Smooth control systems:

State space M and the set of admissible velocities $V \subset TM$. We assume that V is a locally trivial bundle over M and a fiber $V_q = V \cap T_qM$ is a smooth submanifold.

Space of admissible curves:

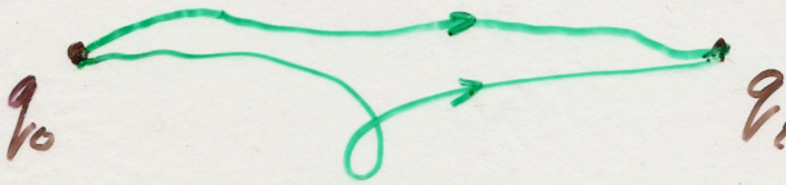
$$\Omega = \{\gamma : [0, t_\infty) \longrightarrow M,\ \dot\gamma(t) \in V_{\gamma(t)}\}.$$

Examples:

1) V is a spherical bundle of a Riemannian manifold
$\Rightarrow \Omega$ is the space of curves parametrized by the length.

2) V is a vector distribution
$\Longrightarrow$ Ω is the space of integral curves.

Control problem:
get q_1 from q_0, where $q_0, q_1 \in M$:



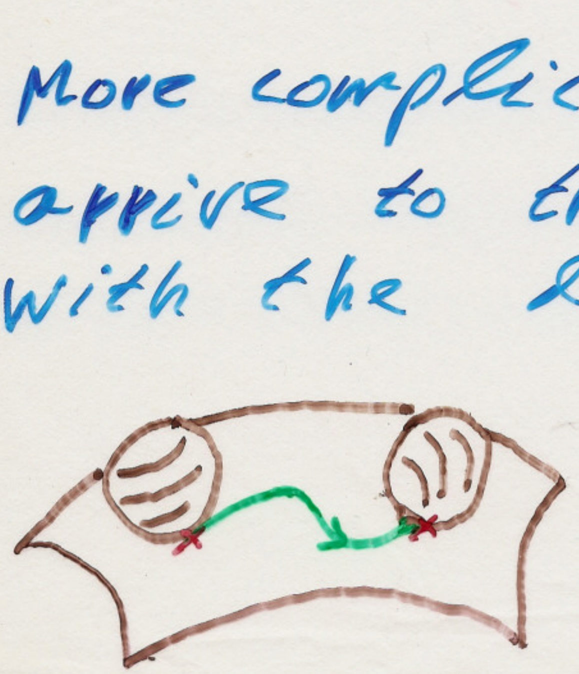
<u>Example.</u> M is the total space of a principal bundle and V is a connection on the bundle; if q_0 and q_1 belong to the same fiber of the bundle, then to get q_1 from q_0 means to find curves on the base with a prescribed holonomy.

Special case:

A rigid body on the plane.

Get a desired orientation by rolling without slipping or twisting. The distribution (Levi Civita connection) is contact $\Longleftrightarrow$ the curvature does not vanish.

More complicated problem: arrive to the desired point with the desired orientation.



Here: $\dim M = 5$, $\dim V_q = 2$.

Boundary mappings

$\partial_t : \Omega \to M \times M, \quad \partial_t : \gamma(\cdot) \mapsto (\gamma(0), \gamma(t)).$

Critical points of ∂_t are <u>singular curves</u> (on the segment $[0, t]$).

Dual objects: $H^\nu \subset T^*M$, $\nu = 0, 1$;

$$H^\nu = \{\lambda \in T_q^*M : \exists v \in V_q, \ \lambda \perp T_v V_q, \ \langle \lambda, v \rangle = \nu\}.$$

Let $\sigma \in \Lambda^2(T^*M)$ be the standard symplectic structure.

Characteristics of $\sigma|_{H^1}$ ($\sigma|_{H^0}$) are called normal (abnormal) extremals.

<u>Prop.</u> Singular curves are projections of extremals to M.

