“Good Lie Brackets” for Control Affine Systems

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We consider a control affine system

$$\dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M, \ u_i \in \mathbb{R}. \tag{1}$$

Here $f_j, \ j = 0, \ldots, k$, are smooth vector fields on the connected manifold M. In this paper, smooth means C^∞. The space of smooth vector fields forms a Lie algebra $\text{Vec}M$ over $\mathbb{R}$, where Lie bracket $[f, g], \ f, g \in \text{Vec}M$, is commutator of the differential operators, $[f, g] = f \circ g - g \circ f$.

Let $\text{Lie}(f_0, \ldots, f_k)$ be Lie subalgebra of $\text{Vec}M$ generated by $f_0, f_1, \ldots, f_k$. We assume that M is equipped with a complete Riemannian metric and all vector fields in $\text{Lie}(f_0, \ldots, f_k)$ have at most linear growth in this metric; in particular, all these vector fields are complete.

Let $u(\cdot) = (u_1(\cdot), \ldots, u_k(\cdot)) \in L_1([0,t]; \mathbb{R}^k)$, $f_u = \sum\limits_{i=1}^{k} u_i f_i$. For any $q_0 \in M$, there exists a unique solution $q(\tau; u(\cdot))$, $0 \leq \tau \leq t$, of the equation $\dot{q} = f_0(q) + f_{u(\tau)}(q)$ such that $q(0; u(\cdot)) = q_0$. Moreover, the map

$$q(0; u(\cdot)) \mapsto q(t; u(\cdot))$$

is a diffeomorphism of M. We address the controllability problem on the group of diffeomorphisms, it concerns the characterization of diffeomorphisms which can be realized, at least approximately, in this way.

We can treat control system as an evolutionary machine, a way to transform a linear time structure into the space structure. Indeed, any control function $u(\cdot)$ (a function of time) produces a flow, a family of transformations of the space M. Zero control provides a prescribed nominal dynamics (no events). A change of control parameters means an event: the fields $f_0 + u_i f_i$ correspond to a short list of available simple immediate events.

It happens that extremely complex transformations can be approximated by a clever use of available simple and predictable elementary events. The complexity of the resulting transformation comes from the complexity of the evolutionary strategy, from the sophisticated choice of the order and timing for the switching between the fields based on the structure of there iterated Lie brackets.

In what follows, we use chronological notations; in particular,

$$q(t; u(\cdot)) = q_0 \circ \overrightarrow{\exp} \int_0^t f_{u(\tau)}\, d\tau.$$

Attainable sets $\mathcal{A}_t \subset \mathrm{Diff}\, M$ are defined as follows:

$$\mathcal{A}_t = \left\{ \overrightarrow{\exp} \int_0^t f_0 + f_{u(\tau)}\, d\tau : u(\cdot) \in L_1([0, t]; \mathbb{R}^k) \right\}.$$

A vector field $V \in \overline{\mathrm{Lie}(f_0, \ldots, f_k)}$ with at most linear grows is called a *good bracket* if $e^{tV} \in \overline{\mathcal{A}}_t$ for any $t > 0$. Here $\overline{S}$ is the closer of the set S; the closure of $\mathrm{Lie}(f_0, \ldots, f_k)$ and of $\mathcal{A}_t$ is taken in the C^∞ topology.

Good brackets provide us with additional control parameters as we can see from the following statement.

Proposition 1. *Let V be a good bracket, $u(\cdot) \in L_1([0,t];\mathbb{R}^k)$ and $v(\cdot)$ be a measurable real functions on $[0,t]$ with values in $[0,1]$. Then the diffeomorphism*

$$\overrightarrow{\exp} \int_0^t v(\tau)(f_0 + f_{u(\tau)}) + (1 - v(\tau))V \, d\tau$$

belongs to $\overline{\mathcal{A}}_t$.

This proposition is a simple corollary of the standard relaxation technique. $\square$

Example 1:

$$\left\{\begin{array}{rcl}\dot{x} & = & u \\ \dot{y} & = & \psi(x,y),\end{array}\right. \qquad u, x \in \mathbb{R},\ \ y \in \mathbb{R}^m,$$

where ψ is a degree n vector-polynomial w.r.t. x.

