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CHRONOLOGICAL SERIES AND THE CAUCHY—KOWALEVSKI THEOREM

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We consider the Cauchy problem for a system of partial differential equations. We prove an existence theorem for a solution of this problem which is analytic in the spatial variable under the assumption of measurability and local integrability of the right side with respect to time only. The solution is represented in the form of a chronological series.

In this paper we consider the Cauchy problem for a system of partial differential equations

$$\begin{gathered}\frac{\partial^m}{\partial t^m} u(t, x)= \\ =f\left(t, x, u(t, x), \ldots, \frac{\partial^{k+|\alpha|} u(t, x)}{\partial t^k, (\partial x^1)^{\alpha_1} \ldots (\partial x^n)^{\alpha_n}}, \ldots\right), \\ \frac{\partial^j}{\partial t^j} u(t, x)|_{t=t_0}=\varphi_j(x), \quad x \in G \subset R^n, \\ j=0, 1, \ldots, m-1\end{gathered} \tag{1}$$

with respect to an N-dimensional function u. Here f is some N-dimensional function depending on t, x, u and all partial derivatives of u of the form

$$\frac{\partial^{k+|\alpha|}}{\partial t^k (\partial x^1)^{\alpha_1} \ldots (\partial x^n)^{\alpha_n}}, \quad |\alpha|+k \leqslant m, \quad k<m,$$

where $\alpha=(\alpha_1, \ldots, \alpha_n)$, $|\alpha|=\alpha_1+\cdots+\alpha_n$.

The classical Cauchy—Kowalevski theorem asserts that if the functions $f, \varphi_k, k=0, 1, \ldots, m-1$, are analytic in all free arguments in the corresponding domains, then there exists a unique analytic solution of this system, defined in some sufficiently small neighborhood of an arbitrary initial point $(t_0, x_0) \in R \times G$.

Despite the fact that this theorem has been known for many years, the subject is sufficiently important that the question of the possibility of strengthening and generalizing it in various directions is constantly found in the field of view of specialists. We shall not

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give here an account of the history of the question, but we refer the reader for details to [4, 7]. We note only [6], in which there is proved an abstract Cauchy—Kowalevski theorem under the hypothesis only of the continuity of the right side of the equation with respect to t.

In the present paper, to solve the Cauchy problem (1) we shall use the technique of chronological series developed in [1, 2].

Representation of the solution in the form of a chronological series gives the possibilities:

1) To prove the existence of a solution of the Cauchy problem (1) under the assumption only of measurability (and local summability) of the right side in t. This is essential, for example, for the theory of optimal control.

2) To get relatively explict formulas, expressing a solution in terms of the initial conditions and the right side.

3) In certain cases to get simple estimates with explicitly calculable constants of the interval of time t on which there exists a solution, of the norm of the solution of the remainder term of the chronological series.

In the proof we first construct a formal solution of the Cauchy problem in the form of a chronological series, and then we prove the convergence of this series.

We consider separately the case of a linear system and the case of a scalar quasilinear equation. This is done for the following reasons.

In the linear case the chronological series can be calculated especially simply, and the proof of its convergence is slightly different from the proof of convergence of the Volterra chronological series, in the case of an ordinary differential equation (cf. [1]). Here we get convenient explicit estimates for the general term of the series. In the scalar quasilinear case the formulas are yet more compact, and furthermore the estimates obtained here are used in the proof of convergence of the general chronological series.

Now we describe the notation used in the present paper.

As usual, by R^n we denote real n-dimensional arithmetic vector space, whose points we treat as column-vectors and always denote by Latin letters; we denote row-vectors by Greek letters. We shall write the scalar product of a row-vector by a column-vector of identical dimensions in the form of matrix multiplication

$$\xi\cdot x=(\xi_1, \dots, \xi_n)\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}=\sum_{\alpha=1}^{n}\xi_\alpha x^\alpha.$$

The Jacobian matrix of an m-dimensional vector function $x\mapsto g(x)$ with respect to the coordinates of the vector $x\in R^n$ we denote by

$$\operatorname{grad} g(x)=\frac{\partial g(x)}{\partial x}=(\partial_\beta g^\alpha(x)),$$

$$\alpha=1, \dots, m, \quad \beta=1, \dots, n, \quad \partial_\beta=\frac{\partial}{\partial x^\beta}.$$

By the modulus of the vector $x\in R^n$ we mean the quantity

$$|x|=\max_{1\leqslant\alpha\leqslant n}|x^\alpha|,$$

correspondingly,

$$|\xi| \doteq \sum_{\beta=1}^{n} |\xi_\beta|$$

is the modulus of the n-dimensional row-vector $\xi=(\xi_1, \ldots, \xi_n)$.

The modulus $|A|$ of the $n\times m$ -matrix $A=(a^\alpha_\beta), \alpha=1, \ldots, n, \beta=1, \ldots, m$, is, by definition,

$$|A|=\sum_{\beta=1}^{m} \max_{1\leqslant\alpha\leqslant n} |a^\alpha_\beta|.$$

By Id_X we denote the identity mapping of the set X. If it is clear from the context which set X is involved, then we write simply Id.

In conclusion we express profound thanks to R. V. Gamkrelidze, who suggested applying the technique of [1] to partial differential equations and for his constant interest with respect to this work.

1. Preparatory Material

In this section we give the initial concepts and facts from analysis and algebra which will be used below. Some of them are commonly known and one can become acquainted with the others in the more detailed account in [1].

1. Differentiations in Algebras. Let $\mathfrak{A}$ be an arbitrary real algebra, i.e., a real vector space in which there is defined a multiplication, satisfying the unique condition of bilinearity. Thus, an algebra $\mathfrak{A}$ can be nonassociative and not have a unit.

By $\mathscr{L}(\mathfrak{A})$ we denote the associative algebra of all linear mappings of the vector space $\mathfrak{A}$ into itself. The product of elements $T_1, T_2\in\mathscr{L}(\mathfrak{A})$ is defined as their composition

$$T_1T_2=T_1\circ T_2 \quad \forall T_1, T_2\in\mathscr{L}(\mathfrak{A}).$$

A linear mapping $\delta\in\mathscr{L}(\mathfrak{A})$ is called a differentiation of the algebra $\mathfrak{A}$ if it satisfies the formal rule of differentiation of a product

$$\delta(ab)=(\delta a)b+a(\delta b) \quad \forall a, b\in\mathfrak{A}. \tag{1.1}$$

The set $\mathrm{Der}(\mathfrak{A})$ of all differentiations of the algebra $\mathfrak{A}$ forms a Lie algebra with the product

$$[\delta_1, \delta_2]=\delta_1\circ\delta_2-\delta_2\circ\delta_1.$$

Let $\delta_j\in\mathrm{Der}(\mathfrak{A}), j=1, 2, \ldots, m$, be arbitrary differentiations of the algebra $\mathfrak{A}$. It turns out that one has the formula

$$\delta_m\circ\ldots\circ\delta_1(ab)=\sum_{\alpha+\beta=m}\sum_{\pi\in S(\alpha,\beta)}$$

$$(\delta_{\pi(m)}\circ\ldots\circ\delta_{\pi(\alpha+1)}(a))(\delta_{\pi(\alpha)}\circ\ldots\circ\delta_{\pi(1)}(b)) \quad \forall a, b\in\mathfrak{A}, \tag{1.2}$$

where $S(\alpha, \beta), \alpha+\beta=m$ is the set of all permutations of the numbers 1, 2, ..., m, preserving the orders of the first α and separately of the last β numbers.

The proof of (1.2) is based on (1.1) and is done by the obvious induction on m.

In the special case when $\delta_j=\delta$, $j=1,\ldots,m$, from (1.2) one gets the Leibniz formula

$$\delta^m(ab)=\sum_{\alpha+\beta=m}\frac{m!}{\alpha!\beta!}(\delta^\alpha a)(\delta^\beta b)\quad \forall a, b\in\mathfrak{A},$$

where $\delta^0=\mathrm{Id}$, $\delta^{k-1}=\delta\circ\delta^k=\delta^k\circ\delta$.

2. One-Parameter Families of Functions. We denote by $\Phi(G)$ the algebra of all infinitely differentiable functions, defined in the domain $G\subset R^n$. By $\Phi^N(G)$ we shall denote the Cartesian product of N copies of $\Phi(G)$.

Let $h\in R^n$. We denote by $\vec{h}$ the first-order linear differential operator

$$\vec{h}=\sum_{\alpha=1}^{n}h^\alpha\partial_\alpha,\quad \partial_\alpha=\frac{\partial}{\partial x^\alpha}$$

and we set for $\forall\varphi\in\Phi(G)$

$$\|\varphi\|_{s,M}=\max_{x\in M}\sum_{k=0}^{s}\frac{1}{k!}\max_{h=1}|\vec{h}^k\varphi(x)|.$$

Here $M\subset G$ is compact and s is an integer $\geqslant 0$.

It is obvious that with the help of the family of seminorms $\|\cdot\|_{s,M}$ we turn $\Phi(G)$ into a Frechet space (completely metrizable and locally convex).

Below we shall see that in those partial differential equations which we consider in this paper, the variables t and x play completely different roles. Hence we shall be especially interested in one-parameter families φ_t, $t\in R$, of elements $\Phi(G)$, to which in the standard way one carries over all the basic constructions of analysis.

