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COMPUTATION OF THE EULER CHARACTERISTIC OF INTERSECTIONS OF REAL QUADRICS

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In this note we present formulas that express the Euler characteristic of the set of real solutions of a system of quadratic equations or quadratic inequalities, and also some results that make it possible to use information about the solution sets of systems of quadratic equations and inequalities to characterize the conditional extremum points of smooth functions.

1. DEFINITION. Let $p: \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^k$ be a symmetric bilinear map. It is called *degenerate* if the origin in $\mathbf{R}^k$ is a critical value of the quadratic map $x \mapsto p(x,x)$, $x \in \mathbf{R}^N \backslash 0$; otherwise the map p is called *nondegenerate*.

Let $\omega \in \mathbf{R}^{k*}$. Then ωp is a scalar quadratic form on $\mathbf{R}^N$. We denote by ωP the corresponding symmetric $N \times N$ matrix. *The map p is degenerate* if and only if for some $\omega \in \mathbf{R}^{k*} \backslash 0$ and $x \in \mathbf{R}^N \backslash 0$ the equalities $\omega P x = 0$ and $p(x,x) = 0$ hold. It is not hard to conclude from this that the degenerate maps form a proper algebraic subset in the space of all symmetric bilinear maps.

More generally, let K be a closed convex cone in $\mathbf{R}^k$ and $K^\circ = \{\omega \in \mathbf{R}^{k*} | \omega y \leq 0 \ \forall y \in K\}$ the dual cone. We call the map p *degenerate relative to K* if for some $\omega \in K^\circ \backslash 0$ and $x \in \mathbf{R}^N \backslash 0$ the relations $\omega P x = 0$ and $p(x,x) \in K$ hold; otherwise the map p is called *nondegenerate relative to K*. Thus, the degeneracy of p is equivalent to the degeneracy of p relative to the cone $K = 0$.

Let S^{N-1} ($\subset \mathbf{R}^N$) be the sphere of unit radius with center at the origin and $\mathbf{P}^{N-1}$ the quotient space of S^{N-1} by the action of the involution $x \mapsto (-x)$. Since $p(-x,-x) = p(x,x)$, the map $\bar{p}: \{x, -x\} \mapsto p(x,x)$ from $\mathbf{P}^{N-1}$ into $\mathbf{R}^k$ is well defined. Assume that p is nondegenerate relative to a cone $K \subset \mathbf{R}^k$. Then, as is not hard to show, $\bar{p}^{-1}(K)$ is a submanifold with boundary of dimension $N - k - 1 + \dim K$ in $\mathbf{P}^{N-1}$ (the boundary, in general, is not smooth) or the empty set. But if K is a linear subspace in $\mathbf{R}^k$ (i.e., $K = -K$), then $\bar{p}^{-1}(K)$ is a real analytic submanifold in $\mathbf{P}^{N-1}$ or is empty.

To an arbitrary scalar quadratic form q on $\mathbf{R}^N$ there corresponds an inertia index $\operatorname{ind} q$, which is equal to the largest dimension of a subspace in $\mathbf{R}^N$ on which the form q is negative-definite. If p is a symmetric bilinear map of $\mathbf{R}^N \times \mathbf{R}^N$ into $\mathbf{R}^k$, then there is defined a function $\operatorname{ind} p: \omega \to \operatorname{ind}(\omega p)$, $\omega \in \mathbf{R}^{k*}$, that takes nonnegative integral values. We fix a closed convex cone $K \subset \mathbf{R}^k$. Let S^{k-1} be the unit sphere in $\mathbf{R}^{k*}$ and $\Omega = S^{k-1} \cap K^0$. We set

$$\Omega_p^n = (\operatorname{ind} p)^{-1}([0,n]) \cap \Omega, \qquad n \geq 0.$$

We note that $K = K^{\circ\circ} = \Omega^\circ$, and so

$$\bar{p}^{-1}(K) = \bar{p}^{-1}(\Omega^\circ) = \{x \in \mathbf{P}^{N-1} | \omega \bar{p}(x) \leq 0, \ \forall \omega \in \Omega\}.$$

Expressions of the form $\chi(\cdot)$ below denote the Euler characteristic of the topological space in parentheses, and all topological spaces that we shall encounter admit a finite triangulation, so that ambiguities in the definition of the Euler characteristic do not arise.

