On the Long-time Behaviour of Dissipative Systems

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A natural mechanical system on a Riemannian manifold M:

trajectories: $\gamma : \mathbb{R} \to M,\ \dot{\gamma} \in T_{\gamma(t)}M$;

kinetic energy: $\frac{1}{2}|\dot{\gamma}(t)|^2$;

potential energy: $V(\gamma(t))$, where $V : M \to \mathbb{R}$;

Hamiltonian: $H(p,q) = \frac{1}{2}|p|^2 + V(q)$,

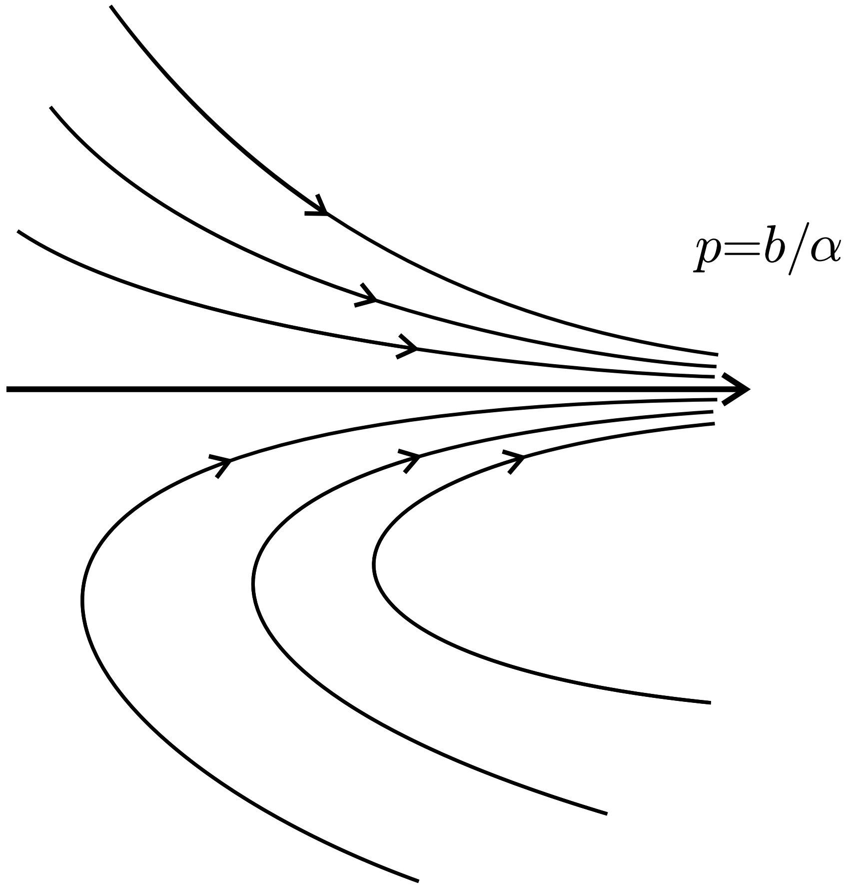
where $p \in T_q^*M,\ |p| = \max\{\langle p, \xi\rangle : \xi \in T_qM,\ |\xi| = 1\}$.

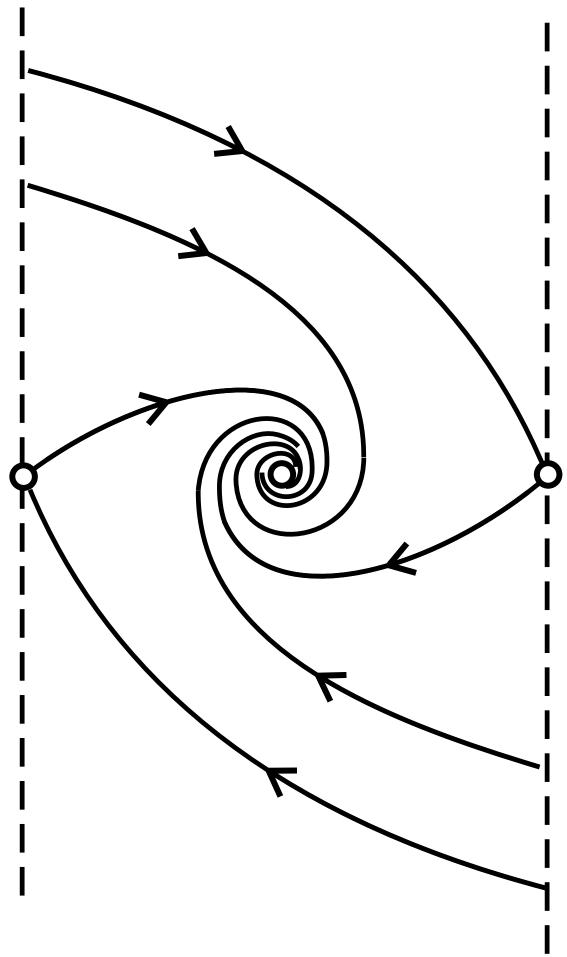
System with an isotropic dissipation:

$$\begin{cases} \dot{p} & = & -\frac{\partial H}{\partial q}(p,q) - \alpha p \\ \dot{q} & = & \frac{\partial H}{\partial p}(p,q), \end{cases}$$