Namely, continuity and differentiability of a family φ_t, $t\in R$, of elements $\Phi(G)$ is defined in the obvious way, by virtue of the fact that $\Phi(G)$ is a linear topological space.

We shall say that the family φ_t, $t\in R$, is measurable if $\forall x\in G$ the scalar function

$$t\mapsto\varphi_t(x)$$

is measurable.

One says that the measurable family φ_t, $t\in R$, is locally integrable if $\forall t_1, t_2\in R$, $s=0,1,\ldots$, and compact $M\subset G$

$$\int_{t_1}^{t_2}\|\varphi_\tau\|_{s,M}\,d\tau<\infty.$$

By the integral of a locally integrable family φ_t, $t\in R$, in the given limits from t_1 to t_2 we mean the function

$$x\mapsto\int_{t_1}^{t_2}\varphi_\tau(x)\,d\tau.$$

One can prove (cf. [1]) that this function belongs to $\Phi(G)$ and one has the formula

$$\vec{h}_1\circ\ldots\circ\vec{h}_k\int_{t_1}^{t_2}\varphi_\tau d\tau=\int_{t_1}^{t_2}\vec{h}_1\circ\ldots\circ\vec{h}_k\varphi_\tau d\tau\quad \forall h_1,\ldots,h_k\in R^n,$$

from which follows the inequality

$$\left\|\int_{t_1}^{t_2}\varphi_\tau d\tau\right\|_{s,M}\leqslant\int_{t_1}^{t_2}\|\varphi_\tau\|_{s,M}\,d\tau\quad\forall s\geqslant 0,\quad M\subset G.$$

The family $\varphi_t,\ t\in R$, is called absolutely continuous if there exists a locally integrable family $\psi_t,\ t\in R$, such that

$$\varphi_t=\varphi_{t_0}+\int_{t_0}^{t}\psi_\tau d\tau.$$

It turns out that in this case for almost all $t\in R$

$$\frac{d\psi_t}{dt}=\psi_t.$$

In fact, it is clear that the topology of the space $\Phi(G)$ can be given with the help of a countable number of seminorms $\|\cdot\|_{s,M_j}$, where $M_j\subset G$ is compact, $j=1,2,\ldots,\ s=0,1,2,\ldots$.

It is easy to see that there exists a set $T_{s,j}$ such that mes $(R\setminus T_{s,j})=0$ and $\forall x\in M_j,\ t\in T_{s,j}$

$$\int_t^{t+\Delta t}\vec{h}^k\psi_\tau(x)\,d\tau=\vec{h}^k\psi_t(x)\,\Delta t+\beta_k(\Delta t;\ x,\ h)\,\Delta t,$$

where $\beta_k(\Delta t;\ x,\ h)\to 0$ as $\Delta t\to 0$ uniformly with respect to $x\in M_j,\ |h|=1$. Consequently,

$$\lim_{\Delta t\to 0}\left\|\frac{1}{\Delta t}\{\varphi_{t+\Delta t}-\varphi_t\}-\psi_t\right\|_{s,M_j}=\lim_{\Delta t\to 0}\max_{x\in M_j}\sum_{k=0}^{s}\frac{1}{k!}\max_{|h|=1}\left|\vec{h}^k\left\{\frac{1}{\Delta t}\int_t^{t+\Delta t}\psi_\tau d\tau-\psi_t\right\}\right|=$$

$$=\lim_{\Delta t\to 0}\max_{x\in M_j}\sum_{k=0}^{s}\frac{1}{k!}\left|\frac{1}{\Delta t}\int_t^{t+\Delta t}\vec{h}^k\psi_\tau d\tau-\vec{h}^k\psi_t\right|=\lim_{\Delta t\to 0}\max_{x\in M_j}\sum_{k=0}^{s}\max_{|h|=1}|\beta_k(\Delta t;\ x,\ h)|\frac{1}{k!}=0.$$

We set $T=\bigcap_{s=0}^{\infty}\bigcap_{j=0}^{\infty}T_{s,j}$ so mes $(R\setminus T)=0$ and $\forall t\in R$ $\frac{d\varphi_t}{dt}=\psi_t$, which is what had to be proved.

All the concepts introduced above (except for absolute continuity) also carry over naturally to families

$$\varphi_{\tau_1,\ldots,\tau_m},\quad \tau_j\in R,\quad j=1,2,\ldots,m,$$

of elements of $\Phi(G)$, depending on m parameters $\tau_1,\ldots,\tau_m$. Moreover, there are defined in an obvious way corresponding concepts for families $\varphi_t,\ t\in R$, of elements of $\Phi^N(G)$. Thus, for example, a family $\varphi_t,\ t\in R$, of elements of $\Phi^N(G)$ is called locally integrable if the family $\varphi_t{}^j,\ t\in R,\ j=1,2,\ldots,N$, of elements of $\Phi(G)$ is locally integrable, and by the integral of the locally integrable family $\varphi_t,\ t\in R$, of elements of $\Phi^N(G)$ is meant the element

$$\int_{t_1}^{t_2}\varphi_\tau d\tau=\begin{pmatrix}\int_{t_1}^{t_2}\varphi_\tau^1\,d\tau\\ \vdots\\ \int_{t_1}^{t_2}\varphi_\tau^N\,d\tau\end{pmatrix}\in\Phi^N(G).$$

In the m-dimensional space of point $\tau^{(m)}=(\tau_1,\ldots,\tau_m)$ by the symbol $\Delta_{t_0,t}(\tau^{(m)})$ we shall denote the simplex

$$\Delta_{t_0,t}(\tau^{(m)}) = \{(\tau_1, \ldots, \tau_m) \mid t_0 \leqslant \tau_m \leqslant \ldots \leqslant \tau_1 \leqslant t\}.$$

If π is an arbitrary permutation of the numbers 1, 2, ..., m, then by the symbol $\Delta_{t_0,t}(\pi\tau^{(m)})$ is denoted the simplex

$$\Delta_{t_0,t}(\pi\tau^{(m)}) = \{(\tau_1, \ldots, \tau_m) \mid t_0 \leqslant \tau_{\pi(m)} \leqslant \ldots \leqslant \tau_{\pi(1)} \leqslant t\}.$$

One has the obvious equality

$$\Delta_{t_0,t}(\tau^{(\alpha)}) \times \Delta_{t_0,t}(\tau^{\beta}) = \bigcup_{\pi\in S(\alpha,\beta)} \Delta_{t_0,t}(\pi^{-1}\tau^{(m)})$$
$$\forall\alpha,\ \beta \quad \alpha+\beta=m.$$

Let $\mathcal{A}=\mathcal{A}(\tau^{(m)})$, $\mathcal{B}=\mathcal{B}(\tau^{(m)})$ be locally integrable functions with values in $\Phi(G)$. Then $\forall\alpha,\ \beta,\ \alpha+\beta=m$ we have:

$$\int\limits_{\Delta_{t_0,t}(\tau^{(\alpha)})} \mathcal{A}(\tau^{(\alpha)})\, d\tau^{(\alpha)} \int\limits_{\Delta_{t_0,t}(\tau^{(\beta)})} \mathcal{B}(\tau^{(\beta)})\, d\tau^{(\beta)} =$$

$$= \int\limits_{\Delta_{t_0,t}(\tau^{(\alpha)})\times\Delta_{t_0,t}(\tau^{(\beta)})} \mathcal{A}(\tau^{(\alpha)})\mathcal{B}(\tau^{(\beta)})\, d\tau^{(\alpha)}\otimes d\tau^{(\beta)} = \sum_{\pi\in S(\alpha,\beta)} \int\limits_{\Delta_{t_0,t}(\tau^{(m)})} \mathcal{A}(\tau_{\pi(1)}, \ldots, \tau_{\pi(\alpha)})\mathcal{B}(\tau_{\pi(\alpha+1)}, \ldots, \tau_{\pi(m)})\, d\tau^{(m)}.$$

Whence follows the important relation

$$\int\limits_{t_0}^{t} d\tau_1 \ldots \int\limits_{t_0}^{\tau_{\alpha-1}} d\tau_{(\alpha)}\mathcal{A}(\tau_1, \ldots, \tau_\alpha)\, d\tau^{(\alpha)} \quad \int\limits_{t_0}^{t} d\tau_1 \ldots \int\limits_{t_0}^{\tau_{\beta-1}} d\tau^{(\beta)}\mathcal{B}(\tau_1, \ldots, \tau_{(\beta)})\, d\tau^{(\beta)} =$$

$$= \sum_{\pi\in S(\alpha,\beta)} \int\limits_{t_0}^{t} d\tau_1 \ldots \int\limits_{t_0}^{\tau_{m-1}} d\tau_m \mathcal{A}(\tau_{\pi(1)}, \ldots, \tau_{\pi(\alpha)}) \quad \mathcal{B}(\tau_{\pi(\alpha+1)}, \ldots, \tau_{\pi(m)})\, d\tau^{(m)}. \tag{1.3}$$

2. Existence of a Solution of the Cauchy Problem

In this section we shall prove the stronger existence theorem for a solution of the Cauchy problem already mentioned in the introduction. We note, first of all, that it suffices to consider systems of first-order partial differential equations, whose right sides do not contain the spatial variable x, i.e., systems of the form

$$\begin{gathered} \frac{\partial u}{\partial t} = f\left(t,\ u,\ \frac{\partial u}{\partial x^1}, \ldots, \frac{\partial u}{\partial x^n}\right), \\ u(t_0,\ x) = u_0(x), \quad x\in G\subset R^n. \end{gathered} \tag{2.1}$$

The reduction of a general system of partial differential equations (1) to such a form is carried out in the usual way, by the introduction of supplementary unknown functions, and subsequent differentiation (cf. e.g., [3]). We shall also have to deal with a system of the form (2) later.

