Fast-oscillating Control and Combinatorics of Permutations

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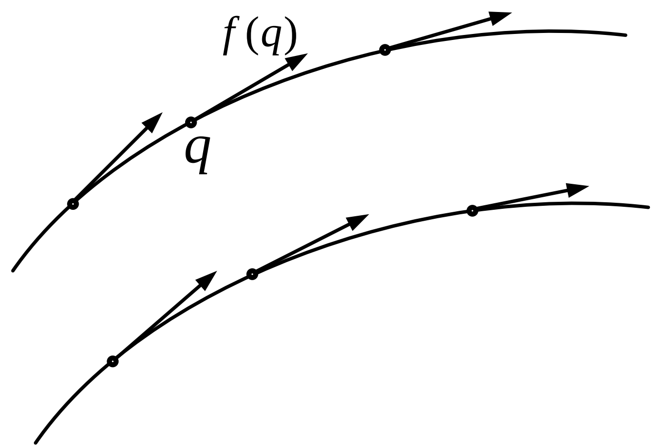
Smooth dynamical system:

$$\dot{q}(t) = f(q(t)), \quad q \in M, \ t \in \mathbb{R},$$



generates a flow

$$P^t : M \to M, \quad P^t : q(0) \mapsto q(t), \quad t \in \mathbb{R}.$$

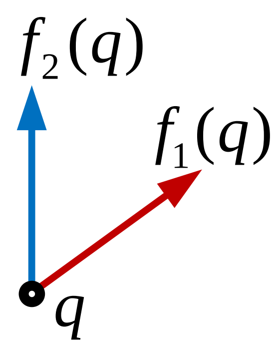
Control system:

$$\dot{q} = f_u(q), \qquad u \in U.$$

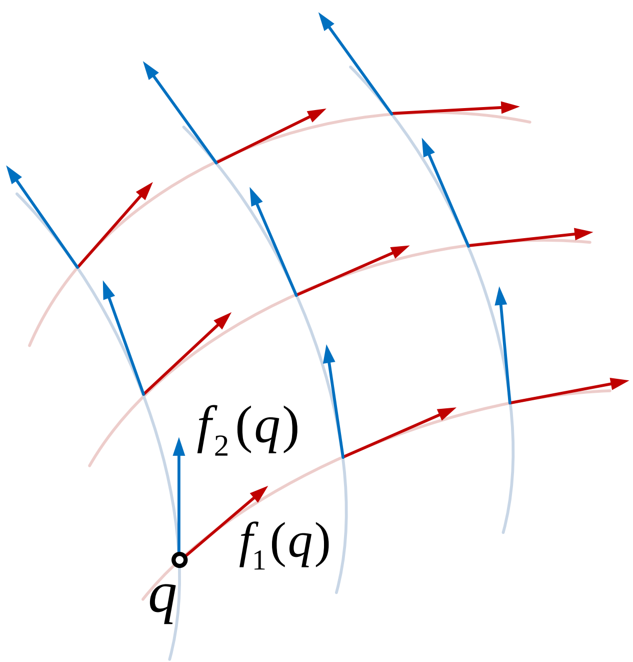
Control: $t \mapsto u(t),\ t \geq 0$.

Trajectory: $t \mapsto q(t)$, where $\dot{q}(t) = f_{u(t)}(q(t))$.

Special case: $U = \{1, 2\}$:



Trajectories:



Fast-oscillating controls allow to uniformly approximate flows generated by:

$$\dot{q} = \nu f_1(q) + (1-\nu) f_2(q), \quad 0 \leq \nu \leq 1.$$

Similarly, for $U = \{1, 2, \ldots, k\}$ we approximate dynamics

$$\dot{q} = \sum_{i=1}^{k} \nu_i f_i(q), \quad \nu_i \geq 0, \quad \sum_i \nu_i = 1.$$

If $0 \in relint\,(conv\, f_U(q))$, then we can much more!

Consider the case

$$f_u = \sum_{i=1}^{k} u^i f_i, \quad u = (u^1, \ldots, u^k) \in U,$$

where U is a neighborhood of $0 \in \mathbb{R}^k$.

Take a sample vector-function $t \mapsto v(t),\ supp\{v(\cdot)\} \subset [0,1]$ and let

$$\dot{q}_\varepsilon(t) = f_{v\left(\frac{t}{\varepsilon}\right)}(q_\varepsilon), \quad q_\varepsilon(0) = q_0.$$

Then, for any "observable" $a : M \mapsto \mathbb{R}$, we have:

$$a(q_\varepsilon(t)) \approx a(q_0) + \sum_{i=1}^{\infty} \varepsilon^i \int \underset{\Delta_i}{\int} f_{v(t_i)} \circ \cdots \circ f_{v(t_1)} a(q_0)\, dt_1 \ldots dt_i$$

$$= a(q_0) + \sum_{i=1}^{\infty} \varepsilon^i \int \underset{\Delta_i}{\int} p_i(v(t_i)), \ldots, v(t_1))\, dt_1 \ldots dt_i,$$

where $\Delta_i = \{(t_1, \ldots, t_i) : 0 \leq t_i \leq \cdots \leq t_1 \leq 1\}$ and p_i is a i-linear form, $p_i(v_i, \ldots, v_1) = \langle \omega_i, v_i \otimes \cdots \otimes v_1 \rangle$.

