Well-posed Infinite Horizon Variational Problems

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Abstract

We give an effective sufficient condition for a variational problem with infinite horizon on a Riemannian manifold M to admit a smooth optimal synthesis, i. e. a smooth dynamical system on M whose positive semi-trajectories are solutions to the problem. To realize the synthesis we construct a well-projected to M invariant Lagrange submanifold of the extremals' flow in the cotangent bundle T^*M. The construction uses the curvature of the flow in the cotangent bundle and some ideas of hyperbolic dynamics.

1 Introduction

This paper is dedicated to the 100th anniversary of Lev Semenovich Pontryagin.

Let M be a smooth compact n-dimensional Riemannian manifold. Given $\alpha \geq 0$, we denote by Ω^α the set of all absolutely continuous curves $\gamma : [0, +\infty) \to M$ such that the integral $\int\limits_0^{+\infty} e^{-\alpha t} |\dot\gamma(t)|^2 \, dt$ converges. Here $|\dot\gamma(t)|$ is the Riemannian length of the tangent vector $\dot\gamma(t) \in T_{\gamma(t)}M$. Given $q \in M$, we set $\Omega^\alpha_q = \{\gamma \in \Omega^\alpha : \gamma(0) = q\}$.

Let $U : M \to \mathbb{R}$ be a smooth function. We set

$$\mathfrak{I}_\alpha(\gamma) = \int\limits_0^{+\infty} e^{-\alpha t} \left(\frac{1}{2} |\dot\gamma(t)|^2 - U(\gamma(t)) \right) dt$$

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and try to minimize $\mathfrak{I}_\alpha$ on Ω_q^α. Given $q_0 \in M$, we say that $\gamma_0 \in \Omega_{q_0}^\alpha$ is a *minimizer* if $\mathfrak{I}_\alpha(\gamma_0) = \min\{\mathfrak{I}_\alpha(\gamma) : \gamma \in \Omega_{q_0}^\alpha\}$. We say that $\mathfrak{I}_\alpha$ defines a well-posed variational problem or *admits smooth optimal synthesis* if for any $q \in M$ there exists a unique minimizer γ_q and the map $(q,t) \mapsto \dot{\gamma}_q(t),\ q \in M,\ t \geq 0$, is of class C^1.

The functional $\mathfrak{I}_0$ is simply the action of a mechanical system on the Riemannian manifold M with potential energy U. If $\alpha > 0$, then $\mathfrak{I}_\alpha$ is the *discounted action* with the *discount factor* α.

It is not hard to show that $\mathfrak{I}_0$ does not define a well-posed variational problem for generic U. In this paper we prove that $\mathfrak{I}_\alpha$ defines a well-posed problem for all sufficiently big α. Moreover, we give the effective sharp estimate for critical α.

The functional $\mathfrak{I}_\alpha$ admits smooth optimal synthesis if and only if there exists a complete vector field V of class C^1 on the manifold M such that any positive semi-trajectory γ of the dynamical system $\dot{q} = V(q)$ is a unique minimizer of $\mathfrak{I}_\alpha$ on $\Omega_{\gamma(0)}^\alpha$. Indeed, assume that $\mathfrak{I}_\alpha(\gamma(\cdot)) = \min\{\mathfrak{I}_\alpha(q(\cdot)) : q(\cdot) \in \Omega_{q_0}^\alpha\}$, then $\mathfrak{I}_\alpha(\gamma(s+\cdot)) = \min\{\mathfrak{I}_\alpha(q(\cdot)) : q(\cdot) \in \Omega_{\gamma(s)}^\alpha\}$ for any $s \geq 0$. Define $V(q_0) = \dot{\gamma}_{q_0}(0)$; then $q_0 \mapsto V(q_0)$ is a vector field of class C^1 on M and $\dot{\gamma}_{q_0}(s) = V(\gamma(s)),\ \forall s \geq 0$.

How to characterize the minimizers? If γ is a minimizer then, obviously, γ minimizes the functional $\int\limits_0^T e^{-\alpha t}\left(\frac{1}{2}|\dot{q}(t)|^2 - U(q(t)\right) dt$ on the space of all $q(\cdot) \in H^1([0,T], M)$ such that $q(0) = q_0$, $q(T) = \gamma(T)$ for any $T > 0$. Hence any solution of our infinite horizon variational problem satisfies the Euler–Lagrange equation of the classical finite horizon variational problem.

Let us consider a simple example. Let $M = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $U(\theta) = \cos\theta,\ \theta \in S^1$, the potential energy of the mathematical pendulum. The Euler–Lagrange equation has the form: $\ddot{\theta} = \sin\theta + \alpha\dot{\theta}$. Write it as a system:

$$\left\{ \begin{array}{rcl} \dot{\theta} &=& \xi \\ \dot{\xi} &=& \sin\theta + \alpha\xi \end{array} \right. . \tag{1}$$

This system has 2 equilibrium points: $(\theta,\xi) = (0,0)$ and $(\theta,\xi) = (\pi,0)$. The equilibrium $(0,0)$ is a saddle for any $\alpha \geq 0$. The equilibrium $(\pi,0)$ is a center for $\alpha = 0$ (see Fig. 1), unstable focus for $0 < \frac{\alpha^2}{4} < 1$ (Fig. 2), and unstable node for $\frac{\alpha^2}{4} > 1$ (Fig. 3).

A solution $\theta(\cdot)$ of the Euler–Lagrange equation belongs to Ω^α if and only if $(\xi(\cdot),\theta(\cdot))$ is an equilibrium or a part of the stable submanifold of the

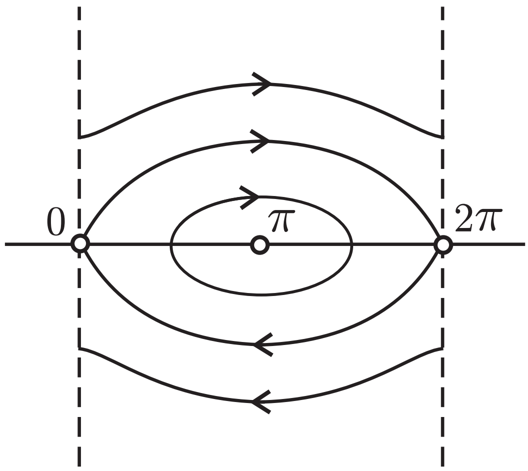

Figure 1:

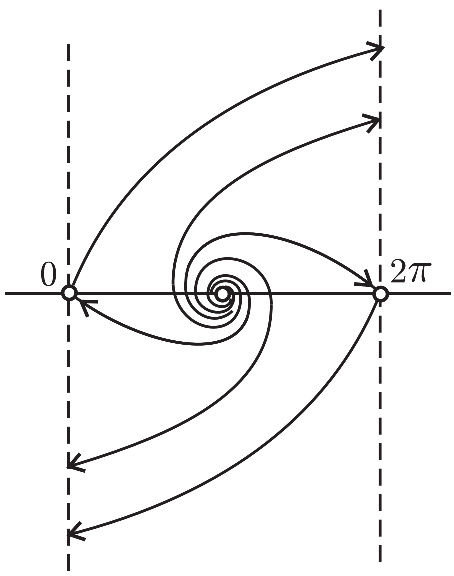

Figure 2:

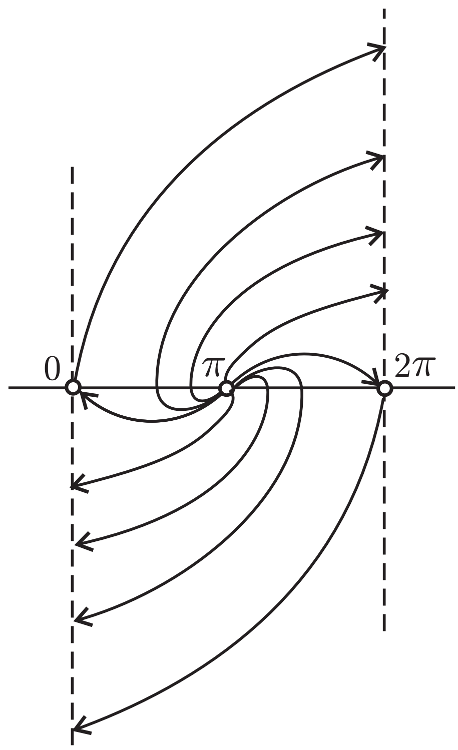

Figure 3:

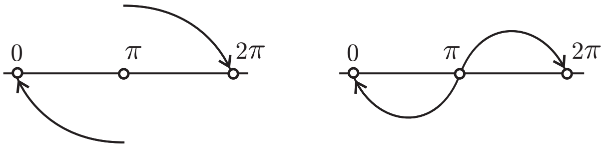

Figure 4:

saddle. The saddle and node equilibria are minimizers while the focus and center are not. If $\frac{\alpha^2}{4} < 1$ then $\mathfrak{I}_\alpha$ does not admit smooth optimal synthesis; if $\frac{\alpha^2}{4} > 1$ then it does admit smooth optimal synthesis. The corresponding to the minimizers trajectories of system (1) in both cases are shown on Figure 4.

In order to formulate our main result we need the following definition.

Definition 1 *The curvature of the Hamiltonian H at $z \in T_q^*M$ is a self-adjoint linear operator $R_z^H : T_q^*M \to T_q^*M$ defined by the formula*

$$R_z^H \zeta = \mathfrak{R}(\zeta, z)z + (\nabla_q^2 U)\zeta, \quad \zeta \in T_q^*M,$$

where ∇ is the covariant derivativation and $\mathfrak{R}$ the Riemannian curvature.