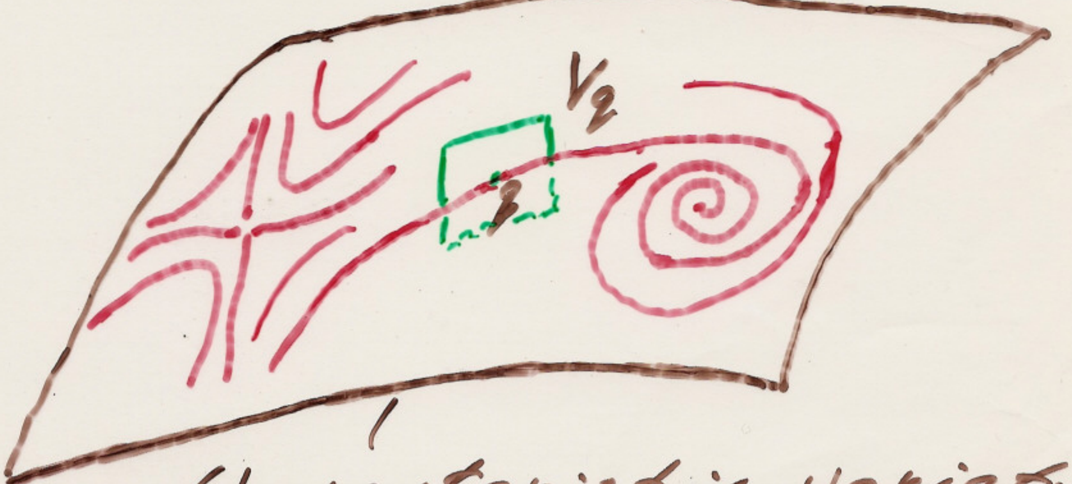
Examples.

1) V is a spherical bundle $\Rightarrow$ H^1 is the spherical bundle (in T^*M), $H^0 = \emptyset$; singular curves are geodesics.

2) V is a vector distribution $\Rightarrow$ $H^1 = \emptyset$, H^0 is the annihilator of the distribution.

a) V is a contact structure $\Rightarrow$ no singular curves.

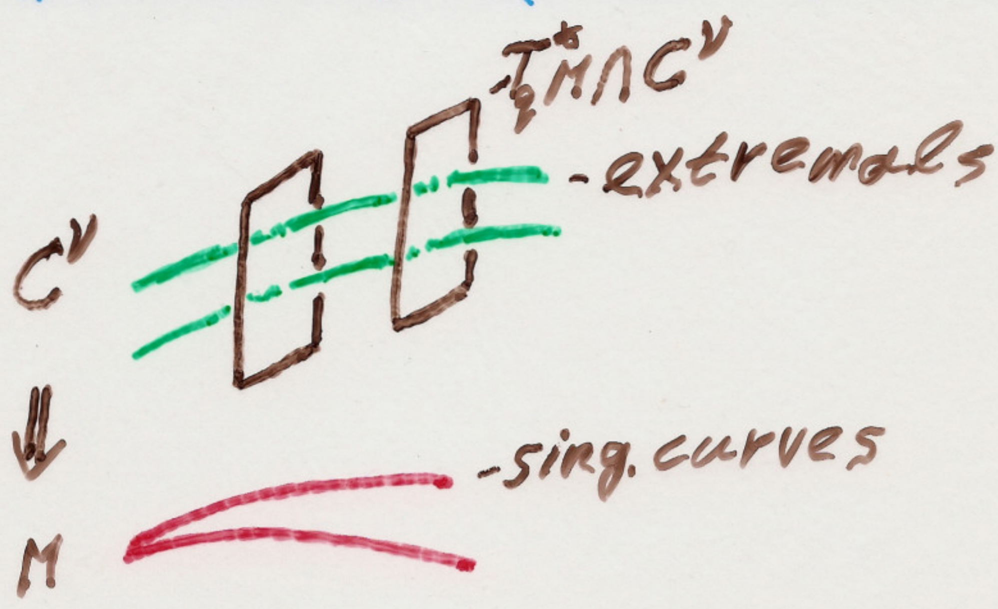
b) V is a generic rank 2 distribution in $\mathbb{R}^3$:



Characteristic variety

c) Rolling bodies rank 2 distribution on a 5-dimensional manifold: singular curves are rollings along geodesics.

General construction:
$\overline{C}^\nu = \{ z \in H^\nu : \ker \sigma_z|_{H^\nu} \neq 0 \}$ is the characteristic variety;
$C^\nu = \{ z \in \overline{C}^\nu : \dim \ker \sigma_z|_{C^\nu} = 1 \}$ is a regular part of the characteristic variety foliated by the extremals.