Extended system:

$$\left\{\begin{array}{rcl}\dot{x} & = & u \\ \dot{y} & = & \sum\limits_{i=0}^{n} \frac{u_i}{i!}\frac{\partial^i \psi}{\partial x^i}(x,y),\end{array}\right. \qquad u_0 = 1,\ \sum_{i,j=0}^{\frac{n}{2}} u_{i+j}\xi_i\xi_j \geq 0,\ \forall\, \xi.\ .$$

Any diffeomorphism produced by the extended system in time t can be C^∞-approximated by a diffeomorhism produced by the original system in the same time.

Example 2:

Vector fields $f_0, f_1, \dots, f_k$ generate a step 3 nilpotent Lie algebra.

Extended system:

$$\dot{q} = f_0 + \sum_{i=1}^{k} \Big(u_i f_i + u_{i0}[f_i, f_0] + \frac{u_{ii}}{2}[f_i, [f_i, f_0]] \Big) +$$

$$\sum_{1 \le i < j \le k} \Big(u_{ij}[f_i, [f_j, f_0]] + v_{ij}[[f_i, f_j], f_0] + w_{ij}[f_i, f_j] \Big),$$

where $\sum\limits_{i,j=0}^{k} u_{ij}\xi_i\xi_j \ge 0, \ \forall\, \xi.,\ u_{00} = 1,\ u_{ij} = u_{ji},$ and $v_{ij}, w_{ij} \in \mathbb{R}$ are free.

Basic formulas:

Let $p_t = \overrightarrow{\exp} \int_0^t f_\tau \, d\tau$, then

$$\overrightarrow{\exp} \int_0^t f_\tau + g_\tau \, d\tau = \overrightarrow{\exp} \int_0^t p_{\tau *}^{-1} g_\tau \, d\tau \circ p_t$$

and

$$p_{t*}^{-1} = \overrightarrow{\exp} \int_0^t \mathrm{ad} f_\tau \, d\tau, \quad (\mathrm{ad} f) g = [f, g].$$

Let us start with small time control. To this end we take a sample control $u(t) = (u_1(t), \dots, u_k(t))$, $t \in [0, 2]$, a small parameter $\varepsilon > 0$ and cook a re-scaled control $\frac{1}{\varepsilon} u(\frac{t}{\varepsilon})$. Now re-scaling time in the equation

$$\dot{q} = f_0(q) + \frac{1}{\varepsilon} \sum_{i=1}^{k} u_i\Big(\frac{t}{\varepsilon}\Big) f_i(q),$$

we obtain the system

$$\frac{dq}{d\tau} = \varepsilon f_0(q) + \sum_{i=1}^{k} u_i(\tau) f_i(q),$$

where $\tau = \frac{t}{\varepsilon}$.

We set $f_u = \sum\limits_{i=1}^{k} u_i f_i(q)$; then

$$q(\varepsilon) = q_0 \circ \overrightarrow{\exp} \int_0^2 \varepsilon f_0 + f_u \, d\tau =$$

$$q_0 \circ \overrightarrow{\exp} \int_0^2 \varepsilon \overrightarrow{\exp} \int_0^\tau \mathrm{ad} f_{u(\theta)} \, d\theta f_0 \, d\tau \circ \overrightarrow{\exp} \int_0^2 f_u \, d\tau.$$

Now assume that $u(t)$ has a form:

$$u(t) = \begin{cases} \frac{v(\tau)}{2}, & 0 \le \tau \le 1 \\ -\frac{v(1-\tau)}{2}, & 1 < \tau \le 2 \end{cases};$$

then $\overrightarrow{\exp} \int_1^2 f_{u(\tau)}\, d\tau = \left(\overrightarrow{\exp} \int_0^1 f_{u(\tau)}\, d\tau \right)^{-1}$ and we get

$$q(\varepsilon) = q_0 \circ \overrightarrow{\exp} \int_0^2 \varepsilon \overrightarrow{\exp} \int_0^\tau \mathrm{ad} f_{u(\theta)}\, d\theta f_0\, d\tau =$$

$$q_0 \circ e^{\varepsilon \left(\int_0^2 \overrightarrow{\exp} \int_0^\tau \mathrm{ad} f_{u(\theta)}\, d\theta f_0\, d\tau + O(\varepsilon) \right)}.$$

