March 27, 2019

SISSA Entrance Examination

PhD in Theoretical Particle Physics

SOLVE TWO OUT OF THE FOLLOWING FOUR PROBLEMS.

Do *not* write your name anywhere on the solutions

Problem 1

Two non-relativistic particles of mass m are placed in a three-dimensional rectangular box of sizes $a > b > c$, in the lowest energy state of the system compatible with the conditions below. The particles interact with each other with a repulsive δ-function potential

$$V = A\,\delta(\vec{r}_1 - \vec{r}_2) \qquad\qquad \text{with } A > 0\ .$$

Using first-order perturbation theory in the limit of small A, compute the energy of the system under the following conditions:

1. Particles not identical.
2. Identical particles of spin zero.
3. Identical particles of spin one-half, with spins in the same direction.
4. Discuss the validity of first order perturbation theory as i) c varies with a and b fixed; and ii) as $a = b = c$.

Problem 2

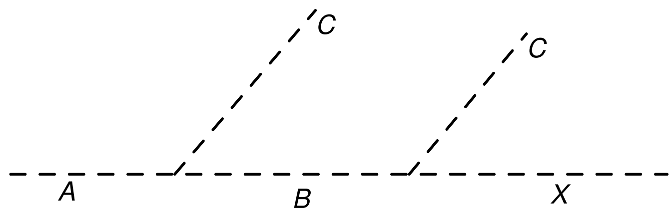

Figure 1:

Consider the following decay chain $A \to CB \to C(B \to CX) \to CCX$ (see Figure 1), where the particle masses satisfy $M_A > M_B > M_X$ and $M_C = 0$. Suppose that at experiment you can observe only the particle C and you want to extract as much information as possible about the particle X. Assume for simplicity that all of the particles in the process are scalars, and all of the particles including intermediate B are on-shell.

1. Find the maximum and the minimum value of the invariant mass of the $C - C$ pair as a function of M_A, M_B, M_X. The invariant mass is defined as follows
$$M_{CC}^2 = (p_C^{(1)} + p_C^{(2)})^2,$$
where the $p_C^{(1,2)}$ are the corresponding four-momenta of the first and the the second C particles. To what directions of the C particles momenta do these values correspond? Remember that the B-particle is on-shell.

2. Find the dependence of the differential decay rate $\frac{d\Gamma}{dM_{CC}}$ on the variable M_{CC}. Ignore the overall coefficient of proportionality.

   Hint: The scalar in its rest frame decays isotropically i.e. $\frac{d\Gamma}{d\cos\theta d\phi} =$ *const, where* ϕ, θ *are azimuthal and polar angles of the decay products.*

3. Suppose the particle A is at rest. Find the dependence of the differential decay rate $\frac{d\Gamma}{d\cos\theta_{CC}}$ on the variable θ_{CC}, where θ_{CC} is the angle between two C particles in the A rest frame. Ignore the overall constant of proportionality. Discuss the results you obtain in two opposite limits $M_A \gg M_B$ and $M_A \to M_B$.

Problem 3

1. Discuss briefly pure and mixed ensembles in quantum mechanics and the notion of density matrix ρ.

2. Consider the state of a spin 1/2 system described in some basis (say, spin component along the z-axis) by the density matrix:

$$\rho = \begin{pmatrix} \frac{1}{2} & a - \frac{1}{2} \\ a - \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \tag{1}$$

where $0 \leq a \leq 1$. Determine for which values of a the ensemble described by eq. (1) is pure or mixed.

3. Consider now the density matrix

$$\rho = \begin{pmatrix} b & 0 \\ 0 & 1-b \end{pmatrix}, \tag{2}$$

where $0 < b < 1$. As in point 2., determine for which values of b the ensemble described by eq. (2) is pure or mixed.

4. Compute the von Neumann entropy

$$S \equiv -\mathrm{tr}(\rho \log \rho)\,,$$

associated to the ensemble (1) with $a = 1$ and to the ensemble (2) with generic b. Compare and discuss the results obtained.

5. Consider a Hilbert state H composed of the tensor product of two spin 1/2 systems A and B. In H, given a density matrix ρ, we can define reduced density matrices ρ_A and ρ_B by taking partial traces over the subsystems B and A, respectively:

$$\rho_A = \mathrm{tr}_B \rho$$
$$\rho_B = \mathrm{tr}_A \rho\,.$$

Consider the two pure states in H of the form

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle_A \otimes |-\rangle_B - |-\rangle_A \otimes |+\rangle_B\Big)\,,$$
$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle_A \otimes |+\rangle_B - |-\rangle_A \otimes |+\rangle_B\Big)\,.$$

Compute the "entanglement" entropy

$$S_E(\rho_A) = -\mathrm{tr}_{\mathrm{A}}\Big(\rho_A \log \rho_A\Big) \tag{3}$$

associated to both the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$. Verify that the same entropy is obtained by computing (3) with the subsystem A replaced by B.

Problem 4

Consider a free complex relativistic scalar quantum field $\phi(x)$, with lagrangian density

$$\mathcal{L}_0 = (\partial_\mu \phi)^*(\partial^\mu \phi) - m^2(\phi^*\phi) \ ,$$

where $m > 0$.

1. What are the global internal symmetries of the system?

Now consider the lagrangian density

$$\mathcal{L}_1 = \mathcal{L}_0 - \frac{\mu^{4-2n}}{2}\phi^{2n} - \frac{(\mu^*)^{4-2n}}{2}\phi^{*2n} \ ,$$

where μ is a complex parameter and n is a positive integer.

2a. What are the global internal symmetries of the system?

2b. For which values of μ and n is the system stable and in that case what is the tree-level mass spectrum of the one-particle states?

Finally, take $n = 1$ and consider the lagrangian density

$$\mathcal{L}_2 = \mathcal{L}_1 - \lambda(\phi^*\phi)^2 \ ,$$

where $\lambda > 0$, m^2 is now allowed to be positive, negative, or zero. Expanding the field potential around its minimum:

3a. Determine the (tree-level) spectrum of the one-particle states.

3b. Determine the global internal symmetries of the system.