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SISSA
Entrance
Examination

PhD in Theoretical Particle Physics
Academic Year 2018/2019

SOLVE TWO AMONG THE FOUR PROBLEMS PRESENTED.

Problem I

Consider the quantum mechanical system of a point particle on a plane which can exist in two spin states. This is described by its position and its spin. The spin states $\{|0>, |1>\}$ are flipped by fermionic creation/annihilation operators as

$$\psi|0>=|1>\,, \qquad \psi^\dagger|1>=|0>\,,$$

where $\psi^\dagger\psi+\psi\psi^\dagger=1$, $\psi^2=0$ and $(\psi^\dagger)^2=0$.

Let us collect the position variables X_1 and X_2 in the complex combination $X=X_1+iX_2$ and let P be the momentum variable canonically conjugated to X, that is $[P,X]=-i$.

Define the fermionic operators

$$Q=\psi\cdot(P+imX^\dagger)\,, \qquad Q^\dagger=\psi^\dagger\cdot(P^\dagger-imX)$$

and the Hamiltonian of the particle as $H=QQ^\dagger+Q^\dagger Q=\left\{Q,Q^\dagger\right\}$. The parameter m is a non zero real number.

1 Show that the rotation $X\to e^{i\phi}X$, $P\to e^{-i\phi}P$, $\psi\to e^{i\phi}\psi$ and $\psi^\dagger\to e^{-i\phi}\psi^\dagger$ is a symmetry of the system, that is write the explicit form of the generator J and check that it commutes with the Hamiltonian.

2 Define the hermitean operator $F=\psi\psi^\dagger$. Show that it is conserved, rotation invariant and determine its spectrum.

3 Define the Grassmann parity $(-1)^F$ and work out its action on the states $|0>$ and $|1>$. Show that for every eigenstate of the Hamiltonian with non zero eigenvalue, there exists another eigenstate with the same eigenvalue and opposite Grassmann parity. Show that the correspondence within bosonic and fermionic non-zero eigenstates is one-to-one.

[*Hint: Do the operators Q and $Q^\dagger$ commute with the Hamiltonian?*]

4 Assuming $m>0$ and $\kappa<0$, compute

$$I=\mathrm{Tr}\left[(-1)^F e^{-\beta H}e^{\kappa J}\right]\,.$$

[*Hints: There is no need to work out the explicit form of H. Not all states in the Hilbert space contribute to I, but those that do are still infinite in number.*]

Problem II

Consider a quantum mechanical one-dimensional non-relativistic system consisting of a particle with mass m in a potential $V(x)$. Suppose that

$$\langle x|\psi_k\rangle = \psi_k(x) = \frac{ika - \tanh(x/a)}{ika + 1} \frac{e^{ikx}}{\sqrt{2\pi}}$$

is an eigenstate of the Hamiltonian for each $k \in \mathbb{R}$, and note that

$$\int_{-\infty}^{+\infty} \psi_k^*(x)\psi_{k'}(x)\, dx = \delta(k - k') \ .$$

1. Determine the potential $V(x)$.
2. Compute explicitly
$$\int_{-\infty}^{+\infty} \langle x|\psi_k\rangle \langle\psi_k|y\rangle\, dk \ . \tag{1}$$
Is the set $\{\psi_k : k \in \mathbb{R}\}$ a basis for the Hilbert space?
[*Hint: The integral can be computed using the residue theorem.*]
3. Show that the system admits exactly one bound state, and determine its wave function $\psi_0(x)$ and energy E_0.
[*Hint: Use the result for eq. (1) to guess the form of $\psi_0(x)$.*]

Problem III

Consider a theory described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + |D_\mu\phi|^2 + |D_\mu\chi|^2 + \lambda_1\left(|\phi|^2 - \frac{v_1^2}{2}\right)^2 + \lambda_2\left(|\chi|^2 - \frac{v_2^2}{2}\right)^2, \quad (1)$$

where ϕ, χ are complex scalar fields coupled to a $U(1)$ gauge field A_μ. The covariant derivatives are defined as follows:

$$\begin{aligned} D_\mu\phi &\equiv (\partial_\mu - ieA_\mu)\phi\,, \\ D_\mu\chi &\equiv (\partial_\mu - ieA_\mu)\chi\,. \end{aligned} \quad (2)$$

1 Consider the limit $e \to 0$ of the theory: what are the symmetries of the action? What will be the spectrum of the particles and the structure of the symmetry breaking?

2 Consider now $e \neq 0$. Identify the mass eigenstates after the spontaneous symmetry breaking and the Goldstone boson "eaten" by the gauge field. Are there remaining massless states and to what do they correspond?

3 Consider the limit $\lambda_1 \ll e, \lambda_2 \ll e$. Find the ratio of the decay width of the two Higgs scalars.

Problem IV

Consider a neutral scalar field φ with Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\,\dot{\varphi}^2 - \frac{1}{2M^2}(\Delta\varphi)^2 - \frac{m^2}{2}\,\varphi^2 - \frac{g}{4!}\,\varphi^4\,, \tag{1}$$

where $\dot{\varphi} \equiv \partial\varphi/\partial x_0$ and $\Delta = \sum_{i=1}^{3}(\partial/\partial x_i)^2$ is the Laplace operator.

1 Discuss the space-time and global symmetries of this scalar theory.

2 Derive the Feynman propagator in momentum space associated to this theory when $g = 0$.

3 Draw the Feynman diagrams that give rise to the one-loop correction to the coupling constant g.

4 Setting the momenta of all external particles to zero (off-shell configuration), show that the one-loop correction in item 3 is finite.

5 Generalizing the theory in d spatial dimensions, find the lowest value of d where the one-loop integral in item 4 diverges.