

Dynamics Control by a Time-varying Feedback

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Abstract

We consider a smooth bracket generating control-affine system in \mathbb{R}^d and show that any orientation preserving diffeomorphism of \mathbb{R}^d can be approximated, in the very strong sense, by a diffeomorphism included in the flow generated by a time-varying feedback control which is polynomial with respect to the state variables and trigonometric-polynomial with respect to the time variable .

1 Introduction

We consider a control-affine system:

$$\dot{q} = f_0(q) + \sum_{i=1}^m u_i f_i(q), \quad q \in \mathbb{R}^d, \quad (1)$$

with $u = (u_1, \dots, u_m) \in \mathbb{R}^m$, where f_i are smooth (i. e. C^∞) vector fields on \mathbb{R}^d . Moreover, we assume that $\{f_1, \dots, f_m\}$ is a bracket generating family of vector fields, i. e. $\text{Lie}_q\{f_1, \dots, f_m\} = \mathbb{R}^d$, for any $q \in \mathbb{R}^d$, where $\text{Lie}_q\{f_1, \dots, f_m\}$ is the linear hull of all iterated Lie brackets of the fields f_1, \dots, f_m evaluated at q .

Feedback control (or time-invariant feedback control) is a mapping

$$v = (v_1, \dots, v_m) : \mathbb{R}^d \rightarrow \mathbb{R}^m.$$

We can put $u_i = v_i(q)$ and obtain a *closed loop* system

$$\dot{q} = f_0(q) + \sum_{i=1}^m v_i(q) f_i(q), \quad q \in \mathbb{R}^d. \quad (2)$$

It is very interesting to know what kind of dynamics we can realize by an appropriate choice of the feedback control. Of course, a smooth or at least Lipschitz feedback is preferable if we want system (2) to correctly define a dynamical system. Unfortunately, we cannot expect too much. In particular, if $f_0 = 0$ then system (2) with a continuous feedback control cannot have locally asymptotically stable equilibria as it was observed by R. Brockett [2].

J.-M. Coron suggested to use time-varying periodic with respect to time feedback controls

$$v : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad v(t+1, q) = v(t, q), \quad q \in \mathbb{R}^d, \quad t \in \mathbb{R},$$

for system (1) and proved that asymptotic stability can be successfully achieved by a C^∞ time-varying feedback (see [4, 5] or [6, section 11.2]).

In this paper, we focus on the transformation $q(0) \mapsto q(1)$ in virtue of the system

$$\dot{q} = f_0(q) + \sum_{i=1}^m v_i(t, q) f_i(q), \quad q \in \mathbb{R}^d. \quad (3)$$

associated to the time-varying feedback control and demonstrate that practically any type of discrete-time dynamics can be realized in this way.

More precisely, let $\Phi_v : q(0) \mapsto q(1)$ be the transformation of \mathbb{R}^d which sends the initial value of any solution of system (3) to its value at $t = 1$. We denote by $\text{Diff}_0 \mathbb{R}^d$ the group of orientation preserving diffeomorphisms of \mathbb{R}^d . Let $P \in \text{Diff}_0 \mathbb{R}^d$, \mathcal{O}_P be a C^∞ -neighborhood of P and N be a positive integer. We prove (Theorem 8) that there exists a polynomial with respect to q and trigonometric polynomial with respect to t time-varying feedback control v such that $\Phi_v \in \mathcal{O}_P$ and the N -jets of Φ_v and P at the origin coincide. Moreover, construction of the time-varying feedback v is surprisingly simple.

Let us fix some notations. We denote by $\text{Diff}(\mathbb{R}^d)$ the group of diffeomorphisms of \mathbb{R}^d and by $\text{Vec } \mathbb{R}^d$ the space of vector fields on \mathbb{R}^d . We assume that $\text{Diff}(\mathbb{R}^d)$, $\text{Diff}_0(\mathbb{R}^d)$, $\text{Vec } \mathbb{R}^d$, and $C^\infty(\mathbb{R}^d)$ are endowed with the standard topology of the uniform convergence of the partial derivatives of any order on any compact of \mathbb{R}^d .

Given a set \mathcal{F} of vector fields on \mathbb{R}^d , we denote by:

$$\text{Gr}\mathcal{F} = \{e^{t_1 f_1} \circ \dots \circ e^{t_k f_k} \mid t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\},$$

the subgroup of $\text{Diff} \mathbb{R}^d$ generated by flows of vector fields in \mathcal{F} , and by:

$$\text{Gr}_S \mathcal{F} = \{e^{a_1 f_1} \circ \dots \circ e^{a_k f_k} \mid a_i \in C^\infty(\mathbb{R}^d), f_i \in \mathcal{F}, k \in \mathbb{N}\},$$

respectively the subgroup of $\text{Diff}\mathbb{R}^d$ generated by flows of vector fields in \mathcal{F} rescaled by smooth functions on \mathbb{R}^d . We consider time-varying vector fields $V_t(q)$ on \mathbb{R}^d that are smooth with respect to $q \in \mathbb{R}^d$ and locally integrable with respect to $t \in \mathbb{R}$. All vector fields under consideration are supposed to satisfy the growth condition $V_t(q) \leq \varphi(t)(1 + |q|)$, where φ is a locally integrable function. This condition guarantees completeness of the vector field.