Moreover, $\overrightarrow{\exp}\int_0^{1+s} f_{u(\tau)}\,d\tau = \overrightarrow{\exp}\int_0^{1-s} f_{u(\tau)}\,d\tau,\ 0 \le s \le 1$, and we obtain:

$$\int\limits_0^2 \overrightarrow{\exp}\int_0^t \mathrm{ad} f_{u(\tau)}\,d\tau f_0\,dt = \int\limits_0^1 \overrightarrow{\exp}\int_0^t \mathrm{ad} f_{v(\tau)}\,d\tau f_0\,dt,$$

where $v(\cdot)$ is any function from $L_1([0,1];\mathbb{R}^k)$. We set:

$$\mathcal{V} = \left\{\int_0^1 \overrightarrow{\exp}\int_0^t \mathrm{ad} f_{v(\tau)}\,d\tau f_0\,dt : v \in L_1([0,1];\mathbb{R}^k\right\}$$

and we can re-write:

$$q(\varepsilon) = q_0 \circ e^{\varepsilon(V+O(\varepsilon))}, \quad V \in \mathcal{V}.$$

Let $t \mapsto V_t \in \mathcal{V}$ be a piecewise constant family of vector fields, $V_t = V_i,\ \forall t \in (\frac{i-1}{\varepsilon}, \frac{i}{\varepsilon}]$. We repeat our procedure n times with an appropriate choice of $v(\cdot)$ for any segment $[\frac{i-1}{\varepsilon}, \frac{i}{\varepsilon}]$ and obtain:

$$q(n\varepsilon) = q_0 \circ e^{\varepsilon(V_\varepsilon + O(\varepsilon))} \circ \cdots \circ e^{\varepsilon(V_{n\varepsilon} + O(\varepsilon))} = q_0 \circ \overrightarrow{\exp} \int_0^{n\varepsilon} V_t + O(\varepsilon)\, dt.$$

It follows that we can arbitrarily well approximate any diffeomorphism of the form $\overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ by the value at the time moment t of a flow generated by control system (1). Moreover, $t \mapsto V_t$ maybe of class L_1, it is not obliged to be piecewise constant.

Of course, we can also approximate any diffeomorphism of the form

$$\overrightarrow{\exp}\int_0^t V_\tau\,d\tau \circ \overrightarrow{\exp}\int_0^1 f_{u(t)}\,dt, \qquad u(t)\in\mathbb{R}^k,\ V_\tau\in\mathcal{V}.$$

The second term in the product is generated by the control linear system (without drift).

Assume for the moment that the fields $f_1,\ldots,f_k$ generate a finite-dimensional Lie algebra, $\mathrm{Lie}\{f_1,\ldots,f_k\}=L$, $\dim L<\infty$. Let $\mathcal{L}\subset \mathrm{Diff}\,M$ be the Lie group generated by L; it is a finite-dimensional Lie subgroup of $\mathrm{Diff}\,M$. Moreover,

$$\mathcal{L}=\left\{\overrightarrow{\exp}\int_0^1 f_{u(t)}\,dt : u(\cdot)\in L_1([0,1];\mathbb{R}^k)\right\}.$$

Proposition 2. $\overline{\mathcal{V}} = \overline{\text{conv}\{p_* f_0 : p \in \mathcal{L}\}}$.

This proposition is easily derived from the fact that any starting from I curve in $\mathcal{L}$ can be uniformly approximated by the curve of the form $p_t = \overrightarrow{\exp} \int_0^t f_{u(\tau)}\, d\tau$. $\square$

It follows that attainable sets of the system

$$\dot{q} = U(q), \quad U \in \text{conv}\{p_* f_0 : p \in \mathcal{L}\} \tag{2}$$

are contained in the closure of the attainable sets of system (1). Now if $\text{conv}\{p_* f_0 : p \in \mathcal{L}\}$ is an affine subspace or it contains an affine subspace of vector fields than we can repeat the whole procedure and obtain more available vector fields in the right hand side.

Roughly speaking, the construction provides an extension of the original affine subspace of admissible vector fields to a convex set in the closer of the Lie subalgebra generated by this affine subspace. The system can be moved in the direction of any field from this convex set that is built of Lie bracket polynomials and series of the original fields f_j. Moreover, the structure of these polynomials and series does not depend on the choice of the original fields, the fields f_j serve just as variables.

In other words, we may speak about a universal convex set in the closure of the free Lie algebra with generators $a_0, a_1, \ldots, a_k$. This is the set of “good” combinations of brackets and any control affine system can be moved in the direction of these combinations.