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THEOREM 1. *Assume that a symmetric bilinear map* $p\colon \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^k$ *is nondegenerate relative to a cone* K. *Then*

$$\chi(\bar{p}^{-1}(K)) = \tfrac{1}{2}(1+(-1)^{N-1}) - \sum_{n=0}^{N-1}(-1)^n \chi(\Omega_p^n). \tag{1}$$

As an example we consider the case when $k = 3$ and $K = 0$. Then $\bar{p}^{-1}(0)$ is the intersection of three quadrics in $\mathbf{P}^{N-1}$. The equation $\det(\omega P) = 0$ defines a real curve of degree N in $\mathbf{P}^2 = \{\{\alpha\omega, \alpha \in \mathbf{R}\}|\omega \in \mathbf{R}^{3*}\backslash 0\}$. We assume that this is a nonsingular curve. If N is odd, then $\bar{p}^{-1}(0)$ is an odd-dimensional manifold and $\chi(\bar{p}^{-1}(0)) = 0$, and for even N the identity (1) can easily be transformed to the form

$$\chi(\bar{p}^{-1}(0)) = 2\chi(\{\bar{\omega} \in \mathbf{P}^2 |\det(\omega P) \le 0\}).$$

From the well-known inequalities of Petrovskiĭ [1] it now follows that

$$|\chi(\bar{p}^{-1}(0))| \le \tfrac{3}{4}N(N-2)+2.$$

An identity analogous to (1) holds for Hermitian maps as well. We call symmetric R-bilinear maps $p_{\mathbf{C}}\colon \mathbf{C}^N \times \mathbf{C}^N \to \mathbf{R}^k$ that satisfy the condition $p_{\mathbf{C}}(iz, iw) = p_{\mathbf{C}}(z,w)$ $\forall z, w \in \mathbf{C}^N$ *Hermitian* (for $k = 1$ not a sesquilinear complex form is obtained, but the real part of such a form).

An Hermitian map $p_{\mathbf{C}}$ is called *nondegenerate* if the map of $\mathbf{R}^{2N} \times \mathbf{R}^{2N}$ into $\mathbf{R}^k$ obtained from $p_{\mathbf{C}}$ by "forgetting" the complex structure in $\mathbf{C}^N$ is nondegenerate. An integral function $\operatorname{ind} p_{\mathbf{C}}$ takes only even values for Hermitian $p_{\mathbf{C}}$.

Let $\mathbf{C}^N \supset S^{2N-1}$ be the unit sphere in $\mathbf{C}^N$. Multiplication of vectors in $\mathbf{C}^N$ by complex numbers equal to one in absolute value defines an action of the group S^1 on S^{2N-1} and the quotient space $\mathbf{CP}^{N-1}$ of the sphere S^{2N-1} by this action. If $p_{\mathbf{C}}\colon \mathbf{C}^N \times \mathbf{C}^N \to \mathbf{R}^k$ is an Hermitian map, then the map $\bar{p}_{\mathbf{C}}\colon \{e^{i\theta}z, \theta \in \mathbf{R}\} \mapsto p(z,z)$ from $\mathbf{CP}^{N-1}$ into $\mathbf{R}^k$ is well defined.

THEOREM $1_{\mathbf{C}}$. *Assume that an Hermitian map* $p_{\mathbf{C}}\colon \mathbf{C}^N \times \mathbf{C}^N \to \mathbf{R}^k$ *is nondegenerate relative to a cone* K. *Then*

$$\chi(\bar{p}_{\mathbf{C}}^{-1}(K)) = N - \sum_{n=0}^{N-1} \chi(\Omega_{p_{\mathbf{C}}}^{2n}).$$