Given a time-varying vector field $V_t(q)$ on \mathbb{R}^d , let

$$P_t : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad t \in \mathbb{R},$$

be the (non-stationary) flow generated by the differential equation $\dot{q} = V_t(q)$. In other words,

$$\begin{cases} \frac{\partial P_t}{\partial t}(q) = V_t(P_t(q)), \\ P_0(q) \equiv q. \end{cases}$$

In the following we will use the ‘‘chronological’’ notation: $P_t = \overrightarrow{\exp} \int_0^t V_\tau d\tau$, to denote such a flow.

Recall that if \mathcal{F} is a bracket generating family of vector fields then by Rashevski–Chow Theorem (see [8, 3]), for every q_0 , the orbit \mathcal{O}_{q_0} of the family is the whole space \mathbb{R}^d moreover, according to the Orbit Theorem of Sussmann([9] or [1, chapter 5]), any smooth vector field can be presented as a linear combination of vector fields from \mathcal{F} transformed by diffeomorphisms from $\text{Gr}\mathcal{F}$. In particular it is possible to take X_1, \dots, X_d linearly independent at a point $q \in \mathbb{R}^d$ and such that $X_i = P^i_* f_i$, $i = 1, \dots, d$ with $P^i \in \text{Gr}\mathcal{F}$ and $f_i \in \mathcal{F}$.

The main result proved in this paper is as follows.

Theorem. *Let $\{f_1, f_2, \dots, f_m\}$ be a bracket generating family of vector fields on \mathbb{R}^d . Consider the control system*

$$\dot{q} = f_0(q) + \sum_{i=1}^m u_i(t, q) f_i(q), \quad q \in \mathbb{R}^d, \quad (4)$$

with controls u_i such that:

- (i) u_i is polynomial with respect to $q \in \mathbb{R}^d$;
- (ii) u_i is a trigonometric polynomial with respect to $t \in [0, 1]$;

for every $i = 1, \dots, m$.

Fix positive integers N and k , $\varepsilon > 0$, and B ball in \mathbb{R}^d . For any $\Phi \in \text{Diff}_0(\mathbb{R}^d)$, there exist controls $u_1(t, q), \dots, u_m(t, q)$ such that, if P is the flow at time 1 of the system, then

$$J_0^N(P) = J_0^N(\Phi) \quad \text{and} \quad \|P - \Phi\|_{C^k(B)} < \varepsilon.$$

Proof is divided into four sections. In Section 2 we consider a bracket generating family of vector fields closed under multiplication by smooth functions on \mathbb{R}^d , say \mathcal{F} , then we prove that the group of diffeomorphisms generated by flows of vector fields in this family is dense in the connected component of the identity of the group of diffeomorphisms. In Section 3 we use classical Implicit Function Theorem to prove that the N -th jet of a diffeomorphism in $\text{Diff}_0(\mathbb{R}^d)$ sufficiently close to the identity can be presented as the N -th jet of an element in $\text{Gr}\mathcal{F}$. Then using Proposition 2 we can extend this result to every diffeomorphism in $\text{Diff}_0(\mathbb{R}^d)$. The results of Section 2 and Section 3 are combined together in Section 4 to prove that it is possible to find an element in the group $\text{Gr}\mathcal{F}$ with the same N -th jet of a given diffeomorphism and also close to it in the C^∞ -topology. This result, as showed in Section 5, implies the main result in the driftless case, namely $f_0 \equiv 0$, and with controls $u_i(t, \cdot)$ that are piecewise constant with respect to t . Therefore we use Brouwer Fixed Point to prove that it is possible to perturb the map

$$(u_1, \dots, u_m) \mapsto J_0^N \left(\overrightarrow{\exp} \int_0^1 \sum_{i=1}^m u_i(t, \cdot) f_i(\cdot) dt \right),$$

without losing surjectivity. This argument leads to the proof of the Theorem.

2 An approximation result

We start with a simple modification of a standard relaxation result (see [1, chapter 8] or [7]). Its proof is done in the appendix for convenience of the reader.

Proposition 1. *Let X_1, \dots, X_k be smooth vector fields on \mathbb{R}^d and \mathcal{A} be a closed subspace of $C^\infty(\mathbb{R}^d)$. Then, for any time-varying vector field of the form*

$$V_t = \sum_{i=1}^k a_i(t, \cdot) X_i,$$

where $a_i(t, \cdot) \in \mathcal{A}$ and $0 \leq a_i(t, q) \leq \varphi(t)$ for some locally integrable φ , $i = 1, \dots, k$, there exists a sequence of time-varying, piecewise constant with respect to t , vector fields Z_t^n such that

$$Z_t^n \in \{aX_i \mid a \in \mathcal{A}, i = 1, \dots, k\}, \quad \text{for any } t \in [0, 1]$$

and

$$\overrightarrow{\exp} \int_0^t Z_\tau^n d\tau \longrightarrow \overrightarrow{\exp} \int_0^t V_\tau d\tau, \quad \text{as } n \rightarrow \infty$$

in the standard topology and uniformly with respect to $t \in [0, 1]$.