Let Ass$(a_0, \ldots, a_k)$ be the free associated algebra over $\mathbb{R}$, its elements are linear combinations of words in the alphabet $\{a_0, \ldots, a_k\}$. Then Ass$(a_0, \ldots, a_k) = \bigoplus_{n=0}^{\infty} A_n$, where the space A_n consists of linear combinations of words with n letters and $A_0 = \mathbb{R}$ corresponds to the empty word. The *closure* of Ass$(a_0, \ldots, a_k)$ is the algebra of formal series

$$\mathfrak{A} = \left\{ \sum_{n=0}^{\infty} x_n : x_n \in A_n \right\}$$

endowed with topology of the term-wise convergence; we write $\mathfrak{A} = \overline{\text{Ass}(a_0, \ldots .a_k)}$

The universal control affine system with k-dimensional control is the system

$$\dot{x} = x\left(a_0 + \sum_{i=1}^{k} u_i a_i\right), \qquad x \in \mathfrak{A},\ \ u_i \in \mathbb{R}. \tag{3}$$

Given control $u(\cdot) \in L_1([0,t];\mathbb{R}^k)$ and initial condition $x(0)$, we can explicitly write the unique solution of (3) that is a curve in $\mathfrak{A}$ whose homogeneous components are absolutely continuous vector functions.

Let W be the set of words in the alphabet $\{a_0, \ldots, a_k\}$ and

$$\Delta^n(t) = \{(\tau_1, \ldots, \tau_n\} : 0 \le \tau_n \le \cdots \le \tau_1 \le t\}$$

be the n-dimensional simplex. Given a word $w = a_{i_n} \cdots a_{i_1}$, we set

$$S_u^w(t) = \int \cdots \int_{\Delta^n(t)} u_{i_n}(\tau_n) \cdots u_{i_1}(\tau_1)\, d\tau_1 \cdots d\tau_n,$$

where $u_0(t) \equiv 1$. Solutions of (3) have a form:

$$x(t) = x(0) \sum_{w \in W} S_u^w(t) w.$$

We keep using chronological notations while working in $\mathfrak{A}$ with the composition "$\circ$" substituted by the product in $\mathfrak{A}$. In what follows, we assume that $x(0) = 1$. More notations:

$$L = \mathrm{Lie}(a_1, \dots, a_k) \subset \mathrm{Ass}(a_1, \dots, a_k), \qquad \mathcal{L} = \left\{ e^V : V \in \overline{L} \right\},$$

$$(\mathrm{Ad}x)V = xVx^{-1}, \qquad a_u = \sum_{i=1}^{k} u_i a_i;$$

then $\mathrm{Ad}e^V = e^{\mathrm{ad}V}$.

We have:

$$\mathcal{L} = \overline{\left\{ \overrightarrow{\exp} \int_0^1 a_{v(t)}\, dt : v(\cdot) \in L_1([0,1]; \mathbb{R}^k) \right\}}.$$

Moreover,

$$\overline{\mathrm{conv}\{(\mathrm{Ad}x)a_0 : x \in \mathcal{L}\}} = \overline{\left\{\int_0^1 \overrightarrow{\exp} \int_0^t \mathrm{ad} a_{v(\tau)}\, d\tau a_0 \, dt : v \in L_1([0,1];\mathbb{R}^k)\right\}}.$$

We may translate all the computations to the universal setting by the substitution of f_j with a_j and we obtain:

Theorem 1. *For any $t > 0$, the product of $\mathcal{L}$ and the attainable set of the system*

$$\dot{x} = xV, \quad V \in \mathrm{conv}\{(\mathrm{Ad}z)a_0 : z \in \mathcal{L}\}$$

at t is contained in the closure of the attainable set of system (3) at t.

□

Let $V_\alpha,\ \alpha = 1, 2, \ldots,$ be a linearly ordered homogeneous additive basis of L. It may be a Hall basis but this is not necessary. It is easy to see that

$$\mathcal{L} = \left\{ \prod_{\alpha=1}^{\infty} e^{v_\alpha V_\alpha} : v_\alpha \in \mathbb{R} \right\}. \tag{4}$$

This is what people call "the 2nd type coordinates" for the Lie group, while the presentation $\mathcal{L} = e^{\overline{L}}$ is the "1st type coordinates.