Given a self-adjoint linear operator A on a Euclidean space and a constant $a \in \mathbb{R}$, the relation $A < aI$ means that all eigenvalues of A are smaller than a. Now we state the main result of this paper:

Theorem 1 *Let $R_z^H < \frac{\alpha^2}{4} I$ for any $z \in T^*M$ such that $H(z) \leq \max_{q \in M} U(q)$; then $\mathfrak{I}_\alpha$ admits smooth optimal synthesis.*

Corollary 1 *Assume that sectional curvature of M does not exceed $r \geq 0$ and*

$$\nabla_q^2 U < \left(\frac{\alpha^2}{4} - 2r(\max U - \min U) \right) I, \quad \forall q \in M.$$

Then $\mathfrak{I}_\alpha$ admits smooth optimal synthesis.

In the next section we use symplectic language to characterize extremals of the variational problem and formulate a more detailed version of the main result, including its Hamilton–Jacobi interpretation. The discount factor appears as a friction coefficient in the extremals' equation and serves as a "smoothing factor" in the Hamilton–Jacobi setting.

In Section 3 we prove the "partial hyperbolicity" of the set filled by the extremals; this is a principal step in the proof of the main result. A good source for the theory of partially hyperbolic systems is book [3]. It seems that this concept is really relevant to our subject: sufficiently strong isotropic friction almost automatically leads to the partial hyperbolicity and this fact concerns much more general systems than ones studied in this paper.

In Section 4 we complete the proof of the main result: optimal synthesis is obtained as a limit of solutions of the horizon $\tau > 0$ problem as $\tau \longrightarrow +\infty$.

Remark 1. In this paper we assume that the manifold M is compact. The result remains valid if the Riemannian manifold M is complete, the sectional curvatures of M and the second covariant derivative of U are uniformly bounded, the quantity $\max\limits_{q\in M} U(q)$ is substituted by $\sup\limits_{q\in M} U(q) \leq +\infty$, and Ω^α is substituted by the space of curves $\gamma : [0, +\infty) \to M$ such that $|\dot\gamma(t)| \leq c_\gamma e^{(\alpha-\varepsilon_\gamma)t}$ for some positive constants $c_\gamma, \varepsilon_\gamma$ and any $t \geq 0$.

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2 Symplectic setting

Let σ be the standard symplectic form on T^*M, $\sigma = ds$, where s is the Liouville form: $s_z = z \circ \pi_*|_{T_z(T^*M)}$, $\forall z \in T^*M$, and $\pi : T^*M \to M$ is the standard projection.

The Hamiltonian (or the energy function) $H : T^*M \to \mathbb{R}$ associated to the Lagrangian $\frac{1}{2}|\dot\gamma(t)|^2 - U(\gamma(t))$ is defined by the formula $H(z) = \frac{1}{2}|z|^2 + U(\pi(z))$, $\forall z \in T^*M$, where $|z| = \max\{\langle z, v\rangle : v \in T_{\pi(z)}M,\ |v| = 1\}$ is the dual norm to the norm defined by the Riemannian structure. Actually, the Riemannian structure is a self-adjoint isomorphism of TM and T^*M and in this paper we freely use without special mentioning the provided by this isomorphism identification of tangent and cotangent vectors.

The Legendre transformation of the time-varying Lagrangian $e^{-\alpha t}\left(\frac{1}{2}|\dot q|^2 - U(q)\right)$ has the form $e^{-\alpha t}H(e^{\alpha t}z)$, hence solutions of the Euler–Lagrange equations are exactly projections to M of the trajectories of the Hamiltonian system on T^*M

$$\dot z = e^{-\alpha t}\vec{H}(e^{\alpha t}z), \tag{2}$$

where $\vec{H}$ is the Hamiltonian vector field associated to H; the field $\vec{H}$ is defined by the identity $dH = \sigma(\cdot, \vec{H})$. Recall that z is a point of the $2n$-dimensional manifold T^*M; local coordinates on M induce local trivialization of T^*M so that z splits in two n-dimensional vectors, $z = (p, q)$, and $\pi : (p, q) \mapsto q$. Then $s = \sum\limits_{i=1}^{n} p^i dq^i$, $\sigma = \sum\limits_{i=1}^{n} dp^i \wedge dq^i$ and system (2) takes the form:

$$\left\{ \begin{array}{rcl} \dot{q} & = & \frac{\partial H}{\partial p}(e^{\alpha t}p, q) \\ \dot{p} & = & -e^{-\alpha t}\frac{\partial H}{\partial q}(e^{\alpha t}p, q) \end{array} \right. .$$

The time-varying change of variables $\xi = e^{\alpha t}p$ gives the system

$$\left\{ \begin{array}{rcl} \dot{q} & = & \frac{\partial H}{\partial \xi}(\xi, q) \\ \dot{\xi} & = & -\frac{\partial H}{\partial q}(\xi, q) + \alpha \xi \end{array} \right.$$

or in the coordinate free form:

$$\dot{\zeta} = \vec{H}(\zeta) + \alpha e(\zeta), \tag{3}$$

where $\zeta(t) = e^{\alpha t}z(t)$ and e is the vertical *Euler vector field* of the vector bundle $\pi : T^*M \to M$.

We set $\mathfrak{h}_\alpha = \vec{H} + \alpha e$ and denote by $e^{t\mathfrak{h}_\alpha}$, $t \in \mathbb{R}$, the flow on T^*M generated by the vector field $\mathfrak{h}_\alpha$. It is easy to see that

$$\left(e^{t\mathfrak{h}_\alpha}\right)^* s = e^{\alpha t}s + da_t, \ \forall t \in \mathbb{R},$$

where a_t is a smooth function on T^*M, $t \in \mathbb{R}$. Indeed, let $s_t = \left(e^{t\mathfrak{h}_\alpha}\right)^* s$; then

$$\frac{d}{dt}s_t = \left(e^{t\mathfrak{h}_\alpha}\right)^* L_{\mathfrak{h}_\alpha} s = \left(e^{t\mathfrak{h}_\alpha}\right)^* d(i_{\vec{H}}s - H) + \alpha s_t = db_t + \alpha s_t,$$

where $b_t = (i_{\vec{H}}s - H) \circ e^{t\mathfrak{h}_\alpha}$. Hence $s_t = e^{\alpha t}s_0 + d\int\limits_0^t e^{\alpha(t-\tau)}b_\tau \, d\tau$.

Recall that Lagrange subspaces of $T_z(T^*M)$ are n-dimensional isotropic subspaces of the form σ_z, $z \in T^*M$, and Lagrange submanifolds of T^*M are n-dimensional submanifolds whose tangent spaces in all points are Lagrange subspaces. In other words, an n-dimensional submanifold $\mathcal{L} \subset T^*M$ is a Lagrange submanifold if and only if $s|_{\mathcal{L}}$ is a closed form. We say that $\mathcal{L}$ is an *exact* Lagrange submanifold if $s|_{\mathcal{L}}$ is an exact form. We obtain that $e^{t\mathfrak{h}_\alpha}$

transforms (exact) Lagrange submanifolds of T^*M into (exact) Lagrange submanifolds.

More notations. Let 0_q be the origin of the space T_q^*M. We set

$$C_H = \{0_q : d_q U = 0,\ q \in M\},$$

the set of equilibrium points of system (3), and

$$B_H = \{z \in T^*M : H(z) \leq \max U\}.$$

In what follows we tacitly assume that $\alpha > 0$. Note that $\frac{d}{dt} H(\zeta(t)) = \alpha|\zeta(t)|^2$ for any solution ζ of (3). Hence H is strictly monotone increasing along any solution that is not an equilibrium.

Lemma 1 *Let $\zeta(t) = e^{t\mathfrak{h}_\alpha}(\zeta(0))$ be a trajectory of system (3). The following statements are equivalent:*

1. $\pi \circ \zeta(\cdot) \in \Omega^\alpha$.
2. $\zeta(\cdot)$ *is a bounded curve in* T^*M.
3. $\zeta(t) \in B_H,\ \forall t \in \mathbb{R}$.

Proof. The implications *3* $\Rightarrow$ *2* $\Rightarrow$ *1* are obvious. Let us prove that *1* implies *3*.

Assume that the trajectory $\zeta(t),\ t \in \mathbb{R}$, is not contained in B_H. Then we may assume, without lack of generality, that $\zeta(0) \notin B_H$. In other words, $H(\zeta(0)) - \max U > 0$. Set $\mu(t) = H(\zeta(t)) - \max U$, then $\mu(0) > 0$. We have:

$$\dot{\mu}(t) = \frac{d}{dt} H(\zeta(t)) = \alpha|\zeta(t)|^2 = 2\alpha \left(H(\zeta(t)) - U(\pi(\zeta(t)))\right) \geq 2\alpha\mu(t).$$

Hence $\mu(t) \geq e^{2\alpha t}\mu(0),\ \forall t \geq 0$. Then $\frac{1}{2}|\zeta(t)\|^2 \geq e^{2\alpha t}\mu(0)$ and $e^{-\alpha t}|\zeta(t)|^2 \longrightarrow +\infty$ as $t \longrightarrow +\infty$. Hence $\zeta(\cdot) \notin \Omega^\alpha$. $\square$

Definition 2 *Extremal locus of the functional $\mathfrak{I}_\alpha$ is the subset $\mathcal{E}_\alpha$ of T^*M filled by those trajectories of the flow $e^{t\mathfrak{h}_\alpha}$ that satisfy one of the conditions 1–3 of Lemma 1 (and hence all these conditions).*

Corollary 2 *$\mathcal{E}_\alpha$ is a compact invariant subset of the flow $e^{t\mathfrak{h}_\alpha},\ t \in \mathbb{R}$.*

The potential energy U is a *Morse function* if U has only non-degenerate critical points, i. e. the Hessian of U at $q \in M$ is a non-degenerate quadratic form for any q such that $d_qU = 0$.