Proposition 2 (Approximation). *Let $\mathcal{F} \subseteq \text{Vec } \mathbb{R}^d$ be a bracket generating family of vector fields on \mathbb{R}^d such that*

$$af \in \mathcal{F} \text{ for any } a \in C^\infty(\mathbb{R}^d), f \in \mathcal{F}. \quad (5)$$

Then, for any orientation preserving diffeomorphism P of \mathbb{R}^d , there exists a sequence $\{P_n\}_n \subset \text{Gr}\mathcal{F}$ such that

$$P_n \longrightarrow P, \quad \text{as } n \rightarrow \infty,$$

in the standard topology.

Proof. First, note that any orientation preserving diffeomorphism of \mathbb{R}^d is isotopic to the identity. Indeed, let P be an orientation preserving diffeomorphism of \mathbb{R}^d . We can suppose without loss of generality that P fixes the origin just taking the isotopy $H^1(t, \cdot) = P - (1-t)P(0)$. Now, rename for simplicity $P := H^1(0, \cdot)$ and consider another isotopy

$$H^2(t, q) = P(tq)/t, \quad t \in (0, 1], \quad \text{and} \quad H^2(0, q) = \lim_{t \rightarrow 0} P(tq)/t.$$

Since P is orientation preserving then $H^2(0, \cdot)$ belongs to the connected component of the identity of the group of linear invertible operators on \mathbb{R}^d , $GL^+(d, \mathbb{R})$.

Let $P^t \subset \text{Diff}_0(\mathbb{R}^d)$ be a path such that $P^0 = \text{Id}$ and $P^1 = P$. Consider the time-varying vector field

$$V_t = (P^t)^{-1} \circ \frac{d}{dt} P^t.$$

We have

$$\overrightarrow{\exp} \int_0^t V_\tau d\tau = P^t$$

Recall that, since \mathcal{F} is bracket generating then it is possible to take X_1, \dots, X_d such that $X_i = P_*^i f_i$ with $P^i \in \text{Gr}\mathcal{F}$, $f_i \in \mathcal{F}$, $i = 1, \dots, d$ and

$$V_t = \sum_{i=1}^d a_i(t, \cdot) X_i,$$

where $a_i(t, \cdot) \in C^\infty(\mathbb{R}^d)$ for any $t \in [0, 1]$.

By Proposition 1 there exists a sequence $Z_t^n \in \{\alpha X_i \mid \alpha \in C^\infty(\mathbb{R}^d), i = 1, \dots, d\}$ such that

$$\overrightarrow{\exp} \int_0^t Z_\tau^n d\tau \rightarrow P^t, \quad \text{as } n \rightarrow \infty,$$

and the convergence is uniform with respect to $t \in [0, 1]$.

Let $P_n := \overrightarrow{\exp} \int_0^1 Z_t^n dt$, then

$$P_n \rightarrow P, \quad \text{as } n \rightarrow \infty.$$

It remains to prove that $P_n \in \text{Gr}\mathcal{F}$ for every n . Since Z_t^n is piecewise constant in t , so, for any fixed $n \in \mathbb{N}$, there exist disjoint segments I_1, \dots, I_{h_n} covering $[0, 1]$ and functions $\alpha_1, \dots, \alpha_{h_n} \in C^\infty(\mathbb{R}^d)$ such that

$$Z_t^n = \alpha_k X_{i_k} \quad \forall t \in I_k, \quad k = 1, \dots, h_n.$$

Hence

$$\begin{aligned} P_n &= \overrightarrow{\exp} \int_0^1 Z_t^n dt, \\ &= e^{|I_1| \alpha_1 X_{i_1}} \circ \dots \circ e^{|I_{h_n}| \alpha_{h_n} X_{i_{h_n}}} \\ &= e^{|I_1| \alpha_1 P_*^{i_1} f_{i_1}} \circ \dots \circ e^{|I_{h_n}| \alpha_{h_n} P_*^{i_{h_n}} f_{i_{h_n}}} \\ &= (P^{i_1})^{-1} \circ e^{|I_1| (\alpha_1 \circ P^{i_1}) f_{i_1}} \circ P^{i_1} \circ \dots \\ &\quad \circ (P^{i_{h_n}})^{-1} \circ e^{|I_{h_n}| (\alpha_{h_n} \circ P^{i_{h_n}}) f_{i_{h_n}}} \circ P^{i_{h_n}}, \end{aligned} \tag{6}$$

now let $\beta_k = |I_k|(\alpha_k \circ P^{i_k})$, then

$$P_n = (P^{i_1})^{-1} \circ e^{\beta_1 f_{i_1}} \circ P^{i_1} \circ \dots \circ (P^{i_{h_n}})^{-1} \circ e^{\beta_{h_n} f_{i_{h_n}}} \circ P^{i_{h_n}},$$

and $P_n \in \text{Gr}\mathcal{F}$ by assumption (5). \square

In other words we proved that if \mathcal{F} is a bracket generating family of vector fields then $\text{Gr}_S\mathcal{F}$ is dense in the connected component of the identity of $\text{Diff}\mathbb{R}^d$ endowed with the standard C^∞ -topology.