We turn first to the consideration of the special case of a linear system.

1. Case of a Linear System. We consider a first-order linear partial differential system

$$\begin{gathered} \frac{\partial v}{\partial t} = B_0(t,\ x)\, v + B_1(t,\ x)\frac{\partial v}{\partial x^1} + \ldots + B_n(t,\ x)\frac{\partial v}{\partial x^n}, \\ v(t_0,\ x) = u_0(x), \quad x\in G\subset R^n. \end{gathered}$$

Below we shall see that the variables x and t play completely different roles in this system. Hence it will be convenient for us to treat the N^2-valued functions

$$B_i: R\times R^n \to R^{N^2}$$

as families

$$B_i=\{A_t^{(i)};\ t\in R\}$$

of infinitely differentiable mappings

$$A_t^{(i)}=B(t,\cdot):\quad R^n\to R^{N^2}.$$

We shall assume that the families $A_t^{(k)}$, $t\in R$, are locally integrable families in $\Phi^{N^2}(G)$. In accord with this we shall write the Cauchy problem we are studying in the form

$$\frac{d}{dt}\,u_t=\sum_{k=0}^{n} A_t^{(k)}\partial_k u_t,\quad u_{t_0}=u_0\in\Phi^N(G), \tag{2.2}$$

where $\partial_k=\frac{\partial}{\partial x^k}$, $\partial_0=\mathrm{Id}$, and consider it as an equation relative to a family u_t, $t\in R$, of infinitely smooth N-dimensional functions. More precisely, by a solution of (2.2) we shall understand an absolutely continuous family u_t, $t\in J$, of elements of $\Phi^N(G')$, where J is an interval of the real axis containing the point t_0,, and G' is a subdomain of the domain G such that for almost all $t\in J$

$$\frac{d}{dt}\,u_t=\sum_{k=0}^{n} A_t^{(k)}\partial_k u_t,$$

and $u_{t_0}=u_0$ in $\Phi^N(G')$

By virtue of the assumption about absolute continuity, the problem (2.2) is equivalent to the integral equation

$$u_t=u_0+\sum_{k=0}^{n}\int_{t_0}^{t} A_\tau^{(k)}\partial_k u_\tau d\tau. \tag{2.3}$$

We shall solve (2.3) by successive substitutions:

$$u_t=u_0+\sum_{k=0}^{n}\int_{t_0}^{t} A_\tau^{(k)}\partial_k u_\tau d\tau=u_0+\sum_{k_1=0}^{n}\int_{t_0}^{t} A_\tau^{(k_1)}\partial_{k_1}u_0 d\tau+\sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\int_{t_0}^{t} d\tau_1\int_{t_0}^{\tau_1} d\tau_2 A_{\tau_1}^{(k_1)}\partial_{k_1}A_{\tau_2}^{(k_2)}\partial_{k_2}u_{\tau_2}=\dots.$$

The formal series which arises

$$F_{t,t_0}(u_0)=\left(\mathrm{Id}+\sum_{m=1}^{\infty}\sum_{k_1=0}^{n}\dots\sum_{k_m=0}^{n}\int_{t_0}^{t} d\tau_1\dots\int_{t_0}^{\tau_{m-1}} d\tau_m A_{\tau_1}^{(k_1)}\partial_{k_1}\ \dots A_{\tau_m}^{(k_m)}\partial_{k_m}\right)u_0 \tag{2.4}$$

is called the chronological series of the Cauchy problem (2.3) or (2.2).

Let V be a complex extension of the domain $G\subset R^n$. We denote by $\Omega(V)$ the Banach space of all functions $\varphi: V\to C$, bounded and analytic in the domain V, with norm

$$\|\varphi\|_V^{C^n}=\sup_{z\in V}|\varphi(z)|.$$

Let $\Omega^k(V)$ be the Cartesian product of k copies of $\Omega(V)$. For $v\in\Omega^k(V)$ we set

$$\|v\|_V^{C^n}=\max_{1\leqslant j\leqslant k}\|v^k\|_V^{C^n}.$$

PROPOSITION 2.1. If $A_t^{(k)}\in\Omega^{N^2}(V)$, $u_0\in\Omega^N(V)$ and the functions

$$t\mapsto\|A_t^{(\alpha)}\|_V^{C^n},\quad \alpha=0,1,\ldots,n,$$

are locally integrable, then for any subdomain V', the distance from the boundary of which to the boundary of the domain V is greater than some $\varepsilon>0$, there exists a $\rho>0$, such that the series

$$F_{t,t_0}(u_0(z))=\left(\mathrm{Id}+\sum_{m=1}^{\infty}\sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}\int_{t_0}^{t}d\tau_1\ldots\int_{t_0}^{\tau_{m-1}}d\tau_m A_{\tau_1}^{(k_1)}(z)\,\partial_{k_1}\ldots A_{\tau_m}^{(k_m)}(z)\,\partial_{k_m}\right)u_0(z)$$

converges absolutely and uniformly for $|t-t_0|<\rho$, $z\in V'$.

Proof. For any $z\in V'$, by virtue of the Cauchy integral formula

$$A_{\tau_\alpha}^{k_\alpha}(z)=\frac{1}{(2\pi i)^n}\int_{C_1}\cdots\int_{C_n}A_{\tau_\alpha}^{(k_\alpha)}(\xi_\alpha)\frac{d\xi_\alpha}{\Pi_{\xi_\alpha}(z)},$$

$$u_0(z)=\frac{1}{(2\pi i)^n}\int_{C_1}\cdots\int_{C_n}\frac{u_0(\xi)}{\Pi_\xi(z)}\,d\xi,$$

where $\Pi_{\xi_\alpha}=(\xi_\alpha^1-z^1)\ldots(\xi_\alpha^n-z^n)$, and C_k is a circle of radius ε with center at the point z^k.

Consequently, we have

$$A_{\tau_1}^{(k_1)}(z)\,\partial_{k_1}A_{\tau_2}^{(k_2)}(z)\ldots A_{\tau_m}^{(k_m)}(z)\,\partial_{k_m}u_0(z)=\frac{1}{(2\pi i)^{n(m+1)}}\int_{C_1}\cdots\int_{C_n}\cdots\int_{C_1}\cdots\int_{C_n}A_{\tau_1}^{(k_1)}(\xi_1)\ldots$$

$$\ldots A_{\tau_m}^{(k_m)}(\xi_m)\,u_0(\xi)\,\chi_{\xi_1,\ldots,\xi_m,\xi}^{k_1,\ldots,k_m}(z)\,d\xi_1\ldots d\xi_m\,d\xi,$$

where

$$\chi_{\xi_1,\ldots,\xi_m,\xi}^{k_1,\ldots,k_m}(z)=\frac{1}{\Pi_{\xi_1}(z)}\,\partial_{k_1}\frac{1}{\Pi_{\xi_2}(z)}\ldots\partial_{k_m}\frac{1}{\Pi_\xi(z)},$$

$$\partial_{k_j}=\frac{\partial}{\partial z^{k_j}},\quad \partial_0=\mathrm{Id}.$$

For $|\xi_k^\alpha-z^\alpha|\geqslant\varepsilon$, $|\xi^\alpha-z^\alpha|\geqslant\varepsilon$, $k=1,2,\ldots,m$, $\alpha=1,2,\ldots,n$, one has the inequality

$$\left|\chi_{\xi_1,\ldots,\xi_m,\xi}^{k_1,\ldots,k_m}(z)\right|\leqslant\frac{2^m m!}{\delta^n\delta^{m(n+1)}},$$

where $\delta=\min\{1,\varepsilon\}$. This estimate is proved by direct induction on m (cf. the analogous estimate in [1]).

