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SISSA
Entrance
Examination

PhD in Theoretical Particle Physics
Academic Year 2017/2018

SOLVE TWO AMONG THE FOUR PROBLEMS PRESENTED.

Problem I

Consider a quantum harmonic oscillator in one spatial dimension with Hamiltonian (in natural units with $\hbar = 1$)

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 + \lambda q^{2n} \,, \tag{1}$$

with n a positive integer and $\lambda > 0$.

1. Using time-independent perturbation theory, compute at first order in λ the correction $\Delta E_n^{(0)}$ to the ground state energy $E_n^{(0)}$.

2. Determine the effective dimensionless expansion parameter associated to the above perturbative expansion.

3. Write the Lagrangian L associated to the system (1) and, with an appropriate rescaling, cast it in a form where the kinetic term is normalized to 1/2 (like in usual conventions used in relativistic quantum field theory). Then, through a Wick rotation in time, pass to its euclidean version L_E.

4. Consider the partition function

$$Z(\lambda) = \int_{q(0)=q(\beta)} \mathcal{D}q(\tau) \exp\Big(- \int_0^\beta d\tau \, L_E[q(\tau)]\Big) \,, \tag{2}$$

where we integrate over all configurations that are periodic over a time β (inverse temperature). Write a formula for the ground state energy $E_n^{(0)}$ in terms of $Z(\lambda)$.

5. Focus now on the case $n = 2$. Draw the Feynman diagram corresponding to the leading order correction in λ to the ground state energy $E_{n=2}^{(0)}$. Knowing the partition function for the unperturbed quadratic harmonic oscillator

$$Z(\lambda = 0) = \frac{1}{2\sinh\left(\frac{\omega\beta}{2}\right)} \,, \tag{3}$$

and the unperturbed propagator $G(\tau, \tau')$ at equal times

$$G(\tau, \tau) = \frac{1}{2\omega} \coth \frac{\omega\beta}{2} \,, \tag{4}$$

calculate the value of this Feynman diagram and rederive the correction $\Delta E_{n=2}^{(0)}$ to the ground state energy $E_{n=2}^{(0)}$ computed in point 1. Show that the two computations match.

[Hint: possible useful relations: $\Gamma(z) = \int_0^\infty dt \, e^{-t} t^{z-1}$, $\Gamma(1/2) = \sqrt{\pi}$*]*

Problem II

Consider a statistical ensemble of $N \times N$ real symmetric matrices ϕ with probability distribution $P(\phi) = \frac{1}{Z} e^{-N \operatorname{Tr} V(\phi)}$, where V is an even polynomial. The normalization factor

$$Z = \int (d\phi)\, e^{-N \operatorname{Tr} V(\phi)}$$

can be seen as the partition function of a field theory in $0+0$ dimensions (no space and no time).

1. Discuss the symmetries of this system.

2. Every symmetric matrix can be decomposed as $\phi = O^T \Lambda O$ where Λ is diagonal, with the eigenvalues of ϕ as entries, and O is orthogonal. We can use Λ and O as new degrees of freedom. Discuss the virtue of this parametrization for Z. Do you see an analogy with other systems?

3. Argue that the Jacobian determinant of the transformation $\phi \to \{\Lambda, O\}$ has the form
$$\det J = \Pi_{a<b}(\lambda_a - \lambda_b)\,.$$
[*Hint: define a metric* $\operatorname{Tr}(\delta\phi\delta\phi)$ *on the space of symmetric matrices. In the basis* $\{\delta\lambda_a, R_{ab}\}$, *with* $R_{ab} = \delta O_{ac} O^T_{cb}$, *it is diagonal. The determinant of this metric is the square of* $\det J$.]

   Rewrite Z as an integral over the eigenvalues. Bring the Jacobian into the exponent and draw an analogy to the partition function of a one-dimensional gas.

4. Now consider the Gaussian matrix models with $V(\phi) = \frac{1}{2} m^2 \phi^2$. The path integral can be evaluated by the method of steepest descent. Derive the equation for the eigenvalues.
In the limit $N \to \infty$ it can be approximated by a continuous equation
$$\frac{1}{2} m^2 \lambda = \Re e\, G(\lambda + i\epsilon)\ ,$$
where
$$G(z) = \int d\mu \frac{\rho(\mu)}{z - \mu}\ .$$

Determine the spectral density ρ (it is known as Wigner's semicircle distribution).