Lemma 2 *Assume that U is a Morse function and $\zeta(t) = e^{t\mathfrak{h}_\alpha}(\zeta(0))$, $t \in \mathbb{R}$. The curve $\zeta(\cdot)$ is bounded if and only if there exists $\lim\limits_{t\to+\infty} \zeta(t) \in C_H$.*

This lemma is a simple corollary of the monotonicity of H along trajectories of the flow $e^{t\mathfrak{h}_\alpha}$. $\square$

Next statement is an expanded version of Theorem 1; it explains the way the smooth optimal synthesis is constructed.

Theorem 2 *If $R^H_z < \frac{\alpha^2}{4} I$, $\forall z \in B_H$, then the flow $e^{t\mathfrak{h}_\alpha}$ has an invariant exact Lagrange submanifold $\Psi \subset T^*M$ of class C^1 such that $\pi|_\Psi$ is a diffeomorphism of Ψ on M and the minimizers are exactly positive semi-trajectories of the dynamical system $\dot q = V(q)$, where $V = \pi_*\left(\mathfrak{h}_\alpha|_\Psi\right)$. Moreover, $C_H \subset \Psi \subset \mathcal{E}_\alpha$; if U is a Morse function then $\Psi = \mathcal{E}_\alpha$.*

Remark 2. Submanifold $\Psi \subset T^*M$ in Theorem 2 is the graph of an exact 1-form on M, i. e. $\Psi = \{d_qu : q \in M\}$, where u is a C^2-function on M. Then u is a solution of the modified Hamilton–Jacobi equation:

$$H(du) - \alpha u = const.$$

Indeed, let $q \in M$; then $T_{d_qu}\Psi$ is a Lagrange subspace of $T_{d_qu}(T^*M)$ and $\mathfrak{h}_\alpha \in T_{d_qu}\Psi$. Hence

$$0 = \sigma(\xi, \mathfrak{h}_\alpha) = \sigma(\xi, \vec{H}) + \alpha\sigma(\xi, e) = \langle d_{d_qu}H - \alpha d_qu, \xi\rangle, \quad \forall \xi \in T_{d_qu}\Psi.$$

We obtain that $d(H(du) - \alpha u) = 0$. The choice of *const* is at our disposal. If we set $const = 0$, then $-u(q) = \min\left(\mathfrak{J}_\alpha|_{\Omega^\alpha_q}\right)$, $\forall q \in M$. Indeed, $\min\left(\mathfrak{J}_\alpha|_{\Omega^\alpha_q}\right) = \mathfrak{J}_\alpha(\gamma)$, where $\gamma(t) = \pi \circ e^{t\mathfrak{h}_\alpha}(q)$. We have: $\mathfrak{J}_\alpha(\gamma) = \int\limits_0^\infty e^{-\alpha t}\left(\langle d_{\gamma(t)}u, \dot\gamma(t)\rangle - H(d_{\gamma(t)}u)\right) dt = \int\limits_0^\infty \left(e^{-\alpha t}\frac{d}{dt}u(\gamma(t)) - e^{-\alpha t}\alpha u(\gamma(t))\right) dt = \int\limits_0^\infty \frac{d}{dt}\left(e^{-\alpha t}u(\gamma(t))\right) dt = -u(\gamma(0))$.

Standard Hamilton–Jacobi equation corresponds to $\alpha = 0$. As we know, in general, this equation does not have smooth solutions, it has only generalized ones. It would be very interesting to study transformations of the solutions for the parameter α running from the indicated by Theorem 2 "smooth area" to 0.

Remark 3. The Hamiltonian H is the energy of a natural mechanical system on the Riemannian manifold and the discount factor α plays the role of a negative friction coefficient. Moreover, the change of variables $z \mapsto -z,\ z \in T_q^*M,\ q \in M$, transforms $\mathfrak{h}_\alpha$ in $-\mathfrak{h}_{-\alpha}$ that allows to apply our analysis of the flow $e^{t\mathfrak{h}_\alpha}$ to the dissipative mechanical system (with a positive friction coefficient). As a byproduct we obtain the description of the subset of T^*M filled by the bounded trajectories in the case when the friction coefficient is greater than certain critical value.

3 Partial hyperbolicity

We start to prove Theorem 2. The Levi Civita connection ∇ on T^*M defines a smooth "horizontal" vector distribution $D = \bigcup\limits_{z \in T^*M} D_z$ on T^*M, where D_z is the subspace of $T_z(T^*M)$ filled by the velocities of the parallel translations of z along the curves in M. We denote: $\Delta_z = T_z(T^*_{\pi(z)}M)$ and call $\Delta = \bigcup\limits_{z \in T^*M} \Delta_z$ the vertical distribution. Then $T_z(T^*M) = \Delta_z \oplus D_z,\ \forall z \in T^*M$.

Note that both Δ_z and D_z are Lagrange subspaces of the symplectic space $T_z(T^*M)$. This is evident for Δ_z; in what concerns D_z, its property to be a Lagrange subspace is just another way to say that the Levi Civita connection is symmetric (i. e. torsion free). A vector distribution on a subset of the symplectic manifold is called a *Lagrange vector distribution* if its fibers are Lagrange subspaces of the tangent spaces.

Let $w \in T_z(T^*M),\ w = w_{ver} + w_{hor}$, where $w_{ver} \in \Delta_z,\ w_{hor} \in D_z$. We set $|w|^2 = |w_{ver}|^2 + |\pi_* w_{hor}|^2$ and thus define a canonical Riemannian structure on T^*M.

Proposition 1 *Assume that $R_z^H < \frac{\alpha^2}{4} I,\ \forall z \in B_H$. Then there exist continuous Lagrange distributions $E^\pm = \bigcup\limits_{z \in \mathcal{E}_\alpha} E_z^\pm$ on $\mathcal{E}_\alpha$ and positive constants $c_\pm$, ε such that:*

1. $e_*^{t\mathfrak{h}_\alpha} E^\pm = E^\pm,\ \forall t \in \mathbb{R}$.
2. $|e_*^{t\mathfrak{h}_\alpha} w_-| \le c_- e^{(\frac{\alpha}{2}-\varepsilon)t} |w_-|,\ |e_*^{t\mathfrak{h}_\alpha} w_+| \ge c_+ e^{(\frac{\alpha}{2}+\varepsilon)t} |w_+|,\ \forall\ w_\pm \in E^\pm,\ t \ge 0$.
3. $E_z^\pm \cap \Delta_z = 0,\ \forall z \in \mathcal{E}_\alpha$.

Proof. Recall that ${e^{t\mathfrak{h}_\alpha}}^*\sigma = e^{\alpha t}\sigma$; hence $e^{t\mathfrak{h}_\alpha}_*$ transforms Lagrange subspaces of the tangent spaces to T^*M in the Lagrange subspaces. We define the Jacobi curves and the curvature operators in the same way as for the Hamiltonian flow (see Appendix). Namely, let $z \in T^*M$; then

$$J^{\mathfrak{h}_\alpha}_z(t) = e^{-t\mathfrak{h}_\alpha}_*\Delta_{\zeta(t)}, \quad \zeta(t) = e^{t\mathfrak{h}_\alpha}(z),$$

is a monotone decreasing curve in the Lagrange Grassmannian $L\left(T_z(T^*M)\right)$ and $R^{\mathfrak{h}_\alpha}_z = R_{J^{\mathfrak{h}_\alpha}_z}(0)$ is a self-adjoint operator on Δ_z. Obviously, $R_{J^{\mathfrak{h}_\alpha}_z}(t) = e^{-t\mathfrak{h}_\alpha}_* R^{\mathfrak{h}_\alpha}_\zeta e^{t\mathfrak{h}_\alpha}_*|_{J^{\mathfrak{h}_\alpha}_z(t)}$ and $R^{\mathfrak{h}_0}_z = R^H_z$.

Lemma 3 $R^{\mathfrak{h}_\alpha}_z = R^H_z - \frac{\alpha^2}{4}I$.

Proof. We set $D^\alpha_z = \left(J^{\mathfrak{h}_\alpha}_z\right)^\circ(0) \in L\left(T_z(T^*M)\right)$, $D^\alpha = \bigcup\limits_{z\in T^*M} D^\alpha_z$, the canonical Ehresmann connection associated to the flow $e^{t\mathfrak{h}_\alpha}$. Then $D^0 = D$, the Levi Civita connection.