Consequently, $\forall z\in V'$, $m=1,2,\ldots$, we have

$$\left|\sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}\int_{t_0}^{t}d\tau_1\ldots\int_{t_0}^{\tau_{m-1}}d\tau_m A_{\tau_1}^{(k_1)}(z)\,\partial_{k_1}A_{\tau_2}^{(k_2)}(z)\ldots A_{\tau_m}^{(k_m)}(z)\,\partial_{k_m}u_0(z)\right|\leqslant$$

$$\leqslant\frac{2^m m!}{\delta^{m(n+1)}}(n+1)^m\int_{t_0}^{t}d\tau_1\ldots\int_{t_0}^{\tau_{m-1}}d\tau_m\|A_{\tau_1}\|_V^{C^n}\ldots\|A_{\tau_m}\|_V^{C^n}\|u_0\|_V^{C^n}.$$

where

$$\| A_{\tau_k} \|_V^{C^n} = \max_{0 \leqslant j \leqslant n} \| A_{\tau_k}^{(j)} \|_V^{C^n}.$$

Since

$$\int_{t_0}^{t} d\tau_1 \ldots \int_{t_0}^{\tau_{m-1}} d\tau_m \| A_{\tau_1} \|_V^{C^n} \ldots \| A_{\tau_m} \|_V^{C^n} = \frac{1}{m!} \left(\int_{t_0}^{t} \| A_\tau \|_V^{C^n} d\tau \right)^m,$$

we have the estimate

$$\left\| \sum_{k_1=0}^{n} \cdots \sum_{k_m=0}^{n} \int_{t_0}^{t} d\tau_1 \ldots \int_{t_0}^{\tau_{m-1}} d\tau_m A_{\tau_1}^{(k_1)} \partial_{k_1} A_{\tau_m}^{(k_m)} \ldots A_{\tau_m}^{(k_m)} \partial_{k_m} u_0 \right\|_V^{C^n} \leqslant \left[\frac{2(n+1)}{\delta^{n+1}} \right]^m \left(\int_{t_0}^{t} \| A_\tau \|_V^{C^n} d\tau \right)^m \| u_0 \|_V^{C^n},$$

from which follows the assertion of Proposition 2.1.

Using the absolute and uniform convergence of the series

$$F_{t,t_0}(u_0(z)) = u_0(z) + \int_{t_0}^{t} d\tau \sum_{k=0}^{n} A_\tau^{(k)}(z) \partial_k u_0(z) + \ldots,$$

we get, if the hypotheses of Proposition 2.1 hold,

$$\frac{d}{dt} F_{t,t_0}(u_0(z)) = \sum_{k=0}^{n} A_t^{(k)}(z) \partial_k \{ u_0(z) + \sum_{m=1}^{\infty} \sum_{k_1=0}^{n} \cdots \sum_{k_m=0}^{n} \int_{t_0}^{t} d\tau_1 \ldots \int_{t_0}^{\tau_{m-1}} d\tau_m \times$$

$$\times A_{\tau_1}^{(k_1)}(z) \partial_{k_1} \ldots A_{\tau_m}^{(k_m)}(z) \partial_{k_m} u_0(z) \} = \sum_{k=0}^{n} A_t^{(k)}(z) \partial_k F_{t,t_0}(u_0(z)),$$

$$F_{t_0,t_0}(u_0(z)) = u_0(z),$$

but this also means that the series $F_{t,t_0}(u_0)$ is a solution of (2.2).

Thus, we have proved the following theorem.

THEOREM 2.1. Let $A_t^{(k)}$, $t \in R$, be a locally integrable family in $\Phi^{N^*}(G)$ and

$$u_0 \in \Omega^N(V), \quad A_t^{(k)} \in \Omega^{N^*}(V) \quad k = 0, 1, 2, \ldots,$$

where the functions

$$t \mapsto \| A_t^{(k)} \|_V^{C^n}, \quad k = 0, 1, \ldots, n$$

are locally Lebesgue integrable. Here $V \subset C^n$ is some domain containing $G \subset R^n$.

Then for any subdomain $V' \subset V$, the distance from the boundary of which to the boundary of the domain V is greater than some $\varepsilon > 0$, the series

$$F_{t,t_0}(u_0(z)) = u_0(z) + \sum_{m=1}^{\infty} \sum_{k_1=0}^{n} \cdots \sum_{k_m=0}^{n} \int_{t_0}^{t} d\tau_1 \ \ldots \int_{t_0}^{\tau_{m-1}} d\tau_m A_{\tau_1}^{(k_1)}(z) \partial_{k_1} \ldots A_{\tau_m}^{(k_m)}(z) \partial_{k_m} u_0(z)$$

converges absolutely and uniformly for all $z \in V'$ and those t for which

$$\left| \frac{2(n+1)}{\delta^{n+1}} \int_{t_0}^{t} \| A_\tau \|_V^{C^n} d\tau \right| \leqslant \rho < 1,$$

where

$$\delta = \min\{1, \varepsilon\}, \quad \| A_t \|_V^{C^n} = \max_{0 \leqslant j \leqslant n} \| A_t^{(j)} \|_V^{C^n}$$

and is a solution of the Cauchy problem

$$\frac{du_t}{dt}=\sum_{k=0}^{n} A_t^{(k)}\partial_k u_t, \quad u_{t_0}=u_0.$$

Here one has the estimate

$$\| F_{t,t_0}(u_0)\|_V^{C^n} \leqslant \frac{\| u_0 \|_V^{C^n}}{1-\frac{2(n+1)}{\delta^{n+1}}\left|\int_{t_0}^{t} \| A_\tau \|_V^{C^n} d\tau\right|}.$$

2. Formal Solution of the Cauchy Problem (General Case). We turn to the consideration of a general system of partial differential equations of type (2.1)

$$\begin{gathered}\frac{\partial v}{\partial t}=g(t, v, \partial_1 v, \ldots, \partial_n v),\\ v(t_0, x)=u_0(x), \quad x\in G\subset R^n.\end{gathered} \tag{2.5}$$

As in the linear case too, the right side g of this equation is convenient for us to consider as a family

$$g=\{f_t;\ t\in R\}$$

of smooth mappings

$$f_t=g(t, \cdot): R^p\to R^N, \text{ where } p=(n+1)N.$$

Hence we shall write (2.5) in the form

$$\begin{gathered}\frac{du_t}{dt}=f_t(u_t, \partial_1 u_t, \ldots, \partial_n u_t),\\ u_{t_0}=u_0\end{gathered} \tag{2.6}$$

and say that an absolutely continuous family u_t, $t\in J$, is a solution of (2.6) if for almost all $t\in J$

$$\frac{du_t}{dt}=f_t(u_t, \partial_1 u_t, \ldots, \partial_n u_t)$$

and $u_{t_0}=u_0$ in $\Phi^N(G')$.

Here J is an interval of the real axis containing the point t_0, and G' is a subdomain of the domain G, possibly coinciding with it.

Our basic assumptions are that $u_0\in\Phi^N(G)$, and the family f_t, $t\in R$, can be expanded in a MacLaurin series*

$$f_t^j(u, \partial_1 u, \ldots, \partial_n u)=\sum_{\alpha^{(0)},\ldots,\alpha^{(n)}} a_t^{j,\alpha^{(0)},\ldots,\alpha^{(n)}}(u)^{\alpha^{(0)}}(\partial_1 u)^{\alpha^{(1)}}\ldots(\partial_n u)^{\alpha^{(n)}}, \quad j=1, 2, \ldots, N,$$

whose radius of convergence $R_t\geqslant R>0$. Here

$$\begin{gathered}\alpha^{(k)}=(\alpha_1^{(k)}, \ldots, \alpha_N^{(k)}),\\ (\partial_k u)^{\alpha^{(k)}}=(\partial_k u^1)^{\alpha_1^{(k)}}\ldots(\partial_k u^N)^{\alpha_N^{(k)}}.\end{gathered}$$

*It is obvious that by change of the unknown function one can reduce to this case the case of an arbitrary system whose right side is analytic in all arguments except t.

We construct a formal solution of the Cauchy problem (2.6). To this end we consider the real algebra $\mathfrak{A}$, whose elements are formal power series with real coefficients in the independent variables $\varepsilon, u^1, \ldots, u^N, \ldots, \partial^\alpha u^k, \ldots,$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an arbitrary multi-index with nonnegative components, k = 1, 2, ..., N. For example,

$$a = a[v_1, \ldots, v_q] = \sum_{i_1, \ldots, i_q \geqslant 0} a_{i_1, \ldots, i_q} (v_1)^{i_1} \ldots v_q^{i_q} \in \mathfrak{A},$$

where $v_j \in \{\varepsilon, u^1, \ldots, u^N, \ldots, \partial^\alpha u^k, \ldots\}$. Addition and multiplication of series is defined in the usual way. An arbitrary linear transformation $\mathcal{A}$ of the algebra $\mathfrak{A}$ is called continuous if it can be brought under the sign of an infinite sum, for example,

$$\mathcal{A}(a[v_1, \ldots, v_q]) = \sum_{i_1, \ldots, i_q \geqslant 0} a_{i_1, \ldots, i_q} \mathcal{A}(v^{i_1} \ldots v^{i_q}).$$

We shall be interested in continuous differentiations of the algebra $\mathfrak{A}$, the collection of which we denote by $\mathrm{Der}_c(\mathfrak{A})$. A differentiation $\mathcal{D} \in \mathrm{Der}_c(\mathfrak{A})$ is uniquely determined by its values on the generators $\varepsilon, \partial^\alpha u^k$, by virtue of the Leibniz rule and the condition of continuity. On the other hand, the values of $\mathcal{D}$ on the generators can be arbitrary.

In the algebra $\mathfrak{A}$ there are natural differentiations $\partial_1, \ldots, \partial_n \in \mathrm{Der}_c(\mathfrak{A})$, which are given on the generators by the formulas:

$$\partial_i \varepsilon = 0, \quad \partial_i (\partial^\alpha u^k) = \partial^{(\alpha_1, \ldots, \alpha_i+1, \ldots, \alpha_N)} u^k$$

for any $\alpha = (\alpha_1, \ldots, \alpha_n)$, $k = 1, \ldots, N$, $i = 1, \ldots, N$.

