary ν-dimensional cycle in $\Lambda_n(A)$ is equal to the linking number of this cycle with $\mathscr{M}_N(A) \backslash \Lambda_n(A)$.

The classes $\gamma_n(A)$ may also be described in the following way: Let $\mathscr{E}_n(A)$ be a vector bundle whose base is $\Lambda_n(A)$ and whose fiber over a point $M \in \Lambda_n(A)$ is the $(n+1)$-dimensional invariant subspace of M corresponding to the eigenvalues $\lambda_1(M), \dots, \lambda_{n+1}(M)$. Then

$$\gamma_n(\mathbf{R}) = w_1(\mathscr{E}_n(\mathbf{R})), \qquad \gamma_n(\mathbf{C}) = c_1(\mathscr{E}_n(\mathbf{C})), \qquad \gamma_n(\mathbf{H}) = p_1(\mathscr{E}_n(\mathbf{H})),$$

where w_1, c_1, and p_1 are Stiefel-Whitney, Chern, and Pontrjagin characteristic classes. It is easy to see that the restriction of the bundle $\mathscr{E}_n(A)$ to the set $\{M \in \mathscr{M}_N(A) \,|\, \lambda_1(M) \neq \lambda_2(M) \neq \cdots \neq \lambda_{n+2}(M)\} \subset \Lambda_n(A)$ decomposes into a direct sum of one-dimensional bundles. The restriction of the class $\gamma_n(A)$ to this set is the sum of characteristic classes of the corresponding one-dimensional bundles, from which the explicit expressions for $\gamma_n(A)$ are easily derived. We shall also assume that $\gamma_j(A) = 0$ for $j < 0$.

Below, as a rule, we omit the argument A where it does not lead to confusion. Let $\delta: H^\nu(\Lambda_n) \to H^{\nu+1}(\mathscr{M}_N, \Lambda_n)$ be a connecting homomorphism in the exact sequence of the pair $(\mathscr{M}_n, \Lambda_n)$, and let $\Gamma_n = \delta(\gamma_n)$. The value of Γ_n at the $(\nu+1)$-dimensional cycle in $(\mathscr{M}_N, \Lambda_n)$ is equal to the intersection index of this cycle with $\mathscr{M}_N \backslash \Lambda_n$. Note that the subset $\mathscr{M}_N \backslash (\Lambda_n \cup \Lambda_{n+1})$ is a pseudomanifold of codimension $3\nu+2$ in $\mathscr{M}_N$, and consequently $\Gamma_n \cup \Gamma_{n+1} \in H^{2\nu+2}(\mathscr{M}_N, \Lambda_n \cup \Lambda_{n+1}) = 0$.

Let $\pi: \omega \mapsto \omega P$, $\omega \in \Omega$; then $\Omega_{n+1} \backslash \Omega_n \subset \pi^{-1}(\Lambda_n)$, and consequently

$$H^i(\Omega_n, \pi^{-1}(\Lambda_n)) \approx H^i(\Omega_{n+1}, \pi^{-1}(\Lambda_n))$$

(an excision). In the following, to avoid overburdening the formulas, we omit notation of excision isomorphisms and of cohomology homomorphisms induced by the inclusion of one pair of sets into another in those cases where they can be uniquely recovered from the context. For example: $\pi^*(\Gamma_n) \in H^2(\Omega, \pi^{-1}(\Lambda_n))$; however, the same symbol, $\pi^*(\Gamma_n)$, may also be used to denote the restriction of this cohomology class to the pair $(\Omega_n, \pi^{-1}(\Lambda_n) \cap \Omega_n)$. If $\xi \in H^i(\Omega_n, \pi^{-1}(\Lambda_n))$, then the same letter may denote both the element of $H^i(\Omega_{n+1}, \pi^{-1}(\Lambda_n))$ corresponding to ξ for the excision isomorphism, as well as the image of ξ in $H^i(\Omega_{n+1})$ for the homomorphism induced by the inclusion $(\Omega_{n+1}, \varnothing) \subset (\Omega_{n+1}, \pi^{-1}(\Lambda_n))$.