Let O_z be a neighborhood of z in T^*M and $v_\alpha(z')$, $z' \in O_z$, a smooth "vertical" vector field. According to the terminology described in the Appendix, the field v_α is parallel for the connection D^α along trajectories of the flow $e^{t\mathfrak{h}_\alpha}$ if and only if $[\mathfrak{h}_\alpha, v_\alpha] \in D^\alpha$. The connection D^α is characterized by the following property: if $v_\alpha \in \Delta$ and $[\mathfrak{h}_\alpha, v_\alpha] \in D^\alpha$, then $[\mathfrak{h}_\alpha, [\mathfrak{h}_\alpha, v_\alpha]] \in \Delta$. Moreover,

$$[\mathfrak{h}_\alpha, [\mathfrak{h}_\alpha, v_\alpha]](z) = -R^{\mathfrak{h}_\alpha}_z v_\alpha(z).$$

Let v be a vertical vector field on O_z that is parallel for the Levi Civita connection D along the trajectories of the flow $e^{t\mathfrak{h}_0}$ and is constant on the vertical rays $(\mathbb{R}_+\zeta_t) \cap O_z$, where $\zeta_t = e^{t\mathfrak{h}_\alpha}(z)$ is any point of the contained in O_z interval of the passing through z trajectory of the flow $e^{t\mathfrak{h}_\alpha}$. Recall that $\tau z' = \exp\left((\ln\tau)e\right)(z')$, $\tau > 0$, $z' \in T^*M$, where e is the Euler vector field. The linearity of the Levi Civita connection implies: $\exp(se)_*D = D$, $\forall s \in \mathbb{R}$. Hence

$$[e, v] = -v, \quad [e, [\mathfrak{h}_0, v]] = 0.$$

Now I claim that the vector field v_α defined by the formula $v_\alpha(\zeta_t) = e^{\frac{\alpha}{2}t}v(\zeta_t)$ is parallel for the connection D^α along the curve ζ_t. Indeed,

$$[\mathfrak{h}_\alpha, v_\alpha](\zeta_t) = [\mathfrak{h}_0 + \alpha e, v_\alpha](\zeta_t) = e^{\frac{\alpha}{2}t}\left([\mathfrak{h}_0, v](\zeta_t) - \frac{\alpha}{2}v(\zeta_t)]\right),$$

$$[\mathfrak{h}_\alpha, [\mathfrak{h}_\alpha, v_\alpha]](\zeta_t) = e^{\frac{\alpha}{2}t}\left(\frac{\alpha^2}{4}I - R^H_{\zeta_t}\right)v(\zeta_t) \in \Delta_{\zeta_t}.$$

We have obtained the desired formula $R_z^{\mathfrak{h}_\alpha} = R_z^H - \frac{\alpha^2}{4}I$ and the following characterization of D^α:

$$D_z^\alpha = span\{[\mathfrak{h}_0, v](z) - \frac{\alpha}{2}v(z) : v \in \mathfrak{V}_0\}, \qquad (4)$$

where $\mathfrak{V}_0$ is the space of vertical vector fields on O_z that are parallel along the flow $e^{t\mathfrak{h}_0}$ for the connection $D = D^0$ and are constant on the vertical rays $\{\tau\zeta_t : \tau, t \in \mathbb{R},\ \zeta_t, \tau\zeta_t \in O_z\}$. $\square$

Next lemma is a simple generalization of the hyperbolicity test of Lewowicz and Wojtkowski (see [4, Th. 5.2]). The proof is almost verbal repetition of the proof from the cited paper and we omit it.

Let N be a Riemannian manifold, $X \in \mathrm{Vec}N$, and $Q : TM \to \mathbb{R}$ a pseudo-Riemannian structure on N (i. e. a smooth field of nondegenerate quadratic forms $Q_z : T_zN \to \mathbb{R},\ z \in N$). Let $L_X q : v \mapsto \frac{d}{dt}Q(e_*^{tX}v)|_{t=0},\ v \in TM$, be the Lie derivative of Q in the direction of X.

Lemma 4 *Let $\beta \in \mathbb{R}$ and $S \subset N$ be a compact invariant subset of the flow $e^{tX},\ t \in \mathbb{R}$. If the quadratic form $L_X Q - \beta Q$ is positive definite on $TN|_S$ then there exist invariant for the flow e^{tX} continuous vector distributions $E^\pm = \bigcup_{z\in\mathcal{E}_\alpha} E_z^\pm$ on $\mathcal{E}_\alpha$ and positive constants $c_\pm, \varepsilon$ such that $(-1)^\pm Q|_{E^\pm} > 0$ and*

$$|e_*^{tX}w_-| \leq c_- e^{(\frac{\beta}{2}-\varepsilon)t}|w_-|,\ |e_*^{tX}w_+| \geq c_+ e^{(\frac{\beta}{2}+\varepsilon)t}|w_+|,\ \forall\ w_\pm \in E^\pm,\ t \geq 0.$$

Next lemma is a generalization of the earlier observation of Piotr Przytycki (see [2, Lemma 2.1]). Let now N be a symplectic manifold endowed with the symplectic form σ, $X \in \mathrm{Vec}M$, $\beta \in \mathbb{R}$, and $L_X\sigma = \beta\sigma$. Let $\Lambda^i = \bigcup_{z\in N} \Lambda_z^i,\ \Lambda_z^i \in L(T_zN),\ i = 0, 1$, be two smooth Lagrange distributions on N. We assume that $\Lambda_z^0 \cap \Lambda_z^1 = 0,\ \forall z \in N$.

Let $v \in T_zN$, then $v = v_0 + v_1$, where $v_i \in \Lambda_z^i$. We define the pseudo-Riemannian structure $Q_{\Lambda^0\Lambda^1}$ on N by the formula:

$$Q_{\Lambda^0\Lambda^1}(v) = \sigma(v_0, v_1), \quad \forall v \in T_zN,\ z \in N.$$

We define the distributions $\Lambda^i(t) = e_*^{-tX}\Lambda^i$; then $t \mapsto \Lambda_z^i(t)$ is a curve in the Lagrange Grassmannian $L(T_zN),\ \forall z \in N,\ i = 0, 1$.

Lemma 5 *Let $S \subset N$ be a compact invariant subset of the flow $e^{tX},\ t \in \mathbb{R}$. If the curve $\Lambda_z^0(\cdot)$ is monotone decreasing and the curve $\Lambda_z^1(\cdot)$ is monotone increasing, $\forall z \in S$, then the form $Q_{\Lambda^0\Lambda^1}$ satisfies conditions of Lemma 4.*

Proof. Let $V \in \mathrm{Vec}N$, $V|_S = V^0 + V^1$, where $V^i \in \Lambda^i$. We set $V(t) = e_*^{-tX}V$, $V^i(t) = e_*^{-tX}V^i$, $V(t) = V(t)_0 + V(t)_1$, $V^i(t) = V^i(t)_0 + V^i(t)_1$, where $V^i(t)_j \in \Lambda^j$, $i,j = 0,1$. We have: $(L_X Q_{\Lambda^0\Lambda^1})(V) = \frac{d}{dt}\sigma(V(t)_1, V(t)_0)|_{t=0}$ and $\beta Q_{\Lambda^0\Lambda^1}(V) = \beta\sigma(V^0, V^1) = \frac{d}{dt}\sigma(V^1(t), V^0(t))|_{t=0}$. Then

$$\sigma(V(t)_1, V(t)_0) = \sigma(V(t)_1, V(t)) = \sigma\big(V(t)_1, V^1(t)\big) + \sigma\big(V(t)_1, V^0(t)\big)$$

$$= \sigma\big(V(t)_1, V^1(t)\big) - \sigma\big(V(t)_0, V^0(t)\big) + \sigma\big(V(t), V^0(t)\big)$$
$$= \sigma\big(V(t)_1, V^1(t)\big) - \sigma\big(V(t)_0, V^0(t)\big) + \sigma\big(V^1(t), V^0(t)\big).$$

The differentiation with respect to t at $t = 0$ gives:

$$(L_X Q_{\Lambda^0\Lambda^1})(V) = \underline{\dot{\Lambda}}^1(V^1) - \underline{\dot{\Lambda}}^0(V^0) + \beta Q_{\Lambda^0\Lambda^1}(V)$$

and the monotonicity assumptions imply that $\underline{\dot{\Lambda}}^0(V^0) < 0$, $\underline{\dot{\Lambda}}^1(V^1) > 0$. $\square$

Note that the manifold $N = T^*M$, vector field $X = \mathfrak{h}_\alpha$, invariant subset $S = \mathcal{E}_\alpha$, constant $\beta = \alpha$, and distributions $\Lambda^0 = \Delta$, $\Lambda^1 = D^\alpha$ satisfy conditions of Lemmas 4 and 5 if $R^H_z < \frac{\alpha^2}{4}I$, $\forall z \in \mathcal{E}_\alpha$. Let $E^\pm$ be the invariant distributions guaranteed by Lemma 4. To complete the proof of Proposition 1 it remains to show that that $E^\pm_z$ are transversal to Δ_z Lagrange subspaces.

We'll prove a little bit more. Namely, we are going to show that:

$$E^\pm_z = \lim_{t\to\mp\infty} J^{\mathfrak{h}_\alpha}_z(t) = \lim_{t\to\mp\infty} \left(J^{\mathfrak{h}_\alpha}_z\right)^\circ(t), \quad \forall z \in \mathcal{E}_\alpha. \tag{5}$$

Indeed, as we know (see Appendix) the limits $\lim\limits_{t\to\pm\infty} J^{\mathfrak{h}_\alpha}_z(t) = J^{\mathfrak{h}_\alpha}_z(\pm\infty)$ and $\lim\limits_{t\to\pm\infty} \left(J^{\mathfrak{h}_\alpha}_z\right)^\circ(t) = \left(J^{\mathfrak{h}_\alpha}_z\right)^\circ(\pm\infty)$ do exist and are transversal to $\Delta_z = J^{\mathfrak{h}_\alpha}_z(0)$. Moreover, $J^{\mathfrak{h}_\alpha}_z(\pm\infty)$ and $\left(J^{\mathfrak{h}_\alpha}_z\right)^\circ(\pm\infty)$ are invariant vector distributions for the flow $e_*^{t\mathfrak{h}_\alpha}$, since $J^{\mathfrak{h}_\alpha}(t+s) = e^{-t\mathfrak{h}_\alpha}J^{\mathfrak{h}_\alpha}(s)$, $\left(J^{\mathfrak{h}_\alpha}\right)^\circ(t+s) = e^{-t\mathfrak{h}_\alpha}\left(J^{\mathfrak{h}_\alpha}\right)^\circ(s)$.