2. We denote by $\mathscr{P}_N$ the space of all scalar quadratic forms on $\mathbf{R}^N$. In $\mathscr{P}_N$ there is an algebraic hypersurface $\Pi = \{q \in \mathscr{P}_N | \ker q \ne 0\}$ of degenerate forms. The set of nonsingular points of the hypersurface Π consists of forms q such that $\dim\ker q = 1$, and the set $\operatorname{sing}\Pi$ of singular points has codimension 3 in $\mathscr{P}_N$. Consequently, Π is a pseudomanifold and it is orientable in many ways, since $\Pi\backslash\operatorname{sing}\Pi$ is disconnected. Let $q \in \Pi\backslash\sin\Pi$, $\operatorname{ind} q = n$. For any sufficiently small neighborhood O_q of q in $\mathscr{P}_N$, the set $O_q\backslash\Pi$ has two connected components, one of which consists of forms of index n, and the other of forms of index $n+1$. In order to choose the orientation of the hypersurface Π at the point q, it is enough to assign one of the components a plus sign, and the other a minus sign. We call the orientation whose component consisting of forms of index n is assigned a $(-1)^n$ sing *canonical.*

Let $m > 0$ and let $G_m^+(\mathscr{P}_N)$ be the manifold of all orientable m-dimensional subspaces in $\mathscr{P}_N$. We denote by $G_m^+(\Pi\backslash\operatorname{sing}\Pi)$ the submanifold (nonclosed) in $\mathscr{P}_N \times G_m^+(\mathscr{P}_N)$ consisting of pairs (q, H) such that $q \in \Pi\backslash\operatorname{sing}\Pi$ and the m-dimensional oriented subspace H is tangent to Π at q. We denote by $G_m^+(\Pi)$ the topological closure of $G_m^+(\Pi\backslash\operatorname{sing}\Pi)$ in $\mathscr{P}_N \times G_m^+(P_N)$. It is not hard to show that $G_m^+(\Pi)$ is an algebraic submanifold of codimension $m+1$ in $\mathscr{P}_N \times G_m^+(\mathscr{P}_N)$ which is, furthermore, an orientable pseudomanifold. The canonical orientation of the manifold $\Pi\backslash\operatorname{sing}\Pi$ induces an orientation of

$G_m^+(\Pi\backslash \operatorname{sing}\Pi)$, which can be extended to a well-defined orientation of the pseudomanifold $G_m^+(\Pi)$ (we note that the set of nonsingular points of $G_m^+(\Pi)$ is connected, and so, unlike Π, there exist only two orientations on $G_m^+(\Pi)$).

Let $p\colon \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^k$ be a symmetric bilinear map such that ωp is a nonzero quadratic form $\forall \omega \neq 0$. If a row $\omega \neq 0$, then its orthogonal complement $\omega^\perp$ is a hyperplane in $\mathbf{R}^{k*}$ with the obvious orientation (which is induced by the standard orientation of $\mathbf{R}^{k*}$); accordingly $(\omega^\perp p) \in G_{k-1}^+(\mathscr{P}_N)$. We set $Tp = \{(\omega p, \omega^\perp p) | \omega \in S^{k-1}\}$—this is a $(k-1)$-dimensional sphere imbedded in $\mathscr{P}_N \times G_{k-1}^+(\mathscr{P}_N)$. The map p is nondegenerate if and only if

$$Tp \cap G_{k-1}^+(\Pi) = \varnothing.$$

Furthermore let $\mathscr{P}_N \ni \iota$ be a quadratic form of the type $\iota\colon x \mapsto |x|^2$. We set $Jp = \{(\iota, \omega^\perp p) | \omega \in S^{k-1}\}$—this is another $(k-1)$-dimensional sphere in $\mathscr{P}_N \times G_{k-1}^+(\mathscr{P}_N)$. It is easy to see that the $(k-1)$-dimensional cycles Tp and Jp are homologous in $\mathscr{P}_N \times G_{k-1}^+(\mathscr{P}_N)$. If p is a nondegenerate map, then the linking coefficient $l(Jp - Tp, G_{k-1}^+(\Pi))$ is defined.

Let Γ be a singular chain in $\mathscr{P}_N \times G_{k-1}^+(\mathscr{P}_N)$ that satisfies the condition $\partial\Gamma = Jp - Tp$. Then $l(Jp - Tp, G_{k-1}^+(\Pi))$ equal the intersection index of Γ with $G_{k-1}^+(\Pi)$ and does not depend on the choice of Γ.