We set $\gamma(t) = \int_0^t u(\tau)\, d\tau$, the i-th order term takes the form:

$$\varepsilon^i \left\langle \omega_i, \int \underset{\Delta_i}{\int} d\gamma(t_i) \otimes \cdots \otimes d\gamma(t_1) \right\rangle.$$

We set:

$$D^n(\gamma) = \int \int_{\Delta_n} d\gamma(t_n) \otimes \cdots \otimes d\gamma(t_1),$$

where γ is a Lipschitz curve in $\mathbb{R}^k$, $\gamma(0) = 0$.

In particular, $D^1(\gamma) = \gamma(1)$. If γ is a closed curve then principal term is $D^2(\gamma)$. Moreover,

$$D^2(\gamma) = \int_0^1 \dot{\gamma}(t) \wedge \gamma(t)\, dt + \frac{1}{2}\gamma(1) \otimes \gamma(1).$$

Let $\Omega_n = \{\gamma : D^1(\gamma) = \cdots = D^{n-1}(\gamma) = 0\}$.

We know that $D^n(\Omega_n) = Lie^n(\mathbb{R}^k) \subset (\mathbb{R}^k)^{\otimes n}$.

If $\gamma \in \Omega_n$ and $D^n(\gamma) = \pi(e_1, \ldots, e_n)$, where π is a "Lie polynomial", then

$$q_\varepsilon(t) = q_0 + \varepsilon^n \pi(f_1, \ldots, f_n)(q_0) + O(\varepsilon^{n+1}).$$

We are looking for symmetries of $D^n\Big|_{\Omega_n}$ in order to better understand the structure of Ω_n.

Let Σ_n be the symmetric group and $\overline{\Sigma}_n = \{\sum\limits_i c_i \sigma_i : \sigma_i \in \Sigma_n\}$ its group algebra. We set:

$$D^n_\sigma(\gamma) = \int\int\limits_{\Delta_n} d\gamma(t_{\sigma(n)}) \otimes \cdots \otimes d\gamma(t_{\sigma(1)}), \quad D^n_{\sum c_i \sigma_i} = \sum c_i D^n_{\sigma_i}.$$

Let $\sigma \in \Sigma_n$, the *monotonicity type* of σ is a word $w_\sigma = s_1 \ldots s_{n-1}$ in the alphabet $\{\alpha, \beta\}$,

$$s_i = \begin{cases} \alpha, & \sigma(i) < \sigma(i+1); \\ \beta, & \sigma(i) > \sigma(i+1). \end{cases}$$

Given a word w, we set $\overline{w} = \sum_{\{w_\sigma = w\}} \sigma$. The *descent subalgebra* of $\overline{\Sigma}_n$:

$$\mathfrak{M}_n = span\left\{\overline{w} : w = s_1 \cdots s_{n-1},\ s_i \in \{\alpha, \beta\}\right\}.$$

It admits a homomorphism:

$$r : \mathfrak{M}_n \to \mathbb{Z}, \quad r(\overline{s_1 \ldots s_{n-1}}) = (-1)^{\#\{i : s_i = \beta\}}.$$

Example:

$$\overline{\alpha\cdots\alpha} = 1, \quad \overline{\beta\cdots\beta} = \begin{pmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1 \end{pmatrix}, \quad r(\overline{\beta\cdots\beta}) = (-1)^{n-1}.$$

Theorem. *A curve γ belongs to Ω_n if and only if*

$$D^n_{\mathfrak{m}}(\gamma) = r(\mathfrak{m}) D^n(\gamma), \quad \forall\, \mathfrak{m} \in \mathfrak{M}.$$

Affine in control system:

$$\dot q = h(q) + \sum_i u^i f_i(q), \quad u = (u^1, \dots, u^k) \in \mathbb{R}^k. \qquad (*)$$

If $h \in Lie\{f_1, \dots, f_k\}$, then we can neutralize the drift h, but this inclusion is violated for many important apparently controllable systems. Examples:

1. *Acceleration control:* $\ddot x = \sum_i u^i g_i(x)$. We rewrite:

$$\dot x = y, \ \dot y = \sum_i u^i g_i(x); \qquad q = (x, y),$$

$$\dot q = h(q) + \sum_i u^i f_i(q), \quad [f_i, f_j] = 0, \ i, j = 1, \dots, k.$$

2. *"Fluid dynamics:"* $\dot y = Ay + B(y, y) + \sum_i u^i g_i$.

Theorem. *Assume that* $[f_i, f_j] = 0,\ i, j = 1, \ldots, k$. *If*

$$conv \left\{ \sum_{i,j} u^i u^j [f_i, [f_j, h]] : u^i, u^j \in \mathbb{R} \right\}$$

is a subspace, then system

$$\dot{q} = h(q) + \sum_{\iota} u^{\iota} f_{\iota}(q) + \sum_{i,j} u^{ij} [f_i, [f_j, h]](q)$$

has "the same control properties" as system (*).

If the fields $[f_i, [f_j, h]], f_{\iota}$ are all commuting then we iterate the theorem etc.

Hint: Use a fast-oscillating control variation:

$$u_\varepsilon^i(t) = \frac{1}{\varepsilon}\sin\left(\frac{t}{\varepsilon^2}\right), \quad u_\varepsilon^j(t) = \frac{1}{\varepsilon}\cos\left(\frac{t}{\varepsilon^2}\right)$$

to single out the desired bracket.

Indeed:

$$\int_0^1 u_\varepsilon^\iota\, dt = O(\varepsilon), \quad \iint_{\Delta_2} u_\varepsilon^i(t_1)u_\varepsilon^j(t_2)\, dt_1 dt_2 = O(1),$$

as $\varepsilon \to 0$.