Take a vector field $V = V^+ + V^-$, where $V^\pm \in E^\pm$. If $V^\pm(z) \neq 0$, $\forall z \in \mathcal{E}_\alpha$, then the component $e_*^{t\mathfrak{h}_\alpha}V^+ \in E^+$ of the vector $e_*^{t\mathfrak{h}_\alpha}V = e_*^{t\mathfrak{h}_\alpha}V^+ + e_*^{t\mathfrak{h}_\alpha}V^-$ dominates as $t \longrightarrow +\infty$ and the component $e_*^{t\mathfrak{h}_\alpha}V^-$ dominates as $t \longrightarrow -\infty$ due to the already proved estimates (see Lemma 4).

Therefore in order to prove (5) it is sufficient to show that $J^{\mathfrak{h}_\alpha}(t) \cap E^\pm = \left(J^{\mathfrak{h}_\alpha}\right)^\circ(t) \cap E^\pm = 0$ for some (and hence for all) $t \in \mathbb{R}$.

We have: $Q_{\Delta D^\alpha}|_{J^{\mathfrak{h}_\alpha}(0)} = Q_{\Delta D^\alpha}|_\Delta = 0$ and $\frac{d}{dt}\Big|_{t=0}\Big(Q_{\Delta D^\alpha}|_{J^{\mathfrak{h}_\alpha}(t)}\Big) < 0$. Hence $Q_{\Delta D^\alpha}|_{J^{\mathfrak{h}_\alpha}(t)} < 0$ for small positive t and $Q_{\Delta D^\alpha}|_{J^{\mathfrak{h}_\alpha}(t)} > 0$ for small negative t. On the other hand, $Q|_{E^+} > 0$ and $Q|_{E^-} < 0$. It follows that $J^{\mathfrak{h}_\alpha}(t) \cap E^+ = J^{\mathfrak{h}_\alpha}(t) \cap E^- = 0$. Similarly $\left(J^{\mathfrak{h}_\alpha}\right)^\circ(t) \cap E^+ = \left(J^{\mathfrak{h}_\alpha}\right)^\circ(t) \cap E^- = 0$. $\square$

Corollary 3 *Under conditions of Proposition 1,* $\mathfrak{h}_\alpha(z) \in E_z^-$, $\forall z \in \mathcal{E}_\alpha$.

Proof. Let $\mathfrak{h}_\alpha = \mathfrak{h}_\alpha^+ + \mathfrak{h}_\alpha^-$, where $\mathfrak{h}_\alpha^\pm(z) \in E_z^\pm$, $\forall z \in \mathcal{E}_\alpha$. Then the length of the vectors $e_*^{t\mathfrak{h}_\alpha}\mathfrak{h}_\alpha^\pm(z) = \mathfrak{h}_\alpha^\pm(e^{t\mathfrak{h}_\alpha}(z))$ is uniformly bounded as t tends to $+\infty$. On the other hand, $|e_*^{t\mathfrak{h}_\alpha}\mathfrak{h}_\alpha^+(z)| \geq c_+ e^{(\frac{\alpha}{2}+\varepsilon)t}|\mathfrak{h}_\alpha^+(z)|$ for all $t \geq 0$. Hence $\mathfrak{h}_\alpha^+(z) = 0$, $\forall z \in \mathcal{E}_\alpha$. $\square$

4 Optimal synthesis

Now we are going to consider variational problems with finite horizons and "free endpoints" and then to study the limit as the horizon tends to infinity. Namely, we study the functionals

$$\mathfrak{I}_\alpha^\tau : \gamma \mapsto \int_0^\tau e^{-\alpha t}\left(\frac{1}{2}|\dot{\gamma}(t)|^2 - U(\gamma(t))\right) dt$$

on the spaces $\Omega_q^{\alpha,\tau} = \{\gamma \in H^1([0,\tau]; M) : \gamma(0) = q\}$. Compactness of M implies the existence of minimizers that are critical points of $\mathfrak{I}_\alpha^\tau$ on $\Omega_q^{\alpha,\tau}$. These critical points are solutions γ of the Euler–Lagrange equations such that the *transversality condition* $\dot{\gamma}(\tau) = 0$ is satisfied. In other words, critical points are projections to M of solutions ζ to equation (3) such that $\zeta(\tau)$ belongs to the zero section of T^*M. Note that $\zeta(t) \in B_H$, $\forall t \in [0,\tau]$; indeed, if a solution leaves B_H, then it never comes back (see the prove of Lemma 2). In the sequel, we identify M with zero section of T^*M, i. e. $M = \{0_q : q \in M\}$, so that $M \subset T^*M$.

Proposition 2 *Assume that* $R_z^H < \frac{\alpha^2}{4}I$, $\forall z \in B_H$. *Then* $e^{-\tau\mathfrak{h}_\alpha}(M)$ *is a smooth section of the bundle* $\pi : T^*M \to M$, $\forall \tau > 0$*; hence for any* $q \in M$ *there exists a unique critical point of* $\mathfrak{I}_\alpha^\tau$ *on* $\Omega_q^{\alpha,\tau}$.

Proof. It is sufficient to show that the map $\pi \circ e^{-t\mathfrak{h}_\alpha}|_M : M \to M$ has no critical points, $\forall t > 0$. Indeed, in this case the maps $\pi \circ e^{-t\mathfrak{h}_\alpha}|_M$ must be coverings of M and $\pi|_M = id$; hence $\pi \circ e^{-t\mathfrak{h}_\alpha}|_M$ are actually diffeomorphisms.

So we have to show that $e_*^{-t\mathfrak{h}_\alpha} T_{0_q} M \cap \Delta_{e^{t\mathfrak{h}_\alpha}(0_q)} = 0$ for any $q \in M$. To this end (and for further limiting procedure) we introduce the Lagrange distribution $\Delta^\alpha = \bigcup\limits_{z \in T^*M} \Delta_z^\alpha \subset T(T^*M)$ as follows:

$$\Delta_z^\alpha = span \left\{ \left(1 + \frac{\alpha}{2}\right) v - [\mathfrak{h}_0, v] : v \in \mathfrak{V}_0 \right\}, \quad z \in T^*M,$$

where $\mathfrak{V}_0$ has the same meaning as in (4). Consider the splitting $T(T^*M) = \Delta^\alpha \oplus D^\alpha$. Symplectic form σ defines the nondegenerate pairing of the subspaces Δ_z^α and D_z^α that gives the identification $\Delta^\alpha = (D^\alpha)^*$.

Then any transversal to Δ_z^α Lagrange subspace $\Lambda \subset T_z(T^*M)$ is identified with the graph of a self-adjoint linear map from D_z^α to $\Delta_z^\alpha = (D^\alpha)^*$ and thus with a quadratic form Q_Λ on D_z^α.

Lemma 6 *We have:* $Q_{D_z^\alpha} = 0,\ Q_{\Delta_z} > 0,\ Q_{D_z} = \frac{\alpha}{\alpha+2} Q_{\Delta_z},\ \forall z \in T^*M$. *In particular,* $Q_{D^\alpha} < Q_D < Q_\Delta$.

Proof. The equality $Q_{D_z^\alpha} = 0$ is obvious. Let $\xi \in D_z^\alpha$, then $\xi = [\mathfrak{h}_0, v](z) - \frac{\alpha}{2} v(z)$ for some $v \in \mathfrak{V}_0$ (see (4)). Assume that $\xi \neq 0$, then $v(z) \neq 0$. Moreover, $v(z) \in \Delta_z$ and $v(z) = \big((1 + \frac{\alpha}{2})v - [\mathfrak{h}_0, v]\big) + \big([\mathfrak{h}_0, v] - \frac{\alpha}{2} v\big)$, where $\big((1 + \frac{\alpha}{2})v - [\mathfrak{h}_0, v]\big) \in \Delta^\alpha$, $\big([\mathfrak{h}_0, v] - \frac{\alpha}{2} v\big) \in D^\alpha$. Then

$$Q_{\Delta_z}(\xi) = \sigma_z \left(\left(1 + \frac{\alpha}{2}\right) v - [\mathfrak{h}_0, v]\,,\ [\mathfrak{h}_0, v] - \frac{\alpha}{2} v \right) = \sigma_z([\mathfrak{h}_0, v], v) > 0.$$

Similarly, $Q_{D_z}(\xi) = \frac{\alpha}{\alpha+2} \sigma_z([\mathfrak{h}_0, v], v)$. $\square$

Note that $T_{0_q} M = D_{0_q},\ \forall q \in M$, since the Levi Civita connection D is a linear connection. Let $z = e^{-\tau\mathfrak{h}_\alpha}(0_q),\ \zeta(t) = e^{t\mathfrak{h}_\alpha}(z)$; then $t \mapsto J_z^{\mathfrak{h}_\alpha}(t) = e_*^{-\tau\mathfrak{h}_\alpha} \Delta_{\zeta(t)}$ is a monotone decreasing and $t \mapsto \left(J_z^{\mathfrak{h}_\alpha}\right)^\circ(t) = e_*^{-\tau\mathfrak{h}_\alpha} D_{\zeta(t)}^\alpha$ a monotone increasing curves in the Lagrange Grassmannian $L\left(T_z(T^*M)\right)$. We'll use simplified notations:

$$\Delta(t) = e_*^{-t\mathfrak{h}_\alpha} \Delta_{\zeta(t)}, \quad D^\alpha(t) = e_*^{-t\mathfrak{h}_\alpha} D_{\zeta(t)}^\alpha, \quad D(t) = e_*^{-t\mathfrak{h}_\alpha} D_{\zeta(t)}.$$

Then $t \mapsto Q_{\Delta(t)}$ is a strongly monotone decreasing and $t \mapsto Q_{D^\alpha(t)}$ a strongly monotone increasing families of quadratic forms. Moreover, $Q_{\Delta(t)} - Q_{D(t)}$ and $Q_{D(t)} - Q_{D^\alpha(t)}$ are nondegenerate quadratic forms since $D(t)$ is transversal to $\Delta(t)$ and $D^\alpha(t)$. It follows that

$$Q_{D^\alpha(0)} < Q_{D^\alpha(t)} < Q_{D(t)} < Q_{\Delta(t)} < Q_{\Delta(0)}, \quad \forall t > 0. \tag{6}$$

In particular, $D(\tau) = e_*^{-\tau\mathfrak{h}_\alpha} D_{0_q}$ is transversal to $\Delta(0) = \Delta_z$. Proposition 2 is proved. $\square$

Given $q \in M$ we denote by $\Phi_\tau(q)$ the value at q of the section $e^{-\tau\mathfrak{h}_\alpha}(M)$ of T^*M; in other words, $\{\Phi_\tau(q)\} = e^{-\tau\mathfrak{h}_\alpha}(M) \cap T_q^*M$. Recall that $\Phi_\tau(q) \in B_H$, $\forall q \in M$, $\tau \geq 0$. In particular, $\tau \mapsto \Phi_\tau(q)$, $\tau \geq 0$, is a uniformly bounded curve in T^*M.