Further, each component of the right side of (2.6) can be considered as an element of the algebra $\mathfrak{A}$:

$$f_t^j(u, \partial_1 u, \ldots, \partial_n u) = \\ = \Sigma a_t^{j, \alpha^{(0)}, \ldots, \alpha^{(n)}} (u)^{\alpha^{(0)}} \ldots (\partial_n u)^{\alpha^{(n)}} \in \mathfrak{A} \quad \forall t \in R, \quad j = 1, \ldots, N.$$

We introduce a one-parameter family of differentiations $\hat{f}_t \in \mathrm{Der}_c(\mathfrak{A})$, giving its action on the generators with the help of the formulas

$$\hat{f}_t \varepsilon = 0,$$
$$\hat{f}_t \partial^\alpha u^k = \partial^\alpha f_t^k (u, \partial_1 u, \ldots, \partial_n u),$$
$$\hat{f}_t u^k = f_t^k (u, \partial_1 u, \ldots, \partial_n u),$$
$$t \in R, \ k = 1, \ldots, N, \ \alpha = (\alpha_1, \ldots, \alpha_n).$$

The most important property for us of the differentiations $\hat{f}_t$, $t \in R$, is that all of them commute with $\partial_1, \ldots, \partial_n$:

$$[\partial_i, \hat{f}_t] = \partial_i \circ \hat{f}_t - \hat{f}_t \circ \partial_i = 0 \quad \forall t \in R, \ i = 1, \ldots, n.$$

In fact, the commutator $[\partial_i, \hat{f}_t]$ is a continuous differentiation, i.e., $[\partial_i, \hat{f}_t] \in \mathrm{Der}_c(\mathfrak{A})$. On the other hand, by virtue of the definitions, $[\partial_i, \hat{f}_t]$ vanishes on the generators. Consequently, $[\partial_i, \hat{f}_t]$ is identically zero.

We construct the way the family of differentiations $\hat{f}_t$, $t \in R$, we have defined acts on elements of the algebra $\mathfrak{A}$

Suppose, for example, $a = a[\varepsilon, u^1, \ldots, u^N, \partial_1 u^1, \partial_2 u^1, \ldots, \partial_n u^N] \in \mathfrak{A}$. Then obviously

$$\hat{f}_t a = \frac{\partial a[\varepsilon, u, \ldots, \partial_n u]}{\partial u} f_t(u, \partial_1 u, \ldots, \partial_n u) + \frac{\partial a[\varepsilon, u, \ldots, \partial_n u]}{\partial(\partial_1 u)} \partial_1 f_t(u, \partial_1 u, \ldots, \partial_n u) + \ldots$$
$$\ldots + \frac{\partial a[\varepsilon, u, \partial_1 u, \ldots, \partial_n u]}{\partial(\partial_n u)} \partial_n f_t(u, \partial_1 u, \ldots, \partial_n u). \tag{2.7}$$

We shall use this formula below for the "determination" of the chronological series corresponding to a quasilinear equation.

Below we shall constantly have to do with one-parameter families of elements of the algebra $\mathfrak{A}$. We say that such a family a_t is measurable, absolutely continuous, etc., if all the coefficients of the corresponding formal series are measurable, absolutely continuous, etc. in t. Integration and differentiation of formal series with respect to t is always done term by term.

Suppose $b = b[v_1, \ldots, v_q] \in \mathfrak{A}$. If $a_1, \ldots, a_q$ are some elements of $\mathfrak{A}$, where the corresponding power series have no free terms, then there is defined the composition $b[a_1, \ldots, a_q] \in \mathfrak{A}$. In what follows, in all formulas where a composition of formal series occurs, one should assume that the "inner" series have no free term.

By a formal solution of the Cauchy problem (2.6) is meant an absolutely continuous one-parameter family a_t of elements of the algebra $\mathfrak{A}$, such that for almost all t one has the equation

$$\frac{d}{dt} a_t = \varepsilon f_t(a_t, \ldots, \partial_n a_t)$$

and $a_{t_0} = u$.

If in the power series corresponding to some formal solution of the Cauchy problem one substitutes in place of the variables $u^1, \ldots, u^N, \ldots, \partial^\alpha u^k, \ldots$ smooth functions $u_0^1(x), \ldots, u^N(x), \ldots, \frac{\partial^\alpha}{\partial x^\alpha} u_0^k(x), \ldots$ and sets $\varepsilon = 1$, then one gets some series of smooth functions. It is clear that in this case when such a series converges uniformly its sum is a solution (not formal) of the Cauchy problem (2.6).

Proposition 2.2. The series

$$F_{t, t_0}(u) = u + \sum_{m=1}^{\infty} \varepsilon^m \int_{t_0}^{t} d\tau_1 \ldots \int_{t_0}^{\tau_{m-1}} d\tau_m \hat{f}_{\tau_m} \circ \ldots \circ \hat{f}_{\tau_2} f_{\tau_1}(u, \partial u)$$

is a formal solution of the Cauchy problem (2.6).

Proof. We note firstly that the series $F_{t, t_0}(u)$ contains only a finite number of terms of each degree, so its "sum" is a well-defined element of the algebra $\mathfrak{A}$. Analogously, the series

$$\hat{\mathcal{P}}_{t, t_0} = \mathrm{Id} + \sum_{m=1}^{\infty} \varepsilon^m \int_{t_0}^{t} d\tau_1 \ldots \int_{t_0}^{\tau_{m-1}} d\tau_m \hat{f}_{\tau_m} \circ \ldots \circ \hat{f}_{\tau_1}$$

gives a continuous linear transformation of the algebra $\mathfrak{A}$.

We note that by the definition itself

$$\hat{\mathcal{P}}_{t, t_0} \circ \partial_k = \partial_k \circ \hat{\mathcal{P}}_{t, t_0}.$$

Moreover, we see that

$$\hat{\mathcal{P}}_{t,t_0}u = F_{t,t_0}(u).$$

An important fact is that the series $\hat{\mathcal{P}}_{t,t_0}$ has the following multiplicative property:

$$\hat{\mathcal{P}}_{t,t_0}(ab) = \hat{\mathcal{P}}_{t,t_0}(a)\,\hat{\mathcal{P}}_{t,t_0}(b), \quad \forall a,\ b \in \mathfrak{A}.$$

In fact, we have on the basis of (1.2) that

$$\hat{f}_{\tau_m}\circ\ldots\circ\hat{f}_{\tau_1}(ab) = \sum_{\alpha+\beta=m}\ \sum_{\pi\in S(\alpha,\beta)} (\hat{f}_{\tau_{\pi(m)}}\circ\ldots\circ\hat{f}_{\tau_{\pi(\alpha+1)}}(a))\,(\hat{f}_{\tau_{\pi(\alpha)}}\circ\ldots\circ\hat{f}_{\tau_{\pi(1)}}(b));$$

$$\forall a,\ b \in \mathfrak{A}.$$

Consequently, using (1.3), we have:

$$\int_{t_0}^{t} d\tau_1 \ldots \int_{t_0}^{\tau_{m-1}} d\tau_m \hat{f}_{\tau_m}\circ\ldots\circ\hat{f}_{\tau_1}(ab) =$$

$$= \sum_{\alpha+\beta=m}\left(\int_{t_0}^{t} d\tau_1\ldots\int_{t_0}^{\tau_{\alpha-1}} d\tau_\alpha \hat{f}_{\tau_\alpha}\circ\ldots\circ\hat{f}_{\tau_1}(a)\right)\left(\int_{t_0}^{t} d\tau_1\ldots\int_{t_0}^{\tau_{\beta-1}} d\tau_\beta \hat{f}_{\tau_\beta}\circ\ldots\circ\hat{f}_{\tau_1}(b)\right),$$

but this means that

$$\hat{\mathcal{P}}_{t,t_0}(ab) = \hat{\mathcal{P}}_{t,t_0}(a)\,\hat{\mathcal{P}}_{t,t_0}(b).$$

From this multiplicative property it also follows that the series $F_{t,t_0}(u)$ is a formal solution of the Cauchy problem.

In fact, $F_{t_0,t_0}(u) = u$ and

$$\frac{d}{dt}F_{t,t_0}(u) = \frac{d}{dt}\left\{u + \varepsilon\int_{t_0}^{t} f_\tau(u, \partial_1 u, \ldots, \partial_n u)\,d\tau + \varepsilon^2\int_{t_0}^{t} d\tau_1\int_{t_0}^{\tau_1} d\tau_2 \hat{f}_{\tau_2} f_{\tau_1}(u, \partial_1 u, \ldots, \partial_n u) + \ldots\right\}$$

$$= \varepsilon\hat{\mathcal{P}}_{t,t_0} f_t(u, \partial_1 u, \ldots, \partial_n u) =$$

$$= \varepsilon\sum_{\alpha(0),\ldots,\alpha(n)} a_t^{\alpha(0),\ldots,\alpha(n)}\,(\hat{\mathcal{P}}_{t,t_0}u)^{\alpha(0)}\ldots(\partial_n\hat{\mathcal{P}}_{t,t_0}u)^{\alpha(n)} = \varepsilon f_t(F_{t,t_0}(u), \partial_1 F_{t,t_0}(u), \ldots, \partial_n F_{t,t_0}(u)),$$

where

$$a_t^{\alpha(0),\ldots,\alpha(n)} = \begin{pmatrix} a_t^{1,\alpha(0),\ldots,\alpha(n)} \\ \vdots \\ a_t^{N,\alpha(0),\ldots,\alpha(N)} \end{pmatrix}.$$

If in the formal series F_{t,t_0} one substitutes in place of $u^1, \ldots, u^N, \ldots, \partial^\alpha u^k, \ldots$ smooth functions $u_0^1(x), \ldots, u^N(x), \ldots, \frac{\partial^\alpha}{\partial x^\alpha}u_0^k(x), \ldots$ and sets $\varepsilon = 1$, then we get a series consisting of smooth vector-functions

$$F_{t,t_0}(u_0) = u_0 + \sum_{m=1}^{\infty}\int_{t_0}^{t} d\tau_1 \ldots \int_{t_0}^{\tau_{m-1}} d\tau_m\, \hat{f}_{\tau_m}\circ\ldots\circ\hat{f}_{\tau_2} f_{\tau_1}(u_0, \partial u_0). \tag{2.8}$$

Such a series we call a chronological series for the Cauchy problem (2.6). We shall prove the convergence of the series (2.8) in the case when u_0 is an analytic function. First we deal with the scalar quasilinear case.