We have:

$$\prod_{\alpha=1}^{\infty} e^{v_\alpha V_\alpha} = 1 + \sum_{\substack{\alpha_1 \le \cdots \le \alpha_m \\ i_1, \ldots, i_m > 0}} \frac{v_{\alpha_1}^{i_1} \cdots v_{\alpha_m}^{i_m}}{i_1! \cdots i_m!} V_{\alpha_1}^{i_1} \cdots V_{\alpha_m}^{i_m}.$$

According to the Poincare–Birkhoff–Wittt theorem, the elements

$$V_{\alpha_1}^{i_1} \cdots V_{\alpha_m}^{i_m}, \quad \alpha_1 \le \cdots \le \alpha_m,\ i_1, \ldots, i_m > 0,\ m \ge 0, \tag{5}$$

form an additive basis of $\mathsf{Ass}\{a_1, \ldots, a_k\}$, hence

$$\overline{\mathsf{span}\mathcal{L}} = \overline{\mathsf{Ass}(a_1, \ldots, a_k)}.$$

Next statement reduces the study of $\mathsf{conv}\{(\mathsf{Ad}z)a_0 : z \in \mathcal{L}\} \subset \overline{L}$ to the study of $\mathsf{conv}\mathcal{L} \subset \overline{\mathsf{Ass}(a_1, \ldots, a_k)}$.

Proposition 3. *Linear map* $Ad_0 : \mathsf{Ass}(a_1, \ldots, a_k) \to \mathsf{Lie}(a_0, a_1, \ldots a_k)$ *defined by its action on the basis:*

$$Ad_0 \left(V_{\alpha_1}^{i_1} \cdots V_{\alpha_m}^{i_m} \right) = (\mathsf{ad}V_{\alpha_1})^{i_1} \cdots (\mathsf{ad}V_{\alpha_m})^{i_m} a_0$$

is injective.

Let t_α, $\alpha = 1, 2, \ldots$, be coordinates on L induced by the basis V_α. In other words, $t_\alpha \in L^*$, $\langle t_\alpha, V_\beta \rangle = \delta_{\alpha,\beta}$. The basis provides the identification of $\bar{L}$ and L^*. According to this identification, a series $\sum_\alpha v_\alpha V_\alpha$ is identified with the linear function $\sum_\alpha v_\alpha t_\alpha$ on L.

Moreover, the monomials $t_{\alpha_1}^{i_1} \cdots t_{\alpha_m}^{i_m}$ are coordinates on the vector space Ass$(a_1, \ldots, a_k)$. A monomial $t_{\alpha_1}^{i_1} \cdots t_{\alpha_m}^{i_m}$ treated as a linear form on Ass$(a_1, \ldots, a_k)$ annihilates all elements of the basis (5) except of $V_{\alpha_1}^{i_1} \cdots V_{\alpha_m}^{i_m}$ and $\langle t_{\alpha_1}^{i_1} \cdots t_{\alpha_m}^{i_m}, V_{\alpha_1}^{i_1} \cdots V_{\alpha_m}^{i_m} \rangle = 1$.

The basis (5) provides the identification of $\overline{\text{Ass}(a_1,\ldots,a_k)}$ with $\text{Ass}(a_1,\ldots,a_k)^*$ and eventually with the space of formal power series on the variables $t_\alpha,\ \alpha = 1,2,\ldots,$. Let $\mathcal{S}$ the space of formal power series and

$$\nu : \overline{\text{Ass}(a_1,\ldots,a_k)} \to \mathcal{S}$$

be the continuous isomorphism of vector spaces that realizes the mentioned identification,

$$\nu : V_{\alpha_1}^{i_1}\cdots V_{\alpha_m}^{i_m} \mapsto t_{\alpha_1}^{i_1}\cdots t_{\alpha_m}^{i_m},$$

where $\alpha_1 \leq \cdots \leq \alpha_m$ as in (5). Linear map ν depends on the choice of the basis V_α and it is not a homomorphism of the algebras.