Lemma 7 *Assume that* $z = \lim\limits_{k\to\infty} \Phi_{\tau_k}(q)$ *for some* $\tau_k \longrightarrow +\infty$ *as* $k \longrightarrow \infty$. *Then* $z \in \mathcal{E}_\alpha$ *and* $E_z^- = \lim\limits_{k\to\infty} \Phi_{\tau_k *}(T_qM)$.

Proof. Let $\gamma(t) = \pi \circ e^{t\mathfrak{h}_\alpha}(z)$, $\gamma_k(t) = \pi \circ e^{t\mathfrak{h}_\alpha}(\Phi_{\tau_k}(q))$. Then $\mathfrak{I}_\alpha^\tau(\gamma_k) \leq \mathfrak{I}_\alpha^\tau(\gamma_k)(const) = \frac{1}{\alpha}(e^{-\alpha t} - 1)U(q) < \frac{1}{\alpha}|U(q)|$. Hence $\mathfrak{I}_\alpha^\tau(\gamma) = \lim\limits_{k\to\infty} \mathfrak{I}_\alpha^\tau(\gamma_k) \leq \frac{1}{\alpha}|U(q)|$, $\forall \tau \geq 0$. We obtain that $\mathfrak{I}_\alpha(\gamma) \leq \frac{1}{\alpha}|U(q)|$. In particular, $\gamma \in \Omega_q^\alpha$ and, according to Lemma 1, $z \in \mathcal{E}_\alpha$.

The subspace $\Phi_{\tau_k *}(T_qM) \subset T_{\Phi_{\tau_k}(q)}(T^*M)$ is the tangent space to the submanifold $e^{-\tau_k\mathfrak{h}_\alpha}(M)$, i.e. $\Phi_{\tau_k *}(T_qM) = e_*^{-\tau_k\mathfrak{h}_\alpha}(D_{0_{\gamma_k(\tau_k)}})$. Now the statement of the lemma follows from (5) and (6). $\square$

Recall that $\Phi_\tau(M)$ is an exact Lagrange submanifold of T^*M, hence $s|_{\Phi_\tau(M)}$ is an exact 1-form, where s is the Liouville form on T^*M. In other words, $\Phi_\tau = da_\tau$ for a smooth scalar function a_τ on M. Lemma 7 together with statement *3* of Proposition 1 imply that second derivatives of a_τ are uniformly bounded for all $\tau \geq 0$.

The functions a_τ are defined up to a constant and we may of course assume that they are uniformly bounded on M. Then there exists a sequence $\tau_k \longrightarrow +\infty$ as $k \longrightarrow \infty$ and function $a_\infty \in C^{1,\infty}(M)$ such that $a_{\tau_k} \longrightarrow a_\infty$ as $k \longrightarrow \infty$ in C^1-topology. Set $\psi(q) = d_q a_\infty$, then $\psi(q) = \lim\limits_{k\to\infty} \Phi_{\tau_k}(q)$, $\forall q \in M$. We obtain that $\psi(q) \in \mathcal{E}_\alpha$ and $\Phi_{\tau_k *}(T_qM) \longrightarrow E_{\psi(q)}^-$ as $k \longrightarrow \infty$, $\forall q \in M$. Hence the function a_∞ is actually of class C^2; the submanifold $\Psi \overset{def}{=} \{\psi(q) : q \in M\}$ is contained in $\mathcal{E}_\alpha$ and $T_z\Psi = E_z^-$, $\forall z \in \Psi$.

According to Corollary 3, $\mathfrak{h}_\alpha(z) \in E_z^-$, hence Ψ is an invariant exact Lagrange submanifold of the flow $e^{t\mathfrak{h}_\alpha}$, $t \in \mathbb{R}$. Moreover, $\Psi \supset C_H$ since $C_H = e^{\tau\mathfrak{h}_\alpha}(C_H) \subset e^{\tau\mathfrak{h}_\alpha}(M)$, $\forall \tau \geq 0$.

If U is a Morse function, then any fixed point $z = 0_q \in C_H$ of the flow $e^{t\mathfrak{h}_\alpha}$ is hyperbolic with real eigenvalues. This is immediately seen after the diagonalization of the Hessian of U in the critical point q by the orthogonal transformation of T_qM. The stable subspace of the linearization of $\mathfrak{h}_\alpha$ at z

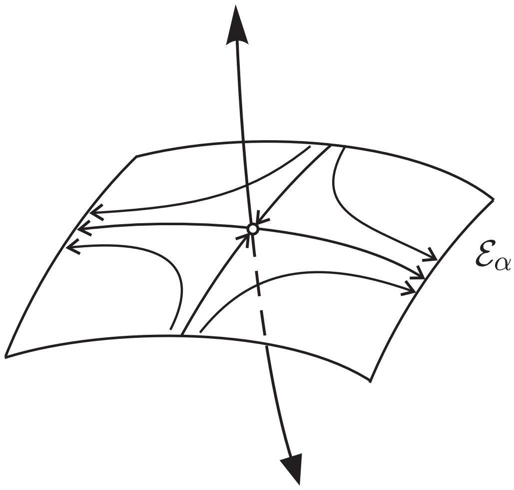

Figure 5:

is contained in $E_z^- = T_z\Psi$. Hence the stable submanifold of the hyperbolic equilibrium z is contained in $\Psi \subset \mathcal{E}_\alpha$. On the other hand, $\mathcal{E}_\alpha$ is the union of the stable submanifolds of all equilibria $z \in C_H$ (see Lemmas $1, 2$). We obtain that $\Psi = \mathcal{E}_\alpha$.

Figure 5 illustrates the structure of $\mathcal{E}_\alpha$ near an equilibrium point z. The stable subspace of the linearized system is always contained in $E_z^- = T_z\mathcal{E}_\alpha$, while the unstable subspace splits in the "less unstable" part that is contained in E_z^- and the "more unstable" part that is equal to E_z^+.

We have proved all good properties of $\mathcal{E}_\alpha$ stated in Theorem 2. It remains only to check that the curves $t \mapsto \pi \circ e^{t\mathfrak{h}_\alpha}(z),\ z \in \mathcal{E}_\alpha$, are minimizers. To do that, we use the classical "fields of extremals" method. We set:

$$\mathcal{L} = \left\{(e^{-\alpha t}z, t) : z \in \mathcal{E}_\alpha,\ t \geq 0\right\} \subset T^*M \times \mathbb{R}_+.$$

Then $\mathcal{L}$ is an n-dimensional submanifold of $T^*M \times \mathbb{R}_+$ and 1-form $(s - e^{-\alpha t}H dt)\Big|_{\mathcal{L}}$ is exact.

Given $q \in M$, there exists a unique $z \in \mathcal{E}_\alpha$ such that $\pi(z) = q$. We set $\gamma(t) = \pi \circ e^{t\mathfrak{h}_\alpha}(z)$ and we have to prove that $\mathfrak{J}_\alpha(\gamma) < \mathfrak{J}_\alpha(\varrho),\ \forall \varrho \in \Omega_q^\alpha$ such that $\varrho \neq \gamma$.

Let

$$\zeta(t) \in \mathcal{E}_\alpha \cap T_{\gamma(t)}M, \qquad \eta(t) \in \mathcal{E}_\alpha \cap T_{\varrho(t)}M$$

be the lifts of γ and ϱ to $\mathcal{E}_\alpha$ and

$$\hat{\zeta}(t) = \left(e^{-\alpha t}\zeta(t), t\right), \qquad \hat{\eta}(t) = \left(e^{-\alpha t}\eta(t), t\right)$$

be the lifts of $\gamma(t)$ and $\varrho(t)$ to $\mathcal{L}$. Then

$$\int\limits_{\hat{\zeta}} (s - e^{-\alpha t} H dt) = \int\limits_0^\infty e^{-\alpha t}(\langle\zeta(t), \dot{\gamma}(t)\rangle - H(\zeta(t)))\, dt = \mathfrak{I}_\alpha(\gamma)$$

and

$$\int\limits_{\hat{\eta}} (s - e^{-\alpha t} H dt) = \int\limits_0^\infty e^{-\alpha t}(\langle\eta(t), \dot{\varrho}(t)\rangle - H(\eta(t)))\, dt < \mathfrak{I}_\alpha(\varrho).$$

Now, $\forall \tau \geq 0$, we define the curves $\hat{\zeta}_\tau : [0, 2\tau] \to \mathcal{L}$ and $\hat{\eta}_\tau : [0, 2\tau] \to \mathcal{L}$ by the formulas:

$$\hat{\zeta}_\tau(t) = \begin{cases} \hat{\zeta}(t), & 0 \leq t \leq \tau, \\ (e^{-\alpha\tau}\zeta(2\tau - t), \tau), & \tau \leq t \leq 2\tau; \end{cases}$$

$$\hat{\eta}_\tau(t) = \begin{cases} \hat{\eta}(t), & 0 \leq t \leq \tau, \\ (e^{-\alpha\tau}\eta(2\tau - t), \tau), & \tau \leq t \leq 2\tau. \end{cases}$$