Consider the factorization:

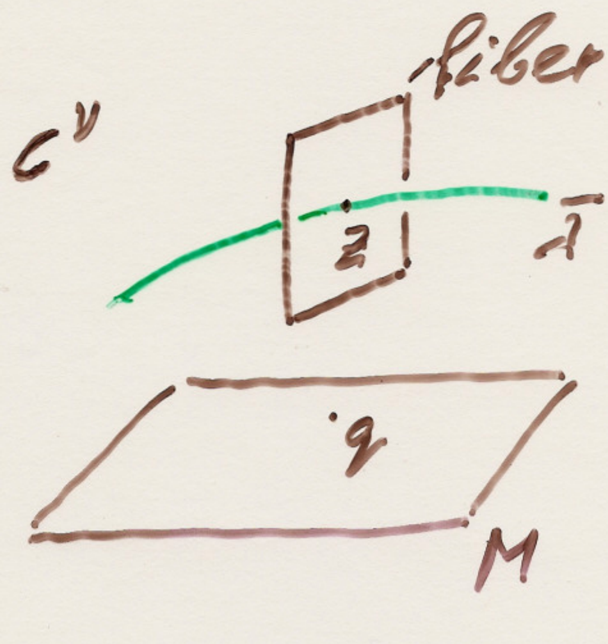
$$F : C^\nu \to C^\nu/\{\text{extremals foliation}\}.$$

Let $\bar\lambda$ be an extremal;

Jacobi curve $J_{\bar\lambda} : \bar\lambda \to \mathrm{Grassm}\left(T_{\bar\lambda} F(C^\nu)\right)$

is a collection of the tangent spaces to the fibers along $\bar\lambda$, $J_{\bar\lambda}(z) = T_{\bar\lambda} F\left(T_q^* M \cap C^\nu\right)$,

where $z \in \bar\lambda \cap T_q^* M$.



Let V be a rank 2 vector distribution, $V^2 = [V, V]$, $V^3 = [V, [V, V]]$, $\dots$;
$\operatorname{rank} V^2 = 3$, $\operatorname{rank} V^3 = 5$, ...
Then $C^\circ = (V^2)^\perp \setminus (V^3)^\perp$,
$\dim C^\circ = 2n - 3$, where $n = \dim M$.
Now let $n = 5$; then V is called *flat* if $V = \operatorname{span}\{X_1, X_2\}$, where X_1, X_2 generate a 5-dimensional nilpotent Lie algebra.
Symmetry group of the flat distribution is 14-dim exceptional simple group G_2 (E. Cartan).

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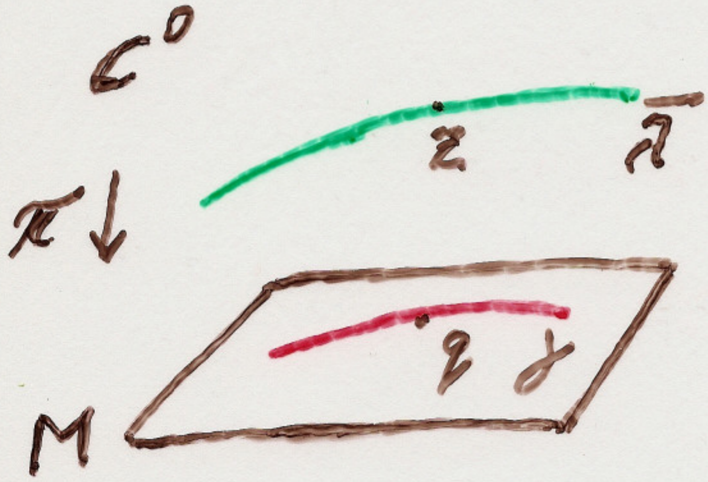
The "rolling bodies" distribution is flat $\Longleftrightarrow$ the bodies are balls of radii r and $3r$.



$$T_{\bar{\lambda}} F(C^0) \cong T_z C^0 / T_z \bar{\lambda}$$

$$J_{\bar{\lambda}_z} : \bar{\lambda} \longrightarrow Gr_2 (T_z C^0 / T_z \bar{\lambda})$$

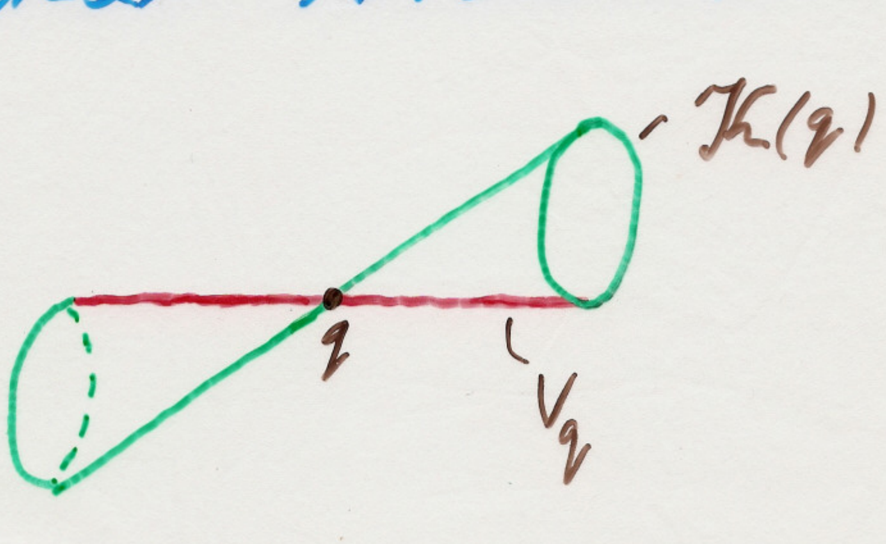
Moreover, $\operatorname{im}(\pi_* \circ J_{\bar{\lambda}_z}) \subset z^{\perp} \subset T_q M$