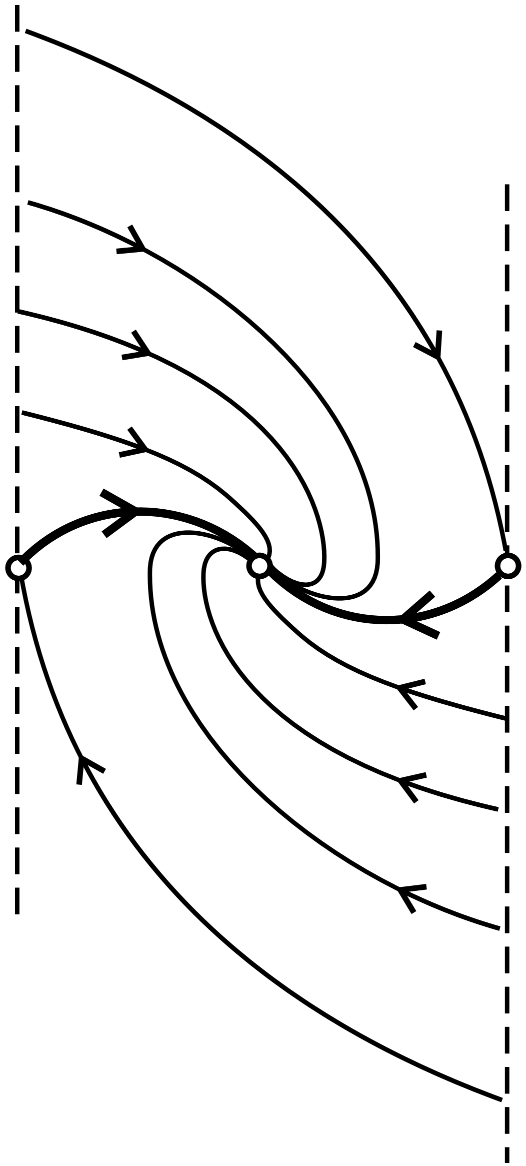
where $\alpha > 0$ is a *friction coefficient*.

$M = \mathbb{R},\ V(q) = bq.$





$V(q) = b\cos q,\ \frac{\alpha^2}{4} < |b|.$



$V(q) = b\cos q,\ \ |b| < \frac{\alpha^2}{4}.$

Definition 1. *"Potential stationary flow" is a gradient vector field* ∇u*, where* $u \in C^2(M)$ *and* $\{d_q u : q \in M\} \subset T^*M$ *is an invariant submanifold of our system.*

In particular, $\dot{\gamma}(t) = \nabla_{\gamma(t)} u$ implies that $t \mapsto d_{\gamma(t)} u$ is a solution.

Definition 2. *The curvature of the Hamiltonian* H *at* $p \in T_q^*M$ *is a self-adjoint linear operator* $R^H_{(p,q)} : T_q^*M \to T_q^*M$ *defined by the formula*

$$R^H_{(p,q)}\xi = \mathfrak{R}(\xi, p)p + (\nabla_q^2 V)\xi, \quad \xi \in T_q^*M,$$

where ∇ *is the covariant derivative and* $\mathfrak{R}$ *the Riemannian curvature.*

Assume that M is complete, $\mathfrak{R}$ and $\nabla^2 V$ are uniformly bounded. Let $\Phi_t : T^*M \to T^*M,\ t \in \mathbb{R}$, be the flow generated by our system, $\Omega_c = \{(p,q) \in T^*M : |p| \le c\}$.

Theorem. *If* $R^H_{(p,q)} < \frac{\alpha^2}{4} I,\ \forall (p,q)$ *s. t.* $H(p,q) \le \max V$, *then* $\exists$ *a potential stationary flow* ∇u *s. t.*

$$\Phi_t(\Omega_c) \to \{d_q u : q \in M\} \ as\ t \to +\infty$$

with an exponential rate, $\forall c > 0$.