3 Get the jet

In this section, given a bracket generating family of vector fields \mathcal{F} , we find a diffeomorphism in the group $\text{Gr}_S\mathcal{F}$ whose N -th jet is exactly the N -th jet of a given diffeomorphism on \mathbb{R}^d . The main tool used is the classical Implicit Function Theorem.

Proposition 3. *Let \mathcal{F} be a bracket generating family of vector fields on \mathbb{R}^d and $N > 0$ a positive integer.*

For any diffeomorphism $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ sufficiently close to the identity there exists $P \in \text{Gr}_S\mathcal{F}$ such that

$$J_0^N(P) = J_0^N(\Phi).$$

Proof. Consider a frame of vector fields linearly independent in $0 \in \mathbb{R}^d$, X_1, \dots, X_d . Let \mathbf{X} be the space of polynomials of degree less or equal than N in d variables and let \mathbf{Y} be the jet-group of N -th order jets at 0 of smooth orientation preserving diffeomorphisms, i.e. $\mathbf{Y} = J_0^N(\text{Diff}_0(\mathbb{R}^d))$. Note that $\dim \mathbf{X} < \infty$ and $\dim \mathbf{Y} < \infty$.

Consider the map

$$F : \begin{array}{ccc} \mathbf{X}^d & \longrightarrow & \mathbf{Y} \\ (u_1, \dots, u_d) & \longmapsto & J_0^N(e^{u_1 X_1} \circ \dots \circ e^{u_d X_d}) \end{array} \quad (7)$$

We want to prove that Implicit Function Theorem applies. Let us compute the differential of F at $0 \in \mathbf{X}^d$

$$\begin{aligned} D_0 F(a_1, \dots, a_d) &= \left. \frac{\partial F}{\partial u_1} \right|_{u_1=\dots=u_d=0} a_1 + \dots + \left. \frac{\partial F}{\partial u_d} \right|_{u_1=\dots=u_d=0} a_d \\ &= a_1 J_0^N(X_1) + \dots + a_d J_0^N(X_d). \end{aligned}$$

We claim that $D_0F : \mathbf{X}^d \rightarrow T_{\text{Id}}\mathbf{Y}$ is invertible. In fact

$$T_{\text{Id}}\mathbf{Y} = T_{\text{Id}}J_0^N(\text{Diff}_0(\mathbb{R}^d)) = J_0^N(T_{\text{Id}}\text{Diff}_0(\mathbb{R}^d)) = J_0^N(\text{Vec}(\mathbb{R}^d)),$$

so for every $V \in J_0^N(\text{Vec}(\mathbb{R}^d))$ there exist b_1, \dots, b_d such that

$$\begin{aligned} V &= J_0^N(b_1X_1 + \dots + b_dX_d) \\ &= J_0^N(b_1)J_0^N(X_1) + \dots + J_0^N(b_d)J_0^N(X_d). \end{aligned}$$

Every element $V \in T_{\text{Id}}\mathbf{Y}$ is image of d polynomials of degree less or equal than N , $a_i = J_0^N(b_i)$. Therefore there exists \mathcal{O} neighborhood of Id in \mathbf{Y} such that F is locally surjective on \mathcal{O} . Namely, for every $\psi \in \mathcal{O}$, there exist $(u_1, \dots, u_d) \in \mathbf{X}^d$ such that $F(u_1, \dots, u_d) = \psi$.

If Φ is sufficiently close to the identity, then $J_0^N(\Phi) \in \mathcal{O}$. Therefore there exist polynomials $v_1, \dots, v_d \in \mathbf{X}^d$ such that

$$J_0^N(e^{v_1X_1} \circ \dots \circ e^{v_dX_d}) = J_0^N(\Phi).$$

It remains to prove that $P = e^{v_1X_1} \circ \dots \circ e^{v_dX_d} \in \text{Gr}_S\mathcal{F}$, but according to Orbit Theorem, for $i = 1, \dots, d$, we have that $X_i = P_*^i f_i$, where $f_i \in \mathcal{F}$ and $P^i \in \text{Gr}\mathcal{F}$. Let $P^i = e^{t_1^i f_1^i} \circ e^{t_2^i f_2^i} \circ \dots \circ e^{t_{s_i}^i f_{s_i}^i}$ with $f_j^i \in \mathcal{F}$. Therefore