[*Hint: the function* G *is analytic in* z *with a cut on an interval* $(-a, a)$, *it is real on the real axis outside the cut and it must behave like* $1/z$ *for large* z *(why?). Based on this information, one can guess that it has the form* $G(z) = Cz + B\sqrt{z^2 - a^2}$. *Fix the constants. The spectral density is then given by the discontinuity of* G *across the cut.*]

Problem III

The Standard Model (SM) of particle physics predicts the following ratio

$$\rho \equiv \frac{M_W^2}{cos^2\theta_W M_Z^2}$$

to be exactly one at tree level. M_W and M_Z are the masses of the charged and neutral gauge bosons, W and Z, and θ_W is the Weinberg angle, $\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$, with g and g' being the gauge couplings of the $SU(2)_L \times U(1)_Y$ gauge symmetry.

1. The Lagrangian describing the interactions between the Higgs field and the gauge boson sector is

$$\mathcal{L}_H = |D_\mu H|^2 + m^2|H|^2 - \lambda(H^\dagger H)^2\,, \tag{1}$$

where

$$D_\mu = \partial_\mu - ig\frac{W_\mu^a\sigma^a}{2} - ig'B_\mu Y$$

is the $SU(2)_L \times U(1)_Y$ covariant derivative and H is the Higgs field, complex doublet of the $SU(2)_L$ symmetry with hypercharge $Y = \frac{1}{2}$. Starting from $\mathcal{L}_H$, prove that $\rho = 1$ at tree level, that is $\rho^{SM}_{tree} = 1$.

[Hint: It might help to write the Higgs field in components, namely

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{2}$$

and assume that in the Standard Model vacuum only ϕ_3 gets a non-vanishing vacuum expectation value.]

2. The condition $\rho^{SM}_{tree} = 1$ is related to the fact that in the "gaugeless" limit $g, g' \to 0$ the global symmetry of the Higgs Lagrangian (1) is larger than $SU(2)_L \times U(1)_Y$. By looking at the Lagrangian (1) in this limit, find such different global symmetry breaking pattern.

[Hint: use again the component expression (2) for the Higgs field.]

3. Suppose the existence of an electroweak scalar triplet field T with hypercharge $Y_T = -1$ added to the SM:

$$T = \begin{pmatrix} t^0 & t^-/\sqrt{2} \\ t^-/\sqrt{2} & t^{--} \end{pmatrix},$$

where the superscript indicates the electric charge of the various complex field components of T. The Lagrangian density associated to this field is

$$\mathcal{L}_T = \mathrm{tr}\,|D_\mu T|^2 - m_T^2\,\mathrm{tr}\,|T|^2 + \Lambda T H^2 + \Lambda (T H^2)^*,$$

where

$$D_\mu T = \partial_\mu T - ig\hat{W}_\mu T - igT\hat{W}^T - ig' Y_T B_\mu T, \quad \hat{W}_\mu \equiv \frac{W_\mu^a \sigma^a}{2},$$
$$TH^2 \equiv T_{ab}\epsilon^{ac}\epsilon^{bd} H_c H_d,$$

and m_T, Λ are new mass scales beyond the SM. For simplicity, we have considered only trilinear interactions between the Higgs doublet H and the triplet field T.

Find the minimum of the scalar potential arising from $\mathcal{L}_T + \mathcal{L}_H$ and the corresponding mass spectrum of the vector bosons W and Z. Using the constraints from $\Delta\rho$, find the bound on the triplet vacuum expectation value, and the constraints on the m_T^2/Λ ratio.

[Hints: Consider only the potential associated to one electrically neutral field component. The Higgs VEV is of order $v \sim 250$ GeV.]

Problem IV

An interesting class of field theories are those invariant under the following transformation

$$x^\mu \to x'^\mu = \lambda\, x^\mu \quad \text{and} \quad \phi(x) \to \phi'(x') = \lambda^{-\Delta}\, \phi(x)\ , \tag{1}$$

where λ is a real positive number, x^μ are space-time coordinates and $\phi(x)$ is a field. The quantity Δ is called *scaling dimension* of the field ϕ.

Consider the following action for a real scalar field in $(n+1)$-dimensions

$$S[\phi] = \int d^{n+1}x \left(-\frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - g\, \phi^r \right)\ , \tag{2}$$

where $r \neq 2$.

1. Find the scaling dimension Δ such that the derivative part of the action, i.e. the action (2) with $m = g = 0$, is invariant under the transformation (1).

2. Determine the value of m and r such that the scaling transformation (1) is a symmetry of the *full* action.

3. Focus on $(3+1)$-dimensions, that is $n = 3$, and take the values of m and r previously found. Using Noether theorem, find the corresponding current j^μ and check explicitly that it is conserved.

4. What do you expect to happen once quantum corrections are taken into account? Motivate your answer.