THEOREM 2. *Let $p\colon \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^k$ be a nondegenerate symmetric bilinear map. Then*

$$\chi(\bar{p}^{-1}(0)) = (-1)^k l(Jp - Tp, G_{k-1}^+(\Pi)) + \tfrac{1}{2}(1 - (-1)^{N(k-1)}).$$

3. In this section quadratic maps are used to investigate a problem concerning arbitrary smooth maps. Let U and M be manifolds of class C^∞, $\dim M \leq \dim U$, $u \in U$, and $F\colon U \to M$ a smooth map. We call u an *extremal point* if $F(u)$ lies on the boundary of the set $F(O_u)$ for some neighborhood O_u of u in U; otherwise we call u an *interior point* of F. The concept of extremal point is the natural geometric analog of the concept of local conditional extremum point for a scalar function subject to restrictions of equality type.

It follows from the implicit function theorem that an extremal point must be critical, i.e., $\operatorname{im} F_u' \neq T_{F(u)}M$, where $F_u'\colon T_uU \to T_{F(u)}M$ is the differential of the map F at the point u. At a critical point u the Hessian of F, i.e. the symmetric bilinear map $F_u''\colon \ker F_u' \times \ker F_u' \to \operatorname{coker} F_u'$, is well defined (we recall that $\operatorname{coker} F_u' = T_{F(u)}M/\operatorname{im} F_u'$).

PROPOSITION 1. *Assume that the map F_u'' is nondegenerate. Then there exists a homeomorphism $\Phi\colon T_uU \to O_u$ of the space T_uU onto some neighborhood O_u of u in U such that for every $v \in T_uU$ the equality $F(\Phi(v)) = F(u)$ is equivalent to the relations $v \in \ker F_u'$ and $F_u''(v,v) = 0$. In addition, the set $F^{-1}(F(u)) \cap O_u \backslash u$ contains no critical points of F.*

If F_u'' is nondegenerate, then zero in $\operatorname{coker} F_u'$ is not a critical value of the quadratic map $v \mapsto F_u''(v,v)$, $v \in \ker F_u' \backslash 0$. Two cases are possible: a) zero is a regular value of this map, or b) zero, in general, is not a value of this map.

COROLLARY. *Under the conditions of Proposition 1, if $F_u''(v,v) = 0$ for some $v \neq 0$, then u is an interior point of F.*

In §§1 and 2 we computed the Euler characteristic of the set of nontrivial zeros of a quadratic map. Equality of this value to zero for the map $\overline{F}_u''$ is a necessary condition for u to be an extremal point.

Let us turn to case b).

PROPOSITION 2. *Let $k^2 \leq N$ and let $\mathscr{P}(N,k)$ be the space of all symmetric bilinear maps of $\mathbf{R}^N \times \mathbf{R}^N$ into $\mathbf{R}^k$. In the space $\mathscr{P}(N,k)$ there exists a proper algebraic subset $\mathfrak{A}$ such that for any $p \in \mathscr{P}(N,k)\backslash\mathfrak{A}$ the following alternative is valid: either $p(x,x) = 0$ for some $x \neq 0$, or the image of the map $x \mapsto p(x,x)$, $x \in \mathbf{R}^N$, does not coincide with $\mathbf{R}^k$.*

PROPOSITION 3. *Assume that F''_u is nondegenerate. If the image of the quadratic map $v \mapsto F''_u(v,v)$, $v \in \ker F'_{u'}$ does not coincide with* $\operatorname{coker} F'_{u'}$ *then u is an extremal point of F.*

REMARK 1. The restriction $k^2 \leq N$ in the formulation of Proposition 2 can apparently be weakened but cannot be elimated entirely. For example, any map close to $z \mapsto z^2$, $z \in \mathbf{C} = \mathbf{R}^2$, is obviously surjective. This also goes for the map $(z,w) \mapsto (z\bar{w}, |w|^2 - |z|^2)$ from $\mathbf{C}^2 = \mathbf{R}^4$ into $\mathbf{C} \oplus \mathbf{R} = \mathbf{R}^3$ that realizes the Hopf bundle.

REMARK 2. We have shown how the problem of characterizing extremal points leads to systems of quadratic equations. If we attempt to characterize local conditional extremum points under restrictions of inequality type, we arrive at systems of quadratic inequalities.

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