THEOREM 1. *Let (E_r, d_r), $r \geq 2$, be a cohomology spectral sequence of the first quadrant, defined by the following relations:*

i) *$E_2^{i,\nu j} = H^i(\Omega_{N-j})$, and $E_2^{i,j} = 0$ for $j \not\equiv 0 \bmod \nu$.*

ii) *The differential $d_{\nu+1}: H^i(\Omega_n) \to H^{i+\nu+1}(\Omega_{n+1})$ is defined by $d_{\nu+1}(\xi) = \pi^*(\Gamma_n) \cup \xi$, $\xi \in H^i(\Omega_n)$.*

iii) *If $\xi \in H^i(\Omega_n)$ is a representative of class $[\xi] \in E_{\nu r+1}^{i,(N-n)\nu}$, $r > 1$, then $d_{\nu r+1}([\xi]) = [\langle \pi^*(\Gamma_{n+r-1}), \dots, \pi^*(\Gamma_n), \xi \rangle]$, where $\langle \pi^*(\Gamma_{n+r-1}), \dots, \pi^*(\Gamma_n), \xi \rangle$, is the result of applying the Massey operation of order r to cohomology classes within the angled brackets.*

Then the spectral sequence (E_r, d_r) converges to $H(A\mathbf{P}^N \backslash \hat{p}^{-1}(K))$. (Recall that for $A = \mathbf{R}$ we consider cohomology groups with coefficients in $\mathbf{Z}_2$.)

Massey operations are partially defined cohomology operations subordinate to cohomology multiplication. The Massey triad $\langle \xi, \eta, \varsigma \rangle$ is defined in the case where $\xi \cup \eta = 0$ and $\eta \cup \varsigma = 0$. If we let cocycles x, y, and z represent cohomology classes ξ, η, and ς, respectively, where $x \cup y = \delta X$ and $y \cup z = \delta Z$, then the cochain $X \cup z - (-1)^{\deg x} x \cup Z$ is a cocycle. Generally speaking, the cohomology class of this cocycle is not uniquely defined by the classes ξ, η, and ς; the family of all cohomology classes ξ, η, and ς obtained by the indicated method are denoted by $\langle \xi, \eta, \varsigma \rangle$. The definition and main properties of higher Massey operations are contained in [2] and [3].

2. From here on, $A = \mathbf{R}$. In this section we shall set out the results of computations of the Betti numbers of the set $\hat{p}^{-1}(K)$ for the case $k = 2$ (i.e., $\hat{p}: \mathbf{RP}^N \to \mathbf{R}^3$). We shall assume that $b_i(\Omega_n) = \dim H_i(\Omega_n; \mathbf{Z}_2)$, $b_i = \dim H_i(\hat{p}^{-1}(K); \mathbf{Z}_2)$, and $\Theta_i: H_i(\hat{p}^{-1}(K), \mathbf{Z}_2) \to H_i(\mathbf{RP}^N, \mathbf{Z}_2)$ are the homomorphisms induced by the embedding $\hat{p}^{-1}(K) \subset \mathbf{RP}^N$, $i \geq 0$. We set $b_j = 1$ for $j < 0$. As before, we suppose that $p \in \mathscr{P}_N^2(\mathbf{R})$ is nondegenerate relative to K. Finally, let

$$m = \min_{\omega \in \Omega} \operatorname{ind} \omega p - 1.$$

THEOREM 2. *Assume either that $K \neq 0$ or that $K = 0$ and $m + 1 < [N/2]$. Then the following assertions are true:*

1) *θ_i is an isomorphism for $i < m - 1$, θ_{m-1} is an epimorphism, and $\theta_j = 0$ for $j > m$.*

2) *$b_{m-1} = b_1(\Omega_{m+1}) + \varepsilon$, $b_m = b_0(\Omega_{m+1}) + b_1(\Omega_{m+2}) + \varepsilon - 1$, and $b_j = b_0(\Omega_{j+1}) + b_1(\Omega_{j+2}) - 1$, where $m < j \leq N$ if $K \neq 0$ and where $m < j \leq [(N-3)/2]$ if $K = 0$; here $\varepsilon = \operatorname{rank} \theta_m$.*

REMARK. If $K = 0$, then $\hat{p}^{-1}(K)$ is a smooth $(N-3)$-dimensional manifold (an intersection of three quadrics in $\mathbf{RP}^N$); consequently, in this case, $b_i = b_{N-3-i}$, $0 \leq i \leq N-3$. In addition, $\Omega = S^2$, and therefore $m + 1 \leq [(N+1)/2]$. The case $K = 0$, $[N/2] \leq m + 1 \leq [(N+1)/2]$ is not taken into account by Theorem 2, and will have to be examined separately.

The equation $\det(\omega P) = 0$ determines a curve of degree $N+1$ in $\mathbf{RP}^2 = \{\{\omega, -\omega\} \mid \omega \in S^2\}$. We will denote this curve by C_p, and will let c_p be the number of connected components of it.

COROLLARY 1. *Under the conditions of Theorem 2, if $K = 0$ and the curve C_p is nonsingular, then*

$$\frac{1}{2} \sum_{i=0}^{N-3} b_i = c_p + 2(m + \varepsilon) - [(N+2)/2].$$

Thus, the sum of the Betti numbers of the manifold $\hat{p}^{-1}(0)$ depends only on m (the number of ovals of the curve C_p), and on one other parameter, ε, equal to either zero or one.