3. Case of a One-Dimensional Quasilinear Equation. We consider a one-dimensional quasilinear first-order partial differential equation

$$\frac{du}{dt}=a_t^{(1)}(u)\,\partial_1 u+\ldots+a_t^{(n)}(u)\,\partial_n u=f_t(u,\partial u)\;=f_t(u,\partial_1 u,\ldots,\partial_n u),$$
$$u_{t_0}=u_0\in\Phi^N(G). \tag{2.9}$$

It turns out that one has the following formula for the integrand of the (m+1)-st term of the series (2.8)

$$\hat{f}_{\tau_m}\circ\ldots\circ\hat{f}_{\tau_2}f_{\tau_1}(u_0,\partial u_0)=\hat{f}_{\tau_m}\circ\ldots\circ\hat{f}_{\tau_2}f_{\tau_1}(u_0,\partial_1 u_0,\ldots,\partial_n u_0)=$$
$$=\sum_{j_1=1}^{n}\ldots\sum_{j_{m-1}=1}^{n}\partial_{j_1}\circ\ldots\circ\partial_{j_{m-1}}\left(a_{\tau_1}^{(j_1)}(u_0)\ldots a_{\tau_{m-1}}^{(j_{m-1})}(u_0)\,f_{\tau_m}(u_0,\partial u_0)\right). \tag{2.10}$$

We shall prove this formula by induction on the number m. For m = 1 it is obvious. Let (2.10) be true for $m\leqslant p$. Then for m = p + 1,

$$\hat{f}_{\tau_{p+1}}\circ\ldots\circ\hat{f}_{\tau_2}f_{\tau_1}(u_0,\partial u_0)=\hat{f}_{\tau_{p+1}}\sum_{j_1=1}^{n}\ldots\sum_{j_{p-1}=1}^{n}\partial_{j_1}\circ\ldots\circ\partial_{j_{p-1}}\left(a_{\tau_1}^{(j_1)}(u_0)\ldots a_{\tau_{p-1}}^{(j_{p-1})}(u_0)\,f_{\tau_p}(u_0,\partial u_0)\right)=$$
$$=\sum_{j_1=1}^{n}\ldots\sum_{j_{p-1}=1}^{n}\partial_{j_1}\circ\ldots\circ\partial_{j_{p-1}}\hat{f}_{\tau_{p+1}}\left(a_{\tau_1}^{(j_1)}(u_0)\;\ldots a_{\tau_{p-1}}^{(j_{p-1})}(u_0)\,f_{\tau_p}(u_0,\partial u_0)\right)=\sum_{j_1=1}^{n}\ldots\sum_{j_{p-1}=1}^{n}\partial_{j_1}\circ\ldots$$
$$\ldots\circ\partial_{j_{p-1}}\left(\frac{\partial}{\partial u}\left(a_{\tau_1}^{(j_1)}(u_0)\ldots a_{\tau_p}^{(j_p)}(u_0)\right)f_{\tau_p}(u_0,\partial u_0)\;+a_{\tau_1}^{(j_1)}(u_0)\ldots a_{\tau_{p-1}}^{(j_{p-1})}(u_0)\sum_{j_p=1}^{n}\frac{\partial}{\partial(\partial_{j_p}u)}f_{\tau_p}(u_0,\partial u_0)\right)=$$
$$=\sum_{j_1=1}^{n}\ldots\sum_{j_{p-1}=1}^{n}\partial_{j_1}\circ\ldots\circ\partial_{j_p}\sum_{j_p=1}^{n}\left(\frac{\partial}{\partial u}\left(a_{\tau_1}^{(j_1)}(u_0)\ldots a_{\tau_p}^{(j_p)}(u_0)\right)\partial_{j_p}u_0 f_{\tau_{p+1}}(u_0,\partial u_0)+\right.$$
$$\left.+a_{\tau_1}^{(j_1)}(u_0)\ldots a_{\tau_p}^{(j_p)}(u_0)\,\partial_{j_p}f_{\tau_{p+1}}(u_0,\partial u_0)\right)\;=\sum_{j_1=1}^{n}\ldots\sum_{j_p=1}^{n}\partial_{j_1}\circ\ldots\circ\partial_{j_p}\left(a_{\tau_1}^{(j_1)}(u_0)\ldots a_{\tau_p}^{(j_p)}(u_0)\,f_{\tau_{p+1}}(u_0,\partial u_0)\right).$$

(2.10) is proved.

Just as in Paragraph 1 Section 2, we shall use the Cauchy integral formula to prove the convergence of the chronological series obtained

$$F_{t,t_0}(u_0)=u_0+\int_{t_0}^{t}f_\tau(u_0,\partial u_0)\,d\tau+\int_{t_0}^{t}d\tau_1\int_{t_0}^{\tau_1}d\tau_2\hat{f}_{\tau_2}(u_0,\partial u_0)+\ldots=u_0+\int_{t_0}^{t}\sum_{j=1}^{n}a_\tau^{(j)}(u_0)\,\partial_j u_0\,d\tau+$$
$$+\int_{t_0}^{t}d\tau_1\int_{t_0}^{\tau_1}d\tau_2\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\partial_{j_1}\left(a_{\tau_1}^{(j_1)}(u_0)\,a_{\tau_2}^{(j_2)}(u_0)\,\partial_{j_2}u_0\right)+\ldots+\int_{t_0}^{t}d\tau_1\ldots\int_{t_0}^{\tau_{m-1}}d\tau_m\sum_{j_1=1}^{n}\ldots\sum_{j_{m-1}=1}^{n}\partial_{j_1}\circ\ldots$$
$$\ldots\circ\partial_{j_{m-1}}\left(a_{\tau_1}^{(j_1)}(u_0)\ldots a_{\tau_m}^{(j_m)}(u_0)\,\partial_{j_m}u_0\right)+\ldots, \tag{2.11}$$

corresponding to the Cauchy problem (2.9) in the analytic case. Namely, one has the following

Proposition 2.3. If V is a complex extension of the domain $G\subset R^n$ and $u_0\in\Omega(V)$,

$$a_t^{(j)}(u_0)\in\Omega(V)\;\forall t\in R,\quad j=1,2,\ldots,n,$$

and the functions $t\mapsto\|a_t^{(j)}(u_0)\|_{V^{C^n}}$ are locally integrable, then for any subdomain V' of the domain V, the distance from the boundary of which to the boundary of the domain V is greater than some $\varepsilon>0$, there exists a $\rho>0$, such that (2.11) converges absolutely and uniformly in V' for $|t-t_0|<\rho$.

Proof. We have $\forall z\in V'$

$$a_{\tau_1}^{(j_1)}(u_0(z))\cdot a_{\tau_\alpha}^{(j_2)}(u_0(z))\dots a_{\tau_m}^{(j_m)}(u_0(z))\,\partial_{j_m}u_0(z)=\frac{1}{(2\pi i)^n}\int_{C_1}\dots\int_{C_n}a_{\tau_1}^{(j_1)}(u_0(\xi))\dots a_{\tau_m}^{(j_m)}(u_0(\xi))\,\partial_{j_m}u_0(\xi)\cdot\frac{d\xi}{\Pi_\xi(z)},$$

where C_α is a circle of radius ε with center z^α,

$$\Pi_\xi(z)=(\xi^1-z^1)\dots(\xi^n-z^n).$$

Consequently, $\forall z\in V'$, $m=1,2,\dots$

$$\hat{f}_{\tau_m}\circ\dots\circ\hat{f}_{\tau_2}f_{\tau_1}(u_0(z),\ \partial u_0(z))=\frac{1}{(2\pi i)^n}\sum_{j_1=1}^{n}\dots\sum_{j_m=1}^{n}\int_{C_1}\dots\int_{C_n}a_{\tau_1}^{(j_1)}(u_0(\xi))\dots a_{\tau_m}^{(j_m)}(u_0(\xi))\,\chi_\xi^{j_1,\dots,j_m}(z)\,d\xi,$$

where

$$\chi_\xi^{j_1,\dots,j_m}(z)=\partial_{j_1}\circ\dots\circ\partial_{j_{m-1}}\frac{1}{\Pi_\xi(z)},$$
$$\partial_{j_\alpha}=\frac{\partial}{\partial z^{j_\alpha}}.$$