$$\begin{aligned} P &= e^{v_1 P_*^1 f_1} \circ \dots \circ e^{v_d P_*^d f_d} \\ &= P^1 \circ e^{(P^1)^{-1}(v_1)f_1} \circ (P^1)^{-1} \circ \dots \circ P^d \circ e^{(P^d)^{-1}(v_d)f_d} \circ (P^d)^{-1} \\ &= \underbrace{e^{t_1^1 f_1^1} \circ \dots \circ e^{t_{s_1}^1 f_{s_1}^1}}_{P^1} \circ e^{(P^1)^{-1}(v_1)f_1} \circ \underbrace{e^{-t_{s_1}^1 f_{s_1}^1} \circ \dots \circ e^{-t_1^1 f_1^1}}_{(P^1)^{-1}} \circ \dots \\ &\quad \circ \underbrace{e^{t_1^d f_1^d} \circ \dots \circ e^{t_{s_d}^d f_{s_d}^d}}_{P^d} \circ e^{(P^d)^{-1}(v_d)f_d} \circ \underbrace{e^{-t_{s_d}^d f_{s_d}^d} \circ \dots \circ e^{-t_1^d f_1^d}}_{(P^d)^{-1}} \\ &= e^{w_1 g_1} \circ \dots \circ e^{w_\ell g_\ell}, \end{aligned} \tag{8}$$

with $g_1, \dots, g_\ell \in \mathcal{F}$ and $\ell = d + 2(s_1 + \dots + s_d)$. Therefore $P \in \text{Gr}_S\mathcal{F}$ and the proposition follows. \square

Now consider any diffeomorphism $\Phi \in \text{Diff}_0(\mathbb{R}^d)$. By Proposition 2 there exists a sequence $\{P_n\}_n \subset \text{Gr}_S\mathcal{F}$ that tends to Φ . So for n sufficiently large last Proposition applies to $P_n^{-1} \circ \Phi$ and we have the following result.

Corollary 4. *Let $\mathcal{F} \subseteq \text{Vec}\mathbb{R}^d$ be a bracket generating family of vector fields and $N > 0$ a positive integer. For every $\Phi \in \text{Diff}_0(\mathbb{R}^d)$ there exists $P \in \text{Gr}_S\mathcal{F}$ such that*

$$J_0^N(P) = J_0^N(\Phi).$$

4 Geometric statement of the main result

The purpose of this section is to link the results of the last two sections in order to find an element in the group $\text{Gr}_S\mathcal{F}$ with the same N -th jet of a given diffeomorphism and also close to it in the C^∞ -topology.

Proposition 5. *Let $\mathcal{F} \subseteq \text{Vec}\mathbb{R}^d$ be a bracket generating family of vector fields. Let N and k be positive integers, $\varepsilon > 0$, and B ball of \mathbb{R}^d . For any $\Phi \in \text{Diff}_0(\mathbb{R}^d)$, there exists $P \in \text{Gr}_S\mathcal{F}$ such that*

$$J_0^N(P) = J_0^N(\Phi) \quad \text{and} \quad \|P - \Phi\|_{C^k(B)} < \varepsilon.$$

Proof. We can suppose that $J_0^N(\Phi) = \text{Id}$. Indeed, by Corollary 4, there exists $Q \in \text{Gr}_S\mathcal{F}$ such that $J_0^N(Q) = J_0^N(\Phi)$. Then we consider, instead of Φ , the diffeomorphism $\Psi = \Phi \circ Q^{-1}$ which has trivial jet.

The idea of the proof is the same of Proposition 2. Since $J_0^N(\Phi) = \text{Id}$, then Φ can be written as

$$\Phi(x) = x + g(x),$$

with $J_0^N(g) = 0$. Consider the one parameter family of diffeomorphisms with trivial jet

$$\Phi_t(x) = x + tg(x).$$

This is a path in $\text{Diff}(\mathbb{R}^d)$ from $\Phi_0 = \text{Id}$ to $\Phi_1 = \Phi$. Let V_t a nonautonomous vector field such that

$$\Phi_t = \overrightarrow{\exp} \int_0^t V_\tau d\tau.$$

Let X_1, \dots, X_d be a frame of vector fields linearly independent at 0 such that $X_i = \text{Ad } P^i f_i, P^i \in \text{Gr}\mathcal{F}, f_i \in \mathcal{F}$. Therefore

$$V_t = \sum_{i=1}^d a_i(t, \cdot) X_i,$$

with $a_i(t, \cdot) \in C^\infty(\mathbb{R}^d)$ for any $t \in [0, 1]$. Note that, since $J_0^N(\Phi_t) = \text{Id}$ and the vector fields X_i are linearly independent, then $J_0^N(a_i(t, \cdot)) = 0$ for any $t \in [0, 1]$.

Now let \mathcal{A} be the closed subspace of $C^\infty(\mathbb{R}^d)$ of smooth functions α such that $J_0^N(\alpha) = 0$. By Proposition 1 there exists a piecewise constant in t sequence $Z_t^n \in \{\alpha X_i \mid \alpha \in \mathcal{A}, i = 1 \dots, d\}$ such that

$$\overrightarrow{\exp} \int_0^t Z_\tau^n d\tau \rightarrow \Phi_t, \quad \text{as } n \rightarrow \infty,$$

in the C^∞ -topology and uniformly with respect to $t \in [0, 1]$.
 So, if $P_n = \overrightarrow{\exp} \int_0^1 Z_\tau^n d\tau$, then