Definition 1. *We say that a nonzero function $\varphi : L \to \mathbb{R}$ is exponential if the restriction of φ to any finite-dimensional subspace of L is continuous and*

$$\varphi(z_1 + z_2) = \varphi(z_1)\varphi(z_2), \quad \forall\, z_1, z_2 \in L.$$

It is easy to see that, written in the coordinates, exponential functions are exactly functions of the form $\varphi(t) = e^{\langle v,t\rangle}$, where

$$v = \{v_\alpha\}_{\alpha=1}^{\infty}, \quad t = \{t_\alpha\}_{\alpha=1}^{\infty}, \quad \langle v, t\rangle = \sum_{\alpha=1}^{\infty} v_\alpha t_\alpha.$$

Recall that an element of L has only a finite number of nonzero coordinates t_α.

The space of exponential functions is denoted by $\mathcal{E}$. The identification of the exponential function with the exponential series gives the inclusion $\mathcal{E} \subset \mathcal{S}$.
Proposition 4. $\nu(\mathcal{L}) = \mathcal{E}$.

Proof. Indeed,

$$\nu\left(\prod_{\alpha=1}^{\infty} e^{v_\alpha V_\alpha}\right) = 1 + \sum_{\substack{\alpha_1 \le \cdots \le \alpha_m \\ i_1,\ldots,i_m > 0}} \frac{v_{\alpha_1}^{i_1} \cdots v_{\alpha_m}^{i_m}}{i_1! \cdots i_m!} \nu\left(V_{\alpha_1}^{i_1} \cdots V_{\alpha_m}^{i_m}\right) =$$

$$1 + \sum_{\substack{\alpha_1 \le \cdots \le \alpha_m \\ i_1,\ldots,i_m > 0}} \frac{v_{\alpha_1}^{i_1} \cdots v_{\alpha_m}^{i_m}}{i_1! \cdots i_m!} t_{\alpha_1}^{i_1} \cdots t_{\alpha_m}^{i_m} = \prod_{\alpha=1}^{\infty} e^{v_\alpha t_\alpha} = e^{\langle v,t\rangle}. \qquad \square$$

We see that $\nu(\mathcal{L})$ does not depend on the choice of the basis V_α, unlikely the isomorphism ν.

The space of formal series $\mathcal{S}$ is the adjoint space to the space of (finite) linear combinations of partial differentials $\frac{\partial^{i_1}}{\partial t_{\alpha_1}^{i_1}} \cdots \frac{\partial^{i_m}}{\partial t_{\alpha_m}^{i_m}}\Big|_{t=0}$, where the pairing of the differential and the series is just the action of the differential on the series.

We set:

$$\mathcal{D} = \operatorname{span}\left\{ \frac{\partial^{i_1}}{\partial t_{\alpha_1}^{i_1}} \cdots \frac{\partial^{i_m}}{\partial t_{\alpha_m}^{i_m}}\Big|_{t=0} : \alpha_1 \leq \cdots \leq \alpha_m,\ i_1, \ldots, i_m > 0,\ m \geq 0 \right\},$$

$\mathcal{S} = \mathcal{D}^*$. To any $\varphi \in \mathcal{S}$ we associate a quadratic form Q_φ by the following formula:

$$Q_\varphi(\eta) = \eta_t \eta_s \varphi(t+s), \quad \eta \in \mathcal{D},$$

where η_t differentiates with respect to t and η_s differentiates with respect to s.

Theorem 2. *Let* $\varphi \in \mathrm{span}\mathcal{E}$, $\varphi(0) = 1$. *Quadratic form* Q_φ *is nonnegative if and only if* $\varphi \in \mathrm{conv}\mathcal{E}$.

Proof. If $\varphi \in \mathcal{E}$, then $\varphi(t+s) = \varphi(t)\varphi(s)$ and

$$Q_\varphi(\eta) = \eta_t\eta_s\varphi(t+s) = (\eta\varphi)^2 \geq 0.$$

Let $\varphi = \sum\limits_{i=1}^{n} c_i\varphi_i$. We may assume that $\varphi_i(t) = e^{\langle w_i, t\rangle}$, $i = 1, \ldots, n$, where $w_1, \ldots, w_n$ are mutually distinct. Of course, there exists $m > 0$ such that the truncation of these infinite vectors to $\mathbb{R}^m$ are also mutually distinct.