For all $|\xi^\alpha-z^\alpha|\geqslant\varepsilon,\ \alpha=1,\dots,n$, it is obvious that one has the estimate

$$|\chi_\xi^{j_1,\dots,j_m}(z)\leqslant\frac{(m-1)!}{\varepsilon^{m-1}\varepsilon^n},$$

from which it follows that $\forall z\in V'$

$$|\hat{f}_{\tau_m}\circ\dots\circ\hat{f}_{\tau_2}f_{\tau_1}(u_0(z),\ \partial u_0(z))|\leqslant\sum_{j_1=0}^{n}\dots\sum_{j_{m-1}=1}^{n}\sum_{j_m=1}^{n}\frac{1}{(2\pi i)^n}\int_{C_1}\dots\int_{C_n}\|a_{\tau_1}^{(j_1)}(u_0)\|_V^{C^n}\dots\|a_{\tau_m}^{(j_m)}(u_0)\|_V^{C^n}\|\partial_{j_m}u_0\|_V^{C^n}\frac{(m-1)!}{\varepsilon^{m-1}\varepsilon^n}\leqslant$$
$$\leqslant\frac{(m-1)!n^{m-1}}{\varepsilon^{m-1}}\|a_{\tau_1}(u_0)\|_V^{C^n}\dots\|a_{\tau_m}(u_0)\|_V^{C^n}\|\operatorname{grad}u_0\|_V^{C^n},$$

where

$$\|a_{\tau_\alpha}\|_V^{C^n}=\max_{1\leqslant j\leqslant n}\|a_{\tau_\alpha}^{(j)}(u_0)\|_V^{C^n}=\max_{1\leqslant j\leqslant n}\max_{z\in V}|a_{\tau_\alpha}^{(j)}(u_0(z))|,$$

$$\|\operatorname{grad}u_0\|_V^{C^n}=\sum_{j=1}^{n}\|\partial_j u_0\|_V^{C^n}=\sum_{j=1}^{n}\max_{z\in V}|\partial_j u_0(z)|,\quad \partial_j=\frac{\partial}{\partial z^j}.$$

Consequently, whence we get that for all

$$\left|\int_{t_0}^{t}d\tau_1\dots\int_{t_0}^{\tau_{m-1}}d\tau_m\hat{f}_{\tau_m}\circ\dots\circ\hat{f}_{\tau_2}f_{\tau_1}(u_0(z),\partial u_0(z))\right|\leqslant\frac{(m-1)!n^{m-1}}{\varepsilon^{m-1}}\int_{t_0}^{t}d\tau_1\dots\int_{t_0}^{\tau_{m-1}}d\tau_m\|a_{\tau_1}\|_V^{C^n}\dots$$
$$\dots\|a_{\tau_m}\|_V^{C^n}\|\operatorname{grad}u_0\|_V^{C^n}=\frac{(m-1)!}{\varepsilon^{m-1}}n^{m-1}\frac{1}{m!}\left(\int_{t_0}^{t}\|a_\tau\|_V^{C^n}d\tau\right)^m\|\operatorname{grad}u_0\|_V^{C^n}=\frac{\varepsilon}{m\cdot n}\left(\frac{n}{\varepsilon}\int_{t_0}^{t}\|a_\tau\|_V^{C^n}d\tau\right)^m\|\operatorname{grad}u_0\|_V^{C^n}.$$

Thus, we have proved the inequality

$$\left\|\int_{t_0}^{t}d\tau_1\dots\int_{t_0}^{\tau_{m-1}}d\tau_m\hat{f}_{\tau_m}\circ\dots\circ\hat{f}_{\tau_2}f_{\tau_1}(u_0,\partial u_0)\right\|_{V'}^{C^n}\leqslant\frac{\varepsilon}{m\cdot n}\left(\frac{n}{\varepsilon}\int_{t_0}^{t}\|a_\tau\|_V^{C^n}d\tau\right)^m\|\operatorname{grad}u_0\|_V^{C^n},$$

from which follows the absolute and uniform convergence of the series (2.11) for all $z\in V'$ and t sufficiently close to t_0.

The proposition is proved.

From Proposition 2.3 and the arguments of Section 2 Paragraph 2 we get the following theorem.

THEOREM 2.2. Let the hypotheses of Paragraph 2 Section 2 hold and

$$u_0 \in \Omega(V), \quad a_t^{(k)}(u_0) \in \Omega(V), \quad \|u_0\|_V^{C^n} \leqslant R_t, \quad k=1,\ldots,n$$

and the functions

$$t \mapsto \| a_t^{(k)}(u_0) \|_V^{C^n}, \quad k=1,2,\ldots,n,$$

be locally summable. Here $V \subset C^n$ is a domain containing $G \subset R^n$.

Then for any subdomain $V' \subset V$ the distance from whose boundary to the boundary of the domain V is greater than $\varepsilon > 0$, the series (2.11) converges absolutely and uniformly in the domain V' for those t for which

$$\left| \frac{n}{\varepsilon} \int_{t_0}^{t} \| a_\tau \|_V^{C^n} d\tau \right| \leqslant \rho < 1$$

and is a solution of the Cauchy problem (2.9). Here one has the estimate

$$\| F_{t,t_0}(u_0) \|_{V'}^{C^n} \leqslant \frac{\varepsilon}{n} \| \operatorname{grad} u_0 \|_V^{C^n} \left(1 + \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{n}{\varepsilon} \int_{t_0}^{t} \| a_\tau \|_V^{C^n} d\tau \right)^m \right) = \frac{\varepsilon}{n} \| \operatorname{grad} u_0 \|_V^{C^n} \left(1 + \ln \left[1 - \frac{n}{\varepsilon} \left| \int_{t_0}^{t} \| a_\tau \|_V^{C^n} d\tau \right| \right] \right).$$

We note in a series of cases one can collapse the series (2.11) and get a solution of the Cauchy problem in closed form. We demonstrate this in the simplest example. Suppose, for example, we consider the Cauchy problem

$$\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x}, \quad u_{t_0}(x) = x \in R.$$

Then according to the account above we get that a solution of this problem is the series

$$F_{t,t_0}(x) = x + \int_{t_0}^{t} x \frac{\partial}{\partial x} x d\tau + \ldots = x + x(t-t_0) + x(t-t_0)^2 + \ldots = \frac{x}{1-(t-t_0)}, \quad |t-t_0| \leqslant \rho < 1,$$

which is verified directly.

To conclude this Paragraph we note that the formula (2.7), used for "determining" the chronological series corresponding to the Cauchy problem, can be reduced to a concrete formula for solving this problem in certain other cases too.

4. Convergence of the General Chronological Series. In this paragraph we give the proof of the convergence of the chronological series corresponding to the quasilinear system

$$\frac{du}{dt} = A_t^{(1)}(u)\, \partial_1 u + \ldots + A_t^{(n)}(u)\, \partial_n u = f_t(u, \partial u) = f_t(u, \partial_1 u, \ldots, \partial_n u), \quad u_{t_0} = u_0$$

in the case when the initial function u_0 is analytic in the domain G. For our goals it suffices to consider only this case, because, as is known (cf. [3]), one can reduce to this case an arbitrary system of partial differential equations of the form (2.5).

We shall prove convergence of the chronological series corresponding to this quasilinear system in the neighborhood of an arbitrary initial point $(t_0, x_0) \in R \times G$.

Since, by assumption, the function u_0 is analytic in G, at the point x_0 it can be expanded in a Taylor series

$$u_0^j(x)=\sum_{k_1,\ldots k_n} b_t^{(j)}(x^1-x_0^1)^{k_1}\ldots(x^n-x_0^n)^{k_n},\quad j=1,2,\ldots,N,$$

converging for all x sufficiently close to x_0.

We set

$$v_0(x)=\sum_{k_1,\ldots,k_n}\max_{1\leqslant j\leqslant n}|b_{k_1,\ldots,k_n}^{(j)}|(x^1-x_0^1)^{k_1}\ldots(x^n-x_0^n)^{k_n}.$$

Then the function v_0 is defined and analytic close to the point x_0 and is a majorant for the initial functions u_0^i $(i=1,2,\ldots,N)$ in a neighborhood of this point.

We recall that an analytic function f, representable by a convergent power series, is a majorant of an analytic function g, if the coefficients of the expansion of g in a power series are in modulus less than or equal to the corresponding coefficients of the expansion of f (assuming that the coefficients of the expansion of f are greater than or equal to zero).

We denote by e the N-dimensional vector

$$e=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

and we set

$$\bar{v}_0=v_0\cdot e.$$

By our assumptions (cf. Sec. 2 Para. 2), the right side of the system can be expanded in a MacLaurin series, i.e., the components $a_{j,k}^i(t,u)$ of the matrices $A_t^{(k)}(u)$, $k=1,2,\ldots,N$, $i,j=1,2,\ldots,N$, are representable in the form of a series

$$a_{j,k}^i(t,u)=\sum_{\alpha_1,\ldots,\alpha_N} a_{j,k,\alpha_1,\ldots,\alpha_N}^i(u^1)^{\alpha_1}\ldots(u^N)^{\alpha_N}$$

converging for $|u|<R_t$.