The curves $\hat{\zeta}_\tau$ and $\hat{\eta}_\tau$ have common endpoints, hence

$$\int\limits_{\hat{\zeta}_\tau} (s - e^{-\alpha t} H dt) = \int\limits_{\hat{\eta}_\tau} (s - e^{-\alpha t} H dt).$$

We have:

$$\int\limits_{\hat{\eta}_\tau} (s - e^{-\alpha t} H dt) = \int\limits_{\hat{\eta}|_{[0,\tau]}} (s - e^{-\alpha t} H dt) - e^{-\alpha\tau} \int\limits_0^\tau \langle\eta(t), \dot{\varrho}(t)\rangle\, dt$$

and

$$\left| e^{-\alpha\tau} \int\limits_0^\tau \langle\eta(t), \dot{\varrho}(t)\rangle\, dt \right| < c e^{-\frac{\alpha}{2}\tau} \int\limits_0^\tau |e^{-\frac{\alpha}{2}t}\dot{\varrho}(t)|\, dt$$

$$\leq ce^{-\frac{\alpha}{2}\tau}\sqrt{\tau}\left(\int\limits_0^\infty e^{-\alpha t}|\dot{\varrho}(t)|^2\,dt\right)^{\frac{1}{2}},$$

where $c = \max\{|z| : z \in \mathcal{E}_\alpha\}$. Hence

$$\lim_{\tau\to+\infty}\int\limits_{\hat{\eta}_\tau}(s - e^{-\alpha t}Hdt) = \int\limits_{\hat{\eta}}(s - e^{-\alpha t}Hdt)$$

and similarly

$$\lim_{\tau\to+\infty}\int\limits_{\hat{\zeta}_\tau}(s - e^{-\alpha t}Hdt) = \int\limits_{\hat{\zeta}}(s - e^{-\alpha t}Hdt).$$

Summing up, we obtain:

$$\mathfrak{I}_\alpha(\gamma) = \int\limits_{\hat{\zeta}}(s - e^{-\alpha t}Hdt) = \int\limits_{\hat{\eta}}(s - e^{-\alpha t}Hdt) < \mathfrak{I}_\alpha(\varrho). \qquad \square$$

Appendix

Here we collect some definitions and geometric facts that are used in the paper; see [1] and references therein for the consistent presentation.

Monotone curves in the Lagrange Grassmannian

Let Σ be a $2n$-dimensional symplectic space endowed with a symplectic form σ. A Lagrange subspace is an n-dimensional subspace $\Lambda \subset \Sigma$ such that $\sigma|_\Lambda = 0$. The set of all Lagrange subspaces forms a compact $\frac{n(n+1)}{2}$-dimensional manifold that is called Lagrange Grassmannian and is denoted $L(\Sigma)$. Symplectic group acts transitively on the Lagrange Grassmannian; moreover, symplectic group acts transitively on the pairs of transversal Lagrange subspaces.

Let $\Pi \in L(\Sigma)$, the symplectic form defines a non-degenerate pairing of Π and Σ/Π: the scalar product of $x \in \Pi$ and the residue class $y + \Pi$ is equal to $\sigma(x, y)$. We obtain a natural identification $\Sigma/\Pi = \Pi^*$. Now set:

$$\Pi^\pitchfork = \{\Lambda \in L(\Sigma) : \Pi \cap \Lambda = 0\};$$

then $\Pi^\pitchfork$ is an affine space over the vector space $\mathrm{Sym}(\Sigma/\Pi)$ of linear self-adjoint maps from Σ/Π to $\Pi = (\Sigma/\Pi)^*$ (or, in other words, over the space of quadratic forms on Σ/Π). Indeed, the sum of $\Lambda \in \Pi^\pitchfork$ and $S \in \mathrm{Sym}(\Sigma/\Pi)$ is defined as follows:

$$\Lambda + S = \{S(y + \Pi) + y : y \in \Lambda\} \in L(\Sigma).$$

An affine space is a vector space "with no origin". Let us take $\Delta \in \Pi^\pitchfork$ and order Δ to be the origin. Then $\Pi^\pitchfork$ turns into $\mathrm{Sym}(\Sigma/\Pi)$. Moreover, obvious isomorphism $\Sigma/\Pi \cong \Delta$ induces the isomorphism of $\mathrm{Sym}(\Sigma/\Pi)$ with $\mathrm{Sym}\Delta$ and the isomorphism of Π with Δ^*. This makes $\Pi^\pitchfork$ a coordinate chart of the manifold $L(\Sigma)$ coordinatized by $\mathrm{Sym}(\Delta)$.

Given $\Lambda_i \in \Pi^\pitchfork$, $i = 0, 1$, let $Q_{\Lambda_i} \in \mathrm{Sym}(\Delta)$ be the coordinate presentation of Λ_i; then $\dim(\Lambda_0 \cap \Lambda_1) = \dim\ker(Q_{\Lambda_0} - Q_{\Lambda_1})$.

Let $V \in T_{\Lambda_0}L(\Sigma)$. To the tangent vector V we associate a quadratic form $\underline{V}$ on Λ_0 (or, in other words, a self-adjoint map from Λ_0 to $\Lambda_0^* = \Sigma/\Lambda_0$) as follows: take a smooth curve $\Lambda_t \in L(\Sigma)$ such that $\dot\Lambda_0 = V$ and a smooth curve $x_t \in \Lambda_t$ and set $\underline{V}(x_0) = \sigma(x_0, \dot x_0)$. Then $V \mapsto \underline{V}$ is a natural isomorphism of $T_{\Lambda_0}L(\Sigma)$ and $\mathrm{Sym}(\Lambda_0)$.

A curve $t \mapsto \Lambda_t$ is *regular* if $\underline{\dot\Lambda}_t$ is a non-degenerate quadratic form. A regular curve is *monotone increasing* (*monotone decreasing*) if $\underline{\dot\Lambda}_t$ is positive definite (negative definite).

Let $\Pi \cap \Lambda_t = 0$, then Λ_t belongs to the affine space $\Pi^\pitchfork$ and the derivative $\dot\Lambda_t$ belongs to the vector space $\mathrm{Sym}(\Sigma/\Pi)$. Obvious isomorphism $\Sigma/\Pi \cong \Lambda_t$ induces the isomorphism $\mathrm{Sym}(\Sigma/\Pi) \cong \mathrm{Sym}\Lambda_t$, which transforms $\dot\Lambda_t$ in $\underline{\dot\Lambda}_t$. In particular, a monotone increasing (decreasing) curve is presented by a strongly monotone increasing (decreasing) family of quadratic forms in any coordinate chart $\Pi^\pitchfork$.

Let $t \mapsto \Lambda_t$ be a regular curve in $L(\Sigma)$ and $\tau \in \mathbb{R}$; then $\Lambda_t \in \Lambda_\tau^\pitchfork$ for all t that are sufficiently close and not equal to τ. We can treat $t \mapsto \Lambda_t$ as a curve in the affine space $\Lambda_\tau^\pitchfork$ with a singularity at $t = \tau$. This singularity is actually a simple pole. We can write the Laurent expansion of Λ_t at $t = \tau$ in this affine space (that is an affine space over the vector space $\mathrm{Sym}(\Sigma/\Lambda_\tau)$). All terms of the Laurent expansion but the free term are elements of the vector space $\mathrm{Sym}(\Sigma/\Lambda_\tau)$, while the free term is an element of the affine space $\Lambda_\tau^\pitchfork$ itself. We denote this free term by Λ_τ° and call the curve $\tau \mapsto \Lambda_\tau^\circ$ the *derivative curve* of the curve $\Lambda_.$.

Given $t \in \mathbb{R}$, the derivative curve defines a splitting: $\Sigma = \Lambda_t \oplus \Lambda_t^\circ$, which induces the identifications

$$\Lambda_t^\circ = \Sigma/\Lambda_t = \Lambda_t^*, \qquad \Lambda_t = \Sigma/\Lambda_t^\circ = \Lambda_t^{\circ *}.$$

Then

$$\underline{\dot{\Lambda}}_t : \Lambda_t \to \Lambda_t^\circ, \qquad \underline{\dot{\Lambda}}_t^\circ : \Lambda_t^\circ \to \Lambda_t$$

are self-adjoint linear maps. The operator $R_\Lambda(t) : \Lambda_t \to \Lambda_t$ defined by the formula

$$R_\Lambda(t) = -\underline{\dot{\Lambda}}_t^\circ \circ \underline{\dot{\Lambda}}_t$$

is the *curvature operator* of the curve $\Lambda_.$.

Assume that $t \mapsto \Lambda_t$ is a monotone curve, then $|\underline{\dot{\Lambda}}_t|$ is a positive definite quadratic form on Λ_t and the curvature operator $R_\Lambda(t)$ is a symmetric operator with respect to the Euclidean structure defined by this quadratic form. In particular, the operator $R_\Lambda(t)$ is diagonalizable and all its eigenvalues are real. We say that the monotone curve $\Lambda_.$ *has a positive (negative) curvature* if all eigenvalues of $R_\Lambda(t)$ are positive (negative) and uniformly separated from 0. If a monotone curve $\Lambda_.$ has a positive (negative) curvature, then the curve $\Lambda_.^\circ$ is also monotone and the direction of monotonicity of $\Lambda_.^\circ$ coincides with (is opposite to) the direction of monotonicity of $\Lambda_.$.

