July 06, 2016

SISSA
Entrance
Examination

PhD in Theoretical
Particle Physics

Solve two out of the five exercises given below.

PROBLEM 1

Consider the quantum system of a particle on the real line and let (p, q) be the standard phase space variables. Let $H_0(p, q) = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} q^2$ and consider the Hamiltonian

$$H_{\alpha,\beta}(p, q) = H_0(p, q) + \alpha q + \frac{\beta}{2}\left(q^2 p + p q^2\right) + \frac{m\beta^2}{2} q^4,$$

where α and β are real parameters.

- Show that the systems $H_{\alpha,\beta}(p, q)$ and $H_{\alpha,\beta'}(p, q)$ for $\beta \neq \beta'$ are unitarily equivalent.
- Compute the diffusion kernel $K\left(x_f, x_i; T\right) = \langle x_f | e^{-iTH} | x_i \rangle$ of the theory defined by the Hamiltonian $H_{\alpha,\beta}(p, q)$.
- Compute the partition function and the energy spectrum of the theory.

Problem 2

We have a Quantum Mechanical system with three degrees of freedom represented by canonical operators q_a, p_b with commutation relations

$$[q_a, p_b] = i\,\delta_{ab}, \quad a, b = 1, 2, 3, \qquad \text{(we set } \hbar = 1\text{)},$$

where each coordinate q_a takes values in $\mathbb{R}$. The dynamics is governed by the Hamiltonian operator

$$H = \frac{1}{2m}\,p_1^2 + \frac{1}{4m}\,p_2^2 + \frac{1}{6m}\,p_3^2 + \frac{k^2}{2}\big(q_1^2 + 2\,q_2^2 + 3\,q_3^2\big),$$

acting on the Hilbert space $\mathscr{H} \equiv L^2(\mathbb{R}^3)$. m and k^2 are positive parameters.

1. Identify the Lie group G of the continuous symmetries of H which act linearly on the canonical operators q_a, p_b.

2. Show that the states at each energy level form an *irreducible* representation of G.

3. Which irreducible representations of G appear in $\mathscr{H}$? How many copies of each irreducible representation are present in $\mathscr{H}$?

4. Consider the perturbed Hamiltonian

$$H_\lambda = H + \lambda\big(q_1^2 p_2^2 + 4 q_2^2 p_1^2 - 2 q_1 p_2 q_2 p_1 - 2 q_2 p_1 q_1 p_2\big).$$

   4a. Which Lie subgroup $G' \subset G$ remains a good symmetry for $\lambda \neq 0$?

   4b. Show that the perturbed energy eigenstates $H_\lambda|E\rangle = E|E\rangle$ are *uniquely* labelled by the eigenvalues of the maximal commuting subset of generators of G'.

   4c. Write the expression of the energy E in terms of the eigenvalues in question 4b.

Problem 3

Consider the following Euclidean Lagrangian in two dimensions

$$\mathcal{L}_E = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^\dagger D^\mu \phi + \frac{e^2}{2}\big(|\phi|^2 - \zeta\big)^2$$

with $\zeta > 0$, for a complex scalar field ϕ and an Abelian gauge field A_μ. The covariant derivative is $D_\mu \phi = (\partial_\mu - iA_\mu)\phi$.

1. Write the equations of motion.

2. List the symmetries of the theory, continuous and discrete. For the continuous symmetries, write the currents and verify that they are conserved.

3. We are interested in field configurations with finite Euclidean action (instantons). Show that the Euclidean action of a configuration is bounded by its topological number:

   $$S \geq 2\pi\zeta\,|k| \;, \qquad k \equiv \frac{1}{2\pi}\int_{\mathbb{R}^2} F \;,$$

   and that among all configurations in the same topological class (*i.e.* with the same topological number k), those that saturate the bound are all and only the solutions to the following "vortex equations":

   $$F_{12} \pm e^2\big(|\phi|^2 - \zeta\big) = 0 \;, \qquad\qquad D_1\phi \pm iD_2\phi = 0 \;, \tag{1}$$

   where the upper (lower) sign is for $k > 0$ ($k < 0$). Solutions are called "vortices".

   Do vortices also solve the equations of motion?

4. Deform the Lagrangian by the term

   $$\delta\mathcal{L}_E = i\frac{\Theta}{4\pi} F_{\mu\nu}\epsilon^{\mu\nu} \;,$$

   where Θ is a real parameter. a) How does Θ affects the equations of motion? b) How does it affect the vortices? c) How does it affect the partition function of the theory? d) Is there a transformation of Θ that leaves the quantum theory invariant?

5. The vortex equations (1) cannot be solved analytically. Consider $k \geq 0$. Write an ansatz for solutions with rotational symmetry, and study the behavior of those solutions at $\rho \to 0$ and $\rho \to \infty$.

   Estimate the size of such vortices with rotational symmetry.

Problem 4

Consider the theory of the electron ($\psi_{(e)}$) and the muon ($\psi_{(\mu)}$) interacting with the photon (A_μ) and an extra massive spin-1 field (A'_μ) described by the following Lagrangian

$$\mathcal{L} = \mathcal{L}_{QED} + \mathcal{L}' \,,$$

where

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}_{(e)}(i\not{D} - m_{(e)})\psi_{(e)} + \bar{\psi}_{(\mu)}(i\not{D} - m_{(\mu)})\psi_{(\mu)} \,,$$

the covariant derivative $D_\mu \equiv \partial_\mu - igA_\mu$ and

$$\mathcal{L}' = -\frac{1}{4}{F'_{\mu\nu}}^2 + \frac{\kappa}{2}F^{\mu\nu}F'_{\mu\nu} + \frac{1}{2}M^2(A'_\mu)^2 + g'A'_\mu(\bar{\psi}_{(\mu)}\gamma^\mu\psi_{(e)} + h.c.) \,.$$

1. Find for which values of the parameters the massive vector decays.
2. Find for which values of the parameters the muon can decay to an electron and the massive vector. Compute the corresponding decay rate $\Gamma_{\mu\to e\gamma'}$ at leading order in perturbation theory.
3. Take the limit $M \to 0$ of the ratio $\Gamma_{\mu\to e\gamma'}/m_{(\mu)}$ and explain the meaning of the result.

Problem 5

Consider the following Lagrangian for a scalar field π coupled to a static point-like source

$$\mathcal{L} = \frac{1}{2}(\partial\pi)^2 - \frac{1}{\Lambda^3}(\partial\pi)^2\Box\pi + \alpha\pi\delta^3(\vec{x}) \tag{2}$$

1. Show how the above Lagrangian transforms under the transformation $\pi \to \pi + b_\mu x^\mu$, where b_μ is an arbitrary constant 4-vector and discuss the physical consequences of the result.
2. Discuss at what distance from the source the non-linearities of the π field become important in the spherically symmetric solution. Consider $\alpha >> 1$.
3. Compute the field $E(r)$ generated by the point-like source in the spherically symmetric solution, where $\vec{E} \equiv \vec{\nabla}\pi \equiv \hat{r}E(r)$.
4. Set $\alpha = 0$. Explain why the operators generated at the quantum level in the theory (2) must have at least two derivatives for each π. What does it change for $\alpha > 0$?