COROLLARY 2. *Let $N \geq 4$. If $\hat{p}^{-1}(0) = \varnothing$ and the curve C_p is nonsingular, then C_p consists of $[(N+1)/2]$ nested ovals. If N is even, then there is one more component, not contractible in $\mathbf{RP}^2$.*

COROLLARY 3. *Let $N \geq 4$. If the curve C_p is nonsingular and does not contain a cluster of $[(N-1)/2]$ nested ovals, then the manifold $\hat{p}^{-1}(0)$ is connected.*

3. In this section $k > 0$ is arbitrary and $p \in \mathscr{P}_N^k(\mathbf{R})$ is nondegenerate relative to the cone $K \subset \mathbf{R}^{k+1}$. Let $V_p^K = \{x \in \mathbf{R}^{N+1} \mid |x| = 1, p(x,x) \in K\}$ be the inverse image of manifold with boundary $\hat{p}^{-1}(K)$ under the standard double covering $S^N \to \mathbf{RP}^N$. The topological invariants of sets of the form V_p^K play an essential role in the local investigation of smooth maps (see [1]). We shall describe the spectral sequence convergent to $H(S^N \backslash V_p^K; \mathbf{Z}_2)$. With the aid of Alexander duality, the cohomology groups of the space V_p^K can be recovered from the cohomology groups $S^N \backslash V_p^K$.

Let $\delta_n: H^i(\Omega_n, \Omega_{n-1}; \mathbf{Z}_2) \to H^{i+1}(\Omega_{n+1}, \Omega_n; \mathbf{Z}_2)$ be the connecting homomorphism in the exact sequence of the triad $(\Omega_{n+1}, \Omega_n, \Omega_{n-1})$. Note that

$$H^i(\Omega_{n+1} \cap \pi^{-1}(\Lambda_n), \Omega_n \cap \pi^{-1}(\Lambda_n)) \approx Y^i(\Omega_{n+1}, \Omega_n)$$

(an excision).

THEOREM 3. *There exists a cohomology spectral sequence* $(\tilde{E}_r, \tilde{d}_r)$, $r \geq 2$, *of the first quadrant converging to* $H(S^N \backslash V_p^K; \mathbf{Z}_2)$, *such that the following assertions are true:*

i) $\tilde{E}_2^{i,j} = H^i(\Omega_{N-j}, \Omega_{N-j-1}; \mathbf{Z}_2)$ *for* $i \geq 0$ *and* $j > 0$.

ii) *There exists a long exact sequence*

$$\cdots \to H^i(\Omega_N; \mathbf{Z}_2) \to \tilde{E}_2^{i,0} \xrightarrow{\mu_i} H^i(\Omega_N, \Omega_{N-1}; \mathbf{Z}_2) \xrightarrow{\Delta_i} H^{i+1}(\Omega_N; \mathbf{Z}_2) \to \cdots,$$

where $\Delta_i: \xi \mapsto \pi^*(\gamma_{N-1}) \cup \xi, i \geq 0$.

iii) *The differential* $\tilde{d}_2: H^i(\Omega_n, \Omega_{n-1}; \mathbf{Z}_2) \to H^{i+2}(\Omega_{n+1}, \Omega_n; \mathbf{Z}_2)$, *where* $0 \leq n \leq N-2$ *and* $i \geq 0$, *is given by*

$$\tilde{d}_2(\xi) = \delta_n(\pi^*(\gamma_{n-1}) \cup \xi) + \pi^*(\gamma_n) \cup \delta_n \xi,$$

and for the differentials $\tilde{d}_2: H^i(\Omega_{N-1}, \Omega_{N-2}; \mathbf{Z}_2) \to \tilde{E}_2^{i+2,0}$, *which take values in the zero row, the relation*

$$\mu_{i+2} \circ \tilde{d}_2(\xi) = \delta_n(\pi^*(\gamma_{N-1}) \cup \xi)$$

is valid.

iv) *All differentials* $\tilde{d}_r, r \geq 2$, *depend only on the filtration* Ω_n, $n \geq 0$, *of the set* Ω *and on the cohomology classes* $\pi^*(\gamma_n) \in H^1(\pi^{-1}(\Lambda_n))$.

REMARKS. There are explicit expressions for the higher differentials $\tilde{d}_r, r > 2$, which shall be considered later in a detailed publication.

The results presented here are the natural extension of joint work conducted with R. V. Gamkrelidze [1], to whom I wish to express my gratitude for our valuable discussions and for his constant support.

All-Union Institute of Scientific and Technical Information
Moscow

Received 1/OCT/86

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Translated by R. LOGAN