Let $t \mapsto \Lambda_t$ be a regular curve and $t \mapsto \Delta_t$ another curve in $L(\Sigma)$ such that $\Lambda_t \cap \Delta_t = 0,\ \forall t \in \mathbb{R}$. We may treat $\{(t, \Lambda_t) : t \in \mathbb{R}\} \subset \mathbb{R} \times \Sigma$ and $\{(t, \Delta_t) : t \in \mathbb{R}\} \subset \mathbb{R} \times \Sigma$ as subbundles of the trivial vector bundle $\mathbb{R} \times \Sigma$; these subbundles define a splitting of the trivial bundle. We say that the "section" $x_t \in \Lambda_t,\ t \in \mathbb{R}$, is *parallel for the splitting* $\Sigma = \Lambda_t \oplus \Delta_t,\ t \in \mathbb{R}$, if $\dot{x}_t \in \Delta_t,\ \forall t \in \mathbb{R}$.

Canonical splitting $\Sigma = \Lambda_t \oplus \Lambda_t^\circ$ can be characterized as follows: the relations $x_t \in \Lambda_t,\ \dot{x}_t \in \Delta_t,\ \forall t \in \mathbb{R}$, imply the relation $\ddot{x}_t \in \Lambda_t$ if and only if $\Delta_t = \Lambda_t^\circ,\ \forall t \in \mathbb{R}$; moreover, $\ddot{x}_t = -R_\Lambda(t)x_t$ in the case of the canonical splitting.

Theorem *Let $\Lambda_t,\ t \in \mathbb{R}$, be a monotone curve in $L(\Sigma)$. If The curve $\Lambda_.$ has negative curvature, then there exist $\lim\limits_{t\to\pm\infty} \Lambda_t = \Lambda_{\pm\infty} = \lim\limits_{t\to\pm\infty} \Lambda_t^\circ$ and $\Lambda_t \cap \Lambda_\tau = 0,\ \forall -\infty \le t < \tau \le +\infty$.*

The existence of the limits and the fact that $\Lambda_t,\ t \in \mathbb{R}$, are mutually transversal can be easily explained. Recall that symplectic group acts transitively on the set of pairs of transversal Lagrange subspaces and that coordinate presentation of a Lagrange subspace $\Lambda \in L(\Sigma)$ is a quadratic form

Q_Λ. Assume that the curve $t \mapsto \Lambda_t$ is monotone decreasing and the curve $t \mapsto \Lambda_t^\circ$ is monotone increasing (the opposite "increasing–decreasing" case is treated similarly). We can always find an appropriate coordinate chart such that $Q_{\Lambda_0} - Q_{\Lambda_0^\circ} > 0$. Moreover, $t \mapsto Q_{\Lambda_t}$ is a strictly monotone decreasing family of quadratic forms, while $t \mapsto Q_{\Lambda_t^0}$ is a strictly monotone increasing family, and $Q_{\Lambda_t} - Q_{\Lambda_t^\circ}$ are non-degenerate forms. Hence Λ_t and Λ_t° never leave our chart for positive t and there exist $\lim\limits_{t\to+\infty} Q_{\Lambda_t} \geq \lim\limits_{t\to+\infty} Q_{\Lambda_t^\circ}$. A more carefull analysis shows that these two limits coincide and convergence to the common limit has an exponential rate. The limiting procedure as $t \longrightarrow -\infty$ is performed similarly: we simply take a coordinate chart such that $Q_{\Lambda_0} - Q_{\Lambda_0^\circ} < 0$.

Jacobi curves

Let M be a smooth n-dimensional manifold and T^*M its cotangent bundle equipped with the standard symplectic structure. Given $q \in M$ and $z \in T_q^*M$ we set $\Sigma_z = T_z(T^*M)$, $\Delta_z = T_z(T_q^*M)$; then Σ_z is a symplectic space and Δ_z is a Lagrange subspace of this symplectic space.

Let $h : T^*M \to \mathbb{R}$ be a smooth (Hamiltonian) function, $\vec{h} \in \mathrm{Vec}(T^*M)$ the associated to h Hamiltonian vector field, and $t \mapsto e^{t\vec{h}}$ the generated by $\vec{h}$ Hamiltonian flow on T^*M. *Jacobi curve* $J_z^{\vec{h}}(t)$ is a curve in the Lagrange Grassmannian $L(\Sigma_z)$ defined by the formula:

$$J_z^{\vec{h}}(t) = e_*^{-t\vec{h}} \Delta_{e^{t\vec{h}}(z)}, \quad t \in \mathbb{R}.$$

We list some easily derived from the definition basic properties of the Jacobi curves. Let $\zeta(t) = e^{t\vec{h}}(z)$, $\zeta(t) \in T_{\gamma(t)}^*M$; then quadratic form $\underline{\dot{J}_z^{\vec{h}}}(0)$ on $J_z^{\vec{h}}(0) = \Delta_z$ is equal to $-d_z^2\left(h|_{T_q^*M}\right)$ and quadratic form $\underline{\dot{J}_z^{\vec{h}}}(t)$ on $J_z^{\vec{h}}(t)$ is obtained from $-d_{\zeta(t)}^2\left(h|_{T_{\gamma(t)}^*M}\right)$ by the linear change of variables $e_*^{-t\vec{h}} : \Delta_{\zeta(t)} \to J_z^{\vec{h}}(t)$. It follows that Jacobi curve $J_z^{\vec{h}}(t)$ is monotone decreasing (increasing), $\forall z \in T^*M$, if and only if $h|_{T_q^*M}$ is strongly convex (concave), $\forall q \in M$. Other properties:

$$J_z^{\vec{h}^\circ}(t) = e_*^{-t\vec{h}}\left(J_{\zeta(t)}^{\vec{h}}\right)^\circ(0), \quad R_{J_z^{\vec{h}}}(t) = e_*^{-t\vec{h}} R_{J_{\zeta(t)}^{\vec{h}}}(0) e_*^{t\vec{h}}|_{J_z^{\vec{h}}(t)}.$$

Lagrange distribution $J_z^{\vec{h}^\circ}(0)$, $z \in T^*M$, on T^*M is called the *canonical connection* associated to $\vec{h}$ and linear operator $R_{J_z^{\vec{h}}}(0) : \Delta_z \to \Delta_z$ is called the *curvature operator* of $\vec{h}$ at $z \in T^*M$. We will use simplified notations:

$$\Delta_z^h = J_z^{\vec{h}^\circ}(0), \qquad R_z^h = R_{J_z^{\vec{h}}}(0);$$

then $\Sigma_z = \Delta_z \oplus \Delta_z^h$, $z \in T^*M$, is the associated to $\vec{h}$ *canonical splitting* of the vector bundle $T(T^*M)$.

Any vector field $f \in \mathrm{Vec}(T^*M)$ splits in the *vertical* and *horizontal* parts as follows: $f = f_{ver} + f_{hor}$, where $f_{ver}(z) \in \Delta_z$, $f_{hor}(z) \in \Delta_z^h$, $\forall z \in T^*M$. Now let $v \in \mathrm{Vec}(T^*M)$ be a vertical vector field, i. e. $v_{hor} = 0$; we say that the field v is *parallel for the connection* Δ^h along trajectories of the flow e^{tf} if $[f, v]_{ver} = 0$.

Horizontal vector fields and parallel vertical vector fields can be defined for any Ehresmann connection (i. e. for any vector distribution $\mathfrak{D}$ on T^*M such that $\Sigma_z = \Delta_z \oplus \mathfrak{D}_z$). The associated to $\vec{h}$ canonical connection is characterized by the following property: if $v_{hor} = 0$ and $[\vec{h}, v]_{ver} = 0$, then $[\vec{h}, [\vec{h}, v]]_{hor} = 0$.

Finally, for any vertical vector field v and any $z \in T^*M$ we have:

$$R_z^h v(z) = -[\vec{h}, [\vec{h}, v]_{ver}]_{hor}(z).$$

Example. Let M be a Riemannian manifold and $H : T^*M \to \mathbb{R}$ the energy function of a natural mechanical system on M, i. e. $H(z) = \frac{1}{2}|z|^2 + U(q)$, $\forall q \in M$, $z \in T^*M$, where $U : M \to \mathbb{R}$ is the potential energy. Then Δ^H is actually standard Levi Civita connection ∇ rewritten as an Ehresmann connection on T^*M. More precisely, Δ_z^h is the subspace of Σ_z filled by the velocities of the parallel translations of the covector z along curves in M. Moreover, Riemannian structure gives the identification $T^*M \cong TM$. Combining this identification with the identification $T_q^*M \cong T_z(T_q^*M) = \Delta_z$ of the vector space T_q^*M with its tangent space we obtain the explicit formula for the curvature operator R_z^H of the Hamiltonian field $\vec{H}$ at $z \in T^*M$:

$$R_z^H = \mathfrak{R}(\cdot, z)z + \nabla_q^2 U,$$

where $\mathfrak{R}$ is the Riemannian curvature and ∇^2 is the second covariant derivative.

References

[1] A. A. Agrachev, *The curvature and hyperbolicty of Hamiltonian systems*, Proceed. Steklov Math. Inst., (2007), **256**, 26–46

[2] A. A. Agrachev, F. C. Chittaro, *Smooth optimal synthesis for infinite horizon variational problems*, J. ESAIM: Control, Optimisation and Calculus of Variations, (2009), **15**, 173–188

[3] Ya. B. Pesin, *Lectures on partial hyperbolicity and stable ergodicity*, EMS, 2004

[4] M. P. Wojtkovski, *Magnetic flows and Gaussian thermostats on manifolds of negative curvature*, Fundamenta Mathematicæ, (2000) **163**, 177–191