and $\pi_* \circ J_{\bar{\lambda}_z} : \bar{\lambda} \longrightarrow \mathbb{P}(z^{\perp} / T_q \gamma)$ is a curve in the projective plane.

<u>Th.</u> The distribution is flat $\Longleftrightarrow$ this curve is a conic, $\forall z \in C^0$.

In general, let $K_z(q)$ be the osculating conic of $\pi_* \circ T_\lambda$; then $K_z(q) \subset z^\perp / T_q \gamma$ is zero locus of a quadratic form on $z^\perp$ of type $(++-0)$.

Finally,
$$\mathcal{K}(q) = \bigcup_{z \in T_q^* M \cap C^0} K_z(q)$$
is zero locus of a $(+++--)$ quadratic form on $T_q M$, an intrinsically defined by the distribution conformal structure on M!



Rolling Bodies model.

Ex: 2-covering of the config. space is diffeom. to $M = S^3 \times S^2$.

Algebra of split-octonions:

$$\hat{\mathbb{O}} = \{w_1 + \ell w_2 : w_i \in \mathbb{H}\},$$

$$(a+\ell b)(c+\ell d) = (ac + d\bar{b}) + \ell(\bar{a}d + cb).$$

Pseudo-norm: $x = w_1 + \ell w_2$,

$$Q(x) = |w_1|^2 - |w_2|^2 \Longrightarrow Q(xy) = Q(x)Q(y),$$

$$x^{-1} = \frac{\bar{x}}{Q(x)}.$$

$$M = \{q = w_1 + \ell w_2 : \bar{w}_1 = -w_1,\ |w_1| = |w_2| = 1\}.$$

Flat case: $V_q = \{x \in T_q M : qx = 0\}$.

$$\mathcal{K}(q) = Q^{-1}(0) \cap T_q M.$$

General theory.

Let $\bar\lambda$ be a parametrized extremal, $J_{\bar\lambda} : t \mapsto J(t)$; $J(t)$ is an isotropic subspace of the symplectic space $T_{\bar\lambda}F(C^\nu)$. In a regular situation, after a reduction, we may assume that $J(t)$ are Lagrangian subspaces (of a, maybe, smaller symplectic space) and $J(t) \cap J(\tilde t) = 0$ if $t \neq \tilde t$ and $|t - \tilde t|$ is small.

Consider the projectors:

$$\pi_{t\tilde t} : J(t) \oplus J(\tilde t) \longrightarrow J(\tilde t),$$

where $J(t) \oplus J(\tilde t) = \Sigma$, a fixed symplectic space.

$J(t)$ and $J(\tau)$ are invariant subspaces of the operator $\frac{\partial^2 \widetilde{\kappa}_{t\tau}}{\partial t \partial \tau}$. We have:

$$tr\left(\frac{\partial^2 \widetilde{\kappa}_{t\tau}}{\partial t \partial \tau}\Big|_{J(\tau)}\right) = -tr\left(\frac{\partial^2 \widetilde{\kappa}_{t\tau}}{\partial t \partial \tau}\Big|_{J(t)}\right) =$$

$= \frac{K}{(t-\tau)^2} + g(t,\tau)$, where

g is a smooth symmetric func.;

$\Rightarrow g(t,t)$ is the generalized Ricci curvature of the parametrized extremal $t \mapsto \lambda(t)$, $g(t,t) \overset{def}{=} \rho_\lambda(t)$. The chain rule:

$\rho_{\lambda\circ\varphi}(\varphi(t)) = \rho_\lambda(\varphi(t))\dot{\varphi}^2 + K\mathbb{S}(\varphi)$.

Schwartzian

relation $\rho_\lambda = 0$ defines a canonical projective structure on $\bar{\lambda}$.