$$P_n \rightarrow \Phi, \quad \text{as } n \rightarrow \infty,$$

in the standard topology. Now, for any n , we have that $P_n \in \text{Gr}_S \mathcal{F}$ for the chain of equalities (6). Moreover P_n has trivial jet. Indeed, since the sequence Z_t^n is piecewise constant, then there exist intervals I_1, \dots, I_h such that

$$Z_t^n = \alpha_i X_{j_i} \quad \text{for any } t \in I_i,$$

with $j_i \in \{1, \dots, d\}$. So

$$\begin{aligned} J_0^N(P_n) &= J_0^N \left(\overrightarrow{\exp} \int_0^1 Z_t^n dt \right) \\ &= J_0^N (e^{|I_1| \alpha_1 X_{j_1}}) \circ \dots \circ J_0^N (e^{|I_h| \alpha_h X_{j_h}}) \\ &= e^{|I_1| J_0^N(\alpha_1) J_0^N(X_{j_1})} \circ \dots \circ e^{|I_h| J_0^N(\alpha_h) J_0^N(X_{j_h})} \\ &= \text{Id}, \end{aligned}$$

and the result is proved. □

5 Main Result

In this last section we prove the main result using Proposition 5 and a fixed point argument. We start giving an equivalent formulation of Proposition 5 in terms of flows of the system:

$$\dot{q} = \sum_{i=1}^m u_i(t, q) f_i(q), \quad q \in \mathbb{R}^d. \quad (9)$$

Suppose that $\mathcal{F} = \{f_1, \dots, f_m\}$ is a bracket generating family of vector field on \mathbb{R}^d . By Proposition 3 there exist smooth functions a_1, \dots, a_k such that

$$J_0^N(\Phi) = J_0^N (e^{a_1 f_{i_1}} \circ \dots \circ e^{a_k f_{i_k}}), \quad (10)$$

with $i_j \in \{1, \dots, m\}$. Now there exist m functions $u_1(t, q), \dots, u_m(t, q)$ piecewise constant in t such that

$$J_0^N(\Phi) = J_0^N \left(\overrightarrow{\exp} \int_0^1 \sum_{i=1}^m u_i(t, \cdot) f_i dt \right). \quad (11)$$

We then proved the following Lemma.

Lemma 6. *Let $\{f_1, f_2, \dots, f_m\}$ be a bracket generating family of vector fields on \mathbb{R}^d . Consider the control system*

$$\dot{q} = \sum_{i=1}^m u_i(t, q) f_i(q), \quad q \in \mathbb{R}^d, \quad (12)$$

with controls u_i piecewise constant with respect to $t \in [0, 1]$ and smooth with respect to $q \in \mathbb{R}^d$, for every $i = 1, \dots, m$.

Let N and k be positive integers, $\varepsilon > 0$, and B ball in \mathbb{R}^d . For any $\Phi \in \text{Diff}_0(\mathbb{R}^d)$, there exist controls $u_1(t, q), \dots, u_m(t, q)$ such that, if P is the flow at time 1 of system (12), then

$$J_0^N(P) = J_0^N(\Phi) \quad \text{and} \quad \|P - \Phi\|_{C^k(B)} < \varepsilon.$$

What remains is to prove last result adding a drift f_0 to system (12). Moreover we want to have a certain regularity for the controls. Both these results can be proved with a fixed point argument. Indeed, let \mathbf{U} the space of m -uples of controls $u(t, q)$ piecewise constant in t and smooth with respect to q . Consider the map

$$\begin{aligned} \tilde{F} : \quad \mathbf{U} &\longrightarrow J_0^N(\text{Diff}_0(\mathbb{R}^d)) \\ (u_1, \dots, u_m) &\longmapsto J_0^N \left(\overrightarrow{\exp} \int_0^1 \sum_{i=1}^m u_i(t, \cdot) X_i dt \right). \end{aligned} \quad (13)$$

This map is continuous and, by last Lemma, it is also surjective. Moreover \tilde{F} has a continuous right inverse. Indeed there is a smooth correspondence between the time-varying feedback controls u_1, \dots, u_m and the functions a_1, \dots, a_k in (10). By Implicit Function Theorem applied to map F in (7), we have that the right inverse of F is continuous and so is the right inverse of \tilde{F} .

In the next Lemma we prove, using a fixed point argument, that every small perturbation of a continuous surjective map with continuous right inverse and with finite dimensional target space is surjective too.

Lemma 7. *Let X be a topological space, $\varepsilon > 0$, and $F : X \rightarrow \mathbb{R}^n$ a continuous and surjective with continuous right inverse. If $G : X \rightarrow \mathbb{R}^n$ is continuous and such that $\sup_{x \in K} |F(x) - G(x)| < \varepsilon$ for any $K \subseteq X$ compact, then G is surjective.*