$\{d_q u : q \in M\}$ *is a normally stable submanifold of* Φ^t.

If M *is compact and* $R^H_{(p,q)} < \frac{(k-1)\alpha^2}{k^2} I$, *then* $u \in C^k(M)$.

The map $(H,\alpha) \mapsto u$ *is continuous in the* C^2*-topology.*

The least action principle:

$$u(q) = -\inf \left\{ \int_{-\infty}^{0} e^{\alpha t} \left(\frac{1}{2} |\dot{\gamma}(t)|^2 - V(\gamma(t)) \right) dt : \gamma(0) = q \right\}.$$

The modified Hamilton–Jacobi equation:

$$H(d_q u, q) + \alpha u(q) = 0.$$

Smaller dissipation:

Consider a Markov process on measures: $A : \mu \mapsto \alpha \int\limits_0^\infty e^{-\alpha t} \Phi_*^t \mu \, dt$.

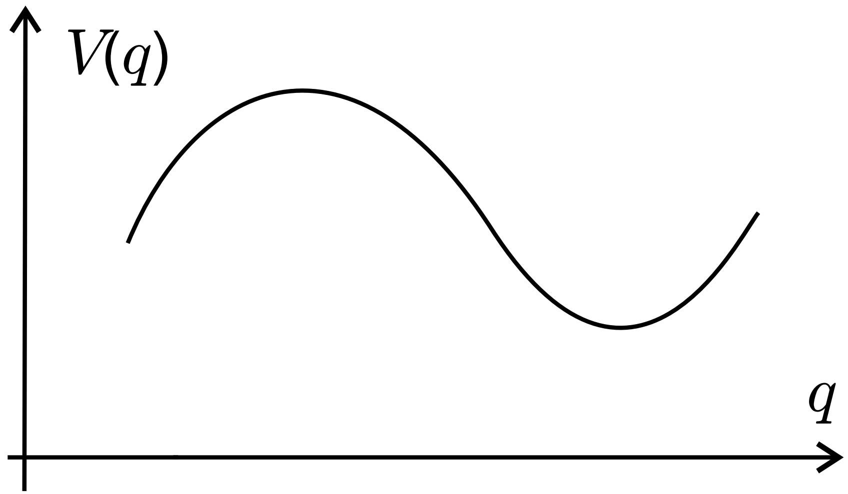
The limiting "velocity distribution" on $T_q^* M$ is limit of conditional probability measures:

$$\nu_q = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left((A^n \mu)\Big|_{O_\varepsilon(T_q^* M)} \Big/ A^n \mu(O_\varepsilon(T_q^* M)) \right),$$

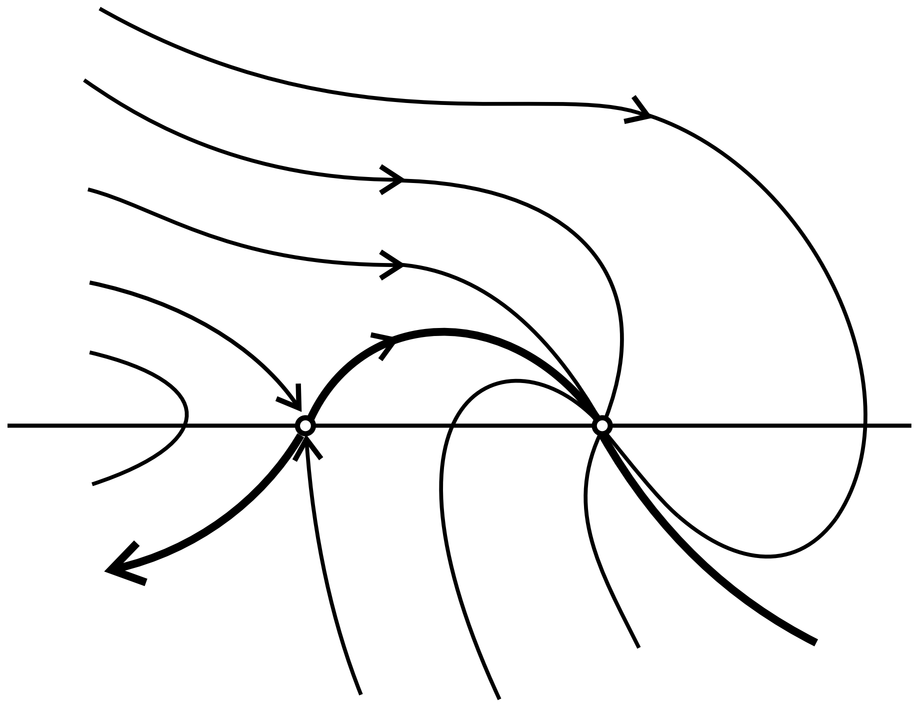
where μ is a volume measure on Ω_c, $q \in M$.

