July 2, 2018

**SISSA**
**Entrance**
**Examination**

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**PhD in Theoretical Particle Physics**
**Academic Year 2018/2019**

SOLVE TWO AMONG THE THREE PROBLEMS PRESENTED.

## Problem I

Consider a theory described by the Lagrangian density

$$
\begin{aligned}
\mathcal{L}_{\mathrm{S}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu S|^2 + \frac{M_S^2}{2}S^\dagger S - \lambda_S (S^\dagger S)^2 + \\
+ \bar{\psi}_L \gamma^\mu D_\mu \psi_L + \bar{\psi}_R \gamma^\mu D_\mu \psi_R + (y_\psi \bar{\psi}_L \psi_R S + h.c.) \ ,
\end{aligned}
\tag{1}
$$

where $S$ is a complex scalar field coupled to a $U(1)_X$ gauge field $A_\mu$ with charge $X_S = 1$, $\psi_L$ and $\psi_R$ are chiral fermions (left-handed and right-handed, respectively) with $U(1)_X$ charges $X_L$ and $X_R$. The covariant derivatives are defined as $D_\mu \phi = (\partial_\mu - i g_X X_\phi A_\mu)\phi$, where $\phi$ represents any field and $X_\phi$ its $U(1)_X$ charge. Take $M_S^2, \lambda_S > 0$ and $y_\psi$ real.

1. Discuss the symmetries of the theory and compute the spectrum.
*[Hint: you can work in the unitary gauge, $S(x) = \frac{v_S + s(x)}{\sqrt{2}}$.]*

Let us now introduce a coupling of this theory with the Standard Model (SM) via the following interaction of $S$ with the SM Higgs boson $H$ (often called *Higgs portal*)

$$
\mathcal{L}_{\mathrm{portal}} = -\lambda_{SH}(S^\dagger S)(H^\dagger H) \ , \tag{2}
$$

where $0 < \lambda_{SH} \ll 1$ is a small coupling. Recall that the SM Higgs potential is given by $\mathcal{L}_{\mathrm{SM}} \supset \mu_0^2(H^\dagger H) - \lambda(H^\dagger H)^2$, with $\mu_0^2, \lambda > 0$. Consider the limit where the physical scalar $s$ and the gauge field $A_\mu$ are much heavier than the electroweak scale, but the fermions $\psi_{L,R}$ are instead light ($y_\psi \ll 1$).

2. Discuss the condition for electroweak symmetry breaking to occur.
*[Hint: focus only on the effects of $\mathcal{L}_{\mathrm{portal}}$ on the SM Higgs potential, given the results from point 1.]*

3. For very heavy $s$ and $A_\mu$, derive the lowest-dimension effective operator describing the interactions of the Higgs with the fermions $\psi_L$, $\psi_R$.
*[Hint: integrate out $s$ considering only its interactions with light fields.]*

4. In the unitary gauge one can write $H(x) = (0, \frac{v+h(x)}{\sqrt{2}})^t$, where $v$ is the Higgs vacuum expectation value. Compute the decay width $\Gamma(h \to \bar{\psi}\psi)$.

   *[Hint: in this case the integral of the two-body phase space reduces to* $\frac{1}{8\pi}\left(\frac{2|\vec{p}|}{m_h}\right)$*, where* $\vec{p}$ *is the momentum of either final-state particle in the center of mass frame.]*

   Bonus point: How could this be tested experimentally at the LHC?

Problem II

Consider a one-dimensional quantum-mechanical system of a particle with mass $m$ and wave function

$$\psi(x) = \frac{N}{\cosh(x/L)} \ ,$$

where $L > 0$ and $N$ is the appropriate normalization factor.

1. Show that this particle is in an eigenstate of an Hamiltonian with classical potential

   $$V(x) = -\frac{\hbar^2}{mL^2 \cosh^2(x/L)} \ ,$$

   with energy eigenvalue $E = -\hbar^2/(2mL^2)$.

2. Show that $\langle x \rangle = 0$ and $\langle p \rangle = 0$.

3. Determine $N$ (take it real and positive) and compute $\langle x^2 \rangle$ and $\langle p^2 \rangle$.

4. Show, by explicit computation, that the Heisenberg inequality, is obeyed.

5. Consider now a two-well potential $V_2(x) = V(x-b)+V(x+b)$, each of the weels defined as $V(x)$ above. Assume the two weels to be very far apart, that is $2b >> L$. Call $\psi_1$ and $\psi_2$ the wave functions associated to $V(x-b)$ and $V(x+b)$, respectively, and define

   $$\psi_+ = \psi_1 + \psi_2 \quad , \quad \psi_- = \psi_1 - \psi_2 \ .$$

   Which wave function between $\psi_+$ and $\psi_-$ has minimal energy? Motivate your answer.

In solving the exercise, you may need the following integrals

$$\int_{-\infty}^{\infty} dx \frac{1}{\cosh^2(x)} = 2 \quad , \quad \int_{-\infty}^{\infty} dx \frac{x^2}{\cosh^2(x)} = \frac{\pi^2}{6} \quad , \quad \int_{-\infty}^{\infty} dx \frac{\sinh^2(x)}{\cosh^4(x)} = \frac{2}{3}$$

## Problem III

Consider a real self-interacting scalar field in four dimensions, with action

$$S_M = \int d^4x \left[ -\frac{1}{2}(\partial\phi)^2 - V(\phi^2) \right] \ . \qquad (1)$$

(Here the signature is $- + ++$).

1. Derive the corresponding Euclidean action.

2. Use the background field expansion $\phi = \bar{\phi} + \eta$ and the functional integral to derive the formal expression for the effective action at one loop:

   $$\Gamma(\bar{\phi}) = S(\bar{\phi}) + \frac{1}{2}\mathrm{Tr}\log(\Delta/\mu^2) \ , \qquad (2)$$

   where $\Delta$ is the appropriate Laplace-type operator and the scale $\mu$ has been inserted for dimensional reasons.

   The effective potential $V_{eff}$ is defined by

   $$\Gamma(\bar{\phi}) = \int d^4x V_{eff}(\bar{\phi}) \ ,$$

   where $\bar{\phi}$ is constant.

   Write the effective potential as a momentum integral by making the form of Tr in (2) explicit.

3. Now assume the classical potential

   $$V = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \ .$$

   Using a momentum cutoff (or another regularization, if preferred) calculate the logarithmically divergent part of the effective potential.

4. Define the renormalized quartic coupling as the coefficient of $\phi^4/4!$ in the effective potential.

   - Calculate the beta function, which is the logarithmic derivative of the renormalized coupling with respect to the scale $\mu$. (Assume that the effects of the mass can be neglected.)

- What does this say about the asymptotic behavior of the theory at very large momentum?

In solving the exercise, you may need the following integral

$$\int_0^x du\, u\, \log(u+A) = \frac{1}{4}\left[2\left(x^2 - A^2\right)\log\left(A+x\right) + 2A^2\log(A) + 2Ax - x^2\right]$$