Proof. Let F^{-1} be the right inverse of F and define, for every \bar{y} in \mathbb{R}^n , the map $\chi_{\bar{y}}(y) = y - G \circ F^{-1}(y) + \bar{y}$. Let $\delta = \varepsilon + \|\bar{y}\|$, then for every $y \in B_\delta = B_\delta(0)$ we have

$$\begin{aligned} \|\chi_{\bar{y}}(y)\| &\leq \|y - G \circ F^{-1}(y)\| + \|\bar{y}\| \\ &\leq \sup_{y \in B_\delta} \|y - G \circ F^{-1}(y)\| + \|\bar{y}\| \\ &\leq \sup_{x \in F^{-1}(B_\delta)} \|F(x) - G(x)\| + \|\bar{y}\| \\ &< \varepsilon + \|\bar{y}\| \\ &= \delta. \end{aligned}$$

So $\chi_{\bar{y}}(B_\delta) \subseteq B_\delta$ and, since the map $\chi_{\bar{y}}$ is continuous, by Brouwer Fixed Point Theorem, there exists $\tilde{y} \in B_\delta$ such that

$$\chi_{\bar{y}}(\tilde{y}) = \tilde{y},$$

namely

$$G \circ F^{-1}(\tilde{y}) = \bar{y}.$$

We proved that, for every $y \in \mathbb{R}^n$, there exists, $x \in X$ such that $y = G(x)$. \square

The Main Result can now be proved.

Theorem 8. *Let $\{f_1, f_2, \dots, f_m\}$ be a bracket generating family of vector fields on \mathbb{R}^d . Consider the control system*

$$\dot{q} = f_0(q) + \sum_{i=1}^m u_i(t, q) f_i(q), \quad q \in \mathbb{R}^d, \quad (14)$$

with controls u_i such that:

- (i) u_i is polynomial with respect to $q \in \mathbb{R}^d$;
- (ii) u_i is a trigonometric polynomial with respect to $t \in [0, 1]$;

for every $i = 1, \dots, m$.

Fix positive integers N and k , $\varepsilon > 0$, and B ball of \mathbb{R}^d . For any $\Phi \in \text{Diff}_0(\mathbb{R}^d)$, there exist controls $u_1(t, q), \dots, u_m(t, q)$ such that, if P is the flow at time 1 of system (14), then

$$J_0^N(P) = J_0^N(\Phi) \quad \text{and} \quad \|P - \Phi\|_{C^k(B)} < \varepsilon.$$

Proof. Proof splits into three steps. First prove that it is sufficient to consider controls that are polynomials with respect to $q \in \mathbb{R}^d$, then we add the drift to the system, and finally we find controls that are trigonometric polynomials with respect to t by smoothing the time dependence of the piecewise constant controls. Let us start with the first step and note that, as a consequence of the density of polynomials in the space of smooth functions on a bounded set and by Lemma 7, we can assume that $u_i(t, q)$ is a polynomial in q for every $t \in [0, 1]$ and for every $i = 1, \dots, m$.

Now set $\mathbf{Y} = J_0^N(\text{Diff}_0(\mathbb{R}^d))$ and consider the family of continuous maps

$$F_\varrho : \quad \mathbf{U} \quad \longrightarrow \quad \mathbf{Y} \\ (u_1, \dots, u_m) \longmapsto J_0^N \left(\overrightarrow{\exp} \int_0^\varrho \varrho f_0 + \sum_{i=1}^m u_i(t, \cdot) X_i dt \right).$$

We claim that, if there exists $\varrho > 0$ such that F_ϱ is surjective then so is F_ϱ for $\varrho = 1$. Indeed

$$F_\varrho(u_1(t, \cdot), \dots, u_m(t, \cdot)) = F_1 \left(\frac{u_1(t/\varrho, \cdot)}{\varrho}, \dots, \frac{u_m(t/\varrho, \cdot)}{\varrho} \right).$$

Similarly the map $\tilde{F}_\varrho(u_1, \dots, u_m) = J_0^N \left(\overrightarrow{\exp} \int_0^\varrho \sum_{i=1}^m u_i(t, \cdot) X_i dt \right)$ is surjective for every $\varrho > 0$ since it is equal to the map \tilde{F} defined in (13) up to rescalings of the time dependence of the controls u_i . We have, for small $\varrho > 0$, that F_ϱ is a small perturbation of \tilde{F}_ϱ . So Lemma 7 applies and F_1 is surjective.

Finally, for any control $u(t, q)$ piecewise constant in t and polynomial in q , we can write

$$u(t, q) = \sum_{|\alpha|=0}^N a_\alpha(t) q^\alpha,$$

with α multi-index and $a_\alpha(t)$ piecewise constant. For every α , the function a_α admits a Fourier expansion of the form

$$a_\alpha(t) = \sum_{j=0}^{\infty} \eta_\alpha^j \cos(2\pi jt) + \xi_\alpha^j \sin(2\pi jt).$$

Consider the trigonometric polynomial

$$a_\alpha^n(t) = \sum_{j=0}^n \eta_\alpha^j \cos(2\pi jt) + \xi_\alpha^j \sin(2\pi jt),$$

then $a_\alpha^n(t) \rightarrow a_\alpha(t)$ as $n \rightarrow \infty$ in $L^1[0, 1]$. So let

$$u^n(t, q) = \sum_{|\alpha|=0}^N a_\alpha^n(t) q^\alpha,$$

then

$$u^n(t, q) \rightarrow u(t, q), \quad \text{as } n \rightarrow \infty, \quad (15)$$

and the convergence is uniform with all derivatives on compact sets of \mathbb{R}^d and in $L^1[0, 1]$ with respect to t .