Proposition. *If M is compact and V is a Morse function, then for a. e. $q \in M$ there exists an atomic ν_q that does not depend on μ and c; $supp(\nu_q) \subset T_q^* M \cap \{unstable\ subman.\ of\ the\ equilibria\}$.*

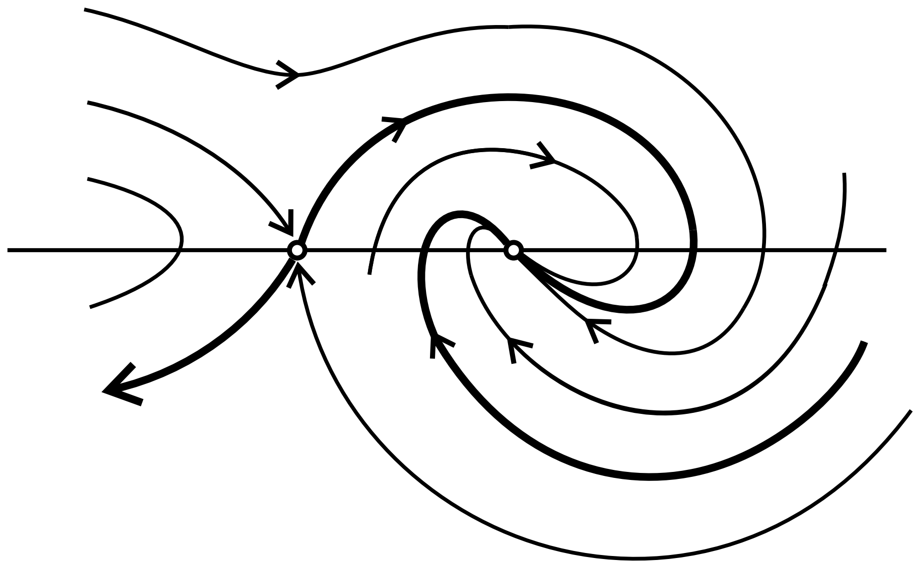
One degree of freedom:



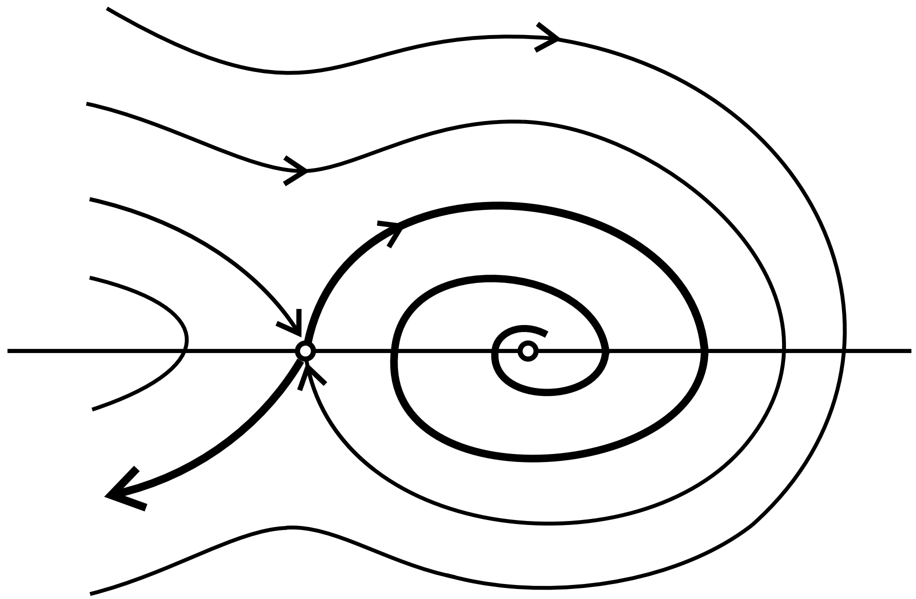
$\max V'' < \frac{\alpha^2}{4}$



$V''(q_{min}) < \frac{\alpha^2}{4} \ll \max V''$



$\frac{\alpha^2}{4} < V''(q_{min})$



$$\lim_{\alpha \to 0} \mu_q^\alpha = \rho_q(p)dp,$$

$\rho_q(p) = c_q Area\{z : H(z) \leq H(p,q)\}$, if $H(p,q) < \max V$; otherwise $\rho_q(p) = 0$.