Chapter 1

Is it possible to recognize Local Controllability in a finite number of differentiations?

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1.1 Description of the problem

Let $f_1, \ldots, f_k,\ k \geq 2$, be real–analytic vector fields defined on a neighborhood of the origin in R^n, and $t > 0$. The point $x \in R^n$ is called attainable from the origin for a time less than t and with no more than N switchings, if there exists a subdivision $0 = t_0 < t_1 < \ldots < t_{N+1} < t$ of the segment $[0, t]$ and solutions $\xi_j(t),\ t \in [t_j, t_{j+1}]$ of the differential equations $\dot{x} = f_{i_j}(x)$, for some $i_j \in \{1, \ldots, k\}$, such that $\xi_0(0) = 0$, $\xi_{j-1}(t_j) = \xi_j(t_j)$ for $j = 1, \ldots, N$, $\xi_N(t_{N+1}) = x$. Let $\mathcal{A}_t(N)$ be the set of all such points x; the set $\mathcal{A}_t = \bigcup\limits_{N>0} \mathcal{A}_t(N)$ is called the attainable set for a time no greater than t.

The family of vector fields $\{f_1, \ldots, f_k\}$ is said to be *Small Time Locally Controllable* (STLC) at $0 \in R^n$, if $\mathcal{A}_t$ contains an open neighborhood of 0 for any positive t, i. e. $0 \in int\, \mathcal{A}_t \quad \forall t > 0$.

Let $\{f_1, \ldots, f_k\}$ be an STLC family of vector fields. Is it true that:

(a) *$\exists\, N_1$ such that any family of vector fields with the same Taylor polynomials of order N_1 at 0 as the Taylor polynomials of $f_1, \ldots, f_k$ is STLC ?*

(b) *$\exists\, N_2$ such that $\mathcal{A}_t$ contains a ball of radius t^{N_2} centered at 0, for any small enough $t > 0$?*

(c) *$\exists\, N_3$ such that $0 \in int\, \mathcal{A}_t(N_3)$, for any $t > 0$?*

The property (c) implies both (a) and (b). The answer to question (c) is positive, if $n \leq 2$, and is negative for some STLC couples $\{f_1, f_2\}$ of polynomial vector fields, if $n \geq 4$. Question (c) is open for $n = 3$. Questions (a) and (b) are open for all $n \geq 3$ for both analytical and polynomial cases.

1.2 Motivations and references

The local controllability problem is classical in Geometric Control Theory. If one has a controlled system, the first question is: in what direction one can move? Small Time Local Controllability is just a possibility of moving in any direction, slower or faster. Problems (a), (b), (c) have the following meaning:

- Is it always possible to recognize an STLC family in a finite number of differentiations?
- Is it possible to move not too slow in any direction?
- How complicated is the control strategy one really needs?

If $\{f_1, \ldots, f_k\}$ is an STLC family, then $\{-f_1, \ldots, -f_k\}$ is also STLC (this is true though not evident); hence $\{f_1, \ldots, f_k\}$ is small time stabilizable at 0 by open-loop controls. See [15] for the connection with the stabilizability by a feedback control.

There are many partial results. If the cone $\{\sum_{i=1}^{k} \alpha_i f_i \mid \alpha_i \geq 0\}$ is a linear space, then a simple characterization of the STLC families given in [1] provides the positive answers to all the questions (see also [13, 14] for uniform estimates of N_2, N_3 in the case of polynomial vector fields of prescribed degree). The symmetrized family $\{f_1, \ldots, f_k, -f_1, \ldots, -f_k\}$ automatically satisfies the previous condition. If this symmetrized family is STLC, then the attainable sets for the original family $\{f_1, \ldots, f_k\}$ have a nonempty interior, although the family may fail to be STLC ([2]). If $\{f_1, \ldots, f_k\}$ is STLC, then for any $t > 0$ there exists N_t such that $0 \in int\, \mathcal{A}_t(N_t)$ (do not confuse with property (c), where N doesn't depend on t !). The positive answer to all the questions in the 2-dimensional case follow from the results of [3]; a counterexample to property (c) in the 4-dimensional case was obtained in [9].

This is a long story, many efforts were made and rather strong sufficient as well as some necessary conditions for Small Time Local Controllability in a space of arbitrary dimension were obtained, see [4—8], [10—12]. Unfortunately (or maybe fortunately for you, the reader of this chapter), the gap between the necessary and sufficient conditions is still big enough to keep open the above formulated fundamental questions.

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