We get

$$a_t(u)=\sum_{\alpha_1,\ldots\alpha_N}\max_{1\leqslant k\leqslant n}\max_{1\leqslant i,j\leqslant N}|a_{j,k,\alpha_1,\ldots,\alpha_N}^i(t)|(u^1)^{\alpha_1}\ldots(u^N)^{\alpha_N}.$$

Then the function $a_t(u)$ is defined for all t and all u sufficiently small in absolute value. Let E be the N^2 matrix consisting of ones:

$$E=\begin{pmatrix}1&1&\ldots&1\\1&1&\ldots&1\\ \vdots&\vdots&&\vdots\\1&1&\ldots&1\end{pmatrix}.$$

We set

$$g_t(u,\partial_1 u,\ldots,\partial_n u)=g_t(u,\partial u)=a_t(u)E\sum_{i=1}^n\partial_i u=a_t(u)\cdot e\sum_{i=1}^n\sum_{j=1}^N\partial_i u^j.$$

By the definition itself, the function

$$(u, \partial_1 u, \ldots, \partial_n u) \mapsto g_t(u, \partial_1 u, \ldots, \partial_n u) = g_t(u, \partial u)$$

majorizes (componentwise) the function

$$(u, \partial_1 u, \ldots, \partial_n u) \mapsto f_t(u, \partial u) = f_t(u, \partial_1 u, \ldots, \partial_n u) = \sum_{k=1}^{n} A_t^{(k)}(u)\, \partial_k u.$$

Since the function $\bar{v}_0$ defined above componentwise majorizes the initial function u_0 in the neighborhood of the point $x_0 \in G$, the function

$$x \mapsto g_t(\bar{v}_0(x), \partial_1 \bar{v}_0(x), \ldots, \partial_n \bar{v}_0(x))$$

majorizes componentwise the function

$$x \mapsto f_t(u_0(x), \partial_1 u_0(x), \ldots, \partial_n u_0(x)).$$

In exactly the same way the function

$$x \mapsto \hat{g}_{\tau_m} \circ \ldots \circ \hat{g}_{\tau_2} g_{\tau_1}(\bar{v}_0(x), \partial_1 \bar{v}_0(x), \ldots, \partial_n \bar{v}_0(x))$$

majorizes componentwise the function

$$x \mapsto \hat{f}_{\tau_m} \circ \ldots \circ \hat{f}_{\tau_2} f_{\tau_1}(u_0(x), \partial_1 u_0(x), \ldots, \partial_n u_0(x)),$$

because in the construction of these functions one uses the same rules which include only differentiation with respect to corresponding arguments, addition, multiplication, and substitution of a series in a series, in one word, rational operations which preserve the relation of majorizing analytic functions.

Everything said above means that the series (2.8), constructed for a quasilinear system

$$\frac{du_t}{dt} = f_t(u_t, \partial u_t) = f_t(u_t, \partial_1 u_t, \ldots, \partial_n u_t) = \sum_{k=1}^{n} A_t^{(k)}(u)\, \partial_k u, \quad u_{t_0} = u_0,$$

converges, at least in that (sufficiently small) neighborhood of the initial point $x_0 \in G$, in which the series

$$G_{t,t_0}(\bar{v}_0(x)) = \bar{v}_0(x) + \int_{t_0}^{t} g_\tau(\bar{v}_0(x), \partial \bar{v}_0(x))\, d\tau + \ldots + \int_{t_0}^{t} d\tau_1 \ldots \int_{t_0}^{\tau_{m-1}} d\tau_m \hat{g}_{\tau_m} \circ \ldots \circ \hat{g}_{\tau_2} g_{\tau_1}(\bar{v}_0(x), \partial \bar{v}_0(x)) + \ldots \quad (2.12)$$

converges.

The convergence of the series (2.12) is easy to prove. Namely it turns out that one has the formula

$$\hat{g}_{\tau_m} \circ \ldots \circ \hat{g}_{\tau_2} g_{\tau_1}(\bar{v}_0(x), \partial \bar{v}_0(x)) = N^m e \sum_{j_1=1}^{n} \ldots \sum_{j_m=1}^{n} \partial_{j_1} \circ \ldots \circ \partial_{j_{m-1}} a_{\tau_1}(\bar{v}_0(x)) \ldots a_{\tau_m}(\bar{v}_0(x))\, \partial_{j_m} v_0(x). \quad (2.13)$$

The proof of (2.13) is done by induction and is hardly different from the proof of (2.10), so we shall not dwell on it.

Let S be a small enough neighborhood of the initial point $x_0 \in G_0$, that in this neighborhood the series for v_0 converges and let V be a complex extension of this neighborhood such that

$$v_0 \in \Omega(V), \quad a_t(\bar{v}_0) \in \Omega(V).$$

Analogous to Proposition 2.2, one has

Proposition 2.4. For any subdomain $V' \subset V$, the distance from whose boundary to the boundary of V is greater than some $\varepsilon > 0$, there exists a $\rho > 0$, such that the series $G_{t,t_0}(\bar{v}_0(z))$ converges absolutely and uniformly for $|t - t_0| < \rho$, $z \in V'$.

The proof follows from the estimate

$$\| \hat{g}_{\tau_m} \circ \dots \circ \hat{g}_{\tau_2} g_{\tau_1}(\bar{v}_0, \partial \bar{v}_0) \|_{V'}^{C^n} \leqslant \frac{N^m n^{m-1}}{\varepsilon^{m-1}} (m-1)! \, \| a_{\tau_m} \|_V^{C^n} \dots \| a_{\tau_1} \|_V^{C^n} \| \operatorname{grad} v_0 \|_V^{C^n},$$

$$\| a_\tau \|_V^{C^n} = \sup_{z \in V} | a_\tau(\bar{v}_0(z)) |,$$

which is proved just like the corresponding estimate from Proposition 2.2.

Summarizing everything said above, we assert that one has the following theorem.

THEOREM 2.3. Let the hypotheses of Sec. 2 Para. 2 hold and let $u_0 \in \Omega^N(N)$, $A_t^{(k)}(u_0) \in \Omega^{N^2}(V)$, $\| u_0 \|_V^{C^n} \leqslant R_t$, where the function $t \mapsto \| a_t(v_0) \|_V^{C^n}$ is locally summable. Here V is some sufficiently small complex neighborhood of an arbitrary point $x_0 \in G$, R_t is the radius of convergence of the MacLaurin series constructed for the right side (cf. Sec. 2 Para. 2), and a_t, $\bar{v}_0$ are the functions constructed above.

Then for any subdomain $V' \subset V$, the distance from whose boundary to the boundary of the domain V is greater than some $\varepsilon > 0$, there exists a $\rho > 0$, such that the series (2.8) converges absolutely and uniformly for $|t - t_0| < \rho$, $z \in V'$ and is a solution of the Cauchy problem

$$\frac{du}{dt} = \sum_{k=1}^{n} A_t^{(k)}(u) \, \partial_k u, \quad u_{t_0} = u_0.$$

5. Uniqueness of the Solution and Comparison Theorem. One has the following

THEOREM 2.4. Let the hypotheses of Sec. 2 Para. 2 hold and

$$u_0 \in \Omega^N(V), \quad \| u_0 \|_V^{C^n} \leqslant R_t,$$

where the function

$$t \mapsto \| f_t(u_0, \partial u_0) \|_V^{C^n}$$

is locally summable. Here $V \subset C^n$ is some domain containing G. Then, if u_t, v_t are two solutions of the Cauchy problem (2.6), belonging to $\Omega^N(V)$, then $u_t = v_t$ in $\Omega^N(V)$ for those t for which they are both defined.

For the proof, we consider the series

$$\hat{Q}_{t,t_0} = \hat{\mathcal{P}}_{t,t_0}^{-1} = \mathrm{Id} + \sum_{m=1}^{\infty} (-1)^m \int_{t_0}^{t} d\bar{\tau}_1 \dots \int_{t_0}^{t} d\bar{\tau}_m \, \hat{f}_{\bar{\tau}_1} \circ \dots \circ \hat{f}_{\bar{\tau}_m}.$$

Direct verification (cf. with [1]) shows that

$$\frac{d}{dt} Q_{t,t_0} \, u_t = 0$$

in a neighborhood of each point $(t_0, x_0) \in R \times V$. Consequently,

$$\hat{Q}_{t,t_0} \, u_t = u_0 \text{ in } \Omega^N(V) \text{ and } u_t = \hat{P}_{tt_0} \circ \hat{Q}_{t,t_0} \, u_t = \hat{P}_{t,t_0} \, u_0.$$

Whence also follows the assertion of the theorem.

A direct consequence of the chronological representation of the solution and the preceding theorem is the following result, which can be called the comparison theorem.

THEOREM 2.5. Let the conditions guaranteeing the existence and uniqueness of solutions of the Cauchy problem (2.6) and the Cauchy problem

$$\frac{dv_t}{dt}=g_t(v_t,\ dv_t),$$
$$v_{t0}=v_0 \in \Omega^N(V), \tag{2.14}$$

with which we were concerned above, hold. Then if the functions g_t and v_0 are componentwise majorized by the functions f_t and u_0, then the solution v_t of the Cauchy problem (2.14) is componentwise majorized by the solution u_t of the Cauchy problem (2.6) for all t for which they are defined.

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