Let G_n be the family of continuous maps

$$\begin{aligned} G_n : \quad \mathbf{U} &\longrightarrow \mathbf{Y} \\ (u_1, \dots, u_m) &\longmapsto J_0^N \left(\overrightarrow{\exp} \int_0^1 f_0 + \sum_{i=1}^m u_i^n(t, \cdot) X_i dt \right). \end{aligned}$$

By the convergence in (15), $G_n \rightarrow F_1$ as $n \rightarrow \infty$ for every $(u_1, \dots, u_m) \in \mathbf{U}$, then there exists n_0 integer for which Lemma 7 applies. Therefore the map G_{n_0} is surjective and Theorem follows. \square

Remark 1. Clearly the statement of Theorem 8 holds also if we consider the jet at a point $q \in \mathbb{R}^d$. It is also clear that it is possible to fix a finite number of points in \mathbb{R}^d , say q_1, \dots, q_ℓ , and find an admissible diffeomorphism, arbitrary close to a given one, that realize its N -th jet at all the points q_1, \dots, q_ℓ at the same time.

6 Appendix

Here we prove Proposition 1. The proof is based on the following well-known fact (see, for instance, [1, Lemma 8.2]).

Lemma 9. *Let Z_t and Z_t^n for $n = 1, 2, \dots$ and $t \in [0, 1]$ be nonautonomous vector fields on M . If*

$$\int_0^t Z_\tau^n d\tau \rightarrow \int_0^t Z_\tau d\tau, \quad \text{as } n \rightarrow \infty,$$

in the standard C^∞ topology and uniformly with respect to $t \in [0, 1]$, then

$$\overrightarrow{\exp} \int_0^t Z_\tau^n d\tau \rightarrow \overrightarrow{\exp} \int_0^t Z_\tau d\tau, \quad \text{as } n \rightarrow \infty,$$

in the same topology.

Proof of Proposition 1. First, note that we can suppose, without loss of generality, that $a_i(t, \cdot)$ is piecewise constant in t for every $i = 1, \dots, k$. Indeed, for any $i = 1, \dots, k$, the sequence

$$a_i^n(t, q) = n \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} a_i(\tau, q) d\tau \chi_j^n(t), \quad (16)$$

where $\chi_j^n(t)$ is the characteristic function of the interval $[\frac{j-1}{n}, \frac{j}{n}]$, is such that

$$\int_0^t \sum_{i=1}^k a_i^n(\tau, \cdot) X_i d\tau \rightarrow \int_0^t V_\tau d\tau, \quad \text{as } n \rightarrow \infty,$$

uniformly with respect to t and in the C^∞ -topology. Therefore Lemma 9 allows us to suppose that $a_i(t, \cdot)$ is piecewise constant in t for every i .

Let ℓ be a positive integer such that V_t is constant on $[\frac{j-1}{\ell}, \frac{j}{\ell}]$ for every $j = 1, \dots, \ell$. We can write

$$a_i(t, q) = \sum_{j=1}^{\ell} a_i^j(q) \chi_j^\ell(t), \quad (17)$$

with $a_i^j(q) \geq 0$ for every $q \in \mathbb{R}^d$. Let

$$\alpha^j = \sum_{i=1}^k a_i^j, \quad (18)$$

and let $\{\varepsilon_n\}$ a sequence of nonnegative smooth functions of \mathbb{R}^d such that $\varepsilon_n(0) = 0$ for every n and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ in the C^∞ -topology. Then $\alpha_n^j = \alpha^j + \varepsilon_n$ is strictly positive on $\mathbb{R}^d \setminus \{0\}$ for every j and n .

Now, for every positive integer n and $j = 1, \dots, \ell$, let $b_n^{j,i} = a_i^j / \alpha_n^j$ and consider the family of intervals:

$$A_n^{j,i} = \bigcup_{m=0}^{n-1} \left[\frac{j-1}{\ell} + \frac{m}{n\ell} + \frac{b_n^{j,1} + \dots + b_n^{j,i-1}}{n\ell}, \frac{j-1}{\ell} + \frac{m}{n\ell} + \frac{b_n^{j,1} + \dots + b_n^{j,i}}{n\ell} \right),$$

for $i = 2, \dots, k$, and

$$A_n^{j,1} = \bigcup_{m=0}^{n-1} \left[\frac{j-1}{\ell} + \frac{m}{n\ell}, \frac{j-1}{\ell} + \frac{m}{n\ell} + \frac{b_n^{j,1}}{n\ell} \right).$$

The sequence of vector fields

$$Z_t^n = \alpha_n^j X_i, \quad \text{if } t \in A_n^{j,i}, \quad (19)$$

is such that

$$\int_0^t Z_\tau^n d\tau \rightarrow \int_0^t V_\tau d\tau, \quad \text{as } n \rightarrow \infty.$$

In the standard topology and uniformly with respect to $t \in [0, 1]$. The statement then follows from Lemma 9. \square

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