We have: $Q_\varphi(\eta) = \sum\limits_{i=1}^{n} c_i(\eta\varphi_i)^2$. If all c_i are nonnegative, then $Q_\varphi \geq 0$. Assume that $c_{i_0} < 0$. On the other hand, Taylor polynomials of order n of $\varphi_1, \ldots, \varphi_n$ at 0 are linearly independent. Hence there exists $\eta_0 \in \mathcal{D}$ such that $\eta_0\varphi_{i_0} = 1$, $\eta_0\varphi_i = 0$, $\forall\, i \neq i_0$, and $Q_\varphi(\eta_0) = c_{i_0}$. $\square$

Given a closed convex cone $K \subset \mathcal{S}$, the dual cone $K^\circ \subset \mathcal{D}$ is defined as follows:

$$K^\circ = \{\xi \in \mathcal{D} : \xi\varphi \geq 0, \ \ \forall\varphi \in K\}.$$

Lemma 1. *Dual cone to the closed convex con generated by $\mathcal{E}$ is the set of differential operators whose symbols are nonnegative polynomials.*

Dual cone to $\{\phi \in \mathcal{S} : Q_\phi \geq 0\}$ is the set differential operators whose symbols are sums of squares of real polynomials.

Theorem 1 implies that any element of the set $\overline{\text{conv}(Ad\mathcal{L})a_0} + \overline{L}$ corresponds to a good bracket. It happens that this set is actually the set of all good brackets for the universal system (4).

Theorem 3. *Let* $V \in \overline{Lie(a_0, \ldots, a_k)}$*; the left-invariant vector field* xV *is a good bracket for system (4) if and only if* $V \in \overline{\text{conv}(\text{Ad}\mathcal{L})a_0} + \overline{L}$

Free Lie algebra is too big. It is more practical and sufficient for many purposes to consider its finite-dimensional nilpotent truncations.

In what follows, we use multi-indices. Given a nonnegative integer m, $\mathbb{Z}_+^m$ is the set of m-dimensional vectors with nonnegative integral coordinates. We extend m-dimensional vectors by zeros and assume that $\mathbb{Z}_+^m \subset \mathbb{Z}_+^{m'}$ if $m \leq m'$; then $\mathbb{Z}_+^\infty = \bigcup\limits_{m \geq 0} \mathbb{Z}_+^m$ is a set of infinite vectors with a finite number nonzero coordinates. If $i = (i_1, \dots, i_m) \in \mathbb{Z}_+^m$, then:

$$t^i = t_1^{i_1} \cdots t_m^{i_m}, \quad |i| = \sum_{j=1}^m i_j, \quad \varphi_0^{(i)} = \left.\frac{\partial^{|i|}\varphi}{\partial t_1^{i_1} \cdots \partial t_m^{i_m}}\right|_{t=0}.$$

Let $I \subset \mathbb{Z}_+^\infty$ be a finite subset, we set:

$$\mathcal{P}(I) = \left\{ \sum_{i \in I} c_i t^i : c_i \in \mathbb{R} \right\},$$

a $\#I$-dimensional space of polynomials. We denote by $\Pi_I : \mathcal{S} \to \mathcal{P}(I)$ the continuous linear projector defined by the rule:

$$\Pi_I(t^i) = \begin{cases} t^i, & \text{if } i \in I; \\ 0, & \text{if } i \in \mathbb{Z}_+^\infty \setminus I. \end{cases}$$

Lemma 2. $\mathrm{span}\Pi_I(\mathcal{E}) = \mathcal{P}(I)$, *for any* $I \subset \mathbb{Z}_+^\infty$ *such that* $\#I < \infty$.

Let $C \subset \mathbb{R}_+^m$ be a convex compact subset; we set: $I_C = C \cap \mathbb{Z}_+^m$.

Theorem 4. *Let C be a convex compact subset of $\mathbb{R}_+^m$ and $0 \in C$. If $\phi \in \mathcal{P}(I_C)$ belongs to $\overline{\Pi_{I_C}(\mathrm{conv}\mathcal{E})}$, then*

$$\phi(0) = 1, \qquad \sum_{i,j \in I_{\frac{1}{2}C}} \phi_0^{(i+j)} \xi_i \xi_j \geq 0, \quad \forall\, \xi_i \in \mathbb{R},\, i \in I_{\frac{1}{2}C}. \tag{6}$$

Moreover, if $m = 1$ or $|i| \leq 2, \forall i \in I_C$, then condition (6) is sufficient for ϕ to belong to $\overline{\Pi_{I_C}(\mathrm{conv}\mathcal{E})}$.